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Absolutely Unstable Differential Equations with Aftereffect

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Abstract: For differential equations with a finite number of delays in a finite-dimensional Banach space, the conditions for the instability of the zero solution are obtained at arbitrary constant delays.

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1 Introduction

A significant part of publications on the theory of oscillations deal with the stability of solutions of evolution equations (see [1]– [5]) and, in particular, the absolute stability of solutions of differential-difference equations (see [6], [7]– [11]). However, for such equations the instability of solutions is no less important. For example, stable evolutionary processes occurring in complex dynamic systems are possible due to the instability of some components of these systems [12]. The coexistence of stability and instability in nonlinear dynamical systems is their characteristic property.

It is natural to pay attention to the study of the absolute instability of solutions of differential equations with aftereffect. For the study of such equations see [10], [13]– [15].

In [13], sufficient conditions are obtained for the absolute instability of the zero solution of a nonlinear differential-difference equation

$$\frac{dx(t)}{dt} = Ax(t) + G(t, x(t-\Delta))$$

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in a Banach space using the essentially approximative spectrum of the operator A. In [10] and [14], necessary and sufficient conditions for the absolute instability of zero solutions to linear scalar differential-difference equations of delay and neutral types and sufficient conditions for absolute instability of solutions to systems of linear differential-difference equations of delay type are obtained. In [15], necessary and sufficient conditions are established for the absolute instability of zero solutions of linear differential-difference equations with self-adjoint operator coefficients and an infinite number of deviations of the argument.

Examples of absolutely stable and absolutely unstable systems are given in [8] and [10].

Let E be a finite-dimensional Banach space over a field $\mathbb{C}$ with a norm $\|\cdot\|_E$ and $L(E,E)$ be a Banach algebra of linear continuous operators $A : E \to E$ with a unit operator I and a norm $\|A\|_{L(E,E)} = \sup_{\|x\|_E=1} \|Ax\|_E$.

Consider the equations

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{n} A_k x(t-\Delta_k), \quad t \geq 0, \tag{1}$$

and

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{n} A_k x(t-\Delta_k) + F(t, x(t), x(t-\Delta_1), \ldots, x(t-\Delta_n)), \quad t \geq 0, \tag{2}$$

where $n \in \mathbb{N}$, $A_0, A_1, \ldots A_n$ are the elements of the algebra $L(E,E)$, $\Delta_1, \ldots, \Delta_n$ are non-negative numbers, and $F : [0,+\infty) \times E^{n+1} \to E$ is a continuous mapping for which $F(t, 0, 0, \ldots, 0) = 0$ for all $t \geq 0$.

The purpose of this paper is to find the conditions for the instability of zero solutions of equations (1) and (2) for arbitrary $\Delta_1 \geq 0, \ldots, \Delta_n \geq 0$. In this case, the zero solutions of equations (1) and (2) will be called absolutely unstable.

2 Preliminaries

We will use the following sets:

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$

$$\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\},$$

$$\mathbb{C}_0 = \{z \in \mathbb{C} : \operatorname{Re} z = 0\},$$

$$\mathbb{C}_\gamma = \{z \in \mathbb{C} : \operatorname{Re} z = \gamma\},$$

$$\mathbb{C}_{(\gamma_1,\gamma_2)} = \{z \in \mathbb{C} : \operatorname{Re} z \in (\gamma_1, \gamma_2)\},$$

$$\mathbb{C}_{[\gamma_1,\gamma_2]} = \{z \in \mathbb{C} : \operatorname{Re} z \in [\gamma_1, \gamma_2]\},$$

$$i\mathbb{C}_+ = \{iz : z \in \mathbb{C}_+\},$$

$$-i\mathbb{C}_+ = \{-iz : z \in \mathbb{C}_+\},$$

$$K^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_l| \leq 1, l = \overline{1,n}\}$$

and

$$T^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_l| = 1, l = \overline{1,n}\}.$$

where $\operatorname{Re} z$ is the real part of the number $z \in \mathbb{C}$, i is the imaginary unit, $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1 < \gamma_2$.

We denote by $\sigma(A)$ the spectrum of the operator $A \in L(E,E)$, and by $\operatorname{co} G$, $\operatorname{int} G$ and ∂G the convex hull, the interior, and the boundary of the set G, respectively.

In the sequel the following two theorems on the properties of the spectrum of values of operator functions are of importance as well as the theorem on the instability of the zero solution of equation (2) in the first approximation.

Theorem 2.1 *Let a function $X(z) = X(z_1, \dots, z_n)$ with values in $L(E,E)$ be continuous with respect to $z = (z_1, \dots, z_n)$ on $\Omega = \Omega_1 \times, \dots, \times \Omega_n$, where $\Omega_1, \dots, \Omega_n$ are bounded closed subsets of the set $\mathbb{C}$, and holomorphic for each variable z_k on $\operatorname{int} \Omega_k$, $i = \overline{1,n}$, for arbitrary $z_l \in \Omega_l$, $l \in \{1, \dots, n\} \setminus \{k\}$.*

Then, $\operatorname{co} \bigcup_{z \in \Omega} \sigma(X(z)) = \operatorname{co} \bigcup_{z \in Q} \sigma(X(z))$, where $Q = \partial\, \Omega_1 \times, \dots, \times \partial\, \Omega_n$.

Note that the statement of Theorem 2.1 is correct if the Banach algebra $L(E,E)$ is replaced by an arbitrary Banach algebra with unit [16]. This statement is a generalization of the maximum principle of module [17].

Theorem 2.2 *Let the following conditions be satisfied:*
(1) $Y(z)$ is a continuous function on $\mathbb{C}_{[\gamma_1,\gamma_2]}$ with values in $L(E,E)$;
(2) $\sigma(Y(z)) \subset \mathbb{C}_+$ for all $z \in \mathbb{C}_{\gamma_1}$;
(3) $\sigma(Y(z)) \subset \mathbb{C}_-$ for all $z \in \mathbb{C}_{\gamma_2}$;
(4) for the set

$$N(y) = \{x + yi : x \in (\gamma_1, \gamma_2),\ \sigma(Y(x+yi)) \cap \mathbb{C}_0 \neq \emptyset\}, \quad y \in \mathbb{R}, \tag{3}$$

the relations

$$N(y_1) \subset i\mathbb{C}_+ \tag{4}$$

and

$$N(y_2) \subset -i\mathbb{C}_+ \tag{5}$$

are satisfied for some numbers $y_1 > 0$ and $y_2 < 0$.

Then there is a point $z_0 \in \mathbb{C}_{(\gamma_1,\gamma_2)}$ for which $0 \in \sigma(Y(z_0))$.

Proof. The spectrum $\sigma(Y(z))$ will be considered as a function defined on the set $\mathbb{C}_{[\gamma_1,\gamma_2]}$ with values in the set of non-empty compact subsets of the set $\mathbb{C}_{[\gamma_1,\gamma_2]}$ using the Hausdorff distance between two sets [18]. By virtue of the first condition of the theorem and the finite dimension of the space E, this function is continuous on the set $\mathbb{C}_{[\gamma_1,\gamma_2]}$ [19]. Also, this function is bounded and uniformly continuous on each compact subset of $\mathbb{C}_{[\gamma_1,\gamma_2]}$. Therefore, by the second and third conditions of the theorem, the set $N(y)$ is a non-empty and compact set for each $y \in \mathbb{R}$.

According to (3), each point $x + yi \in N(y)$ corresponds to a set

$$M(x+yi) \subset \sigma(Y(x+yi)) \cap \mathbb{C}_0$$

containing at least one element. Consider the set

$$N_*(y) = \bigcup_{x+yi \in N(y)} M(x+yi).$$

Due to the uniform continuity of $\sigma(Y(x+yi))$ on

$$\mathbb{C}_{[\gamma_1,\gamma_2]} \cap \{z \in \mathbb{C} : |\mathrm{Im}\, z| \le \max\{y_1, |y_2|\}\},$$

the set $N_*(y)$ continuously depends on y on $[y_2, y_1]$. Considering that by virtue of (4) and (5)

$$N_*(y_1) \subset i\mathbb{C}_+$$

and

$$N_*(y_2) \subset -i\mathbb{C}_+,$$

we conclude that $0 \in N_*(y_0)$ for some $y_0 \in (y_2, y_1)$. Then $0 \in \sigma(Y(x_0 + y_0 i))$ for some $x_0 \in (\gamma_1, \gamma_2)$.

Theorem 2.2 is proven. □

It is obvious that Theorem 2.2 is a generalization of the first Bolzano-Cauchy theorem.

Denote by $B[0, r]$ the closed ball $\{x \in E : \|x\|_E \le r\}$.

Theorem 2.3 *Suppose that*

(1) $\Big\{p \in \mathbb{C}_+ : 0 \in \sigma\Big(-pI + A_0 + \sum\limits_{k=1}^{n} e^{-p\Delta_k} A_k\Big)\Big\} \ne \emptyset$*;*

(2) there are numbers $r > 0$ and $N > 0$ such that the relation

$$\sup_{t\ge 0} \|F(t, x_1, x_2, \ldots, x_{n+1}) - F(t, y_1, y_2, \ldots, y_{n+1})\|_E \le N \max_{l=\overline{1,n+1}} \|x_l - y_l\|_E$$

for all $x_l, y_l \in B[0, r]$, $l = \overline{1, n+1}$, is satisfied;

(3) there are numbers $r > 0$, $b > 0$ and $\mu > 0$ such that the relation

$$\sup_{t\ge 0} \|F(t, x_1, x_2, \ldots, x_{n+1})\|_E \le b \max_{l=\overline{1,n+1}} \|x_l\|_E^{1+\mu},$$

for all $x_l \in B[0, r]$, $l = \overline{1, n+1}$, is satisfied.

Then the zero solution of equation (2) *is unstable.*

Note that the substantiation of Theorems 2.1 and 2.3 is given in papers [16] and [6], respectively.

3 Main Results

Theorem 3.1 *Suppose that*

$$\bigcup_{z\in T^n} \sigma\Big(A_0 + \sum_{l=1}^{n} z_l A_l\Big) \subset \mathbb{C}_+. \tag{6}$$

Then the zero solution of equation (1) *is absolutely unstable.*

Proof. We fix arbitrary $\Delta_1 \ge 0, \ldots, \Delta_n \ge 0$. Consider the characteristic function $\chi : \mathbb{C} \to L(E, E)$, which corresponds to equation (1) and is determined by the equality

$$\chi(p) = -pI + A_0 + \sum_{k=1}^{n} e^{-p\Delta_k} A_k.$$

Theorem 2.2 is applicable to this function in the case of $\gamma_1 = 0$ and $\gamma_2 = 2\sum_{l=0}^{n} \|A_l\|_{L(E,E)}$. Indeed, the function $\chi(p)$, as defined, is continuous in $\mathbb{C}_{[0,\gamma_2]}$. Because of (6) and Theorem 2.1

$$\bigcup_{z\in K^n} \sigma\Big(A_0 + \sum_{l=1}^{n} z_l A_l\Big) \subset \mathbb{C}_+.$$

Therefore, for all $p \in \mathbb{C}_{[0,\gamma_2]}$

$$\sigma\Big(A_0 + \sum_{k=1}^{n} e^{-p\Delta_k} A_k\Big) \subset \mathbb{C}_+.$$

Consequently, according to the Dunford theorem on the spectrum mapping of the operator, [20] $\sigma(\chi(p)) \subset \mathbb{C}_+$ for all $p \in \mathbb{C}_0$ and $\sigma(\chi(p)) \subset \mathbb{C}_-$ for all $p \in \mathbb{C}_{\gamma_2}$. Also, according to the Dunford theorem, the set $N(y) = \{x + yi : x \in (0, \gamma_2),\ \sigma(\chi(x + yi)) \cap \mathbb{C}_0 \neq \emptyset\}$ for $y_1 = \gamma_2$ and $y_2 = -\gamma_2$ satisfies relations (4) and (5).

Thus, for the function $\chi(p)$, the conditions of Theorem 2.2 are satisfied.

Consequently, by Theorem 2.2, there is a $p_0 \in \mathbb{C}_{(0,\gamma_2)}$ for which $0 \in \sigma(\chi(p_0))$. This means that for some normalized vector $a \in E$, the vector function $x(t) = e^{p_0 t}a$ is a solution of equation (1). By virtue of the linearity of equation (1) for each $\varepsilon > 0$, the function $\varepsilon x(t)$ is also a solution of this equation. Since $\mathrm{Re}\, p_0 > 0$, we have $\lim_{t\to+\infty} \|x(t)\|_E = +\infty$. Therefore, the zero solution of equation (1) is unstable. From the arbitrariness of the choice of $\Delta_1 \geq 0, \dots, \Delta_n \geq 0$, it follows that the zero solution of equation (1) is absolutely unstable.

Theorem 3.1 is proven. □

Theorem 3.2 *Suppose that*
(1) the relation (6) *is satisfied;*
(2) the second and third conditions of Theorem 2.3 are satisfied.
Then the zero solution of equation (2) *is absolutely unstable.*

Proof. Fix arbitrary $\Delta_1 \geq 0, \dots, \Delta_n \geq 0$. By virtue of the first condition of the theorem and the proof of Theorem 3.1, the first condition of the Theorem 2.3 is satisfied. Therefore, because of Theorem 2.3 and the second condition of Theorem 3.2, the zero solution of equation (2) is unstable. Due to the arbitrariness of the choice of $\Delta_1 \geq 0, \dots, \Delta_n \geq 0$, this solution is absolutely unstable.

Theorem 3.2 is proven. □

Corollary 3.1 *If* $\sigma(A_0) \subset \mathbb{C}_+$ *and the value of* $\sum_{l=1}^{n} \|A_l\|_{L(E,E)}$ *is sufficiently small, then the zero solutions of equations* (1) *and* (2) *are absolutely unstable.*

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