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Solving a System of Nonlinear Fractional Partial Differential Equations Using the Sinc-Muntz Collocation Method

Mahmood Shareef Ajeel, Morteza Gachpazan * and Ali Reza Soheili

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran.

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Abstract: We present a new numerical method for solving a system of nonlinear fractional partial differential equations (SNFPDEs). This technique is based on the Sinc functions and the fractional Muntz-Legendre polynomials together with the collocation method. The proposed approximation reduces the solution of the SNFPDEs to the solution of a system of nonlinear algebraic equations. In some numerical examples, we show that approximate solutions also agree with exact solutions.

Keywords: *Sinc functions; fractional Muntz-Legendre polynomials; fractional partial differential equations; collocation method; Caputo fractional derivative.*

Mathematics Subject Classification (2010): 26A33, 34A08.

1 Introduction

Fractional partial differential equations (FPDEs) are used in many physical models and engineering research, see [1–3]. Recently, several numerical techniques have been proposed by researchers for solving the FPDEs. For example, Chen, Sun, and Liu [4] used the generalized fractional-order Legendre function for solving FPDEs. Al-Khaled [5] used the Sinc–Legendre collocation method for the non-linear Burger's fractional equation. Abbasbandy et al. [6] applied an operational matrix of fractional-order Legendre functions for solving the time-fractional convection-diffusion equation. Other numerical methods can be found in [7–11].

In this paper, we apply a numerical method for solving a system of nonlinear fractional partial differential equations (SNFPDEs) of the following form:

* Corresponding author: mailto:gachpazan@um.ac.ir

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a\dfrac{\partial v(x,t)}{\partial x} + bv^p(x,t) + cu(x,t) = g(x,t), \\ \dfrac{\partial^\beta v(x,t)}{\partial t^\beta} + d\dfrac{\partial u(x,t)}{\partial x} + eu^q(x,t) + fv(x,t) = h(x,t), \end{cases} \tag{1}$$

with the conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad u(0,t) = u_1(t), \qquad v(0,t) = v_1(t), \tag{2}$$

where $p, q \in \mathbb{N}$, $(x,t) \in \Omega = (0,1)\times(0,1)$, and $0 < \alpha, \beta \leq 1$ are the order of the fractional derivatives in the Caputo sense, the continuous functions g and h are known, and the functions $u(x,t)$ and $v(x,t)$ are unknown and should be determined.

The aim of this paper is to apply the Sinc functions and Muntz–Legendre polynomials to achieve the numerical solution of system (1).

This paper is organized as follows. The review of the Caputo fractional derivative and review of the fractional Muntz–Legendre polynomials are presented in Section 2. In Section 3, we recall the notation of the Sinc functions and their properties. In Sections 4 and 5, we discuss the convergence analysis and the approximate solution of the SNFPDEs based on the Sinc functions and Muntz–Legendre polynomials using the collocation method. In Section 6, we present some examples of the SNFPDEs to show efficiency and accuracy of the proposed method. Finally, a conclusion is expressed in Section 7.

2 Preliminaries and Notation

In this section, we give the definition and some properties of the Caputo fractional derivative and fractional-order Muntz–Legendre polynomials.

2.1 Review of the Caputo fractional derivative

Definition 2.1 The fractional derivative of $y(t)$ in the Caputo sense is defined as

$$D_*^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau$$

for $m-1 < \alpha < m$, $m \in \mathbb{N}$ and $t > 0$.

Definition 2.2 Let $\alpha > 0$. The Riemann–Liouville fractional integral operator J_t^α is defined on $L_1[a,b]$ by

$$J_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau) d\tau.$$

Some properties of the Riemann–Liouville fractional integral operator J_t^α and the Caputo fractional derivative operator D_*^α, which will be used later, are as follows:

1) $D_*^\alpha C = 0$, where C is a constant.

2)
$$D_*^\alpha t^v = \begin{cases} \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & v \in \mathbb{N}_0,\ v \geq \lceil\alpha\rceil, \quad or \quad v \in \mathbb{N},\ v > \lfloor\alpha\rfloor, \\ 0, & v \in \mathbb{N}_0,\ v < \lceil\alpha\rceil, \end{cases} \tag{3}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α, and $\lfloor\alpha\rfloor$ is the largest integer less than or equal to α. Also $\mathbb{N}_0 = \{0, 1, \dots\}$.

3) The Caputo fractional derivative is a linear operation,

$$D_*^\alpha\Big(\sum_{i=1}^{n} a_i y_i(t)\Big) = \sum_{i=1}^{n} a_i D_*^\alpha y_i(t).$$

4)
$$J_t^\alpha(J_t^\beta y(t)) = J_t^\beta(J_t^\alpha y(t)) = J_t^{\alpha+\beta} y(t), \qquad \alpha, \beta > 0.$$

5)
$$J_t^\alpha t^v = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)} t^{\alpha+v}.$$

6)
$$D_*^\alpha(J_t^\alpha y(t)) = y(t).$$

7)
$$J_t^\alpha(D_*^\alpha y(t)) = y(t) - \sum_{i=0}^{n-1} y^{(i)}(0)\frac{t^i}{i!}, \qquad n-1 < \alpha \le n, \ \ t > 0.$$

For more details about the properties of the Caputo fractional derivative operator and Riemann–Liouville fractional integral operator see [2].

2.2 Review of the fractional-order Muntz polynomials

Definition 2.3 (see [6]) The fractional-order Muntz–Legendre polynomials on the interval $[0, T]$ are represented by the formula

$$L_n(t;\alpha) = \sum_{k=0}^{n} C_{n,k}\big(\frac{t}{T}\big)^{k\alpha}, \tag{4}$$

where

$$C_{n,k} = \frac{(-1)^{n-k}}{\alpha^n k!(n-k)!} \prod_{v=0}^{n-1}\big((k+v)\alpha + 1\big).$$

The function $L_k(t;\alpha)$, $k = 0, 1, \ldots, n$, forms an orthogonal basis for $M_{n,\alpha} = \mathrm{Span}\{1, t^\alpha, \ldots, t^{n\alpha}\}$, $t \in [0, T]$. Also it satisfies

$$\begin{aligned} L_0(t;\alpha) &= 1, \\ L_1(t;\alpha) &= \big(\frac{1}{\alpha}+1\big)\big(\frac{t}{T}\big)^\alpha - \frac{1}{\alpha}, \\ b_{1,n}L_{n+1}(t;\alpha) &= b_{2,n}(t)L_n(t;\alpha) - b_{3,n}L_{n-1}(t;\alpha), \end{aligned}$$

where

$$\begin{aligned} b_{1,n} &= a_{1,n}^{0,\frac{1}{\alpha}-1}, \ b_{2,n}(t) = a_{2,n}^{0,\frac{1}{\alpha}-1}\big(2(\frac{t}{T})^\alpha - 1\big), \ b_{3,n} = a_{3,n}^{0,\frac{1}{\alpha}-1}, \\ a_{1,n}^{\alpha,\beta} &= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta), \\ a_{2,n}^{\alpha,\beta}(x) &= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2], \\ a_{3,n}^{\alpha,\beta} &= 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2). \end{aligned}$$

Theorem 2.1 *Let $L_n(t;\alpha)$ be the fractional-order Muntz–Legendre polynomials; then we have the following Caputo fractional derivative of the functions $L_n(t;\alpha)$:*

$$D_*^\alpha L_n(t;\alpha) = \sum_{k=1}^{n} D_{n,k} \big(\frac{t}{T}\big)^{(k-1)\alpha}, \tag{5}$$

where

$$D_{n,k} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha-\alpha)T^\alpha} C_{n,k},$$

and $C_{n,k}$ is defined in $L_n(t;\alpha)$.

Proof. It is a result of equations (3) and (4).

Theorem 2.2 *Let $\alpha > 0$ be a real number and let $t \in [0,1]$. Then*

$$L_n(t;\alpha) = P_n^{(0,\frac{1}{\alpha}-1)}(2t^\alpha - 1),$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials with parameters $\alpha, \beta > -1$, see [12, 13].

Proof. See [14].

3 Sinc Function and its Properties

In this section, we recall the notation and properties of the Sinc function and derive useful formulas that will be used in this paper. The Sinc function is defined on $\mathbb{R}$ as (see [15])

$$\text{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Let $g(x)$ be a function defined on $\mathbb{R}$, and let $h > 0$ be a step size. Consider the Whittaker cardinal function of g defined by the series

$$C(g,h)(x) = \sum_{k=-\infty}^{\infty} g(kh)\, \text{Sinc}\big(\frac{x-kh}{h}\big).$$

This series converges (see [15]), and the kth Sinc function is defined on $\mathbb{R}$ as

$$S(k,h)(x) = \text{Sinc}\big(\frac{x-kh}{h}\big).$$

Now, for the positive integer N, the function g can be approximated by truncating as follows:

$$C_N(g,h)(x) = \sum_{k=-N}^{N} g(kh)\, \text{Sinc}\big(\frac{x-kh}{h}\big).$$

The properties of the Whittaker cardinal expansion have been extensively studied in [15]. These properties are derived in the infinite strip D_S-plane of the complex ω-plane, where, for $d > 0$,

$$D_S = \{w = u + iv : |v| < d \leq \pi/2\}.$$

To construct approximations on the interval $[a, b]$, which are used in this paper, the eye-shaped domain in the z-plane (see [15]),

$$D_E = \{z = u + iv : |arg(\frac{x-a}{b-x})| < d \leq \pi/2\},$$

is mapped conformally onto the infinite strip D_S via

$$\omega = \psi(z) = ln(\frac{x-a}{b-x}).$$

The basic functions on $[a, b]$ are taken to be the translated Sinc functions

$$S_k(x) \equiv S(k,h) \circ \psi(x) = \text{Sinc}(\frac{\psi(x) - kh}{h}), \tag{6}$$

where $S(k, h) \circ \psi(x)$ is defined by $S(k, h)(\psi(x))$. The inverse map of $\omega = \psi(z)$ is

$$z = \psi^{-1}(\omega) = \frac{a + be^{\omega}}{1 + e^{\omega}}.$$

Thus we may define the inverse images of the real line and of the evenly spaced nodes

$$x_k = \psi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Definition 3.1 (see [16]) Let $B(D_E)$ be the class of functions g that are analytic in D_E and satisfy

$$\int_{\psi^{-1}(x+L)} |g(z)|dz \to 0, \quad x \to \pm\infty,$$

where

$$L = \{iy : |y| < d \leq \pi/2\},$$

and those on the boundary of D_E satisfy

$$\int_{\partial D_E} |g(z)|dz < \infty.$$

4 Convergence Analysis

The following expressions show that the Sinc interpolation on $B(D_E)$ converges exponentially.

Theorem 4.1 (see [15, 16]) *Assume that $g\psi' \in B(D_E)$; then, for all x in $[a, b]$,*

$$|g(x) - \sum_{k=-\infty}^{\infty} g(kh)S(k,h) \circ \psi(x)| \leq \frac{2N(g\psi')}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|g(x)| = Ce^{-\gamma|\psi(x)|}, x \in \Gamma$, for some positive constants C and γ and the selection $h = \sqrt{\pi d/\gamma N} \leq 2\pi d/ln(2)$, then

$$|\frac{d^n g(x)}{dx^n} - \sum_{k=-N}^{N} g(kh)\frac{d^n}{dx^n} S(k,h) \circ \psi(x)| \leq kN^{(n+1)/2} e^{-\sqrt{\pi d \gamma N}}$$

for all $n = 0, 1, 2, \ldots, m$.

Also, the nth derivative of the function g at some points x_k can be approximated (see [17]) as follows:

$$\delta^{(0)}_{k,j} = [S(k,h) \circ \psi(x)]|_{x=x_j} = \delta_{k,j},$$

where

$$\delta_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

It has been shown that

$$\delta^{(1)}_{k,j} = \frac{d}{d\psi}[S(k,h) \circ \psi(x)]|_{x=x_j} = \frac{1}{h} \begin{cases} 0, & j = k, \\ \frac{(-1)^{(j-k)}}{j-k}, & j \neq k. \end{cases}$$

So the approximate of a function $u(x)$ by the Sinc expansion is

$$u_N(x,t) \simeq \sum_{i=-N}^{N} c_i S_i(x), \tag{7}$$

where $S_i(x)$ is defined in equation (6). Now, for arbitrary fixed $t_j \in (0,1)$, we define $u(x_k) = u(x_k, t_j)$. Then, to approximate the first derivative at the Sinc nodes x_k, we have

$$\begin{aligned} \frac{\partial u_{N,n}(x_k,t_j)}{\partial x} &= \frac{du(x_k)}{dx} = \frac{du_N(x_k)}{dx} + E_1 = \sum_{i=-N}^{N} c_i \big(\frac{d}{dx}[S_i(x)]\big)_{x=x_k} + E_1 \qquad (8) \\ &= \sum_{i=-N}^{N} c_i \big(\frac{d}{d\psi}[S(i,h) \circ \psi(x)] \frac{d\psi}{dx}\big)_{x=x_k} + E_1 \\ &= \sum_{i=-N}^{N} c_i \delta^{(1)}_{i,k} \frac{d\psi(x_k)}{dx} + E_1, \end{aligned}$$

where

$$E_1 = O(Ne^{-\sqrt{\pi d \gamma N}}).$$

5 Approximate Solution to the S-N-FPDEs

In this section, we approximate the solution of equation (1) by applying the Sinc function and fractional Muntz–Legendre polynomials, which are discussed in the previous sections.

First, we approximate the unknown functions $u(x,t)$ and $v(x,t)$ as follows:

$$u_{N,n}(x,t) \simeq \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} S_i(x) L_j(t;\lambda), \tag{9}$$

$$v_{N,n}(x,t) \simeq \sum_{i=-N}^{N} \sum_{j=0}^{n} b_{ij} S_i(x) L_j(t;\lambda), \tag{10}$$

where $S_i(x)$ and $L_j(t;\lambda)$ are defined in equations (6) and (4), respectively. Also, λ is the parameter such that $\alpha = k_1\lambda, \beta = k_2\lambda$, and k_1, k_2 are the smallest natural numbers.

Moreover, let x_k be the Sinc collocation points. Then we approximate the differential $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial v(x,t)}{\partial x}$, $\dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$, and $\dfrac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}$ as follows:

$$\begin{aligned}\frac{\partial u_{N,n}(x_k,t)}{\partial x} &= \sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}\big(\frac{d}{dx}[S_i(x)]\big)_{x=x_k} L_j(t;\lambda) \\ &= \sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}\big(\frac{d}{d\psi}[S(i,h)\circ\psi(x)]\frac{d\psi}{dx}\big)_{x=x_k} L_j(t;\lambda) \\ &= \sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}\delta^{(1)}_{i,k}\frac{d\psi(x_k)}{dx} L_j(t;\lambda),\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial v_{N,n}(x_k,t)}{\partial x} &= \sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij}\big(\frac{d}{dx}[S_i(x)]\big)_{x=x_k} L_j(t;\lambda) \\ &= \sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij}\big(\frac{d}{d\psi}[S(i,h)\circ\psi(x)]\frac{d\psi}{dx}\big)_{x=x_k} L_j(t;\lambda) \\ &= \sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij}\delta^{(1)}_{i,k}\frac{d\psi(x_k)}{dx} L_j(t;\lambda),\end{aligned} \tag{12}$$

and

$$\frac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} = \sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij} S_i(x) D^{\alpha}_{*} L_j(t;\lambda), \tag{13}$$

$$\frac{\partial^{\beta} v_{N,n}(x_k,t)}{\partial t^{\beta}} = \sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij} S_i(x) D^{\beta}_{*} L_j(t;\lambda), \tag{14}$$

where D^{α}_{*} and D^{β}_{*} are defined in Theorem 2.1.

Substituting equations (9)–(14) into equation (1) and the condition (2), we get

$$\begin{cases} \dfrac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} + a\dfrac{\partial v_{N,n}(x_k,t)}{\partial x} + b v^{p}_{N,n}(x_k,t) + c u_{N,n}(x_k,t) = g(x,t), \\ \dfrac{\partial^{\beta} v_{N,n}(x_k,t)}{\partial t^{\beta}} + d\dfrac{\partial u_{N,n}(x_k,t)}{\partial x} + e u^{q}_{N,n}(x_k,t) + f v_{N,n}(x_k,t) = h(x,t), \end{cases} \tag{15}$$

with the conditions

$$u_{N,n}(x,0) = u_0(x), \quad v_{N,n}(x,0) = v_0(x), \quad u_{N,n}(0,t) = u_1(t), \quad v_{N,n}(0,t) = v_1(t). \tag{16}$$

Now, to find the unknown coefficients a_{ij} and b_{ij} in equations (15) and (16), we use the collocation method with suitable collocation points (x_k, t_r), where $x_k = e^{kh}/(1+e^{kh})$, $h = \sqrt{\pi d/N}$, and $d = \pi/2$ for $k = -N,\dots,N$, (see [15]) and t_r are the Chebyshev–Gauss–Lobatto points with the following relation:

$$t_r = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi r}{n}, \quad r = 1,\dots,n.$$

Substituting these points into equations (15) and (16), we get

$$
\begin{cases}
\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} a_{ij}S_i(x_k)D_*^{\alpha}L_j(t_r;\lambda) + a\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} b_{ij}\delta_{i,k}^{(1)}\dfrac{d\psi(x_k)}{dx}L_j(t_r;\lambda) \\
+b\Big(\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} b_{ij}S_i(x_k)L_j(t_r;\lambda)\Big)^p + c\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} a_{ij}S_i(x_k)L_j(t_r;\lambda) = g(x_k,t_r), \\
\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} b_{ij}S_i(x_k)D_*^{\beta}L_j(t_r;\lambda) + d\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} a_{ij}\delta_{i,k}^{(1)}\dfrac{d\psi(x_k)}{dx}L_j(t_r;\lambda) \\
+e\Big(\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} a_{ij}S_i(x_k)L_j(t_r;\lambda)\Big)^q + f\sum\limits_{i=-N}^{N}\sum\limits_{j=0}^{n} b_{ij}S_i(x_k)L_j(t_r;\lambda) = h(x_k,t_r),
\end{cases}
$$

$$
\begin{aligned}
&\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}S_i(x_k)L_j(0;\lambda)) = u_0(x_k),\\
&\sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij}S_i(x_k)L_j(0;\lambda) = v_0(x_k),\\
&\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}S_i(0)L_j(t_r;\lambda) = u_1(t_r),\\
&\sum_{i=-N}^{N}\sum_{j=0}^{n} b_{ij}S_i(0)L_j(t_r;\lambda) = v_1(t_r).
\end{aligned}
$$

Now, we have a system of nonlinear algebraic equations with unknown coefficients a_{ij} and b_{ij}. By using well-known Newtons method, we can find the approximate solutions given in (9) and (10).

6 Numerical Illustration

In this section, we present some examples of SNFPDEs to show the efficiency of the proposed method. The results will be compared with the exact solutions. The accuracy of the present method is estimated by the absolute errors $E^1_{N,n}$ and $E^2_{N,n}$, which are given as follows:

$$E^1_{N,n}(\alpha,\beta) = |u(x_i,t_j) - u_{N,n}(x_i,t_j)|, \quad E^2_{N,n}(\alpha,\beta) = |v(x_i,t_j) - v_{N,n}(x_i,t_j)|.$$

If $\alpha=\beta=\lambda$, we put $E^i_{N,n}(\alpha,\beta) = E^i_{N,n}(\lambda), i = 1,2$.

Example 6.1 Consider the SNFPDEs

$$
\begin{cases}
\dfrac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + v^2(x,t) + u(x,t) = g(x,t),\\
\dfrac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + u^2(x,t) + v(x,t) = h(x,t),
\end{cases}
$$

with the conditions $u(x,0) = x$, $v(x,0) = x^2$, $u(0,t) = t^{\alpha}$ and $v(0,t) = t^{\beta}$.

Table 1: The maximum absolute errors $\max\{E^1_{N,n}(\lambda)\}$ and $\max\{E^2_{N,n}(\lambda)\}$ for Example 6.1.

$\{N, n\}$	$\max\{E^1_{N,n}(\frac{1}{2})\}$	$\max\{E^2_{N,n}(\frac{1}{2})\}$	$\max\{E^1_{N,n}(\frac{1}{3})\}$	$\max\{E^2_{N,n}(\frac{1}{3})\}$	$\max\{E^1_{N,n}(\frac{3}{5})\}$	$\max\{E^2_{N,n}(\frac{3}{5})\}$
$\{2, 4\}$	$1.11e-10$	$1.63e-10$	$3.85e-02$	$5.10e-02$	$1.03e-02$	$1.04e-02$
$\{3, 6\}$	$1.06e-10$	$4.51e-10$	$3.99e-15$	$3.33e-15$	$3.30e-03$	$3.36e-03$
$\{4, 8\}$	$1.81e-11$	$7.02e-11$	$4.66e-15$	$6.88e-15$	$1.29e-03$	$1.31e-03$
$\{5, 10\}$	$1.33e-15$	$1.55e-15$	$4.88e-15$	$5.32e-15$	$6.15e-04$	$6.25e-04$

The exact solutions are $u(x,t) = x + t^\alpha$ and $v(x,t) = x^2 + t^\beta$. For various values of N, n, α, and β, we obtain an approximate solution of this equation. Table 1 shows the maximum absolute errors for the various values of N, n and $\alpha = \beta = \frac{1}{2}, \frac{1}{3}, \frac{3}{5}$. From Table 1, we see that the error can be reduced by increasing the number of collocation points. Also, the absolute errors are shown in Figure 1.

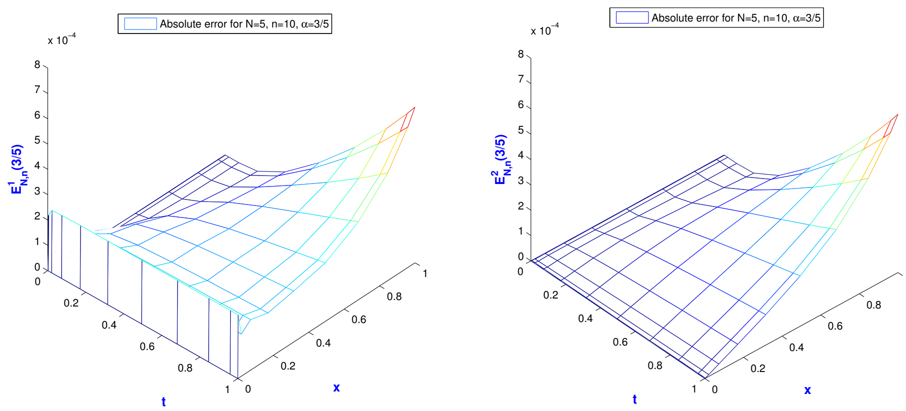

Figure 1: The absolute error functions with $\alpha = \beta = 3/5$ for Example 6.1.

Example 6.2 Consider the SNFPDEs

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \dfrac{\partial v(x,t)}{\partial x} + v^3(x,t) = g(x,t), \\ \dfrac{\partial^\beta v(x,t)}{\partial t^\beta} + \dfrac{\partial u(x,t)}{\partial x} + u^3(x,t) = h(x,t), \end{cases}$$

with the conditions

$$u(x,0) = v(x,0) = u(0,t) = v(0,t) = 0.$$

Table 2: The maximum absolute errors $\max\{E^1_{N,n}(\lambda)\}$ and $\max\{E^2_{N,n}(\lambda)\}$ for Example 6.2.

$\{N,n\}$	$\max\{E^1_{N,n}(\frac{1}{2})\}$	$\max\{E^2_{N,n}(\frac{1}{2})\}$	$\max\{E^1_{N,n}(\frac{1}{3})\}$	$\max\{E^2_{N,n}(\frac{1}{3})\}$
$\{2,4\}$	$3.9767e-02$	$3.5755e-02$	$5.0258e-02$	$3.7833e-02$
$\{3,6\}$	$2.4553e-02$	$2.1419e-02$	$2.2845e-02$	$2.0500e-02$
$\{4,8\}$	$5.8731e-03$	$6.0509e-03$	$5.5623e-03$	$6.3847e-03$
$\{5,10\}$	$6.1088e-03$	$5.9381e-03$	$6.1866e-03$	$5.8873e-03$
$\{6,12\}$	$1.9400e-03$	$1.9239e-03$	$1.8999e-03$	$1.9470e-03$
$\{7,14\}$	$2.1785e-03$	$2.1212e-03$	$2.1709e-03$	$2.1308e-03$
$\{8,16\}$	$9.8963e-04$	$8.4617e-04$	$1.3835e-03$	$1.4523e-04$
$\{9,18\}$	$9.1436e-04$	$8.9726e-04$	$9.0637e-04$	$8.9827e-04$
$\{10,20\}$	$5.1216e-04$	$4.8690e-04$	$5.0375e-04$	$4.9526e-04$

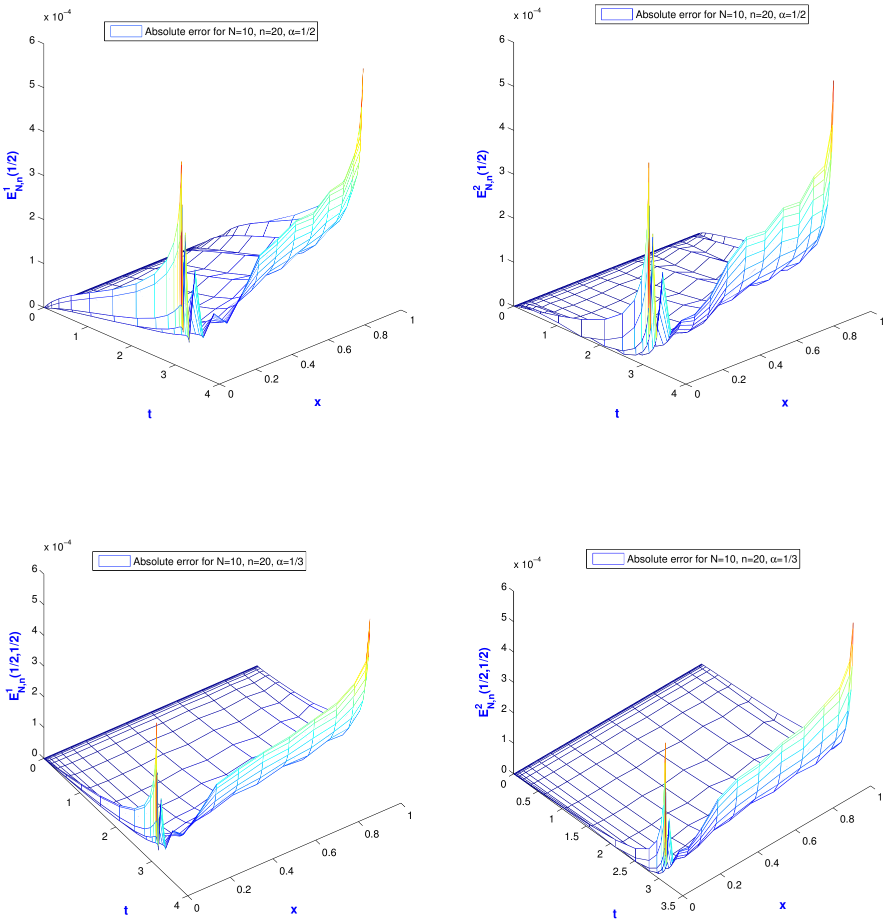

Figure 2: The absolute error functions with $\alpha = \beta = 1/2$ and $\alpha = \beta = 1/3$ for Example 6.2.

Table 3: The computational convergence order for for Example 6.2.

$\{N,n\}_{new}$	$\{N,n\}_{old}$	$\text{Order}^1_{(\frac{1}{2})}$	$\text{Order}^2_{(\frac{1}{2})}$	$\text{Order}^1_{(\frac{1}{3})}$	$\text{Order}^2_{(\frac{1}{3})}$
$\{3,6\}$	$\{2,4\}$	2.3785	2.5275	3.8891	3.0225
$\{4,8\}$	$\{3,6\}$	9.9447	8.7880	9.8214	8.1098
$\{10,20\}$	$\{9,18\}$	11.0020	11.6037	11.1497	11.3019

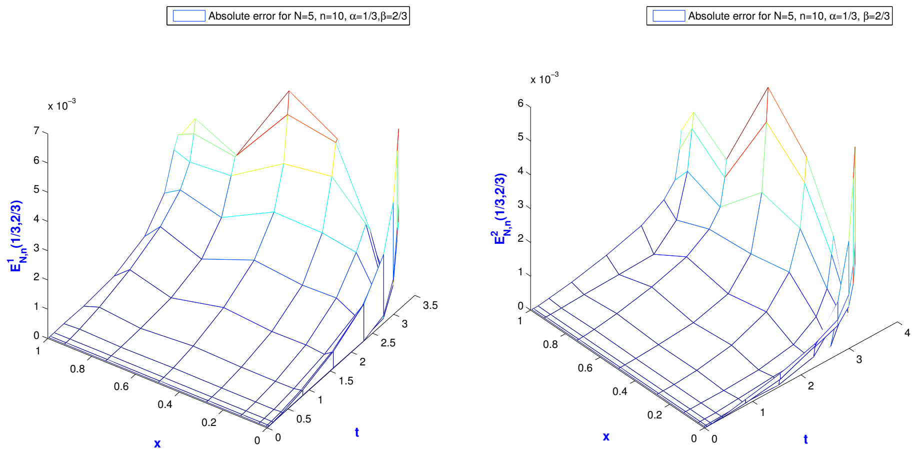

Figure 3: The absolute error functions with $\alpha = 1/3, \beta = 2/3$ for Example 6.2.

The exact solutions are $u(x,t) = t \sin x$ and $v(x,t) = t^2 \sin x$. For various values of N, n, α, and β, we obtain an approximate solution of this equation. The absolute error is shown in Figure 2. Figure 3 shows the maximum absolute error for $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$. Also, Figure 4 and Table 2 show the maximum absolute error for the various values of N, n and $\alpha = \beta = \frac{1}{2}, \frac{1}{3}$. We see that the error can be reduced by increasing the number of collocation points. Also, Table 3 shows the computational convergence orders of the proposed method. We compute the practical orders of convergence as follows:

$$\text{Order}^i_{(\alpha)} = \frac{\log\left(\frac{\max\{E^i_{N,n_{new}}(\lambda)\}}{\max\{E^i_{N,n_{old}}(\lambda)\}}\right)}{\log\left(\frac{h_{new}}{h_{old}}\right)}, \quad i = 1, 2,$$

where

$$h_{new} = \frac{\pi}{\sqrt{2N_{new}}}, \qquad h_{old} = \frac{\pi}{\sqrt{2N_{old}}}.$$

7 Conclusion

In this paper, we applied a basis of the Sinc function and fractional Muntz–Legendre polynomials to obtain the numerical solution of a system of nonlinear fractional partial

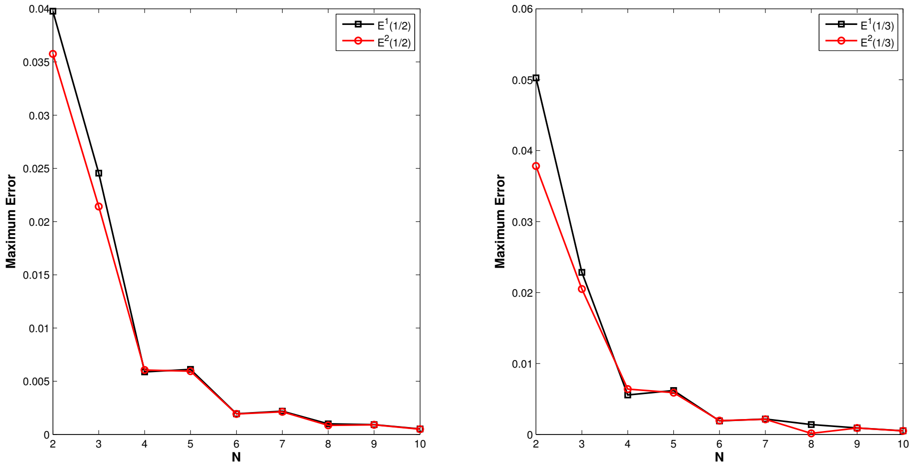

Figure 4: The maximum absolute error convergence of Example 6.2.

differential equations. To get the unknown coefficients of the fractional Muntz–Legendre polynomials, we used the collocation method. The results of the numerical examples showed the efficiency and accuracy of the proposed method.

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