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Asymptotic Behavior in Product and Conjugate Dynamical Systems Using Bi-Shadowing Properties

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Abstract: In this paper, we study the persistence of asymptotic behavior of trajectories generated by the product of discrete-time dynamical systems and also generated by conjugate systems as well, using some shadowing and bishadowing properties. In particular, we establish a relationship between the asymptotic behavior of product systems and their subsystems and give some new results in this direction. We also show that bishadowing is invariant for the systems that are topologically conjugate under certain conditions.

Keywords: *bishadowing; dynamical systems; product systems; conjugate systems.*

Mathematics Subject Classification (2010): 37C50.

1 Introduction

In recent years, the theory of shadowing has become a significant part of qualitative theory of dynamical systems. It plays an important role in the investigation of stability theory and asymptotic behavior of discrete systems, see also [2, 3]. It is usually used to verify computer calculations of the system by ensuring the existence of the true trajectory of the system close to the calculated trajectory (also known as the pseudo trajectory), see [4, 15]. Shadowing firstly appeared in the work of Anosov [6], see also Bowen [7] and is now developed as part of the global theory of dynamical systems, see Palmer [20] and Pilyugin [21]. The relationship between pseudo trajectories and true trajectories became more important particularly in chaotic systems.

Nevertheless, it is natural to pose the inverse problem: given a dynamical system and a family of approximate trajectories, is it possible, for any true trajectory to find

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a close approximate trajectory? In general, the answer to this problem is yes, but in practice, only approximate trajectories with specific properties are considered. This is known as inverse shadowing. In the last three decades, shadowing and inverse shadowing properties have been studied extensively by many authors and various extensions of these concepts have also been obtained and applied for dynamical systems in different ways. For example, the Lipschitz shadowing property [17], limit shadowing property [11], orbital and weak shadowing properties [19], shadowing property for maps on Banach spaces [14], asymptotic shadowing [21], average shadowing property for diffeomorphisms [22], pseudo–orbit tracing property for flows [16], weak inverse shadowing [8], inverse shadowing for set-valued systems [18] and continuous inverse shadowing [12]. A combination of the two concepts of direct and inverse (indirect) shadowing, called bishadowing, was introduced in [10], see also [9]. Bishadowing was considered for set-valued systems with an application to iterated function systems in [1] and for infinite dimensional dynamical systems in [5].

The paper is organized as follows. Some definitions and preliminaries needed throughout the paper will be given in Section 2. The results of bishadowing for the product of systems will be established in Section 3 and those for conjugate systems will be given in Section 4.

2 Definitions and Preliminaries

Let (X, d_X) be a metric space and let $f : X \to X$ be a continuous map from X into itself. In this paper, we shall consider the discrete-time dynamical system on X generated by f along with its iterates. That is,

$$f^0 = id_X \qquad \text{and} \qquad f^{i+1} = f^i \circ f, \qquad i = 0, 1, 2, \cdots$$

We usually identify the map f with the dynamical system that generates.

A sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying $x_{i+1} = f(x_i)$, $i = 0, 1, \cdots$ is called a true trajectory of (the system) f. While a sequence $\{y_i\}_{i=0}^{\infty} \subset X$ satisfying

$$d_X(f(y_i), y_{i+1})) \leq \delta$$

for $i = 0, 1, \cdots$ and for some $\delta > 0$ is called a δ-pseudo-trajectory of f. Note that a true trajectory is also a δ-pseudo trajectory with $\delta = 0$. The totality of all continuous maps from the metric space X into itself will be denoted by $C(X)$.

For a given $\varepsilon > 0$, a δ-pseudo trajectory $\{x_i\}_{i=0}^{\infty}$ of f is said to be ε-traced by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all $i = 0, 1, 2, \cdots$. We say that the dynamical system generated by f has the shadowing property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo trajectory is ε-traced by some point $x \in X$. We say that the dynamical system has the inverse shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any continuous map $\phi \in C(X)$ satisfying $\sup_{x \in X} d(f(x), \phi(x)) < \delta$ and for any $x \in X$ there exists a point $x' \in X$ for which $d(f^i(x'), \phi^i(x)) < \varepsilon$, $i = 0, 1, 2, \cdots$.

We now give the definition of bishadowing of [10] in the context of a metric space.

Definition 2.1 A continuous map $f : X \to X$ is said to be bishadowing on a subset K of X with positive parameters α and β, (or (α, β)-bishadowing), if for any given δ-pseudo trajectory $\mathbf{y} = \{y_i\}_{i=0}^{\infty}$ of f in the set K with $0 \leq \delta \leq \beta$ and any continuous comparison map $\Phi \in C(X)$ satisfying

$$\delta + \sup_{x \in X} d_X(\Phi(x), f(x)) \leq \beta, \tag{1}$$

there exists a trajectory $\mathbf{x} = \{x_i\}_{i=0}^{\infty}$ of Φ in K such that

$$d_X(x_i, y_i) \leq \alpha \left(\delta + \sup_{x \in X} d_X(\Phi(x), f(x)) \right) \tag{2}$$

for all i for which x_i and y_i are defined.

Note that the definition of bishadowing include both the direct shadowing by taking as a special case $\Phi = f$ and the inverse shadowing by taking $\delta = 0$ in Definition 2.1.

3 The Product System

Throughout this section, we consider two compact metric spaces (X, d_X) and (Y, d_Y) and their product metric space $X \times Y$ with metric defined by

$$D((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in X \times Y$. We also consider continuous maps $f : X \to X$, $g : Y \to Y$, and the product $f \times g : X \times Y \to X \times Y$, where the product map is defined by $(f \times g)(x, y) = (f(x), g(y))$.

The proof of the following lemma is straightforward, so will be omitted.

Lemma 3.1 *Let $\psi : X \to X$ and $\varphi : Y \to Y$ be two continuous maps and let $a = \sup_{x \in X} d_X(f(x), \psi(x))$ and $b = \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then for $x \in X$ and $y \in Y$ we have*

$$\sup_{x \in X, y \in Y} \{\max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\}\} = \max\{a, b\}.$$

Theorem 3.1 *Assume that both f and g are (α, β)- bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$, respectively. Then the product system $f \times g$ is (α, β)-bishadowing with respect to the class $C(X) \times C(Y)$.*

Proof. Let $\{(x_i, y_i)\}_{i=0}^{\infty}$ be a δ-pseudo trajectory of the map $f \times g$, with $0 \leq \delta \leq \beta$, and let $\psi \times \varphi \in C(X) \times C(Y)$ satisfying

$$\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x, y)) \leq \beta. \tag{3}$$

Since $\{(x_i, y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of the map $f \times g$ we have, for each $i = 0, 1, \cdots$, that

$$\begin{aligned} \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} &= D((f(x_i), g(y_i)), (x_{i+1}, y_{i+1})) \\ &= D((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta. \end{aligned}$$

Thus, $d_X(f(x_i), x_{i+1}) < \delta$ and $d_Y(g(y_i), y_{i+1}) < \delta$, for $i = 0, 1, \cdots$, which implies that both $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are δ-pseudo trajectories of f and g, respectively. Since both f and g are bishadowing, then for any $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfying

$$\delta + \sup_{x \in X} d_X\left(f(x), \psi(x)\right) \leq \beta \tag{4}$$

and

$$\delta + \sup_{y \in Y} d_Y\left(g(y), \varphi(y)\right) \leq \beta, \tag{5}$$

there exist true trajectories $\{w_i\}_{i=0}^{\infty}$ of ψ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i, w_i) \le \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x) \right) \tag{6}$$

and

$$d_Y(y_i, z_i) \le \alpha \left(\delta + \sup_{y \in Y} d_Y(g(y), \varphi(y) \right), \text{ for } i = 0, 1, \cdots . \tag{7}$$

We may assume $\sup_{x\in X} d_X(f(x), \psi(x)) > \sup_{y\in Y} d_Y(g(y), \varphi(y))$, as the other direction can be treated similarly. We consider the following three cases.

Case 1: For the values of i, for which $d_X(x_i, w_i) > d_Y(y_i, z_i)$, and using the relation (6), we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &= \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} = d_X(x_i, w_i) \\ &\le \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \right). \end{aligned}$$

So, for every $x \in X$, $y \in Y$, and using Lemma 3.1, we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &\le \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \right) \\ &= \alpha \left(\delta + \max \left\{ \sup_{x \in X} d_X(f(x), \psi(x)), \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right\} \right) \\ &= \alpha \left(\delta + \sup_{x \in X, y \in Y} \max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\} \right) \\ &= \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x \times y)) \right). \end{aligned}$$

Case 2: For the values of i, for which $d_X(x_i, w_i) < d_Y(y_i, z_i)$, and using the relation (7), we have

$$\begin{aligned} D((x_i, y_i), (w_i, z_i)) &= \max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} = d_Y(y_i, z_i) \\ &\le \alpha \left(\delta + \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right) \\ &\le \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \right). \end{aligned}$$

From the argument of **Case 1** above we obtain

$$D((x_i, y_i), (w_i, z_i)) \le \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x \times y)) \right).$$

Case 3: For the values of i, for which $d_X(x_i, w_i) = d_Y(y_i, z_i)$, we have the same result as in **Case 1** and **Case 2**.

By combining the three cases, we have

$$D((x_i, y_i), (w_i, z_i)) \le \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x \times y)) \right), \ i = 0, 1, \cdots .$$

Finally, the sequence $\{(w_i, z_i)\}_{i=0}^{\infty}$ is a true trajectory of $\psi \times \varphi$ since

$$(\psi \times \varphi)(w_i, z_i) = (\psi(w_i), \varphi(z_i)) = (w_{i+1}, z_{i+1}), \quad i = 0, 1, \cdots .$$

This means that $f \times g$ is (α, β)- bishadowing with respect to the class $C(X) \times C(Y)$. This ends the proof of Theorem 3.1. $\Box$

For the converse direction of Theorem 3.1, we have the following partial result.

Theorem 3.2 *Assume that f and g are both continuous and that the product system $f \times g$ is (α, β)- bishadowing with respect to the class $C(X) \times C(Y)$. Then at least one of the maps f and g is (α, β)-bishadowing with respect to the corresponding class of comparison maps.*

Proof. Let $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be δ-pseudo trajectories of f and g, respectively, with $0 \leq \delta \leq \beta$, and let $\psi \in C(X)$ and $\varphi \in C(Y)$ satisfy the relations (4) and (5) respectively. Then for $i = 0, 1, \cdots$ we have the following estimates:

$$\begin{aligned} D((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})) &= D((f(x_i), g(y_i)), (x_{i+1}, y_{i+1})) \\ &= \max\{d_X(f(x_i), x_{i+1}), d_Y(g(y_i), y_{i+1})\} < \delta. \end{aligned}$$

Thus, the sequence $\{(x_i, y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of $f \times g$ and since $f \times g$ is bishadowing, for any map $\psi \times \varphi \in C(X) \times C(Y)$ satisfying the relation (3) there exists a true trajectory $\{(w_i, z_i)\}_{i=0}^{\infty}$ of $\psi \times \varphi$ such that

$$D((x_i, y_i), (w_i, z_i)) \leq \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D\left((f \times g)(x, y), (\psi \times \varphi)(x, y)\right) \right), \; i = 0, 1, \cdots .$$

So, for every $x \in X$, $y \in Y$, and using Lemma 3.1, we have

$$\begin{aligned} &\max\{d_X(x_i, w_i), d_Y(y_i, z_i)\} \\ &\quad \leq \alpha \left(\delta + \sup_{(x,y) \in X \times Y} D((f \times g)(x, y), (\psi \times \varphi)(x, y)) \right) \\ &\quad = \alpha \left(\delta + \sup_{(x,y) \in X \times Y} \max\{d_X(f(x), \psi(x)), d_Y(g(y), \varphi(y))\} \right) \\ &\quad = \alpha \left(\delta + \max \left\{ \sup_{x \in X} d_X(f(x), \psi(x)), \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right\} \right). \end{aligned}$$

These estimates imply the following three cases.

Case 1: If $\sup_{x \in X} d_X(f(x), \psi(x)) > \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then we have

$$d_X(x_i, w_i) \leq \alpha \left(\delta + \sup_{x \in X} d_X(f(x), \psi(x)) \right), \quad i = 0, 1, \cdots , \tag{8}$$

and hence f is (α, β)- bishadowing with respect to $C(X)$.

Case 2: If $\sup_{x \in X} d_X(f(x), \psi(x)) < \sup_{y \in Y} d_Y(g(y), \varphi(y))$, then we have

$$d_X(y_i, z_i) \leq \alpha \left(\delta + \sup_{y \in Y} d_Y(g(y), \varphi(y)) \right), \quad i = 0, 1, \cdots , \tag{9}$$

and hence g is (α,β)-bishadowing with respect to $C(Y)$.

Case 3: If $\sup\limits_{x\in X} d_X(f(x),\psi(x)) = \sup\limits_{y\in Y} d_Y(g(y),\varphi(y))$, then the relations in (8) and (9) are both satisfied, and consequently, both f and g are (α,β)-bishadowing with respect to $C(X)$ and $C(Y)$, respectively.

Finally, it should be mentioned that both $\{w_i\}_{i=0}^{\infty}$ and $\{z_i\}_{i=0}^{\infty}$ are true trajectories of ψ and φ, respectively, since

$$(\psi(w_i),\varphi(z_i)) = (\psi\times\varphi)(w_i,z_i) = (w_{i+1},z_{i+1}), \quad i=0,1,\cdots.$$

This completes the proof of Theorem 3.2. □

Now, we introduce the following definition of *mutual* bishadowing for a pair of systems generated by the continuous maps $f: X\to X$ and $g: Y\to Y$ and then establish a result that is related to the converse of Theorem 3.1.

Definition 3.1 The pair of systems (f,g) is called mutually bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$ of f and g, respectively, with positive parameters α and β (or mutually (α,β)-bishadowing), if for any given δ-pseudo trajectories $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ for f and g, respectively, and for any $\psi\in C(X)$ and $\varphi\in C(Y)$ satisfying the relations (4) and (5), respectively, there exist true trajectories $\{w_i\}_{i=1}^{\infty}$ of ψ and $\{z_i\}_{i=1}^{\infty}$ of φ such that

$$d_X(x_i,w_i)\le\alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right) \tag{10}$$

and

$$d_X(y_i,z_i)\le\alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right) \tag{11}$$

for $i=0,1,\cdots$ and $x\in X$, $y\in Y$.

In the context of the preceding definition of mutually bishadowing for a pair of systems generated by the continuous maps $f: X\to X$ and $g: Y\to Y$, we have the following result, which is an improved version of Theorem 3.2.

Theorem 3.3 *The pair of systems (f,g) is mutually (α,β)-bishadowing with respect to the comparison classes $C(X)$ and $C(Y)$ of f and g, respectively, and with positive parameters α and β if and only if $f\times g$ is bishadowing with respect to the comparison class $C(X)\times C(Y)$.*

Proof. Let $\{(x_i,y_i)\}_{i=0}^{\infty}$ be a δ-pseudo trajectory of the map $f\times g$, with $0\le\delta\le\beta$, and let $\psi\times\varphi\in C(X)\times C(Y)$ satisfying the relation (3). It follows from the proof of Theorem 3.1 that both $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are δ-pseudo trajectories of f and g, respectively. Since the pair of systems (f,g) is mutually bishadowing, then for any $\psi\in C(X)$ and $\varphi\in C(Y)$ satisfying the relations (4) and (5) there exist true trajectories $\{w_i\}_{i=0}^{\infty}$ of ψ and $\{z_i\}_{i=0}^{\infty}$ of φ satisfying the relations (10) and (11). Thus, using Lemma 3.1 and from the preceding estimates we obtain for $x\in X$ and $y\in Y$ and for $i=0,1,\cdots$

that

$$\begin{aligned}
&\max\{d_X(x_i,w_i),d_Y(y_i,z_i)\}\\
&\qquad\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right)\\
&\qquad= \alpha\left(\delta+\sup_{(x,y)\in X\times Y}\{\max\{d_X(f(x),\psi(x)),d_Y(g(y),\varphi(y))\}\right)\\
&\qquad= \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right).
\end{aligned}$$

Therefore

$$D((x_i,y_i),(w_i,z_i))\le \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right),\ i=0,1,\cdots.$$

This shows that the map $f\times g$ is (α,β)- bishadowing with respect to the comparison class $C(X)\times C(Y)$.

Conversely, let $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be δ-pseudo trajectories of f and g, respectively, with $0\le\delta\le\beta$, and let $\psi\in C(X)$ and $\varphi\in C(Y)$ satisfy the relations (4) and (5), respectively. From the proof of Theorem 3.2 it follows that the sequence $\{(x_i,y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of the map $f\times g$. Since the map $f\times g$ is (α,β)-bi- shadowing, for any map $\psi\times\varphi\in C(X)\times C(Y)$ satisfying the relation (3) there exists a true trajectory $\{(w_i,z_i)\}_{i=0}^{\infty}$ of $\psi\times\varphi$ such that

$$D((x_i,y_i),(w_i,z_i))\le \alpha\left(\delta+\sup_{(x,y)\in X\times Y} D((f\times g)(x,y),(\psi\times\varphi)(x,y))\right),\ i=0,1,\cdots.$$

Thus

$$\begin{aligned}
\max\{d_X(x_i,w_i),d_Y(y_i,z_i)\}\le \alpha\Bigg(\delta+\max\Bigg\{&\sup_{x\in X} d_X(f(x),\psi(x)),\\
&\sup_{y\in Y} d_Y(g(y),\varphi(y))\Bigg\}\Bigg),\ i=0,1,\cdots.
\end{aligned}$$

This implies that

$$d_X(x_i,w_i)\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right),$$

and

$$d_X(y_i,z_i)\le \alpha\left(\delta+\max\left\{\sup_{x\in X} d_X(f(x),\psi(x)),\sup_{y\in Y} d_Y(g(y),\varphi(y))\right\}\right),\ i=0,1,\cdots.$$

Hence, the pair of systems (f,g) is mutually (α,β)- bishadowing. This ends the proof of Theorem 3.3. □

Finally, for the continuous maps $f:X\to X$ and $g:Y\to Y$, we relate mutual bishadowing of the pair of systems (f,g) with bishadowing of f and g.

Corollary 3.1 *If both f and g are (α, β)- bishadowing with respect to the classes $C(X)$ and $C(Y)$, respectively, then the pair of systems (f, g) is mutually (α, β)-bishadowing with respect to $C(X)$ and $C(Y)$.*

Proof. The proof follows from Theorem 3.1 and Theorem 3.3. □

Corollary 3.2 *If the pair of systems (f, g) is mutually (α, β)-bishadowing with respect to $C(X)$ and $C(Y)$, respectively, then at least one of the two maps f and g is (α, β)-bishadowing with the same corresponding classes.*

Proof. The proof follows from Theorem 3.2 and Theorem 3.3. □

4 Topologically Conjugate Systems

If (X, d_X) and (Y, d_Y) are two metric spaces and $f : X \to X$ and $g : Y \to Y$ are two maps on X and Y, respectively, then we say that f and g are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. If, in addition, h is uniformly continuous, then f and g are called uniformly topologically conjugate, or simply uniformly conjugate. The two classes $C(X)$ and $C(Y)$ are called $(C(X), C(Y))$-topologically conjugate by h in the sense that individual maps in one class are topologically conjugate to a map in the other class by h.

The following two results on invariance of the shadowing and inverse shadowing properties for topologically conjugate systems are standard in the theory of shadowing, see, for example, [23].

Theorem 4.1 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps. If f and g are uniformly conjugate then f has the shadowing property if and only if g has the shadowing property.*

Theorem 4.2 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps. Assume that $C(X)$ and $C(Y)$ are $(C(X), C(Y))$-topologically conjugate. If f and g are uniformly conjugate, then f has the inverse shadowing property with respect to the class $C(X)$ if and only if g has the inverse shadowing property with respect to the class $C(Y)$.*

For the invariance of (α, β)-bishadowing for topologically conjugate systems we have the following result.

Theorem 4.3 *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be two continuous maps that are topologically conjugate by $h : X \to Y$. Assume also that $C(X)$ and $C(Y)$ are $(C(X), C(Y))$-topologically conjugate. If there exists $\lambda \geq 1$ such that*

$$d_X(x', x'') \leq d_Y(h(x'), h(x'')) \leq \lambda\, d_X(x', x''), \qquad \text{for all } x', x'' \in X, \tag{12}$$

then we have

a) *If f is (α, β)-bishadowing with respect to the comparison class $C(X)$, then g is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(Y)$.*

b) *If g has the $(\alpha, \lambda\beta)$-bishadowing with respect to $C(Y)$, then f is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(X)$.*

Proof. a) Let $\{y_i\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of g with $0 \le \delta \le \beta$, which implies that $d_Y(g(y_i), y_{i+1}) < \delta$, and let $\phi \in C(Y)$ satisfy

$$\delta + \sup_{y\in Y} d_Y(g(y), \phi(y)) \le \beta. \tag{13}$$

Note that condition (12) is equivalent to the following condition:

$$d_X(h^{-1}(y'), h^{-1}(y'')) \le d_Y(y', y'') \le \lambda d_X(h^{-1}(y'), h^{-1}(y'')) \qquad \text{for all } y', y'' \in Y. \tag{14}$$

Now,

$$d_X(f(h^{-1}(y_i)), h^{-1}(y_{i+1})) = d_X(h^{-1}(g(y_i)), h^{-1}(y_{i+1})) \le d_Y(g(y_i), y_{i+1}) < \delta.$$

Hence $\{x_i\}_{i=0}^{\infty} = \{h^{-1}(y_i)\}_{i=0}^{\infty}$ is a δ-pseudo trajectory of f. Let $x \in X$ and $\psi \in C(X)$, then using (12) and the conjugacy h, we obtain

$$d_X(f(x), \psi(x)) \le d_Y(h(f(x)), h(\psi(x))) = d_Y(g(h(x)), \phi(h(x))) = d_Y(g(y), \phi(y)),$$

where $y = h(x)$. This implies that

$$\sup_{x\in X} d_X(f(x), \psi(x)) \le \sup_{y\in Y} d_Y(g(y), \phi(y)). \tag{15}$$

From(13) and(15) and for any $\psi \in C(X)$, where $\psi = h^{-1} \circ \phi \circ h$, we have $\delta + \sup_{x\in X} d_X(f(x), \psi(x)) \le \beta$. Since f is (α, β)-bishadowing then there exists a true trajectory $\{w_i\}_{i=1}^{\infty}$ of ψ such that

$$d_X(x_i, w_i) \le \alpha(\delta + \sup_{x\in X} d_X(f(x), \psi(x))) \le \alpha(\delta + \sup_{y\in Y} d_Y(g(y), \phi(y))).$$

Thus, using the second part of (12), we obtain

$$d_Y(y_i, h(w_i)) = d_Y(h(x_i), h(w_i)) \le \lambda\alpha(\delta + \sup_{y\in Y} d_Y(g(y), \phi(y))).$$

Note that $\{h(w_i)\}_{i=0}^{\infty} := \{a_i\}_{i=0}^{\infty}$ is a true trajectory of ϕ since

$$\phi(a_i) = \phi(h(w_i)) = h(\psi(w_i)) = h(w_{i+1}) = a_{i+1}.$$

This ends the proof that g is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(Y)$. $\square$

Proof. b) Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo trajectory of f with $0 \le \delta \le \beta$, which implies that $d_X(f(x_i), x_{i+1}) < \delta$, and let $\phi \in C(Y)$ satisfy

$$\delta + \sup_{x\in X} d_X(f(x), \phi(x)) \le \beta. \tag{16}$$

Now, $d_Y(g(h(x_i)), h(x_{i+1})) = d_Y(h(f(x_i)), h(x_{i+1})) \le \lambda d_X(f(x_i), x_{i+1})$, hence

$$d_Y(g(h(x_i)), h(x_{i+1})) < \lambda\delta.$$

It follows that $\{y_i\}_{i=0}^{\infty} = \{h(x_i)\}_{i=0}^{\infty}$ is a $\lambda\delta$-pseudo trajectory of g. Now let $y \in Y$ and $\psi \in C(Y)$, then using (14) we have

$$\begin{aligned} d_Y(g(y), \varphi(y)) &\le \lambda d_X(h^{-1}(g(y)), h^{-1}(\psi(y))) \\ &= \lambda d_X(f(h^{-1}(y)), \phi(h^{-1}(y))) = \lambda d_X(f(x), \phi(x)), \end{aligned}$$

where $x = h^{-1}(y)$. Therefore $d_Y(g(y), \psi(y)) \leq \lambda d_X(f(x), \phi(x))$ for every $y \in Y$ and $x = h^{-1}(y)$, hence

$$\sup_{y \in Y} d_Y(g(y), \psi(y)) \leq \lambda \sup_{x \in X} d_X(f(x), \phi(x)). \tag{17}$$

From (16) and (17) we get for any $\psi \in C(Y)$, where $\psi = h \circ \phi \circ h^{-1}$, that

$$\lambda\delta + \sup_{y \in Y} d_Y(g(y), \psi(y)) \leq \lambda\beta.$$

Since g has the $(\alpha, \lambda\beta)$-bishadowing, there exists a true trajectory $\{z_i\}_{i=1}^{\infty}$ of ψ such that

$$\begin{aligned} d_Y(y_i, z_i) \leq \alpha(\lambda\delta + \sup_{y \in Y} d_Y(g(y), \psi(y))) &\leq \alpha(\lambda\delta + \lambda \sup_{x \in X} d_X(f(x), \phi(x))) \\ &= \lambda\alpha(\delta + \sup_{x \in X} d_X(f(x), \phi(x))). \end{aligned}$$

Thus, using (13) we obtain

$$d_X(x_i, h^{-1}(z_i)) = d_X(h^{-1}(y_i), h^{-1}(z_i)) \leq \lambda\alpha(\delta + \sup_{x \in X} d_X(f(x), \phi(x))),$$

Note that $\{h^{-1}(z_i)\}_{i=0}^{\infty} = \{b_i\}_{i=0}^{\infty}$ is a true trajectory of ϕ since

$$\phi(b_i) = \phi(h^{-1}(z_i)) = h^{-1}(\psi(z_i)) = h^{-1}(z_{i+1}) = b_{i+1}.$$

This shows that f is $(\lambda\alpha, \beta)$-bishadowing with respect to $C(X)$. $\square$

Remark 4.1 In the preceding theorem, if $\lambda = 1$, then we conclude that the map $f : X \to X$ is (α, β)-bishadowing if and only if $g : Y \to Y$ is (α, β)-bishadowing.

5 Conclusion

We have obtained some results regarding the asymptotic behavior in product and conjugate dynamical systems using bishadowing properties in Theorems 3.1, 3.2, 3.3, Corollaries 3.1, 3.2 and Theorem 4.3. These results generalize some existing results, for example, Theorems 4.1 and 4.2. This is due to the fact that the concept of bishadowing relies on two ways of comparing the trajectories of the system, unlike the case of using the direct shadowing or the inverse shadowing only.

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