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Natural Daftardar-Jafari Method for Solving Fractional Partial Differential Equations

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Abstract: In this paper we introduce a new method, the natural Daftardar-Jafari method for solving fractional differential equations. This method is a combination of the natural transform and an iterative technique. The fractional derivative is considered in the Caputo sense.

Keywords: *fractional partial differential equations, natural transform, Daftardar-Jafari method, Caputo fractional derivative.*

Mathematics Subject Classification (2010): 34A08, 35R11.

1 Introduction

Fractional Differential Equations(FDEs) have received so much attention in the past two decades due to their ability to model well situations that arise in different fields such as engineering, science and medicine [1]. The importance of FDEs has prompted researchers to look into the methods of their solution that are easy to implement and possess a considerable degree of accuracy. However, despite some significant progress that has been made in terms of the methods for solving FDEs, the fact remains that there are no agreed upon universal methods to solve them.

The Laplace transform method was used in [1] to solve linear ordinary and partial differential equations of fractional order. The Adomian decomposition method (ADM) was used in [2] to solve a system of non linear fractional differential equations. The fractional reduced differential transform method (FRDTM) was applied to the Klein-Gordon differential equation of fractional order in [3].

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Then, fairly recently, researchers have explored the possibility of blending integral transforms and decomposition methods, this adventure has proved to be useful. The Laplace decomposition method (LDM), a combination of the Laplace transform and the ADM, was used in [4] to solve fractional diffusion and fractional wave equations. The natural decomposition method (NDM), a combination of the natural transform and the ADM, was used in [5] to solve fractional models.

The uniqueness and existence of fractional differential equations have also been explored extensively by many researchers. In [6], the uniqueness and existence of Sobolev type differential equations of fractional order were studied. In [7], the uniqueness and existence of fractional reaction diffusion equation were also studied.

In this paper, we explore the feasibility of blending the natural transform and an iterative technique suggested by Daftardar and Jafari to come up with the natural transform Daftardar and Jafari method (NDJM). The natural transform was used to solve Maxwell's equations in [8] and to solve a fluid flow problem in [9].

The rest of the paper is structured as follows: firstly, we give some mathematical framework, secondly, we give a general description of our proposed method, thirdly, we offer examples to demonstrate the use of the method, and lastly, we draw up a conclusion.

2 Preliminaries, Analysis of the Method and Examples

In this section we give the mathematical framework that will form the basis of the discussions in this paper, we then provide a description of our proposed method and give some examples.

2.1 Preliminaries

Definition 2.1 The natural transform of the function $y(t)$ is defined as [9]

$$\mathcal{N}[y(t)] = \psi(s,u) = \int_0^\infty e^{-st} y(ut)dt, \qquad t, s, u > 0, \tag{1}$$

s and u are natural transform parameters.

To get the original function $y(t)$ we take the inverse natural transform as

$$y(t) = \mathcal{N}^{-1}[\psi(s,u)].$$

The natural transform has the following important properties [9]:

(i) It is a linear operator. Given functions $y_1(t)$ and $y_2(t)$ with defined natural transforms and constants c_1,$c_2 \in \mathbb{R}$, then

$$\mathcal{N}[c_1 y_1(t) + c_2 y_2(t)] = c_1 \mathcal{N}[y_1(t)] + c_2 \mathcal{N}[y_2(t)],$$

(ii) It exhibits time scaling property,

$$\mathcal{N}[y(ct)] = \frac{1}{c}\psi\left(\frac{s}{c}, u\right) \qquad t > 0, \quad c \in \mathbb{R}, \quad c \neq 0,$$

(iii) The natural transform of the p^{th} derivative of the function $y(t)$ is

$$\mathcal{N}[y^{(p)}(t)] = \left(\frac{s}{u}\right)^p \psi(s,u) - \sum_{i=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{p-i} y^{(i)}(0).$$

Table 2.1 provides some useful information on the natural transforms of some basic functions.

$y(t)$	$\psi(s,u)$
1	$\frac{1}{s}$
$\sin \omega t$	$\frac{u\omega}{s^2+u^2\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+u^2\omega^2}$
e^{ct}	$\frac{1}{s-cu}$
$\sinh ct$	$\frac{cu}{s^2-c^2u^2}$
$\cosh ct$	$\frac{s}{s^2-c^2u^2}$
$\frac{1}{\Gamma(n)}t^{n-1}$	$u^{n-1}s^{-n}$

Table 1: The short table of natural transforms of basic functions.

Definition 2.2 The Caputo fractional derivative of the function $y(t)$ of order μ is defined as [10]

$$\mathcal{D}_t^{\mu} y(t) = \begin{cases} \frac{1}{\Gamma(p-\mu)} \int\limits_0^t \frac{y^{(p)}(s)ds}{(t-s)^{\mu-p+1}} & \text{if} \quad p-1 < \mu \leq p, \quad p \in \mathbb{N}; \\ \frac{d^p}{dt^p} y(t) & \text{if} \quad \mu = p. \end{cases} \tag{2}$$

Definition 2.3 The Mittag-Leffler function in two parameters is defined as [10]

$$E_{\mu,\alpha}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\mu k + \alpha)} \qquad \mu, \alpha > 0. \tag{3}$$

If $\alpha = 1$ in the above definition, we get the one parameter Mittag-Leffler function.

Definition 2.4 The natural transform of the Caputo fractional derivative of order μ of the function $y(t)$ is defined as [10]

$$\mathcal{N}[\mathcal{D}_t^{\mu} y(t)] = \left(\frac{s}{u}\right)^{\mu} \psi(s,u) - \sum_{i=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-i} y^{(i)}(0), \qquad \mu \in (p-1;p]. \tag{4}$$

2.2 Analysis of the method

$$\mathcal{D}_t^{\mu} y(\xi,t) + R(y(\xi,t)) + F(y(\xi,t)) = \eta(\xi,t) \tag{5}$$

with the initial conditions

$$y^{(i)}(\xi,0) = \frac{\partial^i y(\xi,0)}{\partial t^i}, \quad i = 0,1,2,...,p-1. \tag{6}$$

$\mathcal{D}_t^\mu$ is the Caputo fractional derivative with respect to t, $R(y(\xi,t))$ represents the linear operator, $F(y(\xi,t))$ represents non linear terms and $\eta(\xi,t)$ is taken as the source term.

The first step in the NDJM is to take the natural transform on both sides of (5)

$$\mathcal{N}[\mathcal{D}_t^\mu y(\xi,t)] + \mathcal{N}[R(y(\xi,t))] + \mathcal{N}[F(y(\xi,t))] = \mathcal{N}[\eta(\xi,t)], \tag{7}$$

simplifying the above equation and applying the initial conditions yield

$$\begin{aligned} \psi(\xi,s,u) &= \left(\frac{u}{s}\right)^\mu \sum_{i=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-i} y^{(i)}(0) + \left(\frac{u}{s}\right)^\mu \mathcal{N}[\eta(\xi,t)] \\ &\quad - \left(\frac{u}{s}\right)^\mu [\mathcal{N}[R(y(\xi,t))] + \mathcal{N}[F(y(\xi,t))]]. \end{aligned} \tag{8}$$

In the second step, we take the inverse natural transform on both sides of (8) to get

$$\begin{aligned} y(\xi,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu \sum_{i=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-i} y^{(i)}(0) + \left(\frac{u}{s}\right)^\mu \mathcal{N}[\eta(\xi,t)]\right] \\ &\quad -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu [\mathcal{N}[R(y(\xi,t))] + \mathcal{N}[F(y(\xi,t))]]\right], \end{aligned}$$

the above equation can be rewritten as

$$y(\xi,t) = \mathcal{Q}(\xi,t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu [\mathcal{N}[R(y(\xi,t))] + \mathcal{N}[F(y(\xi,t))]]\right], \tag{9}$$

where $\mathcal{Q}(\xi,t)$ is the term due to the initial conditions and the source term.

In the third and final step, we apply an iterative method suggested by Daftardar and Jafari known as the Daftardar-Jafari method (DJM) [11], the solution to (5)-(6) is written as an infinite series,

$$y(\xi,t) = \sum_{n=0}^{\infty} y_n(\xi,t), \tag{10}$$

substituting (10) into (9) gives

$$\sum_{n=0}^{\infty} y_n(\xi,t) = \mathcal{Q}(\xi,t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu \left[\mathcal{N}\left[R\left(\sum_{n=0}^{\infty} y_n\right)\right] + \mathcal{N}\left[F\left(\sum_{n=0}^{\infty} y_n\right)\right]\right]\right]. \tag{11}$$

The non linear term is decomposed as in [11],

$$F\left(\sum_{n=0}^{\infty} y_n(\xi,t)\right) = F(y_0(\xi,t)) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} y_k\right) - F\left(\sum_{k=0}^{n-1} y_k\right)\right]. \tag{12}$$

Substituting (12) into (11) gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(\xi,t) &= \mathcal{Q}(\xi,t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu \mathcal{N}\left[R\sum_{n=0}^{\infty} y_n(\xi,t)\right]\right] \\ &- \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^\mu \mathcal{N}\left[F(y_0(\xi,t)) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} y_k\right) - F\left(\sum_{k=0}^{n-1} y_k\right)\right]\right]\right]. \end{aligned} \tag{13}$$

The following iteration is then deduced:

$$
\begin{aligned}
y_0(\xi,t) &= \mathcal{Q}(\xi,t),\\
y_1(\xi,t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[R(y_0(\xi,t))]\right]-\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[F(y_0(\xi,t))\right]\right],\\
y_2(\xi,t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[R(y_1(\xi,t))]\right]-\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[F(y_1+y_0)-F(y_0)\right]\right],\\
y_n(\xi,t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[R(y_{n-1})]\right]\\
&\quad -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[F(y_0+...+y_{n-1})-F(y_0+...+y_{n-2})\right]\right],\ n=3,4,...
\end{aligned}
$$

Our $n+1$ term approximate solution of (5)-(6) is given by $y(\xi,t)=y_0+y_1+...+y_n$.

2.3 Examples

Example 2.1 Consider the one dimensional time fractional diffusion equation with given initial condition [12]

$$
\mathcal{D}_t^{\mu}y(\xi,t)=\frac{1}{2}\xi^2 y_{\xi\xi}(\xi,t),\quad \mu\in(1,2],\quad (\xi,t)\in[0,1]\times[0,1], \tag{14}
$$

$$
y(\xi,0)=\xi,\quad y_t(\xi,0)=\xi^2.
$$

$R(y(\xi,t))=\frac{1}{2}\xi^2 y_{\xi\xi}(\xi,t)$, $F(y(\xi,t))=0$ and $\eta(\xi,t)=0$, we take the natural transform on both sides of (14) and use the initial conditions, this yields

$$
\psi(\xi,s,u)=\frac{\xi}{s}+\frac{u\xi^2}{s^2}+\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left(\frac{1}{2}\xi^2 y_{\xi\xi}(\xi,t\right). \tag{15}
$$

We then take the inverse natural transform of (15) to get

$$
y(\xi,t)=\xi+t\xi^2+\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\left[\mathcal{N}\left(\frac{1}{2}\xi^2 y_{\xi\xi}\right)\right]\right]. \tag{16}
$$

The NDJM then leads to the following terms:

$$
\begin{aligned}
y_0 &= \xi+t\xi^2,\\
y_1 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\left[\mathcal{N}\left(\frac{1}{2}\xi^2 v_{0\xi\xi}\right)\right]\right]=\frac{\xi^2 t^{\mu+1}}{\Gamma(\mu+2)},\\
y_2 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\left[\mathcal{N}\left(\frac{1}{2}\xi^2 v_{1\xi\xi}\right)\right]\right]=\frac{\xi^2 t^{2\mu+1}}{\Gamma(2\mu+2)},\\
y_3 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\left[\mathcal{N}\left(\frac{1}{2}\xi^2 v_{2\xi\xi}\right)\right]\right]=\frac{\xi^2 t^{3\mu+1}}{\Gamma(3\mu+2)},\\
&\vdots\\
y_n &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\left[\mathcal{N}\left(\frac{1}{2}\xi^2 v_{(n-1)\xi\xi}\right)\right]\right]=\frac{\xi^2 t^{n\mu+1}}{\Gamma(n\mu+2)}.
\end{aligned}
$$

Then our solution to (14) is given by

$$
\begin{aligned}
y(\xi,t) &= \xi+\xi^2\left(1+\frac{x^2 t^{\mu+1}}{\Gamma(\mu+2)}+\frac{\xi^2 t^{2\mu+1}}{\Gamma(2\mu+2)}+\frac{\xi^2 t^{3\mu+1}}{\Gamma(3\mu+2)}+...\right)\\
&= \xi+\xi^2\left(\sum_{n=0}^{\infty}\frac{t^{n\mu+1}}{\Gamma(n\mu+2)}\right).
\end{aligned}
$$

We obtain the same solution as in [12] where the Laplace homotopy perturbation method was used to solve the problem.

Example 2.2 Consider the one dimensional time fractional diffusion equation with given initial condition [13]

$$\begin{aligned} &\mathcal{D}_t^{\mu} y(\xi,t) = y_{\xi\xi}(\xi,t), \quad \mu \in (0,1], \quad x \in \mathbb{R}, \quad t > 0, \\ &y(\xi,0) = \sin(\xi). \end{aligned} \tag{17}$$

In this example, $R(y(\xi,t)) = y_{\xi\xi}(\xi,t)$, $F(y(\xi,t)) = 0$ and $\eta(\xi,t) = 0$. We take the natural transform on both sides of (17) and utilise the initial condition to get

$$\psi(\xi,s,u) = \frac{\sin(\xi)}{s} + \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{\xi\xi}(\xi,t)]. \tag{18}$$

We then take the inverse natural transform on both sides of the above equation, this yields

$$v(x,t) = \sin(\xi) + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[v_{\xi\xi}(\xi,t)]\right], \tag{19}$$

from (19), the NDJM then leads to

$$\begin{aligned} y_0 &= \sin(\xi), \\ y_1 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{0\xi\xi}(\xi,t)]\right] = -\frac{t^{\mu}\sin(\xi)}{\Gamma(\mu+1)}, \\ y_2 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{1\xi\xi}(\xi,t)]\right] = \frac{t^{2\mu}\sin(\xi)}{\Gamma(2\mu+1)}, \\ y_3 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{2\xi\xi}(\xi,t)]\right] = -\frac{t^{3\mu}\sin(\xi)}{\Gamma(3\mu+1)}, \\ &\vdots \\ y_n &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{(n-1)\xi\xi}(\xi,t)]\right] = \frac{(-t^{\mu})^n\sin(\xi)}{\Gamma(n\mu+1)}. \end{aligned}$$

Thus our solution to (17) is given by

$$\begin{aligned} y(x,t) &= \sin(\xi)\left(1 - \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)} - \frac{t^{3\mu}}{\Gamma(3\mu+1)} + \ldots\right) \\ &= \sin(\xi)\sum_{n=0}^{\infty}\frac{(-t^{\mu})^n}{\Gamma(n\mu+1)} = \sin(\xi)E_{\mu}(-t^{\mu}). \end{aligned}$$

This is the same solution as that obtained in [13] using the DJM.

Example 2.3 Consider the following one dimensional nonlinear time fractional diffusion equation with the given initial condition [13]:

$$\begin{aligned} &\mathcal{D}_t^{\mu} y(\xi,t) = y_{\xi\xi}(\xi,t) + 2y^2(\xi,t), \quad \mu \in (0,1], \quad x \in \mathbb{R}, \quad t > 0, \\ &y(\xi,0) = e^{-\xi}. \end{aligned} \tag{20}$$

$R(y(\xi,t)) = y_{\xi\xi}(\xi,t)$, $F(y(\xi,t)) = 2y^2(\xi,t)$ and $\eta(\xi,t)=0$, we take the natural transform on both sides of (20) and utilize the initial condition to get

$$\psi(\xi,s,u) = \frac{e^{-\xi}}{s} + \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{\xi\xi}(\xi,t)] + \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[2y^2(\xi,t)], \tag{21}$$

$$y(\xi,t) = e^{-\xi} + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{\xi\xi}(\xi,t)]\right] + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[2\xi^2(\xi,t)]\right]. \tag{22}$$

The NDJM then entails that we have the following iteration:

$$\begin{aligned}
y_0 &= e^{-\xi},\\
y_1 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{0\xi\xi}(\xi,t)]\right] + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[F(y_0)]\right],\\
&= \frac{e^{-\xi}t^{\mu}}{\Gamma(\mu+1)} + \frac{2e^{-2\xi}t^{\mu}}{\Gamma(\mu+1)},\\
y_2 &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[y_{1\xi\xi}(\xi,t)]\right] + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[F(y_1+y_0) - F(y_0)]\right],\\
&= \frac{e^{-\xi}t^{2\mu}}{\Gamma(2\mu+1)} + \frac{8e^{-2\xi}t^{2\mu}}{\Gamma(2\mu+1)} + \frac{4e^{-2\xi}t^{2\mu}}{\Gamma(2\mu+1)} + \frac{8e^{-3\xi}t^{2\mu}}{\Gamma(2\mu+1)}\\
&+ \frac{2\Gamma(2\mu+1)e^{-2\xi}t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)} + \frac{8\Gamma(2\mu+1)e^{-3\xi}t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)} + \frac{8\Gamma(2\mu+1)e^{-4\xi}t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)}.
\end{aligned}$$

Our approximate solution to (20) is then given by $y(\xi,t) = y_0 + y_1 + y_2$, we note that this is the same solution as in [13] where the DJM was used.

3 Conclusion

In this paper we managed to successfully introduce a new method, the natural Daftardar-Jafari method for solving partial differential equations of fractional order. We have shown that this new technique produces the same results as the other methods that are already in existence. This method is also applicable to ordinary and partial differential equations of integer order.

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