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# Numerical Approximation of the Exact Control for the Vibrating Rod with Improvement of the Final Error by Particle Swarm Optimization

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**Abstract:** The paper studies the numerical approximation of the exact boundary controllability for the vibrating rod by the Hilbert uniqueness method (HUM). This study is based on the knowledge of the asymptotic behavior of the control governing the system at time $T$. This is the idea developed in this work concerning the Dirichlet boundary case. More precisely, an approximate control shall be found which returns the system under consideration to rest at time $T$ with an estimation of the final state error and the improvement of it by using the particle swarm optimization algorithm (PSO).

**Keywords:** *exact controllability; vibrating rod; Hilbert uniqueness method; asymptotic behavior; Dirichlet control; particle swarm optimization.*

**Mathematics Subject Classification (2010):** 93B05, 74K10, 65K10, 65N25, 65M06, 78M32.

## 1 Introduction

Controllability is a classical problem in control theory. The idea that motivated this work is that control theory is certainly, at present, one of the most interdisciplinary areas of research. It is nowadays a rich crossing point of engineering and mathematics. Many problems of control theory such as optimal control and stabilizability may be solved under assumption that the system is controllable, see [9, 16, 19]. Controllability means that it is possible to drive a dynamic system from an arbitrary initial state to an arbitrary

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final state using the set of admissible controls. There are two basic concepts of controllability in the case of infinite dimensional systems: approximate and exact controllability. Approximate controllability allows to drive the dynamic system to arbitrary small neighbourhood of final state while exact controllability means that the dynamic system can be driven to arbitrary final state.
In the case of finite dimensional systems, these concepts are equivalent, see [18].

Approximate controllability has been studied for different types of semilinear dynamical systems, see, for example, [1, 14] and the references therein.

Regarding the problem of exact controllability of linear systems, great efforts have been devoted to its study both theoretically and numerically.

The theory of solving this problem has been introduced in [17] by the use of the semigroup approach. In [15], a new approach called the Hilbert uniqueness method (HUM) has been proposed to solve this problem for hyperbolic systems. Another approach has been proposed to solve this problem for parabolic systems, see, for instance, [2].

Numerically, the problem has been studied in [6, 10–12] through the numerical implementation of the Hilbert uniqueness method.

This method leads to the resolution of the equation

$$\Lambda\{\psi^0, \psi^1\} = \{u^1, -u^0\}\,, \tag{1}$$

where $u^0$ and $u^1$ are the initial conditions of the system and $\Lambda$ is an isomorphism between the Hilbert space $F$ and its dual $F'$ .

The conjugate gradient method was introduced in [6], later on this method was developed in [10] in order to solve (1). The approximate solutions obtained do not converge to the exact solutions as the temporal and spatial grid sizes tend to zero. Methods of regularization including the Tikhonov regularization that result in convergent approximations were introduced in the papers on HUM-based methods. This method shows that this technique improves the last results.

In [11, 12], an alternative to the Tikhonov regularization procedure based on spectral analysis is presented. It was shown that this approach improves the method described in [6, 10].

Another computational method for boundary controllability of the wave equation is the one based on the method proposed in [13]. This approach permits to directly solve an optimization problem in which the equations of the linear system act as equality constraints. To resolve this problem, two methods are proposed. The first one is based on the Lagrange multiplier method. The second one transforms the constrained optimization problem to an unconstrained optimization problem and uses the conjugate gradient method for its resolution. The computational results show that this method provides convergent approximations for problems in which existing methods produce divergent approximations unless they are regularized in some manner. Therefore, this method improves the results found in [6].

The numerical methods cited studied the exact controllabilty of hyperbolic systems. For parabolic systems, see [4].

Motivated by the existence in the literature of these numerical studies, we want to feed it by the numerical study of a system which is neither hyperbolic nor parabolic. More precisely, we study numerically the exact Dirichlet boundary controllability of the vibrating rod.

This study goes through the numerical resolution of equation (1), which determines explicit formulas for $\psi^0$ and $\psi^1$ and therefore, the approximate control that steers the system under consideration to rest time $T$ with an estimation of the final state error.

By a particular example, we present the graphics of the approximate control, the cost function, and the final state error when the points are equidistant and improve it by using a stochastic optimization algorithm named the particle swarm optimization (PSO).

The remaining of the paper is organized as follows.
Section 2 defines the exact Dirichlet boundary controllability of the vibrating rod.
Section 3 describes the HUM approach. In Section 4, we present the method of solving the problem under consideration. In Section 5, we give explicit formulas. In Section 6, experimental results are presented. In Section 7, a stochastic optimization algorithm is used to improve the final error. The results obtained confirm it. Section 8 concludes the paper.

## 2 The Problem under Study

Let $T$ be a given positive number, $u^0(x)$ and $u^1(x)$ denote given functions defined on $\Omega = ]0, L[$. Let $\Sigma$=$\{0, L\}\times]0, T[$, $Q = ]0, L[\times]0, T[$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-2}(\Omega)$. The exact Dirichlet boundary controllability problem for the vibrating rod is as follows.
Find a control function $g$ defined on $\Sigma$ such that $u$ satisfies the system

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } Q, \\ u(x,0) = u^0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u^1(x) & \text{in } \Omega, \\ u(x,T) = 0, \quad \dfrac{\partial u}{\partial t}(x,T) = 0 & \text{in } \Omega, \\ u(0,t) = 0, \quad u(L,t) = 0, & t \in [0,T], \\ \dfrac{\partial u}{\partial x}(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(L,t) = \mathrm{g(t)}, & t \in [0,T]. \end{cases} \tag{2}$$

The first equation in (2) represents the vibrations of the rod. It models the vertical motion of a thin, horizontal rod with small displacements from the rest position. It is neither hyperbolic nor parabolic. $u(x,t)$ denotes the displacement of the point $x$ of the rod, at the instant $t$. $u^0(x)$ and $u^1(x)$ represent, respectively, the initial position and the initial velocity of the rod. The third equation in (2) is the final condition and is called the equilibrium condition. It is well known that when $T > 0$, the exact controllability problem (2) admits at least one state-control solution pair (u,g); furthermore, the exact controller g having minimum boundary $L^2$ norm is unique, see [15, 20].

Our work consists in solving numerically the exact boundary controllability of the vibrating rod when the control is of the Dirichlet type. For this purpose, we consider the control given by the HUM approach and we develop some techniques which allow computation of the control that steers the system at hand to rest at time $T$ with a final error

$$\|\xi\|^2 = \|u(x,T)\|^2_{L^2(\Omega)} + \left\| \frac{\partial u(x,T)}{\partial t} \right\|^2_{L^2(\Omega)}. \tag{3}$$

## 3 Choice of the Control

We recall briefly how the control which steers the system (2) to rest at time $T$ is found. Let $F = H_0^2(\Omega) \times L^2(\Omega)$ and $F' = H^{-2}(\Omega) \times L^2(\Omega)$. For any $\{\psi^0, \psi^1\} \in F$, solve the system

$$\begin{cases} \psi_{tt} + \psi_{xxxx} = 0 & \text{in } Q, \\ \psi(x,0) = \psi^0(x), \quad \dfrac{\partial \psi}{\partial t}(x,0) = \psi^1(x) & \text{in } \Omega, \\ \psi = \dfrac{\partial \psi}{\partial x} = 0 & \text{on } \Sigma, \end{cases}$$

and then resolve the reverse system in $\Theta$:

$$\begin{cases} \Theta_{tt} + \Theta_{xxxx} = 0 & \text{in } Q, \\ \Theta(x,T) = \dfrac{\partial \Theta}{\partial t}(x,T) = 0 & \text{in } \Omega, \\ \Theta = \dfrac{\partial^2 \psi}{\partial x^2} & \text{on } \Sigma. \end{cases}$$

This enables us to (implicitly) define a linear operator $\Lambda$ by

$$\Lambda\{\psi^0, \psi^1\} = \{\frac{\partial \Theta}{\partial t}(x,0), -\Theta(x,0)\}\,.$$

So, for convenient $\psi^0, \psi^1$ and $T$, if one can solve the equation (1), then it is possible to obtain the control $g$ explicitly.

We obtain the corresponding unique minimum $L^2$-norm control by setting $g = \frac{\partial^2 \psi}{\partial x^2}$. It is proved in [15] that $\lambda$ is an isomorphism from $F$ to $F'$. Consequently, for any initial data $u^0, u^1$, such that $\{u^1, -u^0\} \in F'$, equation (1) has a unique solution $\{\psi^0, \psi^1\} \in F$. The $\Theta$ system is, in fact, the u one (reverse) and the state $\{0, 0\}$ is reached at time $T$. See [15] for more details.

## 4 Presentation of the Resolution Method

In this section, we will show how to solve equation (1) and give expressions for $\psi^0$ and $\psi^1$ which can be used for numerical simulations.
Using the techniques of standard optimization, we know that solving (1) is equivalent to solving the minimization problem

$$\underset{\{\psi^0,\psi^1\}\in F}{Inf} \; J\left(\{\psi^0, \psi^1\}\right)\,, \tag{4}$$

where

$$J(\{\psi^0, \psi^1\}) = \frac{1}{2}\int_0^T \left[\frac{\partial^2 \psi(L,t)}{\partial x^2}\right]^2 dt - \int_\Omega \left[\psi^0 u^1 - \psi^1 u^0\right] dx. \tag{5}$$

In the problems of controllability, the knowledge of the asymptotic behavior of the control governing the system at time $T$ may be used for its calculation. This idea will be used to determine explicit formulas for $\psi^0$ and $\psi^1$.

Let $\{\psi_T^0, \psi_T^1\}$ be the solution of (4). We introduce a T factor to transform the functional (5) in the following way:

$$T.J(\{\psi^0, \psi^1\}) = \frac{T}{2}\int_0^T \left[\frac{\partial^2 \psi(L,t)}{\partial x^2}\right]^2 dt - \int_\Omega \left[u^1 T\psi^0 - u^0 T\psi^1\right] dx.$$

Let $\phi = T.\psi$, where $\phi$ is the solution of the system

$$\begin{cases} \dfrac{\partial^2\phi}{\partial t^2} + \dfrac{\partial^4\phi}{\partial x^4} = 0 & \text{in } Q, \\ \phi(x,0) = \phi^0, \quad \dfrac{\partial\phi}{\partial t}(x,0) = \phi^1 & \text{in } \Omega, \\ \phi = \dfrac{\partial\phi}{\partial x} = 0 & \text{on } \Sigma. \end{cases}$$

$$T.J(\{\psi^0,\psi^1\}) = \frac{1}{2T}\int_0^T \left[\frac{\partial^2\phi(L,t)}{\partial x^2}\right]^2 dt - \int_\Omega \left[u^1\phi^0 - u^0\phi^1\right] dx = J\left(\{\phi^0,\phi^1\}\right). \quad (6)$$

The problem (4) becomes

$$Inf\ J\left(\{\phi^0,\phi^1\}\right). \quad (7)$$

Assume the solution of (7) is $\{\phi_T^0,\phi_T^1\}$, then we have $\phi_T^0 = T.\psi_T^0$ and $\phi_T^1 = T.\psi_T^1$. Consider

$$\phi^0 = \lim_{T\to+\infty}\phi_T^0, \quad \phi^1 = \lim_{T\to+\infty}\phi_T^1,$$

then, according to [3], it is possible to determine explicitly $(\phi^0,\phi^1)$. Numerical approximations useful for calculations are determined by this approach. Then, it will be possible to calculate $\psi_T^0$ and $\psi_T^1$ by using

$$\psi_T^0 = \frac{1}{T}\phi^0, \quad \psi_T^1 = \frac{1}{T}\phi^1.$$

## 5 Resolution of the Problem

Denote the orthonormal eigenfunctions by $\omega_j(x)$ and the eigenvalues by $\lambda_j^2$ of $\frac{d^4}{dx^4}$ with the homogeneous Dirichlet condition. Consider (6) and look for $\lim\limits_{T\to+\infty} J\left(\{\phi^0,\phi^1\}\right)$.

Let

$$u^0 = \sum_{j=1}^{\infty} u_j^0\omega_j\ ,\ u^1 = \sum_{j=1}^{\infty} u_j^1\omega_j$$

with $u_j^0 = (u^0,\omega_j)$ and $u_j^1 = (u^1,\omega_j)$. Then

$$\int_\Omega u^0\phi^1 dx = \sum_j (u^0,\omega_j)(\phi^1,\omega_j), \quad \int_\Omega u^1\phi^0 dx = \sum_j (u^1,\omega_j)(\phi^0,\omega_j).$$

We have, in the same way,

$$\phi(x,t) = \sum_j \phi_j(t)\omega_j(x)\,,$$

where

$$\phi_j(t) = (\phi^0,\omega_j)cos(\lambda_j t) + \frac{1}{\lambda_j}(\phi^1,\omega_j)sin(\lambda_j t)\,.$$

Thus

$$\begin{aligned} \frac{1}{2T}\int_0^T \left[\frac{\partial^2\phi(L,t)}{\partial x^2}\right]^2 dt &= \frac{1}{2T}\int_0^T \left[\sum_{j,l}\phi_j(t).\phi_l(t)\frac{d^2\omega_j(L)}{dx^2}.\frac{d^2\omega_l(L)}{dx^2}\right] dt \\ &= \frac{1}{2}\sum_{j,l}\frac{d^2\omega_j(L)}{dx^2}.\frac{d^2\omega_l(L)}{dx^2}\left[\frac{1}{T}\int_0^T \phi_j(t).\phi_l(t)dt\right]. \end{aligned}$$

By developing, we obtain

$$\frac{1}{T}\int_0^T \phi_j(t).\phi_l(t)dt$$

$$= \frac{1}{T}\int_0^T \left\{(\phi^0,\omega_j)cos(\lambda_j t) + (\phi^1,\omega_j)\frac{sin(\lambda_j t}{\lambda_j}\right\}\left\{(\phi^0,\omega_l)cos(\lambda_l t) + (\phi^1,\omega_l)\frac{sin(\lambda_l t}{\lambda_l}\right\}dt$$

$$= \frac{1}{T}\int_0^T \left[(\phi^0,\omega_j)(\phi^0,\omega_l)cos(\lambda_j t)cos(\lambda_l t)\right]dt + \frac{1}{T}\int_0^T \left[(\phi^0,\omega_j)(\phi^1,\omega_l)cos(\lambda_j t)\frac{sin(\lambda_l t}{\lambda_l}\right]dt$$
$$+\frac{1}{T}\int_0^T \left[(\phi^1,\omega_j)(\phi^0,\omega_l)cos(\lambda_l t)\frac{sin(\lambda_j t}{\lambda_j}\right]dt + \frac{1}{T}\int_0^T \left[(\phi^1,\omega_j)(\phi^1,\omega_l)\frac{sin(\lambda_j t).sin(\lambda_l t)}{\lambda_j.\lambda_l}\right]dt$$

For $j \neq l$, the calculation gives

$$\frac{1}{T}\int_0^T \phi_j(t).\phi_l(t)dt \longrightarrow 0 \ as \ T \longrightarrow \infty.$$

and for $j = l$, we have

$$\frac{1}{T}\int_0^T \phi_j(t).\phi_l(t)dt \longrightarrow \left[\frac{1}{2}(\phi^0,\omega_j)^2 + \frac{1}{2\lambda_j^2}(\phi^1,\omega_j)^2\right] as \ T \longrightarrow \infty.$$

Finally,

$$\frac{1}{T}\int_0^T \phi_j(t).\phi_l(t)dt \longrightarrow \delta_j l\left[\frac{1}{2}(\phi^0,\omega_j)^2 + \frac{1}{2\lambda_j^2}(\phi^1,\omega_j)^2\right] as \ T \longrightarrow \infty\,,$$

where $\delta_j l = 1 \ if \ j = l$ and $\delta_j l = 0 \ if \ j \neq l$, and then

$$\frac{1}{2T}\int_0^T \left[\frac{\partial^2\phi(L,t)}{\partial x^2}\right]^2 dt \longrightarrow \frac{1}{4}\sum_j \left[(\phi^0,\omega_j)^2 + \frac{1}{\lambda_j^2}(\phi^1,\omega_j)^2\right]\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2.$$

The initial problem (7) is transformed to the minimization problem according to $\phi^0$ and $\phi^1$

$$\begin{aligned}\frac{1}{4}\sum_j \left[(\phi^0,\omega_j)^2 + \frac{1}{\lambda_j^2}(\phi^1,\omega_j)^2\right]\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2 - \int_\Omega (u^1\phi^0 - u^0\phi^1)dx \\ = \sum_j \left[\frac{1}{4}(\phi^0,\omega_j)^2\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2 - u_j^1(\phi^0,\omega_j)\right] \\ + \sum_j \left[\frac{1}{4\lambda_j^2}(\phi^1,\omega_j)^2\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2 + u_j^0(\phi^1,\omega_j)\right]\end{aligned} \tag{8}$$

We see that in this equality, the first term does not depend on $\phi^1$ and the second does not depend on $\phi^0$. Therefore, the minimization of (8) conducts to the minimization of

$$\frac{1}{4}(\phi^0,\omega_j)^2\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2 - u_j^1(\phi^0,\omega_j) \ \ according \ to \ \ \phi^0$$

and

$$\frac{1}{4\lambda_j^2}(\phi^1,\omega_j)^2\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2+u_j^0(\phi^1,\omega_j)\ \ according\ to\ \ \phi^1.$$

The minimum is determined by

$$\frac{1}{2}(\phi^0,\omega_j)\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2-u_j^1=0,$$

$$\frac{1}{2\lambda_j^2}(\phi^1,\omega_j)\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2+u_j^0=0.$$

Finally, when $T\longrightarrow\infty$, we obtain

$$\phi^0=\sum_{j=1}^{\infty}\frac{2.(u^1,\omega_j).\omega_j}{\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2},$$

$$\phi^1=-\sum_{j=1}^{\infty}\frac{2\lambda_j^2(u^0,\omega_j).\omega_j}{\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2}.$$

We conclude the following approximations:

$$\psi_T^0=\frac{2}{T}\sum_{j=1}^{n}\frac{(u^1,\omega_j).\omega_j}{\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2},\tag{9}$$

$$\psi_T^1=\frac{-2}{T}\sum_{j=1}^{n}\frac{\lambda_j^2(u^0,\omega_j).\omega_j.}{\left[\frac{d^2\omega_j(L)}{dx^2}\right]^2}\tag{10}$$

are explicit formulas.

## 6 Computational Results

In this section, we determine the graphs of the approximate control, the cost function and the final error (3) at the instant $t=T$ and the initial data
$u^0(x)=Ax(1+x)$; $u^1(x)=(1+A)u^0(x)$. A is a coefficient chosen by numerical considerations. The control steering the system (2) to rest at time $T$ is given by $g*=\frac{\partial^2\psi}{\partial x^2}$, where $\psi$ is the solution of the system

$$\begin{cases}\dfrac{\partial^2\psi(x,t)}{\partial t^2}+\dfrac{\partial^4\psi(x,t)}{\partial x^4}=0 & \text{in Q },\\ \psi(x,0)=\psi_T^0,\quad \dfrac{\partial\psi(x,0)}{\partial t}=\psi_T^1 & \text{in }\Omega,\\ \psi=\dfrac{\partial\psi}{\partial x}=0 & \text{on }\Sigma,\end{cases}\tag{11}$$

and $\psi_T^0$ , $\psi_T^1$ are the initial conditions given in (9) and (10).

We use the following algorithm for the implementation.

**Algorithm 6.1** 1: Choice of the initial data $u^0$ and $u^1$.
2: Choice of the order $n$.
3: Calculation of the explicit formulas $\psi_T^0$ and $\psi_T^1$ using (9) and (10).
4: Resolution of the system (11).
5: Calculation of the explicit control $g*$.
6: Calculation of the cost function $\|g*\|^2$.
7: Resolution of the system (2) using the explicit control $g*$.
8: Calculation of the final state error $\|\xi\|^2$.
Return to 2.

**Remark 6.1** The numerical method for resolution of systems (2) and (11) is based on a symmetric finite difference scheme, see [7]. This scheme leads to the resolution of a linear system whose matrix is pentadiagonal symmetric positive.

We have introduced a new approach to determine explicitly the control driving the system (2) to rest at time $T$ with an estimation of the final state error. Particular attention is paid to the system which is neither hyperbolic nor parabolic.

Our paper presents a new view for the numerical approximation of the exact boundary controllability.

We have the following graphs with $L = 1$, $n = 4$ and $T = 1$.

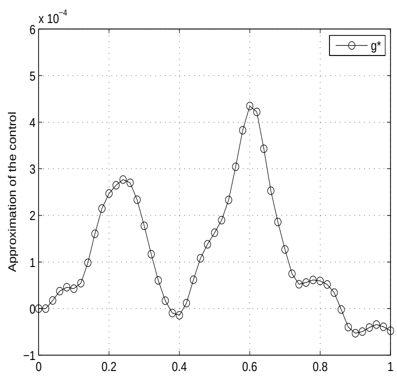


**Figure 1**: The form of approximate control $g^*$ steering the system (2) to rest at time $T$.

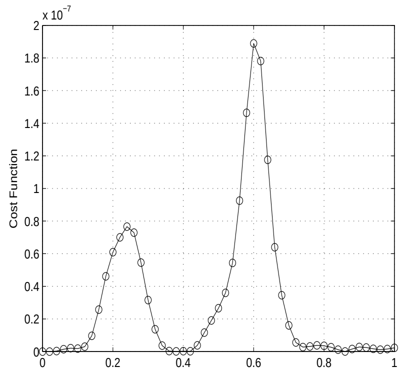


**Figure 2**: Illustration of the cost function obtained by the corresponding unique minimum $L^2$-norm control.

**Remark 6.2** The results obtained are satisfactory althought many approximations have been made (asymptotic aspect, truncation, etc.). However, we think that increasing the value of n increases the efficiency of formulas (9), (10) and allows to make the final state error close to zero. In this perspective, we are trying in the following section to improve the result of the final state error so that the control steers the considered system (2) to rest at time $T$.

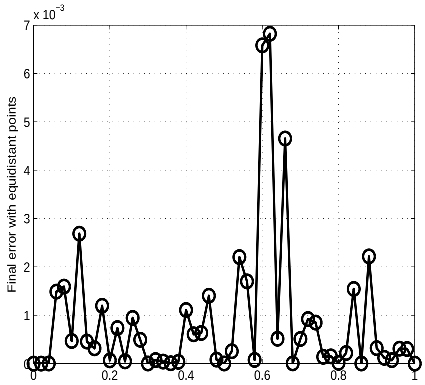


**Figure 3**: Estimation of the final state error with equidistant points so that the system (2) steers to rest at time $T$.

## 7 Improvement of the Final State Error

In this section, the calculation of the final state error is improved to make it as small as possible. The problem that we consider is to minimize (3) using the particle swarm optimization algorithm (PSO) and taking the same example treated in Section 6. For this, we try to determine $x_i$ by the PSO so that the final state error (3) is close to zero. This method is an effective way to improve the final state error.

### 7.1 Basic particle swarm optimization algorithm

The particle swarm optimization (PSO) is a non deterministic method simulated by social behavior of bird flocking or fish schooling, that can be used to optimize a function objective, and was described in [5, 8]. In the PSO algorithm, each individual is called the "particle", which represents a potential solution in a swarm.

We present in the following, the main steps of the basic PSO algorithm. Three $M-$ dimensional vectors compose each particle: the current position $Y_j = (y_{j1}, y_{j2}, ..., y_{jM})$, the velocity $V_j = (v_{j1}, v_{j2}, ..., v_{jM})$ which represents its direction of searching, and the previous best position that it has individually found $P_j = (p_{j1}, p_{j2}, ..., p_{jM})$, called (pbest), the subscipt $j$ ranges from 1 to $s$, where s indicates the size of swarm. Habitually, each particle stores its position and its best value so far (pbest), and therefore recognizes the best value in the swarm, called (sbest) between the set of values (pbest).

The following system shows the displacement of each particle $j$:

$$v_{jl}^{k+1} = w_{jl} v_{jl}^{k} \quad + \quad c_1 r_1^k [(pbest)_{jl}^k - x_{jl}^k], + c_2 r_2^k [(sbest)_{jl}^k - x_{jl}^k] \tag{12}$$

$$x_{jl}^{k+1} = v_{jl}^{k+1} + x_{jl}^{k}\,, \tag{13}$$

where $v_{jl}^{k+1}, x_{jl}^{k+1}$ are the velocity and the position of particle $j$, respectively, at iteration $k+1$, $w_{jl}$ is the inertia weight with its value that ranges from 0.9 to 1.2, $c_1$ and $c_2$ are two parameters situated in the range of 2 to 4, called the acceleration coefficients and $r_1^k, r_2^k$ are two random numbers uniformly distributed in the range $[0, 1]$. In the double subscript in the equations (12) and (13) the first subscript stands for the particle $j$ and the second one for the dimension $l$. The (basic) process for imlementing the PSO is in the algorithm below.

**Algorithm 7.1** Particle Swarm Optimisation.
1: Set the dimension $M$, and the size $s$ of the swarm.
2: Set the iteration number $k$ to zero.
3: Evaluate, for each particle, the velocity vector using its memory and equation (12), where pbest and sbest can be modified.
4: Move each particle to its new position, according to equation (13).
5: Let $k = k + 1$.
6: Go to step 2, and repeat until convergence condition is satisfied.

**Remark 7.1** This section was the subject of a personal communication entitled "Particle Swarm Optimization Algorithm to Improve the Final State Error of the Exact Boundary Controllability", presented at the Sixth International Conference on Metaheuristics and Nature Inspired Computing that was organized in Marrakech (Morocco) in October 2016.

From Figure 4 shown below, it can be seen that the final state error is improved.

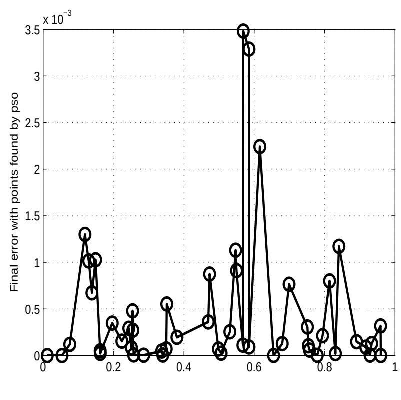


**Figure 4**: Estimation of the final state error with points chosen by the PSO so that the system (2) steers to rest at time $T$.

## 7.2 Discussion

Figure 4 illustrates the decrease of the final state error in comparison with Figure 3 of Section 6. It can be said that the particle swarm optimization (PSO) algorithm has improved the value of the final state error.

The main limitation of the experimental result is the non-comparison of the final error provided by the PSO with other metaheuristics such as the ABC (Artificial Bee Colony), FWA (Fireworks Algorithm), FPA algorithm (Flower Pollination Algorithm).

## 8 Conclusion

The numerical implementation of the Hilbert uniqueness method allowed us to approximate the exact control for the vibrating rod with an estimation of the final state error. The calculation of this error when the selected points are equidistant is compared with the error when points are chosen by the PSO. The results show the improvement of the final error in the second case compared to the first one. In the future, we intend to study the comparison of the final error provided by the PSO with other metaheuristics and study the case of dimension two of the same system.


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