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Existence of Weak Solutions for Nonlinear p-Elliptic Problem by Topological Degree

A. Abbassi, C. Allalou* and A. Kassidi

Laboratory LMACS, FST of Beni Mellal, Sultan Moulay Slimane University, Morocco

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Abstract: In this work, we establish the existence results on weak solutions via the recent Berkovits topological degree for the following nonlinear p-elliptic problems :

$$-div(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{q-2}u + f(x,\ u,\ \nabla u)$$

in a bounded set $\Omega \in \mathbb{R}^N$, where the vector field f is a Carathéodory function.

Keywords: *weighted Sobolev spaces; Hardy inequality; topological degree; Berkovits topological degree; p-elliptic problems.*

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary if $N \geq 2$, and let p, q be real numbers such that $2 < q < p < \infty$, and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions on Ω, i.e., each $w_i(x)$ is measurable a.e. positive on Ω. Let $W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector w. Our objective is to prove the existence of weak solutions to the following nonlinear p-elliptic problem :

$$\begin{cases} -div(a(x,\nabla u)) = \lambda|u|^{q-2}u + f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $a(x,\nabla u) = |\nabla u|^{p-2}\nabla u$. We shall suppose that the following degenerate ellipticity condition is satisfied for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$:

$$a(x,\xi)\cdot\xi \geq \gamma \sum_{i=1}^{N} w_i|\xi_i|^p, \tag{2}$$

* Corresponding author: mailto:chakir.allalou@yahoo.fr

such that γ is a positive constant, and λ is a real parameter. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies only the growth condition, for a.e $x \in \Omega$ and all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^N$,

$$|f(x,\ \mu,\ \xi)| \leq \beta(M(x) + \sigma^{\frac{1}{p'}} |\mu|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\xi_j|^{p-1}), \tag{3}$$

where β is a positive constant, $M \in L^{p'}(\Omega), M(x) \geq 0$ and q is a real number such that $2 < q < p$.

We use, in this paper, the framework of the recent Berkovits topological degree. This notion was introduced by J. Berkovits [7] in the study of the solvability of abstract Hammerstein type equations and variational inequalities. The topological degree theory was introduced for the first time by Leray and Schauder [17] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder [8] constructed a topological degree for the operators of class (S_+) in reflexive Banach spaces with the Galerkin method, see also [20, 22, 23]. This notion was then extented by Berkovits [7] to the classical Leray-Schauder degree for the operators of generalized monotone type. Roughly speaking, the class of operators for the extended degree is essentially obtained by replacing the compact perturbation by a composition of operators of monotone type. We refer to [10, 24] for more details.

For $f \equiv 0$ and $p = q$, Melián et. al. (in [19]) study the eigenvalues of the problem, and there the differentiability with respect to the domain of the first Dirichlet eigenvalue of the minus $p-$Laplacian is shown for the first time. S. Liu [18], by using the Morse theory, has established the existence of weak solutions to the equation $\triangle_p u = f(x,\ u)$ with the Dirichlet boundary conditions. It should be mentioned that the results in this paper are generalised to the case of the p-Laplacian results obtained in [14] for the strongly nonlinear case using the Berkovits topological degree. One of the motivations for studying (1) comes from the applications to such models of fluid mechaincs (see [5,6,11]), nonlinear diffusion(see [21]) and nonlinear elasticity (see [4]). We note that the case $1 < p < 2$ relates to the elastic-plastic models.

This paper is divided into four sections. In the next section, we give some preliminaries and the definition of weighted Sobolev spaces and we recall some classes of mappings of generalized (S_+) type and the recent Berkovits degree. In the third section, we discuss the p-Laplace operator. Finally, we give some existence results for weak solutions of problem (1).

2 Preliminaries

In order to discuss problem (1), we need some theories on topological degree and on spaces $W^{1,p}(\Omega, w)$ which we call the weighted Lebesgue–Sobolev spaces. Firstly, we state some classes of mappings and topological degree, secondly, we give basic properties of spaces $W^{1,p}(\Omega, w)$ which will be used later.

2.1 Classes of mappings and topological degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle \cdot , \cdot \rangle$ between X^* and X in this order, and for a nonempty subset Ω of X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. The symbol $\to$ $(\rightharpoonup)$ stands for strong (weak) convergence.

Definition 2.1 Let Y be another real Banach space. An operator $F : \Omega \subset X \to Y$ is said to be

1. bounded if it takes any bounded set into a bounded set.
2. demicontinuous if for any $(u_n) \subset \Omega,\ u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
3. compact if it is continuous and the image of any bounded set is relatively compact.

Definition 2.2 A mapping $F : \Omega \subset X \to X^*$ is said to be

1. of class (S_+) if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty} \langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
2. quasimonotone if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 2.3 Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

1. F satisfies condition $(S_+)_T$ if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ y_n := Tu_n \rightharpoonup y$ and $\limsup\limits_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.
2. F has the property $(QM)_T$ if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ y_n := Tu_n \rightharpoonup y$, we have $\limsup\limits_{n\to\infty} \langle Fu_n, y - y_n\rangle \geq 0$.

Let $\mathcal{O}$ be the collection of all bounded open sets in X. For any $\Omega \subset X$, we consider the following classes of operators:

$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and satifies condition}(S_+)\}$,
$\mathcal{F}_{T,B}(\Omega) := \{F : \Omega \to X | F \text{ is bounded, demicontinuous and satifies condition}(S_+)_T\}$,
$\mathcal{F}_T(\Omega) := \{F : \Omega \to X | F \text{ is demicontinuous and satifies condition}(S_+)_T\}$,
$\mathcal{F}_B(X) := \{F \in \mathcal{F}_{T,B}(\overline{G}) | G \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{G}})\}$.

Throughout the paper $T \in \mathcal{F}_1(\overline{G})$ is called an essential inner map to F.

Lemma 2.1 *([7], Lemmas 2.2 and 2.4) Let $T \in \mathcal{F}_1(\overline{G})$ be continuous and $S : D_S \subset X^* \to X$ be demicontinuous such that $T(\overline{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

1. If S *is quasimonotone, then* $I + SoT \in \mathcal{F}_T(\overline{G})$, *where* I *denotes the identity operator.*
2. *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{G})$.*

Definition 2.4 Suppose that G is a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{G})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{G})$. The affine homotopy $H : [0, 1] \times \overline{G} \to X$ defined by

$$H(t,\ u) := (1 - t)Fu + tSu \qquad \text{for} \qquad (t,\ u) \in [0, 1] \times \overline{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.1 *[7] The above affine homotopy satisfies the condition* $(S_+)_T$.

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$, for more details see [7].

Theorem 2.1 *There exists a unique degree function*

$$d : \{(F,\ G,\ h)|G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_{T,B}(\overline{G}),\ h \notin F(\partial G)\} \longrightarrow \mathbb{Z}$$

that satisfies the following properties:

1. *(Normalization) For any* $h \in G,$ we have $d(I,\ G,\ h) = 1$.
2. *(Additivity) Let* $F \in \mathcal{F}_{T,B}(\overline{G})$. *If* G_1 *and* G_2 *are two disjoint open subsets of* G *such that* $h \notin F(\overline{G}\backslash(G_1 \cup G_2))$, *then we have*
$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$
3. *(Homotopy invariance) If* $H : [0, 1] \times \overline{G} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* h*:* $[0, 1] \to X$ *is a continuous path in* X *such that* $h(t) \notin H(t,\ \partial G)$ *for all* $t \in [0, 1]$, *then the value of* $d(H(t, \cdot),\ G,\ h(t))$ *is constant for all* $t \in [0, 1]$.
4. *(Existence) If* $d(F, G, h) \neq 0$, *then the equation* $Fu = h$ *has a solution in* G.

2.2 The weighted Sobolev space

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, p be a real number such that $1 < p < \infty$, and $\omega = \{\omega_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $\omega_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose for any $0 \leq i \leq N$ in all our considerations that

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{4}$$
$$w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega). \tag{5}$$

We denote $\partial_i = \dfrac{\partial}{\partial x_i}$. The weighted Sobolev space, denoted by $W^{1,p}(\Omega, w)$, is defined as follows :

$$W^{1,p}(\Omega, w) = \left\{ u \in L^p(\Omega, w_0) \quad \text{and} \quad \partial_i u \in L^p(\Omega, w_i), \quad i = 1, ..., N \right\}.$$

Note that the derivatives $\dfrac{\partial u}{\partial x_i}$ are understood in the sense of distributions. This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx \right]^{1/p}. \tag{6}$$

The condition (4) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and, consequently, we can introduce the subspace

$$X = W_0^{1,p}(\Omega, w)$$

of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (6). Moreover, condition (5) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are the reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0, \dots, N\}$ and where p' is the conjugate of p; i.e., $p' = \frac{p}{p-1}$(for more details we refer to [1–3]).

Let us define the norm on X equivalent to the norm (6) by

$$|||u|||_X = \Big(\sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p \, w_i(x)\, dx \Big)^{1/p}. \tag{7}$$

We can find a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega) \quad \text{and} \quad \sigma^{-p/(q-p)} \in L^1(\Omega) \tag{8}$$

with $q' = \frac{q}{q-1}$. Then the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma \, dx \Big)^{1/q} \le c \Big(\sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p w_i(x)\, dx \Big)^{1/p}, \tag{9}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, otherwise the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{10}$$

expressed by the inequality (9), is compact. Note that $(X, |||.|||_X)$ is a uniformly convex (and thus, reflexive) Banach space.

Remark 2.2 If we suppose that $w_0(x) \equiv 1$, the integrability condition holds : there exists $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega), \text{ for all } i = 1.....N, \tag{11}$$

and note that the assumption *(11)* is stronger than *(5), then*

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\partial_i u|^p \, w_i(x)\, dx \Big)^{1/p} \tag{12}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and equivalent to *(6),* and also, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega) \tag{13}$$

is compact for all $1 \le q \le p_1^*$ if $p\nu < N(\nu + 1)$, and for all $q \ge 1$ if $p\nu \ge N(\nu + 1)$, where $p_1 = p\nu/\nu + 1$ *and* p_1^* *is the Sobolev conjugate of* p_1 *(see [12], pp.30-31).*

3 Notions of Solutions and Properties of p-Laplace Operator

In this section, we give the definition of a weak solution for problem (1), and we discuss the p-Laplace operator $\triangle_p u := div(|\nabla u|^{p-2} \nabla u)$.

Definition 3.1 A point $u \in W_0^{1,p}(\Omega, w)$ is said to be a weak solution of (1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega (\lambda |u|^{q-2}u + f(x,\ u,\ \nabla u))v\, dx, \quad \forall v \in W_0^{1,p}(\Omega, w).$$

Let us consider the following functional :

$$K\, u = \int_\Omega \frac{1}{p}|\nabla u|^p\, dx, \qquad u \in X := W_0^{1,p}(\Omega, w)\ .$$

In view of [9], we have $K \in C^1(X, \mathbb{R})$, and the p-Laplace operator is the derivative operator of K in the weak sense. We denote $L = K' : X \to X^*$, then

$$\langle Lu\, ,\ v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx \qquad \forall v,\ u \in X.$$

Lemma 3.1 *i)* $L : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*

ii) L *is a mapping of type* (S_+)*;*

iii) $L : X \to X^*$ *is a homeomorphism.*

Proof. *i*) It is obvious that L is continuous and bounded. For all $\xi, \eta \in \mathbb{R}^N$, we obtain the following inequalities (see [15]), from which we can get the strict monotonicity of L :

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq (\frac{1}{2})^p|\xi - \eta|^p, \qquad p \geq 2. \tag{14}$$

ii) From (*i*), if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle L\, u_n - L\, u\, ,\, u_n - u\rangle \leq 0$, then

$$\lim_{n\to\infty}\langle L\, u_n\, ,\, u_n - u\rangle = \lim_{n\to\infty}\langle L\, u_n - L\, u\, ,\, u_n - u\rangle = 0.$$

In view of (14), ∇u_n converges in measure to ∇u in Ω, so we get a subsequence denoted again by ∇u_n satisfying $\nabla u_n(x) \to \nabla u(x)$, a.e. $x \in \Omega$.

Since $u_n \rightharpoonup u$ in $X = W_0^{1,p}(\Omega, w)$, one has $(u_n)_n$ is bounded. Therefore, the sequence $(|\nabla u_n|^{p-2}\nabla u_n)_n$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,\ w_i^*)$ and $|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u$ a.e. in Ω, according to Lemma 2.1 in [2] we have

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } \prod_{i=1}^N L^{p'}(\Omega,\ w_i^*) \quad \text{and a.e. in } \Omega.$$

We set $\overline{y}_n = |\nabla u_n|^p$ and $\overline{y} = |\nabla u|^p$. As in [13] (Lemma 5) we can write

$$\overline{y}_n \to \overline{y} \quad \text{in } L^1.$$

By (2) we have

$$\gamma \sum_{i=1}^N w_i |\partial_i u_n|^p \leq |\nabla u_n|^p.$$

Let $z_n = \sum_{i=1}^{N} w_i |\partial_i u_n|^p, \ z = \sum_{i=1}^{N} w_i |\partial_i u|^p, y_n = \frac{\overline{y}_n}{\gamma}$ and $y = \frac{\overline{y}}{\gamma}$. Then, by Fatou's theorem, we obtain

$$\int_\Omega 2y dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z| dx,$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z| dx$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z| dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z| dx \leq 0.$$

This implies

$$\nabla u_n \to \nabla u \quad \text{in} \quad \prod_{i=1}^{N} L^p(\Omega, \ w_i).$$

Hence $u_n \to u$ in $W_0^{1,p}(\Omega, w)$, i.e., L is of type (S_+).

iii) By the strict monotonicity, L is an injection. Since

$$\lim_{|||u|||\to\infty} \frac{\langle L\, u_n, \, u_n - u\rangle}{|||u|||} = \lim_{|||u|||\to\infty} \frac{\int_\Omega |\nabla u|^p \, dx}{|||u|||} = \infty,$$

L is coercive, thus L is a surjection in view of the Minty – Browder theorem (see [24], Theorem 26A) Hence L has an inverse mapping $L^{-1} : X^* \to X$. Therefore, the continuity of L^{-1} is sufficient to ensure L to be a homeomorphism.

If $f_n, \ f \in X^*, \ f_n \to f$, let $u_n = L^{-1} f_n, \ u = L^{-1} f$, then $L\, u_n = f_n, \ L\, u = f$.

So $(u_n)_n$ is bounded in X. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $f_n \to f$, we have

$$\lim_{n\to\infty} \langle Lu_n - Lu_0, \ u_n - u_0\rangle = \lim_{n\to\infty} \langle f_n, u_n - u_0\rangle = 0. \tag{15}$$

Since L is of type (S_+), $u_n \to u_0$, we conclude that $u_n \to u$, so L^{-1} is continuous.

4 Existence of Solutions

In this section, we study the strongly nonlinear problem (1) based on the degree theory in Section 2.

Lemma 4.1 *Under assumption* (3)*, the operator* $S : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*)$ *set by*

$$\langle S\, u, \ v\rangle = -\int_\Omega (\lambda |u|^{q-2} u + f(x, \ u, \ \nabla u)) v dx, \qquad \forall u, \ v \in W_0^{1,p} \ (\Omega, w),$$

is compact.

Proof. **Step 1.** Let $\psi : W_0^{1,p}(\Omega, w) \to L^{p'}(\Omega)$ be the operator defined by

$$\psi\, u(x) := -\lambda |u(x)|^{q-2} u(x) \quad \text{for} \quad u \in W_0^{1,p}(\Omega, w) \quad \text{and} \quad x \in \Omega.$$

It is obvious that ψ is continuous. We prove that ψ is bounded.

We pose $\alpha = (q-1)p'$ with $1 < \alpha < p$. For each $u \in W_0^{1,p}(\Omega, w)$, by using the continuous embedding $L^p(\Omega) \hookrightarrow L^\alpha(\Omega)$ and the Hölder and the Hardy inequality, we have

$$\begin{aligned}
\|\psi\, u\|_{p'}^{p'} &= \int_\Omega |-\lambda |u|^{q-2}u|^{p'} dx \\
&\le \lambda^{p'} \int_\Omega |u|^{(q-1)p'} dx \\
&\le C\, \lambda^{p'} \int_\Omega |u|^p dx \\
&\le C\, \lambda^{p'} \int_\Omega |u|^p \sigma^{\frac{p}{q}}\, \sigma^{-\frac{p}{q}} dx \\
&\le C\, \lambda^{p'} \left(\int_\Omega |u|^q \sigma dx \right)^{\frac{p}{q}} \left(\int_\Omega \sigma^{-\frac{p}{q-p}} dx \right)^{\frac{q-p}{q}} \\
&\le c' \Big(\sum_{i=1}^N \int_\Omega |\partial_i u|^p\, w_i\, dx \Big) \Big(\int_\Omega \sigma^{-\frac{p}{q-p}} dx \Big)^{\frac{q-p}{q}} \\
&\le C' \||u|\|^p. \quad (\text{ Due to } (8))
\end{aligned}$$

This implies that $\|\psi\, u\|_{p'} \le C'\, \||u|\|^{p-1}$. Finally, ψ is bounded on $W_0^{1,p}(\Omega, w)$.

Step 2. Let $\varphi : W_0^{1,p}(\Omega, w) \to L^{p'}(\Omega)$ be an operator defined by

$$\varphi u(x) := -f(x,\ u,\ \nabla u) \quad \text{for} \quad u \in W_0^{1,p}(\Omega, w) \quad \text{and} \quad x \in \Omega.$$

We show that φ is bounded and continuous. For any $u \in W_0^{1,p}(\Omega, w)$, we have by the growth condition (3) and the Hardy inequality (9) that

$$\begin{aligned}
\|\varphi\, u\|_{p'} &\le \left(\int_\Omega |f(x,\ u_n,\ \nabla u_n)|^{p'} dx \right)^{\frac{1}{p'}} \\
&\le \beta \left(\int_\Omega \big(M(x) + \sigma^{\frac{1}{p'}} |u|^{\frac{q}{p'}} + \sum_{i=1}^N w_i^{\frac{1}{p'}} |\partial_i u|^{p-1} \big)^{p'} dx \right)^{\frac{1}{p'}} \\
&\le C_1 \beta \left(\int_\Omega (M(x))^{p'} dx + \int_\Omega |u|^q \sigma dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx \right)^{\frac{1}{p'}} \\
&\le C_1 \beta \left(\int_\Omega (M(x))^{p'} dx \right)^{\frac{1}{p'}} + C_1 \beta \left(\int_\Omega |u|^q \sigma dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx \right)^{\frac{1}{p'}} \\
&\le C_3 \left(\sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx \right)^{\frac{1}{p'}} \\
&\le C_4 \||u|\|^{\frac{p}{p'}}.
\end{aligned} \tag{16}$$

This implies that φ is bounded on $W_0^{1,p}(\Omega, w)$.

Now, we prove that φ is continuous, let $u_n \to u$ in $W_0^{1,p}(\Omega, w)$. Then $u_n \to u$ in $L^p(\Omega, w_0)$ and $\nabla u_n \to \nabla u$ in $L^p(\Omega, w)$. Hence there exist a subsequence still denoted by (u_n) and measurable functions h in $L^p(\Omega, w_0)$ and g in $\prod_{i=1}^{N} L^{p_i}(\Omega, w_i)$ such that

$$u_n(x) \to u(x) \quad \text{and} \quad \nabla u_n(x) \to \nabla u(x),$$

$$|u_n(x)| \leq h(x) \quad \text{and} \quad |\nabla u_n(x)| \leq |g(x)|$$

for a.e., $x \in \Omega$ and all $n \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain that

$$f(x,\ u_n(x),\ \nabla u_n(x)) \to f(x,\ u(x),\ \nabla u(x)) \quad \text{a.e. } x \in \Omega. \tag{17}$$

According to (3) we get

$$|f(x,\ u_n(x),\ \nabla u_n(x))| \leq \beta(M(x) + \sigma^{\frac{1}{p'}} |h(x)|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |g_j(x)|^{p-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$M(x) + \sigma^{\frac{1}{p'}} |h(x)|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |g_j(x)|^{p-1} \in L^{p'}(\Omega),$$

and using (17), we have

$$\int_\Omega |f(x,\ u_k(x),\ \nabla u_k(x)) - f(x,\ u(x),\ \nabla u(x))|^{p'} dx \longrightarrow 0.$$

The dominated convergence theorem implies that

$$\varphi u_k \to \varphi u \quad \text{in} \quad L^{p'}(\Omega).$$

Thus, the entire sequence (φu_n) converges to φu in $L^{p'}(\Omega)$, and then φ is continuous.

Step 3. As the embedding $I : W_0^{1,p}(\Omega, w) \to L^{p'}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'}(\Omega) \to W^{-1,p'}(\Omega, w^*)$ is also compact. Therefore, the compositions $I^* o\varphi$ and $I^* o\varphi : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*)$ are compact. We conclude that $S = I^* o\varphi + I^* o\varphi$ is compact. This completes the present proof.

Theorem 4.1 *Assume that hypothesis* (3) *is satisfied. Then problem* (1) *has a weak solution* u *in* $W_0^{1,p}(\Omega, w)$.

Proof. Let $S : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*)$ be as defined in Lemma 4.1 and $L : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*)$, set by

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx, \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega, w).$$

Then $u \in W_0^{1,p}(\Omega, w)$ is a weak solution of (1) if and only if

$$Lu = -Su. \tag{18}$$

By dint of the properties of the operator L given in Lemma 3.1 and in view of the Minty-Browder theorem (see [24], Theorem 26 A), the inverse operator $T := L^{-1}$:

$W^{-1,p'(x)}(\Omega) \to W_0^{1,p}(\Omega, w)$ is bounded, continuous and satisfies condition (S_+). Furthermore, note that by Lemma (4.1) the operator S is bounded, continuous and quasi-monotone.

Consequently, equation (18) is equivalent to

$$u = Tv \quad \text{and} \quad v + SoTv = 0. \tag{19}$$

According to the terminology of [24], the equation $v + SoTv = 0$ is an abstract Hammerstein equation in the reflexive Banach space $W^{-1,p'}(\Omega, w^*)$.

To solve equations (19), we will apply the degree theory introduced in Section 2. To do this, we first show that the set

$$B := \Big\{ v \in W^{-1,p'}(\Omega, w^*) \mid v + tSoTv = 0 \quad \text{for some} \quad t \in [0,1] \Big\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$.

According to (2), (8), (9) and (16), the Hölder inequality, the Young inequality and the continuous embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$, we get

$$\begin{aligned}
|||Tv|||^p = |||u|||^p &= \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i \, dx \\
&\le \frac{1}{\gamma} \sum_{i=1}^N \int_\Omega |\partial_i u|^p dx = \frac{1}{\gamma}\langle Lu, u\rangle = \frac{1}{\gamma}\langle v, Tv\rangle \\
&\le \frac{t}{\gamma} |\langle S \circ Tv, Tv\rangle| \\
&\le \frac{t\lambda}{\gamma} \int_\Omega |u|^q dx + \frac{t}{\gamma}\int_\Omega |f(x,\ u,\ \nabla u)| u dx \\
&\le C_1 \int_\Omega |u|^p dx + C_2\Big(\int_\Omega |f(x,\ u,\ \nabla u)|^{p'} dx\Big)^{\frac{1}{p'}} + C_3\Big(\int_\Omega |u|^p dx\Big)^{\frac{1}{p}} \\
&\le C_1 \int_\Omega |u|^p \sigma^{\frac{p}{q}} \sigma^{-\frac{p}{q}} dx + C_2\Big(\int_\Omega |f(x,\ u,\ \nabla u)|^{p'} dx\Big)^{\frac{1}{p'}} \\
&\quad + C_3\Big(\int_\Omega |u|^p \sigma^{\frac{p}{q}} \sigma^{-\frac{p}{q}} dx\Big)^{\frac{1}{p}} \\
&\le C_1\Big(\int_\Omega |u|^q \sigma dx\Big)^{\frac{p}{q}} \Big(\int_\Omega \sigma^{\frac{-p}{q-p}} dx\Big)^{\frac{q-p}{q}} + C_2\|\varphi u\|_{p'} \\
&\quad + C_3\Big(\int_\Omega |u|^q \sigma dx\Big)^{\frac{1}{q}} \Big(\int_\Omega \sigma^{\frac{-p}{q-p}} dx\Big)^{\frac{q-p}{pq}} \\
&\le C_1' \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i dx + C_2' |||Tv|||^{\frac{p}{p'}} + C_3'\Big(\sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i dx\Big)^{\frac{1}{p}} \\
&\le C_1' |||Tv|||^p + C_2' |||Tv|||^{\frac{p}{p'}} + C_3' |||Tv|||.
\end{aligned}$$

It follows that $\{Tv \mid v \in B\}$ is bounded.

Since the operator S is bounded, it is obvious from (19) that the set B is bounded in $W^{-1,p'}(\Omega, w^*)$. Consequently, we can now choose a positive constant R such that

$$\|v\|_{W^{-1,p'}(\Omega,w^*)} < R \quad \text{for all} \quad v \in B.$$

As a result

$$v + tSoTv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

Notice by Lemma 2.1 it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = LoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + SoT$ is also bounded. We conclude that

$$I + SoT \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \quad \text{and} \quad I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider an affine homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'}(\Omega, w^*)$ given by

$$H(t,\ v) := v + tSoTv \quad \text{for} \quad (t,\ v) \in [0,1] \times \overline{B_R(0)}.$$

From the homotopy invariance and normalization property of the degree d stated in Theorem 2.1, we have

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

then there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

Finally, we conclude that $u = Tv$ is a weak solution of (1). This completes the proof.

Example. Let us consider the following special case :

$$a(x, \nabla u) = |\nabla u|^{p-2}\nabla u \quad \text{and} \quad f(x, s, \xi) = \sum_{i=1}^{N} w_i |\xi_i|^{p-1} sign(\xi_i).$$

It is easy to show that the Carathéodory function $f(x, s, \xi)$ satisfies the growth condition (3). In particular, let us use a special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $w(x) = d^{\varrho(x)}, \sigma(x) = d^{\mu}(x)$. Finally, the hypotheses of Theorem 4.1 are satisfied. Therefore, the following problem:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega (\lambda |u|^{q-2}u + f(x, u, \nabla u))v\, dx, \quad \forall v \in W_0^{1,p}(\Omega, w),$$

has at least one weak solutions.

5 Conclusion

In this paper, the existence of weak solutions to the stated problem (1) is proved in the weighted Sobolev space, by using the Berkovits topological degree theory. All this, after transforming this Dirichlet boundary value problem related to the p-Laplacian with nonlinearity into a new one governed by a Hammerstein equation. We intend to apply the method for higher order and higher dimensional PDEs of physical interest.

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