Nonlinear Dynamics and Systems Theory, **20** (5) (2020) 535–541

Robust Stability of Markovian Jumping Neural Networks with Time-Varying Delays

Bandana Priya [1], Ganesh Kumar Thakur [2], Sudesh Kumar Garg [1] and M. Syed Ali [3*]

[1] *Department of Applied Scieces, G. L. Bajaj Institute of Technology and Management, Greater Noida.*
[2] *Department of Applied Sciences, Krisna Engineering College, Ghaziabad, Uttar Pradesh.*
[3] *Department of Mathematics, Thiruvalluvar University, Vellore, Tamilnadu.*

Received: May 31, 2018; Revised: November 17, 2020

Abstract: In this paper, global stability of recurrent neural networks with time-varying delays is considered. The uncertainity is considered in all the parameters of the concerned neural networks. A novel LMI-based stability criterion is obtained by using the Lyapunov functional theory to guarantee the asymptotic stability of recurrent neural networks with time-varying delays. Finally, a numerical example is given to demonstrate the correctness of the theoretical results.

Keywords: *Lyapunov functional; linear matrix inequality; recurrent neural networks; time-varying delays.*

Mathematics Subject Classification (2010): 34K20, 34K50, 92B20, 94D05.

1 Introduction

A recurrent neural network naturally involves dynamic elements in the form of feedback connections used as internal memories. Unlike the feedforward neural network whose output is a function of its current inputs only and is limited to static mapping, the recurrent neural network performs dynamic mapping. Recurrent networks are needed for the problems where there exists at least one system state variable which cannot be observed. Most of the existing recurrent neural networks are obtained by adding trainable temporal elements to the feedforward neural networks (such as multilayer perceptron networks [5] and radial basis function networks [2]) to make the output history sensitive. Like feedforward neural networks, this network function as block boxes and the meaning

* Corresponding author: mailto:syedgru@gmail.com

of each weight in these nodes is not known. They play an important role in applications such as classification of patterns, associate memories and optimization etc. (see [2], [5] and the references therein). Thus, research on the properties of an especial stability problem and relaxed stability problem of recurrent neural networks, has become a very active area in the past few years (see for example [3], [8], [9]).

It is well known that time delays are inevitably encountered in neural networks which are usually a main source of oscillation and instability, which brings to the neural network divergence and instability and needs much attention to be payed. According to the finite switching speed of amplifiers in electronic networks, time delay is either constant or time-varying. The stability criteria of neural networks with time-varying delays are classified into two categories, i.e., delay-independent [11] and delay-dependent [14]. The delay-independent stability conditions are usually more conservative than delay-dependent conditions due to the fact that they include less information concerning the time delays, especially for the time delays which are relatively small. Recently, many important results on the delay-dependent stability analysis have been reported for neural networks with time-varying delays [10], [12], [13].

In this paper, we study stability of recurrent neural networks with time-varying delays. By using the Lyapunov functional technique, global robust stability conditions for the recurrent neural networks are given in terms of LMIs, which can be easily calculated by MATLAB LMI toolbox [4]. The main advantage of the LMI based approaches is that the LMI stability conditions can be solved numerically using the effective interior-point algorithms [1]. Numerical examples are provided to demonstrate the effectiveness and applicability of the proposed stability results.

Notations: Throughout the manuscript we will use the notation $A > 0$ (or $A < 0$) to denote that the matrix A is a symmetric and positive definite (or negative definite) matrix. The shorthand *diag* $\{\cdots\}$ denotes the block diagonal matrix. $\|\cdot\|$ stands for the Euclidean norm. Moreover, the notation * always denotes the symmetric block in one symmetric matrix. Let $r(t), t \geq 0$ be a right-continuous Markov chain on a complete probability space $(\Upsilon, \mathfrak{F}, \mathcal{P})$ taking values in a finite space $S = 1, 2, ...N$ with operator $\Lambda = \Pi_{ij}(n \times n)$ given by $P\{r(t+\Delta(t)) = j|r(t) = i\} = \begin{cases} \Pi_{ij}\Delta(t) + o(\Delta(t)), i \neq j, \\ 1 + \Pi_{ij}\Delta(t) + o(\Delta(t)), i = j, \end{cases}$ where $\Delta(t) > 0$ and $\lim_{\Delta(t)\to 0} \frac{o(\Delta(t))}{\Delta(t)} = 0$, $\Pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while Π_{ii}=$-\sum_{j=1, j\neq 1}^{N} \Pi_{ij}.$, $i, j \in S$.

2 System Description and Preliminaries

Consider the following uncertain recurrent neural network with time-varying delays described by

$$\dot{v}_i(t) = -a_i(r(t))v_i(t) + \sum_{j=1}^{n} b_{ij}(r(t)) + G_j(v_j(t)) + \sum_{j=1}^{n} c_{ij}(r(t))G_j(v_j(t-\tau_j(t))) + I_i, \quad (1)$$

in which $v_i(t)$ is the activation of the i^{th} neuron. Positive constant a_i denotes the rates with which the cell i resets their potential to the resting state when isolated from the other cells and inputs. b_{ij} and c_{ij} are the connection weights at the time t, I_i denotes the external input and $G_j(\cdot)$ is the neuron activation function of j^{th} neuron. $\tau_j(t)$ is the bounded time varying delay in the state and satisfies

$0 \le \tau_j(t) \le \bar{\tau},\ 0 \le \dot{\tau}_j(t) \le d < 1,\ i,j = 1,2,..,n.$

The following assumption is made on the activation function.

(A) The neuron activation function $G_j(\cdot)$ in (1) is bounded and satisfies the following Lipschitz condition:

$$|G_j(x) - G_j(y)| \le |L_j(x-y)|$$

for all $x, y \in R, i, j = 1, 2, ..., n$, where $L_j \in R^{n\times n}$ are known constant matrices.

Assume that $v^* = (v_1^*, v_2^*, ..., v_n^*)^T$ is the equilibrium point of the system, then we shift the equilibrium points to the origin by the transformation $x_i(t) = v_i(t) - v_i^*,\ f_j(x_j(t)) = G_j(u_j(t)) - G_j(u_j^*)$. Then the transformed system is given by

$$\dot{x}_i(t) = -a_i(r(t))x_i(t) + \sum_{j=1}^{n} w_{ij}(r(t))f_j(x_j(t)) + \sum_{j=1}^{n} h_{ij}(r(t))f_j(x_j(t-\tau_j(t))). \tag{2}$$

Conveniently, we can write (2) in the form

$$\dot{x}(t) = -A(r(t))x(t) + B(r(t))f(x(t)) + C(r(t))f(x(t-\tau(t))),$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $A = diag\{a_1, a_2, \ldots, a_n\}$, $B = [(b_{ij})_{n\times n}]^T$, $C = [(c_{ij})_{n\times n}]^T$, $f(x(t)) = [f_1(x_1(t)),\ f_2(x_2(t)), \ldots, f_n(x_n(t))]^T$ and $\tau(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T$.

Then we have

$$f^T(x(t))f(x(t)) \le x^T(t)L^T Lx(t),$$

where $L = diag\{L_1, L_2, \ldots, L_n\}$. For convinience we denote, $r(t) = i$.

Lemma 2.1 *(Schur complement [1]). Let M, P, Q be given matrices such that $Q > 0$, then*

$$\begin{bmatrix} P & M^T \\ M & -Q \end{bmatrix} < 0 \iff P + M^T Q^{-1} M < 0.$$

The following Lemmas will be essential for the proofs in the next section.

Lemma 2.2 *Let $x \in R^n,\ y \in R^n$ and $\epsilon > 0$. Then we have $x^T y + y^T x \le \epsilon x^T x + \epsilon^{-1} y^T y$.*

Proof. The proof follows immediately from the inequality $(\epsilon^{1/2}x - \epsilon^{-1/2}y)^T(\epsilon^{1/2}x - \epsilon^{-1/2}y) \ge 0$.

Lemma 2.3 *[6] For any constant matrix $M \in R^{n\times n}, M = M^T > 0$, scalar $\eta > 0$, vector function $\Gamma : [0,\eta] \to R^n$ such that the integrations are well defined, the following inequality holds:*

$$\left[\int_0^\eta \Gamma(s)ds\right]^T M \left[\int_0^\eta \Gamma(s)ds\right] \le \eta \int_0^\eta \Gamma^T(s) M \Gamma(s) ds.$$

3 Stability Results

In this section, some sufficient conditions of stability for system (2) are obtained.

Theorem 3.1 *Under the assumption* (A) *the system* (2) *is robustly asymptotically stable in the mean square if there exist symmetric positive definite matrices* $P_i > 0, Q > 0$, $R > 0$, $S > 0$, *positive scalars* $\gamma_j, (j = 0, 1, 2, 3, 4)$ *and positive diagonal matrix* $M = diag\{m_1, m_2, \ldots, m_n\} > 0$ *such that feasible solutions exist for*

$$\Psi = \begin{bmatrix} \Sigma & 0 & \gamma_2 P_i B_i & \gamma_4 P_i C_i & L^T C_i & \gamma_3 L^T M & \gamma_4 L^T M & \gamma_5 L^T M \\ * & L^T C_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\gamma_5 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_1 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma_4 I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma_3 I & 0 & 0 \\ * & * & * & * & * & * & -\gamma_4 I & 0 \\ * & * & * & * & * & * & * & -\gamma_5 I \end{bmatrix} < 0, \quad (3)$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ * & \Sigma_2 & 0 \\ * & * & -\bar{\tau}^{-1} S \end{bmatrix},$$

$\Sigma_1 = -A_i^T P_i - P_i^T A_i + L^T R L + \bar{\tau} L^T S L + \gamma_1^{-1} L^T L + Q + \gamma_3^{-1} A_i^T A_i + \sum_{j=1}^{N} \Pi_{ij} P_j$,
$\Sigma_2 = -(1-d)Q - (1-d)L^T R L + \gamma_2^{-1} L^T L.$

Proof: We consider the following Lyapunov functional to derive the stability result:

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) + V_4(t, x(t)),$$

where

$$V_1(t, x(t)) = x^T(t)\ P_i\ x(t),$$

$$V_2(t, x(t)) = 2 \sum_{i=1}^{n} m_i \int_0^{x_i} f_i(s) ds,$$

$$V_3(t, x(t)) = \int_{t-\tau(t)}^{t} [x^T(s) Q x(s) ds + f^T(x(s)) R f(x(s))] ds,$$

$$V_4(t, x(t)) = \int_{t-\bar{\tau}}^{t} (s - t + \bar{\tau}) f^T(x(\theta)) S f(x(\theta)) d\theta ds.$$

We can calculate the derivative of V along the trajectories of the system (2), then we have

$$\dot{V}(t, x(t)) = \dot{V}_1(t, x(t)) + \dot{V}_2(t, x(t)) + \dot{V}_3(t, x(t)) + \dot{V}_4(t, x(t)),$$

where

$\dot{V}_1(t, x(t)) = 2x^T(t) P \dot{x}(t) = 2x^T(t) P_i[-A_i x(t) + B_i f(x(t)) + C_i f(x(t - \tau(t)))]$

$+ \sum_{j=1}^{N} \Pi_{ij} x^T(t) P_j x(t),$

$\dot{V}_2(t, x(t)) = 2 \sum_{i=1}^{n} m_i f_i(x_i(t)) \dot{x}_i(t)$

$= f^T(x(t))[-2MA_i x(t) + 2f^T(x(t))MB_i f(x(t)) + 2f^T(x(t))MC_i f(x(t-\tau(t))),$

$\dot{V}_3(t,x(t)) = x^T(t)Qx(t) - (1-d)x^T(t-\tau(t))Qx(t-\tau(t)) + f^T(x(t))Rf(x(t))$

$-(1-d)f^T(x(t-\tau(t)))Rf(x(t-\tau(t))),$

and using Lemma 2.3, we have

$\dot{V}_4(t,x(t)) = \bar{\tau} f^T(x(t))Sf(x(t)) - \int_{t-\bar{\tau}}^{t} f(x(s))Sf(x(s))ds$

$\leq \bar{\tau} f^T(x(t))Sf(x(t)) - \big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\big)^T \ \bar{\tau}^{-1}S \ \big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\big).$

It follows from Lemma 2.2 that

$$\begin{aligned}
2x^T(t)P_iBf(x(t)) &\leq \gamma_1 x^T(t)P_iB_iB_i^TP_ix(t) + \gamma_1^{-1}x^T(t)L^TLx(t),\\
2x^T(t)P_iC_if(x(t-\tau(t))) &\leq \gamma_2 x^T(t)P_iC_iC_i^TP_ix(t) + \gamma_2^{-1}x^T(t-\tau(t))L^TLx(t-\tau(t)),\\
-2f^T(x(t))MA_ix(t) &\leq \gamma_3 x^T(t)L^TMM^TLx(t) + \gamma_3^{-1}x^T(t)A_i^TA_ix(t),\\
2f^T(x(t))MW_if(x(t)) &\leq \gamma_4 x^T(t)L^TMM^TLx(t) + \gamma_4^{-1}x^T(t)L^TB_iB_i^TLx(t),\\
2f^T(x(t))MC_if(x(t-\tau(t))) &\leq \gamma_5 x^T(t)L^TMM^TLx(t) + \gamma_5^{-1}x^T(t-\tau(t))L^TC_iC_i^TLx(t-\tau(t)).
\end{aligned}$$

We obtain

$$\begin{aligned}
\dot{V} \leq x^T(t)\Big[&- A_i^TP_i - P_iA_i + \gamma_1 P_iB_iB_i^TP_i + \gamma_1^{-1}L^TL + \gamma_3^{-1}A_i^TA_i + \gamma_4^{-1}L^TB_iB_i^TL \quad (4)\\
&+ \gamma_4 P_iC_iC_i^TP_i + (\gamma_3+\gamma_4+\gamma_5)L^TMM^TL + Q + L^TRL + \bar{\tau}L^TSL\Big]x(t)\\
&- \Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big)^T \ \bar{\tau}^{-1}S \ \Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big) + x^T(t-\tau(t))[\gamma_2^{-1}L^TL\\
&+ \gamma_5^{-1}L^TC_iC_i^TL - (1-d)Q - (1-d)L^TRL]x(t-\tau(t))\Big\}\\
&\leq \xi^T(t)\Gamma\xi(t), \qquad (5)
\end{aligned}$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & 0 & 0 \\ * & \Gamma_{22} & 0 \\ * & * & -\bar{\tau}^{-1}S \end{bmatrix}, \qquad \Gamma_{11} = -A_i^TP_i - P_iA_i + \gamma_1 P_iB_iB_i^TP_i$$

$+\gamma_1^{-1}L^TL + \gamma_3^{-1}A_i^TA_i + \gamma_4^{-1}L^TB_iB_i^TL + \gamma_2 P_iC_iC_i^TP_i + (\gamma_3+\gamma_4+\gamma_5)L^TMM^TL + Q + L^TRL + \bar{\tau}L^TSL + \sum_{j=1}^{N}\Pi_{ij}P_j, \Gamma_{22} = \gamma_2^{-1}L^TLL - (1-d)Q - (1-d)L^TRL + \gamma_5^{-1}L^TC_iC_i^TL,$

$$\xi^T(t) = [x^T(t) \ x^T(t-\tau(t)) \ \Big(\int_{t-\bar{\tau}}^{t} f(x(s))ds\Big)^T]^T.$$

By using the Schur complement (Lemma 2.1), Σ can be written as $\Omega < 0$. We get

$$\dot{V}(t,x(t)) \leq [\xi^T(t)\Psi\xi(t)],$$

which indicates, from the Lyapunov stability theory [7], that the dynamics of the neural network (2) is asymptotically stable, which completes the proof.

4 Numerical Example

Consider the recurrent neural network (2) with (s=2) of the following form:

$$\dot{x}(t) = -A_i \ x(t) + B_i f(x(t)) + C_i \ f(x(t-\tau(t))),$$

where $f(x) = tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Obviously, $tanh(\cdot)$ satisfies $tanh(x) < 1$ for every $x \in R$, further

$$|tanh(x) - tanh(y)| = \left|\frac{d(tanh(z))}{dt}\right|_{z=\zeta} |x - y| = \left|\frac{4}{(e^{\zeta} + e^{-\zeta})^2}\right| |x - y| \leq |x - y|,$$

for every $x, y \in R$, and $\zeta \in (x, y)$ or $\zeta \in (y, x)$. Thus, $L = diag(1, 1)$. The membership functions for Rule 1 and Rule 2 are $\eta^1 = \frac{1}{e^{-2u_1(t)}}$, $\eta^2 = 1 - \eta^1$,

$$A_1 = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 & -0.02 \\ -0.10 & 0.01 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.01 & -0.02 \\ -0.10 & 0.01 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.02 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.02 \end{bmatrix}.$$

By using the Matlab LMI toolbox [4], we solve the LMI (3) for $\gamma_i > 0$, $(i = 1, 2, ..., 11)$, $\bar{\tau} = 0.5$ and $d = 0.5$, the feasible solutions are

$$P_1 = \begin{bmatrix} 3.8210 & -0.3112 \\ -0.1312 & 0.6531 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.8210 & -0.3112 \\ -0.1312 & 0.6531 \end{bmatrix},$$

$$Q = R = 10^3 \begin{bmatrix} 1.7364 & 0.0024 \\ 0.0024 & 1.7364 \end{bmatrix},$$

$$M = 10^3 \begin{bmatrix} 1.3264 & 0 \\ 0 & 3.3264 \end{bmatrix}, \qquad S = 10^3 \begin{bmatrix} 2.0110 & -0.0002 \\ -0.0002 & 2.0110 \end{bmatrix}.$$

Therefore, the concerned neural network with time-varying delays is asymptotically stable.

5 Conclusion

In this paper, we have performed the robust stability analysis for a class of uncertain recurrent neural networks with time varying delays and uncertainties. Some new stability criteria have been presented to guarantee the recurrent neural network to be robustly asymptotically stable. The linear matrix inequality (LMI) approach has been used to solve the underlying problem. The applicability of the derived results has been demonstrated through the numerical examples for the effectiveness of less conservative numerical solutions.

References

[1] B. Boyd, L. Ghoui, E. Feron and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadephia, PA, 1994.

[2] S. A. Billings and C.F. Fung. Recurrent radial basis function networks for adaptive noise cancellation. *Neural Networks* **8** (1995) 273–290.

[3] J. Cao and J. Wang. Global exponential stability and periodicity of recurrent neural networks with time delays, *IEEE Tans. Circ. Syst. I* **52**(5) (2005) 920–931.

[4] P. Gahinet, A. Nemirovski, A. Laub and M. Chilali. *LMI Control Toolbox User's Guide.* Massachusetts, The Mathworks, 1995.

[5] C.L. Giles, G.M. Kuhn and R.J. Williams. Dynamic recurrent neural networks: Theory and applications. *IEEE Trans. Neural Networks* **5** (1994) 153–156.

[6] K. Gu. An Integral inequality in the stability problem of time-delay systems. *Proceedings of 39th IEEE CDC.* Sydney, Philadelphia, 1994.

[7] J. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations.* Springer, New York, 1993.

[8] S. Hu and J. Wang. Global asymptotic stability and global exponential stability of continuous time recurrent neural networks. *IEEE Trans. Autom. Contr.* **47** (2002) 802–807.

[9] J. Liang and J. Cao. Boundedness and stability for recurrent neural networks with variable coefficients and time-varying delays. *Phys. Lett. A* **318** (2003) 53–64.

[10] P. Balasubramaniam and S. Lakshmanan. Delay-interval-dependent robust-stability criteria for neutral stochastic neural networks with polytopic and linear fractional uncertainties. *Int. J. Comput Math.* **88** (10) (2011) 2001–2015.

[11] O. Faydasicok and S. Arik. Robust stability analysis of a class of neural networks with discrete time delays. *Neural Networks* **29-30** (2012) 52–59.

[12] H. Shen, S. Huo, J. Cao and Tingwen Huang. Generalized State Estimation for Markovian Coupled Networks Under Round-Robin Protocol and Redundant Channels. *IEEE Transactions on Cybernetics* **49**(4) (2019) 1292–1301.

[13] H. Shen, F. Li, H. Yan, H.R. Karimi and H.K. Lam. Finite-time event-triggered H∞ control for T-S fuzzy Markov jump systems. *IEEE Transactions on Fuzzy Systems* **26**(5) (2018) 3122–3135.

[14] M. Syed Ali, R. Vadivel, O. M. Kwon and K. Murugan, Event Triggered Finite Time Boundedness of Uncertain Markov Jump Neural Networks with Distributed Time Varying Delays. *Neural Processing Letters* **49**(3) (2019) 1649–1680.

[15] M. Syed Ali, R Saravanakumar, CK Ahn, HR Karimi, Stochastic H_∞ filtering for neural networks with leakage delay and mixed time-varying delays. *Information Sciences* **388** (2017) 118–134.

[16] R Saravanakumar, MS Ali, HR Karimi, Robust H_∞ control of uncertain stochastic Markovian jump systems with mixed time-varying delays *International Journal of Systems Science* **48** (4) (2017) 862–872.

[17] M. Syed Ali, R. Saravanakumar and Q. Zhu. Less conservative delay-dependent H_∞ control of uncertain neural networks with discrete interval and distributed time-varying delays. *Neurocomputing* **166** (2015) 84–95.