Nonlinear Dynamics and Systems Theory, **21** (4) (2021) 367–380

Existence of Solutions for the Debye-Hückel System with Low Regularity Initial Data in Critical Fourier-Besov-Morrey Spaces

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Received: April 4, 2021; Revised: July 4, 2021

Abstract: This paper is devoted to studying the existence of solutions for the Cauchy problem of the Debye-Hückel system with low regularity initial data in critical Fourier-Besov-Morrey spaces. We show that there exists a unique local solution if the initial data belong to the Fourier-Morrey-Besov space $\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, and furthermore, if the initial data are sufficiently small, then the solution is global.

Keywords: *Debye-Hückel system; local existence; global existence; Littlewood-Paley theory; Fourier-Morrey-Besov spaces.*

Mathematics Subject Classification (2010): 35K45, 35Q99, 70k99, 93-00.

1 Introduction

In this paper, we consider the following Cauchy problem for the Debye-Hückel system in $\mathbb{R}^n \times \mathbb{R}^+$:

$$\begin{cases} \partial_t v = \Delta v - \nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w = \Delta w + \nabla \cdot (w \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are the initial data. Throughout this paper, we assume that $n \geq 2$.

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Notice that the function ϕ is determined by the Poison equation in the third equation of (1), and it is given by $\phi(x,t) = (-\Delta)^{-1}(w-v)(x,t)$. So, the system (1) can be rewritten as the following system:

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot \left(v\nabla(-\Delta)^{-1}(w-v)\right) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w - \Delta w = \nabla \cdot \left(w\nabla(-\Delta)^{-1}(w-v)\right) & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2}$$

W. Nernst and M. Planck introduced the Debye-Hückel system at the end of the nineteenth century as a fundamental model for the ion diffusion in an electrolyte [9]. It can also be derived from the mathematical modeling of semiconductors [19], plasma physics [12], and chemotaxis [8]. Thus, (1) has been studied by many researchers. Ogawa, Takayoshi, and S. Shimizu in [18] established the local well-posedness for large initial data and the global well-posedness for small initial data in the critical Hardy space $\mathcal{H}^1\left(\mathbb{R}^2\right)$. In 2008, Kurokiba and Ogawa in [16] obtained similar results for the initial data in subcritical and critical Lebesgue and Sobolev spaces.

In the context of Besov spaces, Karch in [15] proved the existence of global solution of the system (1) with small initial data in the critical Besov space $\dot{\mathcal{B}}_{p,\infty}^{-2+\frac{n}{p}}\left(\mathbb{R}^n\right)$ with $\frac{n}{2} \leq p < n$. Later, Deng and Li [10] showed that the system (1) is well-posed in $\dot{\mathcal{B}}_{4,2}^{-\frac{3}{2}}\left(\mathbb{R}^2\right)$, and ill-posed in $\dot{\mathcal{B}}_{4,r}^{-\frac{3}{2}}\left(\mathbb{R}^2\right)$ for $2 < r \leq \infty$. Zhao, Liu, and Cui [20] established the existence of global and local solution of the system (1) in the critical Besov space $\dot{\mathcal{B}}_{p,r}^{-2+\frac{n}{p}}\left(\mathbb{R}^n\right)$ with $1 < p < 2n$ and $1 \leq r \leq \infty$ (see also [1–3]).

Inspired by the work [20], the purpose of this paper is to establish the existence of local solution to (1) for large initial data and global solution for small initial data in the critical Fourier-Besov-Morrey space $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$.

Let us firstly recall the scaling property of the systems: if (v,w) solves (1) with the initial data (v_0,w_0) (ϕ can be determined by (v,w)), then (v_γ,w_γ) with $(v_\gamma(x,t),w_\gamma(x,t)) := \left(\gamma^2 v\left(\gamma x,\gamma^2 t\right),\gamma^2 w\left(\gamma x,\gamma^2 t\right)\right)$ is also a solution to (1) with the initial data

$$(v_{0,\gamma}(x),w_{0,\gamma}(x)) := \left(\gamma^2 v_0(\gamma x),\gamma^2 w_0(\gamma x)\right) \tag{3}$$

(ϕ_γ can be determined by (v_γ,w_γ)).

Definition 1.1 A critical space for the initial data of the system (1) is any Banach space $E \subset \mathcal{S}'\left(\mathbb{R}^n\right)$ whose norm is invariant under the scaling (3) for all $\gamma > 0$, i.e.,

$$\|(v_{0,\gamma}(x),w_{0,\gamma}(x))\|_E \approx \|(v_0(x),w_0(x))\|_E\,.$$

In accordance with these scales, we can show that the space pairs $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ are critical for (1).

Throughout the paper, we use $\mathcal{FN}_{p,\lambda,q}^{s}$ to denote the homogenous Fourier-Besov-Morrey spaces, $(v,w) \in X$ to denote $(v,w) \in X \times X$ for a Banach space X $\Big($the product $X \times X$ will be endowed with the usual norm $\|(v,w)\|_{X\times X} := \|v\|_X + \|w\|_X$ $\Big)$, $\|(v,w)\|_X$ to denote $\|(v,w)\|_{X\times X}$, $V \lesssim W$ means that there exists a constant $C > 0$ such that $V \leq CW$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

Now, our main results are stated below.

Theorem 1.1 *Let* $n \geq 2, \rho_0 > 2,\ \ \max\{n-(n-1)p, 0\} \leq \lambda < n,\ 1 \leq p < \infty,$ $q \in [1, \infty],\ (v_0, w_0) \in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ *and* $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1.$
Then there exists $T \geq 0$ *such that the system* (1) *has a unique local solution* $(v, w) \in X_T$*, where*

$$X_T = \mathfrak{L}^{\rho_0}\left(0, T; \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(0, T; \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right),$$

and

$$(v, w) \in \mathcal{C}\left(0, T; \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right).$$

Besides, there exixts $K \geq 0$ *such that if* (v_0, w_0) *satisfies* $\|(v_0, w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \leq K$*, then the above assertion holds for* $T = \infty$*; i.e., the solution* (v, w) *is global.*

2 Preliminaries

In this section, we give some notations and recall basic properties of Fourier-Besov-Morrey spaces, which will be used throughout the paper. The Fourier-Besov-Morrey spaces, presented in [11], are constructed by using a type of localization on Morrey spaces. The function spaces M_p^λ are defined as follows.

Definition 2.1 [11] Let $1 \leq p \leq \infty$ and $0 \leq \lambda < n$. The homogeneous Morrey space M_p^λ is the set of all functions $f \in L^p(B(x_0, r))$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{4}$$

where $B(x_0, r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$. When $p = 1$, the L^1 -norm in (4) is understood as the total variation of the measure f on $B(x_0, r)$ and M_p^λ as a subspace of Radon measures. When $\lambda = 0$, we have $M_p^0 = L^p$.

The proofs of the results discussed in this work are based on a dyadic partition of unity in the Fourier variables, known as the homogeneous Littlewood-Paley decomposition. We present briefly this construction below. For more detail, we refer the reader to [4].
Let $f \in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ and $\text{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{ for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k \leq j-1} \varphi_k(\xi)$$

and

$$h(x)=\mathcal{F}^{-1}\varphi(x), \quad g(x)=\mathcal{F}^{-1}\psi(x).$$

We now present some frequency localization operators

$$\dot{\Delta}_j f=\varphi_j(D)f=2^{dj}\int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)dy$$

and

$$\dot{S}_j f=\sum_{k\leq j-1}\dot{\Delta}_k f=\psi_j(D)f=2^{dj}\int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y)dy.$$

From the definition, one easily derives that

$$\begin{aligned}\dot{\Delta}_j\dot{\Delta}_k f&=0, \quad \text{if } |j-k|\geq 2,\\ \dot{\Delta}_j\left(\dot{S}_{k-1}f\dot{\Delta}_k f\right)&=0, \quad \text{if } |j-k|\geq 5.\end{aligned}$$

The following Bony paraproduct decomposition will be applied throughout the paper:

$$uv=\dot{T}_u v+\dot{T}_v u+R(u,v),$$

where

$$\dot{T}_u v=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_j v, \quad \dot{R}(u,v)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u\tilde{\Delta}_j v, \quad \tilde{\Delta}_j v=\sum_{|j'-j|\leq 1}\dot{\Delta}_{j'}v.$$

Lemma 2.1 *[11] Let $1\leq p_1,p_2,p_3<\infty$ and $0\leq\lambda_1,\ \lambda_2,\ \lambda_3<n$.*
(i) (Hölder's inequality) Let $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{\lambda_3}{p_3}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}}\leq\|f\|_{\mathrm{M}_{p_1}^{\lambda_1}}\|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}. \tag{5}$$

(ii) (Young's inequality) If $\varphi\in L^1$ and $g\in\mathrm{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi*g\|_{\mathrm{M}_{p_1}^{\lambda_1}}\leq\|\varphi\|_{L^1}\|g\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{6}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.2 *[11] Let $1\leq q\leq p<\infty$, $0\leq\lambda_1,\lambda_2<n$, $\frac{n-\lambda_1}{p}\leq\frac{n-\lambda_2}{q}$ and let γ be a multi-index. If supp$(\widehat{h})\subset\{|\xi|\leq A2^j\}$, then there is a constant $C>0$ independent of h and j such that*

$$\|\widehat{D_\xi^\gamma h}\|_{\mathrm{M}_q^{\lambda_2}}\leq C2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{h}\|_{\mathrm{M}_p^{\lambda_1}}. \tag{7}$$

We have now prepared all of the ingredients required to define the function spaces $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$, see [11].

Definition 2.2 (Homogeneous Fourier-Besov-Morrey spaces)
Let $1 \leq p, q \leq \infty$, $0 \leq \lambda < n$ and $s \in \mathbb{R}$. The homogeneous Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials such that the norm $\|f\|_{\mathcal{FN}^s_{p,\lambda,q}}$ is finite, where

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} \overset{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\|\varphi_j \hat{f}\right\|^q_{\mathrm{M}^\lambda_p}\right)^{\frac{1}{q}} & for \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j \hat{f}\right\|_{\mathrm{M}^\lambda_p} & for \quad q = \infty. \end{cases} \tag{8}$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (8) is a Banach space. Since $\mathrm{M}^0_p = L^p$, we have $\mathcal{FN}^s_{p,0,q} = F\dot{B}^s_{p,q}$.

The definition of mixed space-time spaces is given below.

Definition 2.3 Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \Big\{\sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}^\lambda_p)}\Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I, \mathcal{FN}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^\rho(I,\mathcal{FN}^s_{p,\lambda,q})}$ norm.

According to the Minkowski inequality, it is easy to verify that

$$\begin{aligned} L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right) \quad \text{if } \rho \leq q, \\ \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) \quad \text{if } \rho \geq q, \end{aligned}$$

where $\|u(t,x)\|_{L^\rho(I;\mathcal{FN}^s_{p,\lambda,q})} := \left(\int_I \|u(\tau,\cdot)\|^\rho_{\mathcal{FN}^s_{p,\lambda,q}} d\tau\right)^{1/\rho}$.

At the end of this section, we will recall an existence and uniqueness result for an abstract operator equation in a Banach space that will be used to show Theorem 1.1 in the sequel. For the proof, we refer the reader to [4].

Lemma 2.3 *Let X be a Banach space with norm $\|.\|_X$ and $B : X \times X \longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \leq \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ so that $\|y\|_X \leq \varepsilon$, the equation $x := y + B(x, x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \leq 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \leq 2\varepsilon$, then

$$\|\overline{x} - x'\|_X \leq \frac{1}{1 - 4\varepsilon\eta} \|y - y'\|_X .$$

3 Linear Estimates in Fourier-Besov-Morrey Spaces

In this part, we give some crucial estimates in the proof of our main results.

Lemma 3.1 *[7] Let I=(0, T), $s \in \mathbb{R}$, $p,\ q,\ \rho \in [1,\infty]$ and $0 \leq \lambda < n$. There exists a constant $C > 0$ such that*

$$\|e^{t\Delta(.)}u_0\|_{\mathcal{L}^{\rho}\left([0,T),\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|u_0\|_{\mathcal{FN}^{s}_{p,\lambda,q}}, \tag{9}$$

where $u_0 \in \mathcal{FN}^{s}_{p,\lambda,q}$.

Lemma 3.2 *[7] Let I=(0, T), $s \in \mathbb{R}$, $p,\ q,\ \rho \in [1,\infty]$ $0 \leq \lambda < n$ and $1 \leq r \leq \rho$. There exists a constant $C > 0$ such that*

$$\left\|\int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau\right\|_{\mathcal{L}^{\rho}\left(I;\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|f\|_{\mathcal{L}^{r}\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)} \tag{10}$$

for all $f \in \mathcal{L}^{r}\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)$.

4 Bilinear Estimates in Fourier-Besov-Morrey Spaces

Lemma 4.1 *Let $I = (0,T)$, $p,\ q \in [1,\infty]$, $\max\{n-(n-1)p, 0\} < \lambda < n$, $\rho_0 > 2$ and $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$. There exists a constant $C > 0$ such that*

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^{1}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \leq C\Bigg[&\|f\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|g\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\\ &+ \|g\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|f\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\Bigg]\end{aligned}$$

for all $f \in \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$
and $g \in \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)$.

Proof. Applying Bony's paraproduct decomposition and a quasi-orthogonality property for the Littlewood-Paley decomposition, for a fixed j, we obtain

$$\begin{aligned}\dot{\Delta}_j(f\nabla g) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g) + \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}\nabla g\dot{\Delta}_k f)\\ &+ \sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k f\widetilde{\dot{\Delta}}_k\nabla g)\\ &= I_j^1 + I_j^2 + I_j^3\,.\end{aligned}$$

Then, by the triangle inequalities in M_p^λ and in $l^q(\mathbb{Z})$, we have

$$\begin{aligned}
\|\nabla.(f\nabla g)\|_{\mathfrak{L}^1\left(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\right)} &\leq \|f\nabla g\|_{\mathfrak{L}^1\left(I;\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\right)} \\
&\leq \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{\dot{\Delta}_j(f\nabla g)}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&\leq \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^1}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&+ \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^2}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&+ \{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^3}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}} \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

By using the Young inequality in Morrey spaces and the Bernstein-type inequality with $|\gamma| = 0$, we have

$$\left\|\varphi_j \hat{f}\right\|_{L^1} \leq C 2^{j(\frac{n}{p'}+\frac{\lambda}{p})} \left\|\varphi_j \hat{f}\right\|_{M_p^\lambda}.$$

Then

$$\begin{aligned}
\|\widehat{I_j^1}\|_{L^1(I,M_p^\lambda)} &\leq \sum_{|k-j|\leq 4} \|\widehat{(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g)}\|_{L^1(I,M_p^\lambda)} \\
&\leq \sum_{|k-j|\leq 4} \|\varphi_k\widehat{\nabla g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} \|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,L^1)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(\frac{n}{p'}+\frac{\lambda}{p})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{l\leq k-2} 2^{(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})l}2^{(2-\frac{2}{\rho_0})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^k\Big(\sum_{l\leq k-2} 2^{l(2-\frac{2}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^{k(3-\frac{2}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)},
\end{aligned}$$

where we have used the fact that $\rho_0 > 2$ in the last inequality.

Thus, by using the Young inequality, we have

$$\begin{aligned}
J_1 &\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \left\{ \sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q} \Big(\sum_{|k-j|\le 4} 2^{k(3-\frac{2}{\rho_0})} \|\varphi_k \widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \Big)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \\
&\quad \times \left\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{|k-j|\le 4} 2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})q} 2^{k(2+\frac{n}{p'}+\frac{\lambda}{p}-\frac{2}{\rho_0})} \|\varphi_k \widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)} \Big)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|g\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})},
\end{aligned}$$

where we have used $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$.

Similarly, we get

$$J_2 \lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.$$

For J_3, first we use the Young inequality in Morrey spaces, the Bernstein inequality ($|\gamma|=0$) together with the Hölder inequality, to get

$$\begin{aligned}
\|\widehat{I_j^3}\|_{L^1(I,M_p^\lambda)} &\le \sum_{k\ge j-3} \|\widehat{(\dot{\Delta}_k f \tilde{\dot{\Delta}}_k \nabla g)}\|_{L^1(I,M_p^\lambda)} \\
&\le \sum_{k\ge j-3} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{|l-k|\le 1} \|\varphi_l \widehat{\nabla g}\|_{L^{\rho_0}(I,L^1)} \\
&\lesssim \sum_{k\ge j-3} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \sum_{|l-k|\le 1} 2^l 2^{l(\frac{n}{p'}+\frac{\lambda}{p})} \|\varphi_l \widehat{g}\|_{L^{\rho_0}(I,M_p^\lambda)} \\
&\lesssim \sum_{k\ge j-3} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \Big(\sum_{|l-k|\le 1} 2^{l(1-\frac{2}{\rho_0})q'} \Big)^{\frac{1}{q'}} \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{k\ge j-3} 2^{k(1-\frac{2}{\rho_0})} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}.
\end{aligned}$$

Then, applying the Hölder inequality for the series, we obtain

$$\begin{aligned}
J_3 &\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \left\{ \sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q} \Big(\sum_{k\ge j-3} 2^{k(1-\frac{2}{\rho_0})} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \Big)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \left\{ \sum_{j\in\mathbb{Z}} \Big(\sum_{k\ge j-3} 2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})} 2^{k(\frac{n}{p'}+\frac{\lambda}{p}-\frac{2}{\rho_0})} \|\varphi_k \widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)} \Big)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})} \sum_{i\le 3} 2^{i(-1+\frac{n}{p'}+\frac{\lambda}{p})} \\
&\lesssim \|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})},
\end{aligned}$$

where we have used the condition $\lambda > n-(n-1)p$ to ensure that the series $\sum_{i\leq 3} 2^{i(-1+\frac{n}{p'}+\frac{\lambda}{p})}$ converges. Thus, we finished the proof of Lemma 4.1.

5 Proof of Theorem 1.1

To ensure the existence of the global and local solution of the system (1), we will use Lemma 2.3 with the linear and bilinear estimate that we have established in Sections 3 and 4. Let $\rho_0 > 2$ be any given real number and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. Note that the space X_T defined in Theorem 1.1 is a Banach space equipped with the norm

$$\|u\|_{X_T} = \|u\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})} + \|u\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q})}.$$

For $T>0$ to be determined later. Given $(v,w)\in X_T$, we define $\mathfrak{F}(v,w)=(\bar v,\bar w)$ to be the solution of the following initial value problem:

$$\begin{cases} \partial_t\bar v - \Delta\bar v = -\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right), & \bar v(x,0)=v_0(x),\\ \partial_t\bar w - \Delta\bar w = \nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right), & \bar w(x,0)=w_0(x). \end{cases} \tag{11}$$

Obviously, (v,w) is a solution of (1) if and only if it is a fixed point of $\mathfrak{F}$.

Lemma 5.1 *Let $(v,w)\in X_T$. Then $(\bar v,\bar w)\in X_T$. Moreover, there exist two constants $C_0>0$ and $C_1>0$ such that*

$$\|(\bar v,\bar w)\|_{X_T} \leq C_0\,\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C_1\|(v,w)\|^2_{X_T}. \tag{12}$$

Proof. By Duhamel's principle, the system (2) is equivalent to the following integral system:

$$\begin{cases} \bar v(t) = e^{t\Delta}v_0 - \displaystyle\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau\\ \bar w(t) = e^{t\Delta}w_0 + \displaystyle\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau. \end{cases} \tag{13}$$

Set

$$B_1(v,w) := -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

$$B_2(v,w) := \int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

then the equivalent integral system (13) can be rewritten as

$$(\bar v(t),\bar w(t)) = (e^{t\Delta}v_0, e^{t\Delta}w_0) + (B_1(v,w),B_2(v,w)). \tag{14}$$

According to Lemma 3.1 with $s=-2+\frac{n}{p'}+\frac{\lambda}{p}$, $I=[0,\infty)$ and $\rho=\rho_0$ (*or* ρ_0'), we obtain

$$\|e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \lesssim \|v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}$$

and

$$\|e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \lesssim \|v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}},$$

which implies

$$\|e^{t\Delta}v_0\|_{X_T} \lesssim \|v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Similary,

$$\|e^{t\Delta}w_0\|_{X_T} \lesssim \|w_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Thus

$$\|(e^{t\Delta}v_0, e^{t\Delta}w_0)\|_{X_T} \leq C_0 \|(v_0, w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}. \tag{15}$$

Applying Lemma 3.2 with $s = -2 + \frac{n}{p'} + \frac{\lambda}{p}$ and $r = 1$, and Lemma 4.1, we obtain

$$\begin{aligned}
&\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \\
&= \|\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \\
&\lesssim \|\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)\|_{\mathfrak{L}^{1}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\qquad + \|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|w-v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\qquad + \|w-v\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|v\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \\
&\lesssim \left(\|(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} \times \|(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\right) \\
&\lesssim \|(v,w)\|^2_{X_T}.
\end{aligned}$$

Analogously, we get

$$\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)} \lesssim \|(v,w)\|^2_{X_T}.$$

Thus, we obtain

$$\|B_1(v,w)\|_{X_T} \lesssim \|(v,w)\|^2_{X_T}.$$

Similary,

$$\|B_2(v,w)\|_{X_T} \lesssim \|(v,w)\|^2_{X_T}.$$

Finally,

$$\|(B_1(v,w), B_2(v,w))\|_{X_T} \leq C_1 \|(v,w)\|^2_{X_T}. \tag{16}$$

Combining (15) and (16), we obtain

$$\|(\bar{v},\bar{w})\|_{X_T} \leq C_0 \,\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C_1\|(v,w)\|^2_{X_T}.$$

And we have completed the proof of Lemma 5.1, as desired.

The last lemma ensures that $\mathfrak{F}$ is well-defined and maps X_T into itself.

We begin by showing the global existence for small initial data. For this purpose, we choose $T=\infty$. We have from Lemma 5.1

$$\begin{aligned}\|\mathfrak{F}(v,w)\|_{X_\infty} &\leq C_0 \,\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C_1\|(v,w)\|^2_{X_\infty}\\ &\leq C_0 \,\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + 4C_1\varepsilon^2.\end{aligned}$$

Choosing $\varepsilon < \frac{1}{8\max\{C_0,C_1\}}$ for any $(v_0,w_0)\in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ with $\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} < \frac{\varepsilon}{8\max\{C_0,C_1\}}$, we get

$$\|\mathfrak{F}(v,w)\|_{X_\infty} < \varepsilon.$$

Finally, by using Lemma 2.3 we can obtain a unique global solution for small initial data in the closed ball $\bar{B}(0,2\varepsilon)=\{x\in X_\infty \;:\; \|x\|_{X_\infty}\leq 2\varepsilon\}$.

For the local existence, we shall decompose the initial data v_0 into two terms

$$v_0=\mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{v}_0\right)+\mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{v}_0\right):=v_{0,1}+v_{0,2},$$

where $\delta=\delta(v_0)>0$ is a real number. Similary, we decompose w_0:

$$w_0=\mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{w}_0\right)+\mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{w}_0\right):=w_{0,1}+w_{0,2}.$$

Since

$$\begin{cases} v_{0,2}\longrightarrow 0 \;\text{ in }\; \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \;\text{ when }\quad \delta\to+\infty,\\ w_{0,2}\longrightarrow 0 \;\text{ in }\; \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \;\text{ when }\quad \delta\to+\infty,\end{cases}$$

there exists δ large enough such that

$$C_0\|(v_{0,2},w_{0,2})\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \leq \frac{\varepsilon}{2}.$$

We get

$$\left\|\left(e^{t\Delta}v_0,e^{t\Delta}w_0\right)\right\|_{X_T} \leq \left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{X_T}+\frac{\varepsilon}{2}. \tag{17}$$

We have

$$\begin{aligned}&\left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{X_T}\\ &=\left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q})}+\left\|\left(e^{t\Delta}v_{0,1},e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho'_0}(I,\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho'_0}}_{p,\lambda,q})}.\end{aligned}$$

Using the fact that $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we obtain

$$\begin{aligned}
&\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\\
&= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})q}\|\varphi_j \widehat{e^{t\Delta}v_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\\
&\qquad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})q}\|\varphi_j \widehat{e^{t\Delta}w_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\\
&= \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{2}{\rho_0})q}\|\varphi_j |\xi|^2\chi_{B(0,\delta)}\hat{v_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\\
&\qquad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{2}{\rho_0})q}\|\varphi_j |\xi|^2\chi_{B(0,\delta)}\hat{w_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\\
&\lesssim \delta^{2+\frac{2}{\rho_0}}\Big(\Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j \hat{v_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\\
&\qquad + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j \hat{w_0}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q}\Big)\\
&\lesssim \delta^{2+\frac{2}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.
\end{aligned}$$

Thus

$$\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \le C_2\delta^{2+\frac{2}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Similary,

$$\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})} \le C_2\delta^{2+\frac{2}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Hence, $$\begin{aligned}\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{X_T} &\le C_2\delta^{2+\frac{2}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\\ &\quad + C_2\delta^{2+\frac{2}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.\end{aligned}$$

We choose T small enough so that

$$\begin{cases} C_2\delta^{2+\frac{2}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \le \frac{\varepsilon}{4}\\ and\\ C_2\delta^{2+\frac{2}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \le \frac{\varepsilon}{4}.\end{cases}$$

So, if

$$T \le \min\Big\{\Big(\frac{\varepsilon}{4C_2\delta^{2+\frac{2}{\rho_0}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}}\Big)^{\rho_0}, \Big(\frac{\varepsilon}{4C_2\delta^{2+\frac{2}{\rho_0'}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}}\Big)^{\rho_0'}\Big\},$$

then $\left\|\left(e^{t\Delta}v_{0,1}, e^{t\Delta}w_{0,1}\right)\right\|_{X_T} \leq \frac{\varepsilon}{2}$. This result with (5.2) yields that $\left\|\left(e^{t\Delta}v_0, e^{t\Delta}w_0\right)\right\|_{X_T} \leq \varepsilon$. Therefore, applying Lemma 2.3 again, we get a fixed point of $\mathfrak{F}$ in the closed ball $\bar{B}(0,2\varepsilon) = \{x \in X_T \ : \ \|x\|_{X_T} \leq 2\varepsilon\}$. Thus, for any arbitrary $(v_0, w_0) \in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, (1) has a unique local solution in $\bar{B}(0,2\varepsilon)$.

Regularity: We khow if $(v,w) \in X_T \times X_T$ is a solution of (1), then we can show that

$$\nabla \cdot (v\nabla\phi)\,, \ \ \nabla \cdot (w\nabla\phi) \in \mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right).$$

By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned}
&\|v(t_1) - v(t_2)\|^q_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\leq \sum_{j\leq N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^{\lambda}_{p}}\right)^q + 2\sum_{j>N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t)\|_{L^\infty\left(I,\mathrm{M}^{\lambda}_{p}\right)}\right)^q,
\end{aligned}$$

where $\hat{v_j} = \varphi_j\hat{v}$. For any small constant $\varepsilon > 0$, let N be large enough so that

$$\sum_{j>N} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\hat{v}_j(t)\|^q_{L^\infty\left(I,\mathrm{M}^{\lambda}_{p}\right)} \leq \frac{\varepsilon}{4}.$$

According to Taylor's formula and using the same arguments as in [20], we get

$$\begin{aligned}
&\sum_{j\leq N}\left(2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^{\lambda}_{p}}\right)^q \\
&\lesssim |t_1 - t_2|^q \sum_{j\leq N} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\left\|(\widehat{\partial_t u})_j\right\|^q_{L^1\left(I,\mathrm{M}^{\lambda}_{p}\right)} \\
&\lesssim |t_1 - t_2|^q \times \|\partial_t u\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\
&\lesssim |t_1 - t_2|^q \times \left(\|\Delta v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right) \\
&\lesssim |t_1 - t_2|^q \times \left(\|v\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right) \\
&\lesssim |t_1 - t_2|^q \times \left(\|v_0\|^q_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + 2\|\nabla\cdot(v\nabla\phi)\|^q_{\mathfrak{L}^1\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\right).
\end{aligned}$$

Thus, we obtain the continuity of v in time t.

Similarly, we use the same discussion to get the continuity of w in time t. Hence $(v,w) \in C\left(0,T;\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)$, and we are done.

6 Conclusion

In this work, we considered the Debye-Hückel system. The homogeneous Littlewood-Paley decomposition and Bony's paraproduct decomposition are the important tools to obtain the (global and local) well-posedness result for such system. Our results extend and complement the previous ones of Zhao, Jihong, Liu, and Cui [20].

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