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Solving Laplace Equation within Local Fractional Operators by Using Local Fractional Differential Transform and Laplace Variational Iteration Methods

H. K. Jassim [1], J. Vahidi [2,3,*] and V. M. Ariyan [3,4]

[1] *Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq*
[2] *Department of Mathematics, Iran University of Science and Technology, Tehran, Iran*
[3] *Department of Mathematical Sciences, University of South Africa, UNISA 0003, South Africa*
[4] *Department of Mathematical Sciences, Mangosuthu University of Technology, Umlazi, South Africa*

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Abstract: In this paper, we utilize the local fractional differential transform (LFDTM) and Laplace variational iteration methods (LFLVIM) to obtain approximate solutions for the Laplace equation (LE) within local fractional derivative operators (LFDOs). The efficiency of the considered methods is illustrated by some examples. The results obtained by the LFDTM are compared with the results obtained by the LFLVIM. We demonstrate that the two approaches are very effective and convenient for finding the approximate analytical solutions of PDEs with LFDOs.

Keywords: *Laplace equation; local fractional differential transform method; local fractional Laplace variational iteration method; approximate solutions.*

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* Corresponding author: mailto:jvahidi@iust.ac.ir

1 Introduction

The LFDTM and LFVIM are powerful approximate methods for various kinds of linear and nonlinear PDEs with LFDOs. For example, the Laplace variational iteration method (LFLVIM) has been applied to PDEs in physics and mathematics. Jassim et al. applied this method to diffusion and wave equations [1] and the Laplace equation [2]. Furthermore, Liu et al. [3] used the LFLVIM for a fractal vehicular traffic flow, and Li et al. to a fractal heat conduction problem [4]. Furthermore, the LFDTM has been applied to solve ordinary and partial differential equations on the Cantor sets. Jafari et al. utilized this method to find the approximate solution of ODEs [5–7]. Yang et al. applied the LFDTM to solve a two dimensional diffusion equation [8].

Our aim is to extend the applications of the proposed methods to obtain the analytical approximate solutions to the Laplace equation within local fractional derivative operators of the form

$$\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\kappa^{2\vartheta}}+\frac{\partial^{2\vartheta}\psi(\eta,\kappa)}{\partial\eta^{2\vartheta}}=0 \tag{1}$$

with

$$\psi(\eta,0)=\phi_1(\eta), \quad \frac{\partial^{\vartheta}}{\partial\kappa^{\vartheta}}\psi(\eta,0)=\phi_2(\eta), \tag{2}$$

where $\phi_1(\eta)$ and $\phi_2(\eta)$ are given functions.

There are many approximate and numerical methods utilized to solve PDEs within LFDOs, namely, the LFFDM [9], LFDM [10], LFSEM [11,12], LFVIM [13–15], LFLDM [16], RDTM [17] and SVIM [18].

2 Local Fractional DTM

In the following the basic definitions and fundamental operations of the LFDTM are shown [8].

The two dimensional differential transform of the LF analytic function $\psi(\eta,\kappa)$ via LFDOs is

$$\Psi(\beta,\varepsilon)=\frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=\eta_0,\kappa=\kappa_0}, \tag{3}$$

where $\beta,\varepsilon=0,1,\dots,n$ and $0<\vartheta\le 1$.

The 2D differential inverse transform of $\Psi(\beta,\varepsilon)$ via LFDOs is

$$\psi(\eta,\kappa)=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)(\eta-\eta_0)^{\beta\vartheta}(\kappa-\kappa_0)^{\varepsilon\vartheta}. \tag{4}$$

By combining (3) and (4), it can be obtained that

$$\psi(\eta,\kappa)=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=\eta_0,\kappa=\kappa_0}(\eta-\eta_0)^{\beta\vartheta}(\kappa-\kappa_0)^{\varepsilon\vartheta}. \tag{5}$$

If $\eta_0=0$ and $\kappa_0=0$, then (3) is shown as follows:

$$\Psi(\beta,\varepsilon)=\frac{1}{\Gamma(1+\beta\vartheta)}\frac{1}{\Gamma(1+\varepsilon\vartheta)}\left[\frac{\partial^{(\beta+\varepsilon)\vartheta}\psi(\eta,\kappa)}{\partial\eta^{\beta\vartheta}\partial\kappa^{\varepsilon\vartheta}}\right]_{\eta=0,\kappa=0}, \tag{6}$$

and (4) is expressed as follows:

$$\psi(\eta,\kappa)=\sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)\eta^{\beta\vartheta}\kappa^{\varepsilon\vartheta}. \tag{7}$$

Theorem 2.1 *Suppose that $\psi(\eta,\kappa),\varphi(\eta,\kappa)$ and $\theta(\eta,\kappa)$ are local fractional analytic functions and $\Psi(\beta,\varepsilon)$, $\Phi(\beta,\varepsilon)$and $\Theta(\beta,\varepsilon)$ are their corresponding local fractional differential transforms with order of fraction ϑ, then we have*

1. *If $\psi(\eta,\kappa)=\varphi(\eta,\kappa)+\theta(\eta,\kappa)$, then $\Psi(\beta,\varepsilon)=\Phi(\beta,\varepsilon)+\Theta(\beta,\varepsilon)$.*
2. *If $\psi(\eta,\kappa)=\varphi(\eta,\kappa)+\theta(\eta,\kappa)$, then $\Psi(\beta,\varepsilon)=\sum_{r=0}^{\beta}\sum_{s=0}^{\varepsilon}\Phi(\beta,\varepsilon-s)\Theta(\beta-r,\varepsilon)$.*
3. *If $\psi(\eta,\kappa)=a\varphi(\eta,\kappa)$, where a is a constant, then $\Psi(\beta,\varepsilon)=\Phi(\beta,\varepsilon)$.*
4. *If $\psi(\eta,\kappa)=\dfrac{\partial^{\vartheta}}{\partial\eta^{\vartheta}}\varphi(\eta,\kappa)$, then $\Psi(\beta,\varepsilon)=\dfrac{\Gamma(1+(\beta+1)\vartheta)}{\Gamma(1+\beta\vartheta)}\Phi(\beta+1,\varepsilon)$.*
5. *If $\psi(\eta,\kappa)=\dfrac{\partial^{\vartheta}}{\partial\kappa^{\vartheta}}\varphi(\eta,\kappa)$, then $\Psi(\beta,\varepsilon)=\dfrac{\Gamma(1+(\varepsilon+s)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Phi(\beta,\varepsilon+1)$.*
6. *If $\psi(\eta,\kappa)=\dfrac{\partial^{(r+s)\vartheta}}{\partial\eta^{r\vartheta}\partial\kappa^{s\vartheta}}\varphi(\eta,\kappa)$, then*
   $\Psi(\beta,\varepsilon)=\dfrac{\Gamma(1+(\beta+r)\vartheta)}{\Gamma(1+\beta\vartheta)}\dfrac{\Gamma(1+(\varepsilon+s)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Phi(\beta+r,\varepsilon+s)$.
7. *If $\psi(\eta,\kappa)=\dfrac{(\eta-\eta_0)^{r\vartheta}}{\Gamma(1+r\vartheta)}\dfrac{(\kappa-\kappa_0)^{s\vartheta}}{\Gamma(1+s\vartheta)}$, $\Psi(\beta,\varepsilon)=\dfrac{\delta_{\vartheta}(\beta-r)}{\Gamma(1+r\vartheta)}\dfrac{\delta_{\vartheta}(\varepsilon-s)}{\Gamma(1+s\vartheta)}$,*
   where the local fractional Dirac delta function is given by

$$\delta_{\vartheta}(\beta-r)=\begin{cases}1, & \beta=r,\\ 0, & \beta\neq r,\end{cases} \quad \text{and} \quad \delta_{\vartheta}(\varepsilon-s)=\begin{cases}1, & \varepsilon=s,\\ 0, & \varepsilon\neq s.\end{cases}$$

3 Local Fractional LVIM

Let us consider the following local fractional PDEs on the Cantor sets with LFDOs:

$$L_{\vartheta}\varphi(\eta,\kappa)+R_{\vartheta}\varphi(\eta,\kappa)+N_{\vartheta}\varphi(\eta,\kappa)=\omega(\eta,\kappa), \tag{8}$$

where $L_{\vartheta}=\dfrac{\partial^{m\vartheta}}{\partial\kappa^{m\vartheta}}$ denotes the linear LFDO, R_{ϑ} is the remaining linear operator, N_{ϑ} represents the general nonlinear LFDO, and ω is the source term.

According to the rule of LFVIM, the correction local fractional functional for (8) is [13–15]

$$\begin{aligned}&\varphi_{n+1}(\kappa)=\varphi_n(\kappa)+\\ &\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{\sigma(\kappa-\xi)^{\vartheta}}{\Gamma(1+\vartheta)}\left(L_{\vartheta}\left[\varphi_n(\xi)\right]+R_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right]+N_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right]-\omega(\xi)\right)(d\xi)^{\vartheta},\end{aligned} \tag{9}$$

where $\frac{\sigma(\kappa-\xi)^{\vartheta}}{\Gamma(1+\vartheta)}$ is a fractal Lagrange multiplier.

For initial value problems of (8), we can start with

$$\varphi_0(\eta,\kappa) = \varphi(\eta,0) + \frac{\kappa^{\vartheta}}{\Gamma(1+\vartheta)}\varphi^{(\vartheta)}(\eta,0) + \cdots + \frac{\kappa^{(m-1)\vartheta}}{\Gamma(1+(m-1)\vartheta)}\varphi^{((m-1)\vartheta)}(\eta,0). \quad (10)$$

We now take the local fractional Laplace transform for (9), we get

$$\begin{aligned}&\widetilde{L}_{\vartheta}\left\{\varphi_{n+1}(\kappa)\right\} = \widetilde{L}_{\vartheta}\left\{\varphi_{n}(\kappa)\right\} + \quad (11)\\ &\widetilde{L}_{\vartheta}\left\{\frac{1}{\Gamma(1+\vartheta)}\int_0^{\kappa}\frac{\sigma(\kappa-\xi)^{\vartheta}}{\Gamma(1+\vartheta)}\left(L_{\vartheta}\left[\varphi_n(\xi)\right] + R_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right] + N_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right] - \omega(\xi)\right)(d\xi)^{\vartheta}\right\},\end{aligned}$$

or, equivalently,

$$\begin{aligned}\widetilde{L}_{\vartheta}\left\{\varphi_{n+1}(\kappa)\right\} &= \widetilde{L}_{\vartheta}\left\{\varphi_{n}(\kappa)\right\} + \widetilde{L}_{\vartheta}\left\{\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\right\}\times \quad (12)\\ &\quad\widetilde{L}_{\vartheta}\left\{L_{\vartheta}\left[\varphi_n(\xi)\right] + R_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right] + N_{\vartheta}\left[\widetilde{\varphi}_n(\xi)\right] - \omega(\xi)\right\}.\end{aligned}$$

Take the local fractional variation of (12), which is given by

$$\begin{aligned}&\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\varphi_{n+1}(\kappa)\right\}\right) = \delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\varphi_{n}(\kappa)\right\}\right) + \quad (13)\\ &\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\right\}\widetilde{L}_{\vartheta}\left\{\left(L_{\vartheta}\left[\varphi_n(\kappa)\right] + R_{\vartheta}\left[\widetilde{\varphi}_n(\kappa)\right] + N_{\vartheta}\left[\widetilde{\varphi}_n(\kappa)\right] - \omega(\kappa)\right)\right\}\right).\end{aligned}$$

By using the computation of (13), we get

$$\begin{aligned}\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\varphi_{n+1}(\kappa)\right\}\right) &= \delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\varphi_{n}(\kappa)\right\}\right) + \widetilde{L}_{\alpha}\left\{\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\right\}\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{L_{\vartheta}\left[\varphi_n(\kappa)\right]\right\}\right)\\ &= 0. \quad (14)\end{aligned}$$

Hence, from (14) we get

$$1 + \widetilde{L}_{\vartheta}\left\{\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\right\}s^{m\vartheta} = 0, \quad (15)$$

where

$$\begin{aligned}\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{L_{\vartheta}\left[\varphi_n(\kappa)\right]\right\}\right) &= \delta^{\vartheta}\left(s^{m\vartheta}\widetilde{L}_{\vartheta}\left\{\varphi_n(\kappa)\right\} - s^{(m-1)\vartheta}\varphi_n(0) - \cdots - \varphi_n^{((m-1)\vartheta)}(0)\right)\\ &= s^{m\vartheta}\delta^{\vartheta}\left(\widetilde{L}_{\vartheta}\left\{\varphi_n(\kappa)\right\}\right). \quad (16)\end{aligned}$$

Therefore, we have

$$\widetilde{L}_{\vartheta}\left\{\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)}\right\} = -\frac{1}{s^{m\vartheta}}. \quad (17)$$

Taking the inverse version of the Yang-Laplace transform into (17), we have

$$\frac{\sigma(\kappa)^{\vartheta}}{\Gamma(1+\vartheta)} = \widetilde{L}_{\vartheta}\left(-\frac{1}{s^{m\vartheta}}\right) = -\frac{\kappa^{(m-1)\vartheta}}{\Gamma(1+(m-1)\vartheta}. \quad (18)$$

Hence, we have the following iteration algorithm:

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_{n+1}(\kappa)\right\} &= \widetilde{L}_\vartheta\left\{\varphi_n(\kappa)\right\} - \frac{1}{s^{m\vartheta}}\widetilde{L}_\vartheta\left\{L_\vartheta\left[\varphi_n(\kappa)\right] + R_\vartheta\left[\varphi_n(\kappa)\right] + N_\vartheta\left[\varphi_n(\kappa)\right] - \omega(\kappa)\right\} \\
&= \widetilde{L}_\vartheta\left\{\varphi_n(\kappa)\right\} - \frac{1}{s^{m\vartheta}}\widetilde{L}_\vartheta\left\{s^{m\vartheta}\varphi_n(\kappa) - \cdots - \varphi_n^{((m-1)\vartheta}(0)\right\} \\
&\quad -\frac{1}{s^{m\vartheta}}\widetilde{L}_\vartheta\left\{R_\vartheta\left[\varphi_n(\kappa)\right] + N_\vartheta\left[\varphi_n(\kappa)\right] - \omega(\kappa)\right\} \\
&= \frac{1}{s^{\vartheta}}\varphi_n(0) - \frac{1}{s^{2\vartheta}}\varphi_n^{(\vartheta)}(0) - \cdots - \frac{1}{s^{m\vartheta}}\varphi_n^{((m-1)\vartheta}(0) \\
&\quad -\frac{1}{s^{m\vartheta}}\widetilde{L}_\vartheta\left\{R_\vartheta\left[\varphi_n(\kappa)\right] + N_\vartheta\left[\varphi_n(\kappa)\right] - \omega(\kappa)\right\},
\end{aligned} \tag{19}$$

where the initial value reads as

$$\widetilde{L}_\vartheta\left\{\varphi_0(\eta,\kappa)\right\} = \frac{1}{s^{\vartheta}}\varphi(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi^{(\vartheta)}(\eta,0) + \cdots + \frac{1}{s^{m\vartheta}}\varphi^{((m-1)\vartheta)}(\eta,0). \tag{20}$$

Therefore, the local fractional series solution of (8) is

$$\varphi(\eta,\kappa) = \lim_{n\to\infty}\widetilde{L}_\vartheta^{-1}\left(\widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\}\right). \tag{21}$$

4 Applications

In this section, an example for the Laplace equation involving LFDOs is presented in order to demonstrate the simplicity and the efficiency of the above methods.

Example 4.1 Let us consider the Laplace equation within LFDOs:

$$\frac{\partial^{2\vartheta}\varphi(\eta,\kappa)}{\partial\kappa^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi(\eta,\kappa)}{\partial\eta^{2\vartheta}} = 0, \tag{22}$$

$$\varphi(\eta,0) = -E_\vartheta(\eta^\vartheta), \quad \frac{\partial^{\vartheta}\varphi(\eta,\kappa)}{\partial\kappa^{\vartheta}} = 0. \tag{23}$$

I. Below we present the LFDTM.
Using the LFDTM on both sides of (22), we can write

$$\frac{\Gamma(1+(\varepsilon+2)\vartheta)}{\Gamma(1+\varepsilon\vartheta)}\Phi(\beta,\varepsilon+2) + \frac{\Gamma(1+(\beta+2)\vartheta)}{\Gamma(1+\beta\vartheta)}\Psi(\beta+2,\varepsilon) = 0. \tag{24}$$

The transformed initial conditions are

$$\Phi(\beta,0) = -\frac{1}{\Gamma(1+\beta\vartheta)}, \quad \Phi(\beta,1) = 0. \tag{25}$$

In view of (24) and (25), the results are listed as follows:

$$\Psi(0,0)=-1,\quad \Psi(0,1)=0,\quad \Psi(0,2)=\frac{1}{\Gamma(1+2\vartheta)},\quad \Psi(0,3)=0,$$
$$\Psi(0,4)=\frac{1}{\Gamma(1+4\vartheta)},\quad \Psi(0,5)=0,\quad \Psi(0,6)=\frac{1}{\Gamma(1+6\vartheta)},\quad \Psi(1,0)=-\frac{1}{\Gamma(1+\vartheta)},$$
$$\Psi(1,1)=0,\quad \Psi(1,2)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+2\vartheta)},\quad \Psi(1,3)=0,$$
$$\Psi(1,4)=-\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+4\vartheta)},\quad \Psi(1,5)=0,\quad \Psi(1,6)=\frac{1}{\Gamma(1+\vartheta)}\frac{1}{\Gamma(1+6\vartheta)},$$
$$\Psi(2,0)=-\frac{1}{\Gamma(1+2\vartheta)},\quad \Psi(2,1)=0,$$

$$\Psi(2,2)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+2\vartheta)},\ \Psi(2,3)=0,\quad \Psi(2,4)=-\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+4\vartheta)},$$
$$\Psi(2,5)=0,\quad \Psi(2,6)=\frac{1}{\Gamma(1+2\vartheta)}\frac{1}{\Gamma(1+6\vartheta)},\ \Psi(3,0)=-\frac{1}{\Gamma(1+3\vartheta)},\ \Psi(3,1)=0,$$
$$\Psi(3,2)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+2\vartheta)},\quad \Psi(3,3)=0,\quad \Psi(3,4)=-\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+4\vartheta)},$$
$$\Psi(3,5)=0,\quad \Psi(3,6)=\frac{1}{\Gamma(1+3\vartheta)}\frac{1}{\Gamma(1+6\vartheta)},\ \Psi(4,0)=-\frac{1}{\Gamma(1+4\vartheta)},\ \Psi(3,1)=0,$$
$$\Psi(4,2)=\frac{1}{\Gamma(1+4\vartheta)}\frac{1}{\Gamma(1+2\vartheta)},\quad \Psi(4,3)=0,\quad \Psi(4,4)=-\frac{1}{\Gamma(1+4\vartheta)}\frac{1}{\Gamma(1+4\vartheta)},$$
$$\Psi(4,5)=0,\quad \Psi(4,6)=\frac{1}{\Gamma(1+4\vartheta)}\frac{1}{\Gamma(1+6\vartheta)},\cdots$$

and so on. Hence, $\psi(\eta,\kappa)$ is evaluated as follows:

$$\begin{aligned}\psi(\eta,\kappa) &= \sum_{\beta=0}^{\infty}\sum_{\varepsilon=0}^{\infty}\Psi(\beta,\varepsilon)\eta^{\beta\vartheta}\kappa^{\varepsilon\vartheta} \qquad (26)\\ &= -\left[1+\frac{\eta^{\vartheta}}{\Gamma(1+\vartheta)}+\frac{\eta^{2\vartheta}}{\Gamma(1+2\vartheta)}+\cdots\right]\left[1-\frac{\kappa^{2\vartheta}}{\Gamma(1+2\vartheta)}+\frac{\kappa^{4\vartheta}}{\Gamma(1+4\vartheta)}-\cdots\right],\end{aligned}$$

which is exactly the same as the solution obtained by the LFFDM [11] and it converges to the closed form solution:

$$\psi(\eta,\kappa)=-E_{\vartheta}(\eta^{\vartheta})\cos_{\vartheta}(\kappa^{\vartheta}). \qquad (27)$$

II. As the next step we apply the LFLVIM.

In view of (19) and (22), we get the following iterative formula:

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_{n+1}(\eta,\kappa)\right\} &= \widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\} - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_n}{\partial\kappa^{2\vartheta}} + \frac{\partial^{2\vartheta}\varphi_n}{\partial\eta^{2\vartheta}}\right\} \\
&= \widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\} - \frac{1}{s^{2\vartheta}}\left[s^{2\vartheta}\widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\} - s^{\vartheta}\varphi_n(\eta,0) - \varphi_n^{(\vartheta)}(\eta,0)\right] \\
&\quad -\frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_n(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\} \\
&= \frac{1}{s^{\vartheta}}\varphi_n(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi_n^{(\vartheta)}(\eta,0) - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_n(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\}. \qquad (28)
\end{aligned}$$

From (23), the initial value reads

$$\varphi_0(\eta,\kappa) = -E_\vartheta(\eta^\vartheta). \qquad (29)$$

Hence, we get the first approximation, namely,

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_1(\eta,\kappa)\right\} &= \frac{1}{s^{\vartheta}}\varphi_0(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi_0^{(\vartheta)}(\eta,0) - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_0(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\} \\
&= -\frac{1}{s^{\vartheta}}E_\vartheta(\eta^\vartheta) + \frac{1}{s^{3\vartheta}}E_\vartheta(\eta^\vartheta). \qquad (30)
\end{aligned}$$

The second approximation reads

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_2(\eta,\kappa)\right\} &= \frac{1}{s^{\vartheta}}\varphi_1(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi_1^{(\vartheta)}(\eta,0) - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_1(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\} \\
&= -\frac{1}{s^{\vartheta}}E_\vartheta(\eta^\vartheta) + \frac{1}{s^{3\vartheta}}E_\vartheta(\eta^\vartheta) - \frac{1}{s^{5\vartheta}}E_\vartheta(\eta^\vartheta). \qquad (31)
\end{aligned}$$

The other approximations are written as

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_3(\eta,\kappa)\right\} &= \frac{1}{s^{\vartheta}}\varphi_2(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi_2^{(\vartheta)}(\eta,0) - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_2(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\} \\
&= -\frac{1}{s^{\vartheta}}E_\vartheta(\eta^\vartheta) + \frac{1}{s^{3\vartheta}}E_\vartheta(\eta^\vartheta) - \frac{1}{s^{5\vartheta}}E_\vartheta(\eta^\vartheta) + \frac{1}{s^{7\vartheta}}E_\vartheta(\eta^\vartheta). \qquad (32)
\end{aligned}$$

Proceeding in this manner, we can derive the following formula:

$$\begin{aligned}
\widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\} &= \frac{1}{s^{\vartheta}}\varphi_{n-1}(\eta,0) + \frac{1}{s^{2\vartheta}}\varphi_{n-1}^{(\vartheta)}(\eta,0) - \frac{1}{s^{2\vartheta}}\widetilde{L}_\vartheta\left\{\frac{\partial^{2\vartheta}\varphi_{n-1}(\eta,\kappa)}{\partial\eta^{2\vartheta}}\right\} \\
&= \sum_{r=0}^{n}(-1)^{r+1}\frac{1}{s^{(2r+1)\vartheta}}E_\vartheta(\eta^\vartheta). \qquad (33)
\end{aligned}$$

Consequently, the LF series solution is

$$\begin{aligned}
\varphi(\eta,\kappa) &= \lim_{n\to\infty}\widetilde{L}_\vartheta^{-1}\left(\widetilde{L}_\vartheta\left\{\varphi_n(\eta,\kappa)\right\}\right) = \widetilde{L}_\vartheta^{-1}\left[\sum_{r=0}^{\infty}(-1)^{r+1}\frac{1}{s^{(2r+1)\vartheta}}E_\vartheta(\eta^\vartheta)\right] \\
&= -E_\vartheta(\eta^\vartheta)\left[\sum_{r=0}^{\infty}(-1)^{r}\frac{\kappa^{2r\vartheta}}{\Gamma(1+2r\vartheta)}\right] = -E_\vartheta(\eta^\vartheta)\cos_\vartheta(\kappa^\vartheta), \qquad (34)
\end{aligned}$$

from Eqs. (27) and (34), the approximate solution of the Laplace equation (22) by using the LFLVIM is the same result as that obtained by the LFDTM and it clearly appears that the approximate solution remains closed form to the exact solution.

5 Conclusions

In this work, the LFDTM and LFLVIM have been successfully applied to finding the approximate analytical solutions for the Laplace equation with LFDOs. The solutions obtained by the proposed methods are an infinite power series for the appropriate initial condition, which can, in turn, be expressed in a closed form to the exact solution. The example shows that the results of the LFDTM are in excellent agreement with the results given by the LFLVIM and local fractional function decomposition method.

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