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Some Generalizations of Lyapunov's Approach to Stability and Control

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Abstract: In the paper presents a brief survey of some new developments in Lyapunov's approach including the generalized perturbation equation and its applications; the use of nonanalytic Lyapunov functions; an extension of the Barbashin-Krasovskii theorem related to asymptotic stability assured by a Lyapunov function with nonpositive derivative; the consistency condition for a time-space mosaic that constitutes a discontinuous Lyapunov function valid for investigation of stability; the introduction of non sign-definite functions for use in control (carrying surfaces); the extremal set construction for control, stabilization, and nonlinear asymptotic observer design.

Keywords: *nonanalytic Lyapunov function; nonperiodic systems; control and identification; discontinuous Lyapunov function.*

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1 Introduction

After the seminal work of Lyapunov [1], stability theory was recognized as an independent and important field of knowledge. Since that time of 1892, it counted spectacular achievements such as Chetaev's instability theorem, Malkin's reduction principle, Krasovskii-Lyapunov functionals for delay differential equations, stability with respect to a part of variables, absolute stability of control systems, vector Lyapunov functions, matrix Lyapunov functions, to name just a few. These fundamental developments and some other important results can be found in [1 − 24], see also references therein.

This survey of some relatively recent developments concentrates on directions where the author personally participated.

The bibliography of the survey is limited to the topics considered which are presented in the order that relates to the areas of application and seems convenient for the reader. Efforts have been made to make the paper self-contained.

2 Generalized Perturbation Equation

This concept was proposed in the joint work [25] with V.V.Rumyantsev. In the classical stability theory, for a given nonlinear system

$$x' = \frac{dx}{dt} = f(x,t), \quad x \in R^n, \quad t \geq a \geq 0, \quad x(a) = b \tag{2.1}$$

under standard conditions ensuring the existence, uniqueness and extendibility of solutions in some region of initial data, the Lyapunov methods [1] can be applied to investigate stability of a certain particular solution of interest that corresponds to

$$x^*(t) = x(a,b,t), \quad x^*(a) = b. \tag{2.2}$$

Stability of solutions (2.2) is studied with the use of the perturbation equation which is obtained from (2.1), (2.2) by the transformation

$$x(t) = x^*(t) + w(t). \tag{2.3}$$

Substituting (2.3) into (2.1) and assuming the function f in (2.1) to be analytic with respect to x , one can use the expansion

$$\frac{dx^*}{dt} + \frac{dw}{dt} = f(x^* + w, t) = f(x^*, t) + \nabla f(x^*, t)w + g(w,t) \tag{2.4}$$

yielding, after cancellation of the first terms, the perturbation equation

$$w' = A(t)w + g(w,t), \quad g(0,t) = 0, \quad t \geq a. \tag{2.5}$$

Here $A(t)$ is the Jacobian matrix of $f(x,t)$, (2.1), calculated on the solution $x^*(t)$, (2.2), and $g(w,t)$ are higher order terms with all partial derivatives calculated on the same solution (2.2).

According to (2.3), the unperturbed motion $x^*(t)$ of (2.2) corresponds to the trivial solution $w(t) = 0$ of the perturbation equation (2.5). This allows us to substitute the problem of stability of the motion $x^*(t)$, (2.2), of the nominal equation (2.1) by the problem of stability of trivial solution $w = 0$ of the perturbation equation (2.5). This approach led to the powerful and elegant methods that constitute the classical stability theory, see, e.g. [1 – 16] and further references therein.

Consideration of perturbation equation (2.5) with all its comfort of using linear approximation $dw/dt = A(t)w$ and then, if necessary, successive higher order terms (in critical cases) has, however, some specific qualities.

First, if a particular solution $x^*(t)$, (2.2), is not given as an explicit function of a, b, t (i.e. as a formula), then perturbation equation (2.5) cannot be determined.

Second, if the solution (2.2) and, thus, the perturbation equation (2.5) are known, then the results of stability on that basis are applicable to that particular solution only.

To bypass these difficulties, let us not fix $x(a)$ in (2.1) and consider $x^*(t)$ of (2.2) as unknown parameter-function. Then the deviation $w(t)$ is governed by the equation

$$w' = \frac{dw}{dt} = f(x^* + w, t) - f(x^*, t) = q(w, x^*, t), \quad t \geq a \tag{2.6}$$

that follows from the first equality of (2.4). In contrast with equations (2.4), (2.5), the composite function q in (2.6) contains an unknown solution $x^*(t)$ as its argument. On the other hand, $q(0,x^*,t)=0$ for all $x^*(t)$, t, thus $w(t)=0$ is the solution of (2.6) for any $x^*(t)$. It means that trivial solution $w=0$ can be put in correspondence to any particular solution $x^*(t)$, serving therefore, the whole region of possible initial data.

If $f(.)$ in (2.1) is analytic with respect to x, then $q(.)$ of (2.6) is analytic with respect to w, yielding the generalized perturbation equation

$$w' = A(x^*(t),t)w + g(w,x^*(t),t), \qquad g(0,x^*(t),t)=0, \quad t\ge a. \tag{2.7}$$

For some particular $x^*(t)$, it is, of course, identical to (2.5) with corresponding $A(t)$, $g(w,t)$, where we use the same notation A, g for different functions. However, without fixing $x^*(t)$, it represents a bundle of equations given on a continuum of different particular solutions. With this meaning, we shall drop sometimes the indication of a particular solution, writing simply

$$w' = A(x,t)w + g(w,x,t), \qquad t\ge a \tag{2.8}$$

with the understanding that (2.8) is a corresponding perturbation equation for every solution $x(t)$ of (2.1). It means that the form (2.8) is conserved while the terms are different for different $x(t)$.

Example 2.1 To illustrate the point, consider an example from [3, Sections 4, 44]:

$$x' = x(\alpha^2 - x^2), \quad \alpha>0, \quad t\ge a. \tag{2.9}$$

According to (2.6), we have

$$w' = (\alpha^2-3x^2)w - 3xw^2 - w^3, \qquad t\ge a \tag{2.10}$$

which is the generalized perturbation equation (2.8) in our case of (2.9).

With (2.10) we can do the standard stability analysis for (2.9) as follows. Equation (2.9) has three stationary solutions $x_1=0$, $x_{2,3}=\pm\alpha$. Substituting those solutions in (2.10), we immediately obtain instability for $x_1=0$ and asymptotic stability for $x_{2,3}=\pm\alpha$, all by the first approximation in (2.10). These results can also be established by considering the Lyapunov function $V=w^2/2$ which has the following derivative on trajectories of (2.10)

$$V' = w^2(\alpha^2 - 3x^2 - 3xw - w^2). \tag{2.11}$$

For $x_1=0$, we have from (2.11) that $V'>0$ if $\alpha^2-w^2>0$, asserting instability and yielding domain of repulsion $w\in(-\alpha,\alpha)$ with respect to nominal solution $x_1(t)=0$.

For $x_{2,3}=\pm\alpha$, we have from (2.11)

$$V'_{2,3} = w^2(-2\alpha^2 \mp 3\alpha w - w^2), \tag{2.12}$$

asserting asymptotic stability of both solutions for small w. To find domain of attraction for $x_2=\alpha$, we take the upper sign in (2.12) and solve the inequality $w^2+3\alpha w+2\alpha^2>0$, yielding $w>-\alpha$ or $w<-2\alpha$, which in coordinates $t0x$ corresponds to $x>0$ or $x<-\alpha$ since in this case $w=x-x_2=x-\alpha$. However, in the region $x<-\alpha$ there is another attractor, namely, $x_3=-\alpha$; hence, domain of attraction for $x_2=\alpha$ is

$x \in (0, \infty)$. For $x_3 = -\alpha$, the same arguments with the lower sign in (2.12) yield domain of attraction $x \in (-\infty, 0)$; details are left to the reader.

We see that generalized perturbation equation can be used for all known solutions of the nominal equation. Moreover, it can be used for stability analysis of solutions that cannot be expressed as explicit integrals and for which one cannot write the specific perturbation equation (2.5) corresponding to a particular solution $x^*(t)$ (see (2.2) – (2.3)), *not given as a formula*. In such cases, the generalized perturbation equation represents a new and important tool for stability analysis.

Example 2.2 Use of bundles of first integrals [25].

Chetaev's method of construction of Lyapunov functions in the form of bundles of first integrals [2] (see also [5, Section 10] and further references therein) can be used with the generalized perturbation equation, that is, for stability analysis of sets of solutions. Consider the classical example of the Euler case in the motion of a rigid body around its fixed center of mass without external forces. Equations of such motion are usually written in the form

$$Ap' + (C - B)qr = 0, \tag{2.13}$$

$$Bq' + (A - C)rp = 0, \tag{2.14}$$

$$Cr' + (B - A)pq = 0, \tag{2.15}$$

where $t \geq a$ and p, q, r are projections of the vector of angular velocity on coordinate axes taken as principal axes of the ellipsoid of inertia, and A, B, C are principal moments of inertia of the rigid body.

Suppose that $p^*(t)$, $q^*(t)$, $r^*(t)$ is some particular solution of (2.13) – (2.15). Substituting $p = p^* + \xi$, $q = q^* + \eta$, $r = r^* + \zeta$ into (2.13) – (2.15), eliminating terms that are cancelled by virtue of nominal equations (2.13) – (2.15) and dropping the superscript, we obtain the generalized perturbation equations

$$A\xi' = (B - C)(r\eta + q\zeta + \eta\zeta), \tag{2.16}$$

$$B\eta' = (C - A)(p\zeta + r\xi + \zeta\xi), \tag{2.17}$$

$$C\zeta' = (A - B)(q\xi + p\eta + \xi\eta). \tag{2.18}$$

Here the prime ($\prime$) denotes time derivative, and p, q, r are fixed particular solutions of (2.13) – (2.15) defined by certain initial conditions $p(a) = p_0$, $q(a) = q_0$, $r(a) = r_0$.

By inspection, one can see that equations (2.13) – (2.15) have the following first integrals

$$T = Ap^2 + Bq^2 + Cr^2 = \text{const}, \tag{2.19}$$

$$M = A^2p^2 + B^2q^2 + C^2r^2 = \text{const}. \tag{2.20}$$

Case 1 $A = B = C$. In this case all solutions are stationary, $p = p_0$, $q = q_0$, $r = r_0$, and all are stable.

Case 2 $p = q = r = 0$. Equations (2.16) – (2.18) coincide with (2.13) – (2.15). Therefore, integrals T, M with ξ, η, ζ instead of p, q, r are also first integrals of perturbed motions. Being positive definite, they can be used as Lyapunov functions to conclude about stability of this trivial solution (at rest).

Case 3 $A \neq B \neq C \neq A$, and $p_0^2 + q_0^2 + r_0^2 > 0$. In this case, and taking into account (2.13) – (2.15), generalized perturbation equations (2.16) – (2.18) have the following first integrals

$$T^* = A(p+\xi)^2 + B(q+\eta)^2 + C(r+\zeta)^2 = \text{const}, \tag{2.21}$$

$$M^* = A^2(p+\xi)^2 + B^2(q+\eta)^2 + C^2(r+\zeta)^2 = \text{const}, \tag{2.22}$$

where constants T^*, M^* are defined by initial data p_0, q_0, r_0 and initial perturbations ξ_0, η_0, ζ_0. Since T^*, M^* do not vanish at $\xi = \eta = \zeta = 0$, they cannot be taken as Lyapunov functions.

Consider the function

$$V = (T^* - T)^2 + (M^* - M)^2. \tag{2.23}$$

This function is nonnegative, $V \geq 0$; vanishes if $\xi = \eta = \zeta = 0$, and its total derivative on trajectories of perturbed motions (2.16) – (2.18) of the system (2.13) – (2.15) is zero, $V' = 0$, since V is a bundle of integrals. If V were positive definite, one would conclude about stability of all motions. Unfortunately, this is not the case.

If ξ, η, ζ are not all zero, $|\xi| + |\eta| + |\zeta| > 0$, then $V = 0$ if and only if $T^* = T$ and $M^* = M$. To find the manifold on which $V = 0$, we can write, by virtue of (2.19) – (2.22)

$$T^* - T = A(2p\xi + \xi^2) + B(2q\eta + \eta^2) + C(2r\zeta + \zeta^2) = 0, \tag{2.24}$$

$$M^* - M = A^2(2p\xi + \xi^2) + B^2(2q\eta + \eta^2) + C^2(2r\zeta + \zeta^2) = 0. \tag{2.25}$$

Denoting parentheses in (2.24), (2.25) as x, y, z, we obtain for the case $A \neq B \neq C \neq A$ the integral-invariant manifold in the $\xi\eta\zeta$-space

$$\frac{x}{BC(B-C)} = \frac{y}{CA(C-A)} = \frac{z}{AB(A-B)} = \lambda(t). \tag{2.26}$$

Physically, it means that ξ, η, ζ satisfying (2.26) do not affect the energy nor the angular momentum of the body.

From conservation property of integrals at the left-hand side of (2.24), (2.25), it follows that perturbed trajectories either lie entirely on the manifold (2.26) or do not intersect it at all. If for nominal motions $p(t)$, $q(t)$, $r(t)$ there are no perturbed trajectories that lie on the manifold (2.26), then those motions are stable by Lyapunov's theorem on stability [1, Section 16] with the function V of (2.23) which is positive definite if (2.26) does not contain perturbed trajectories. Referring the reader to [25] for details, the conclusion is as follows.

Summary The rest $p = q = r = 0$ and all motions in trivial case $A = B = C$ are stable. The motion $p = q = 0$, $r(t) = r_0 = \text{const}$ in cases $A \leq B < C$ or $A \geq B > C$ (i.e. constant rotation around extreme axis C, including circular ellipsoids of inertia) is also stable. From the above analysis, we see that all other motions in the case $A \neq B \neq C \neq A$ are unstable. In the case of circular ellipsoid of inertia (ellipsoid of revolution, $A \neq B = C$), constant rotation around an equatorial axis is unstable and all other motions are stable.

Remark 2.1 The term "stability of sets of solutions" may sometimes be misinterpreted and confused with the notion usually referred to as "stability of sets", see, e.g., [26, 27] and references therein. The term "globally asymptotically stable set" means the existence of a globally contracting Lyapunov function acting outside of the set and bringing every trajectory from the exterior of the set onto that set. Such "stability sets" are also "viability sets", i.e., sets from which a trajectory cannot escape (the last term does not imply the global attraction of outside trajectories).

The stability of a set in this sense does not mean stability of solutions within that set. The use of Lyapunov functions to establish the global attraction of trajectories to some set has nothing to do with stability in the sense of Lyapunov. It means, in fact, a control application, proving certain quality referred to as ultimate boundedness, viability, practical stability, with some variations in terminology and definitions used by different authors. The level sets $V(x) \leq c$ can be used for construction of so-called overvaluing or comparison systems $dz/dt = h(t, z)$ with the property $z(t, t_0, z_0) \geq x(t, t_0, x_0)$ if $z_0 \geq x_0$.

In contrast, the generalized perturbation equation serves to establish stability of solutions in the sense of Lyapunov that start in some region of initial conditions.

3 Nonanalytic Lyapunov Functions

When N.N.Krasovskii (then my Ph.D. thesis supervisor) suggested the use of nonanalytic regulators for stabilization of nonlinear systems [28, 29], this naturally led to the introduction of nonanalytic Lyapunov functions.

Since the right-hand sides of perturbation equation are represented as convergent Maclaurin series around the trivial solution $x(t) = 0$, so the nonanalytic Lyapunov functions are also taken as finite sums of special power terms, containing absolute values and sign-functions of critical variables see [28 − 30]. Those sums are finite since asymptotic stability and instability are usually decided by terms up to a certain finite order.

Clearly, nonanalytic Lyapunov functions can be used also for other purposes. For example let us find the stability (viability) set in Example 1 of [27, p.248] for the system:

$$\frac{dx}{dt} = x(1 - x^2 - y^2) + yf(t, x, y), \qquad |f(.)| \leq 1;$$
$$\frac{dy}{dt} = xg(t, x, y) + y(1 - x^2 - y^2), \qquad |g(.)| \leq 1.$$

Taking $V = |x| + |y|$, we obtain on trajectories of the system

$$\begin{aligned}\frac{dV}{dt} &= (|x| + |y|)(1 - x^2 - y^2) + yf(.)\,\text{sign}\; x + xg(.)\,\text{sign}\; y \\ &\leq (|x| + |y|)(1 - x^2 - y^2) + |y| + |x| = V(2 - x^2 - y^2) \leq 0,\end{aligned}$$

if $x^2 + y^2 \geq 2$ which yields the circle of radius $\sqrt{2}$ as the global asymptotic stability set for the above system.

4 Extension of the Barbashin-Krasovskii Theorem unto Nonperiodic Systems

This theorem presents a sufficient condition for establishing asymptotic stability making use of a Lyapunov function $V(x) > 0,\ x \neq 0;\ V(0) = 0$ with nonpositive derivative $dV/dt \leq 0$ on the trajectories of the perturbation equation in a neighborhood of the origin. Such Lyapunov functions are usually constructed in practical cases of nonlinear systems. We reproduce the theorem in a simple formulation given by Barbashin [14, p.25].

Theorem 4.1 *If there is a positive definite function $V(x)$ such that $dV/dt < 0$ outside of a set M and $dV/dt \leq 0$ on M, where M is a set not containing entire trajectories (except for the origin), then the solution $x = 0$ is asymptotically stable.*

Note that it is easy to verify that M does not contain entire semitrajectories of a differential equation. Indeed, if a system is of the form

$$x' = g(x,t), \quad x(t_0) = x_0, \quad x \in R^n, \quad t \geq t_0 \tag{4.1}$$

and a surface M is given by $F(x) = 0$, then M does not contain entire trajectories if for some $t > T \geq t_0$ we have

$$\frac{dF}{dt} = \nabla F g(x,t) \neq 0.$$

For stationary systems $x' = g(x)$, not depending explicitly on t, the theorem (for the case of stability in the large) has been proved in [31] and is known as the Barbashin-Krasovskii theorem. For systems (4.1) where $g(x,t)$ is periodic in t, this theorem is proved in [4, Section 14] and is known as Krasovskii's theorem.

Further extension of this theorem follows from Theorem 4.1 for systems of class A, see [32, pp.21−27], as described below.

Definition 4.1 *System* (4.1) is said to be *of class A* if and only if the function $g(x,t)$ is such that for every solution $x(.,x_0,t_0)$ of the equation (4.1) and for any fixed $\bar{t} > t_0$ there is a sequence

$$\alpha_s > 0, \qquad \lim \alpha_s = 0, \tag{4.2}$$

such that there exists a sequence

$$\tau_s = \tau_s(x_0, t_0, \bar{t}, \alpha_s) > 0, \quad \tau_{s+1} > \tau_s, \quad s = 1, 2, \dots, \quad \lim \tau_s = \infty \tag{4.3}$$

for which

$$\|x(\bar{t}, x_s, t_0) - x(\bar{t} + \tau_s, x_0, t_0)\| \leq \alpha_s, \qquad s = 1, 2, \dots, \tag{4.4}$$

where

$$x_s = x(t_0 + \tau_s, x_0, t_0), \qquad s = 1, 2, \dots. \tag{4.5}$$

Remark 4.1 If one makes a drawing to illustrate conditions (4.2) to (4.5), it can be seen that those conditions, in application to solutions of differential equations, resemble the Cauchy criterion: a sequence $x_m \in R^n$ has a finite limit $x_0 = \lim x_m$ if and only if for every $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that $\|x_p - x_q\| < \varepsilon$ whenever $p > N(\varepsilon)$ and $q > N(\varepsilon)$. In the above conditions, the role of ε is played by α_s of (4.2), the role of $N(\varepsilon)$ is played by τ_s of (4.3), and p, q, are played by $t_0 + \tau_s$ and $\bar{t} + \tau_s$ of (4.5), (4.4). Thus, class A contains systems with asymptotically contracting translations of every trajectory in some region, and if that region is a neighborhood of the origin, the Barbashin-Krasovskii Theorem follows. Conversely, if the Barbashin-Krasovskii Theorem is valid for some systems, those systems must be of the class A defined by (4.2) to (4.5).

Definition 4.2 If in the context of Definition 4.1 we can take $\alpha_s = 0,\ s = 1, 2, \ldots,$ in (4, 6), then the *system* (4.1) is said to be *of class* A_0.

Definition 4.3 If in the context of Definition 4.1 we can take $\alpha_s = 0$ and $\tau_s = s\omega$, $s = 1, 2, \ldots,$ with $\omega = \text{const} > 0$ defined by the function $g(x,t)$ in (4.1) but independent of x_0, t_0, $\bar{t}$, then the *system* (4.1) is said to be *of class* A^*.

It is clear that

$$A^* \subseteq A_0 \subseteq A. \tag{4.6}$$

Lemma 4.1 *The class* A^* *is nonempty and contains, in particular, all stationary systems and all systems where* $g(x,t)$ *is periodic in* t.

It is interesting and important that, in fact, classes A^*, A_0, A do not coincide: $A^* \neq A_0 \neq A$. Let us denote by G the general class of systems in (4.1) such that $g(x,t)$ satisfies only standard conditions of existence, uniqueness and extendibility.

Lemma 4.2 *Strictly:* $A^* \subset A_0 \subset A \subset G$.

Proof It is sufficient to provide examples, which are given in [32].

In the theorem that follows, notation θ denotes a closed neighborhood containing the origin, the sets Ω^-, Ω^+ are closed neighborhoods such that $\Omega^- \subseteq \theta \subset \Omega^+$, the closed set $C\theta = \Omega^+ - \Omega^-$, where Ω^- is open, other sets are closed and the set $\Omega_0(t) \subseteq C\theta$ plays the role of M as in the Barbashin-Krasovskii Theorem above.

Theorem 4.2* *If the system (4.1) is of class* A *and there is a function* $V(x)$ *such that for all* $(x,t) \in C\theta \times [t_0, \infty)$ *we have:*

$$\nabla V \cdot g(x,t) \leq 0, \tag{4.7}$$

where the equality is valid only at points of a set $\Omega_0(t) \subseteq C\theta,\ t \in [t_0, \infty)$*, that contains no semitrajectories of (4.1), then there exists* $T(x_0, t_0) > 0$ *such that*

$$\begin{aligned} x(t, x_0, t_0) \in \Omega^- \subseteq \theta \quad &\textit{for all} \quad x_0 \in \Omega^+ - \theta \\ &\textit{and all} \quad t \in [t_0 + T(x_0, t_0), \infty). \end{aligned} \tag{4.8}$$

The *proof* of this theorem which is cast in the context of differential games can be found in [32, pp.25−27]. Considering $V(x) > 0,\ x \neq 0,\ V(0) = 0$ in the case $g(0,t) = 0$, $\{0\} \in \theta$, and letting $\theta \to \{0\}$, we obtain the case of asymptotic stability for systems of class A of which stationary and periodic systems present particular cases of the smaller class A^*, $A^* \subset A_0 \subset A$. Thus, the Barbashin-Krasovskii Theorem is valid for far more general systems than stationary and periodic ones.

Example 4.1 Let

$$S\colon x' = -xt(1 + \sin 2t), \quad x(0) = x_0, \quad t \geq 0. \tag{4.9}$$

Consider $V = x^2$, then on trajectories of (4.9) we have

$$V' = 2xx' = -2x^2 t(1 + \sin 2t) \leq 0, \quad t \geq 0. \tag{4.10}$$

**Acknowledgement* — Fruitful discussions with George Leitmann, especially with respect to Lemma 4.1 and Theorem 4.2, are gratefully acknowledged.

Except for $x = 0$, which point is excluded from the complement $C\theta$ for any $\theta \to \{0\}$, derivative $V' = 0$ only at isolated points $t = 0$ and $t_n = \pi/2 + \pi n$, $n = 0, 1, \dots$, thus, trivial solution $x = 0$ is asymptotically stable. Equation (4.9) has separable variables, and it can be verified directly by Definitions 4.1, 4.2 that $S \in A$, $S \notin A_0$.

5 Lyapunov's Approach in Use for Control and Identification

Lyapunov's methods have been applied to control problems of different nature, see, e.g. [5, 9, 10, 18, 19, 21, 26 − 30, 32, 33, 35 − 45] and references therein. An interesting generalization for control of motion is developed in the joint work with J.M.Skowronski [33].

Consider a non-linear differential equation with controls:

$$x' = \frac{dx}{dt} = F(x, t, u), \quad x \in R^N, \quad u \in U \subset R^m, \quad t \in [0, t_f], \tag{5.1}$$

$$x(t_0) = x_0 \in \Delta_1 \subseteq \Delta \subset R^N, \quad t_0 \in [0, t_f], \tag{5.2}$$

$$u = u(x, t) \in U \subset R^m, \quad t \in [0, t_f]. \tag{5.3}$$

Equation (5.1) with control (5.3) takes the form

$$x' = \frac{dx}{dt} = f(x, t), \quad f(x, t) = F(x, t, u(x, t)), \quad t \in [0, t_f]. \tag{5.4}$$

We assume that the functions F, u and the sets U, Δ in (5.1) − (5.3) are such that the function f in (5.4) satisfies standard conditions for the existence and uniqueness of a solution $x(t)$ with values in Δ, given initial condition (5.2) and a control function $u(\cdot)$ with values in U. The sets U, Δ, Δ_1 are open connected sets (domains) and the set of control functions $\{u(\cdot)\}$ contains the function $u(\cdot) = 0$. We allow $t_f = \infty$.

With these hypotheses, the above relations are well defined and may be regarded in two ways:

(a) as nominal equations of a dynamical system with the motion $x(t) \in \Delta$, in phase coordinates;

(b) as perturbation equations of certain dynamical system, whereby $f(0, t) = 0$ and $x(t) \in \Delta$ represents a deviation from some unperturbed nominal motion which is not explicitly given; the nominal equations of the system are not written, but $x(t) = 0$ designates precisely its nominal motion.

In his doctoral dissertation [1] A.M.Lyapunov gave a thorough study of the problem (b). The principal idea of the approach is decomposition of motion $x(t)$ into two motions: a motion along a certain surface V and a motion of the surface V itself. This idea is not related to the kind of equation (the nominal or perturbation one), nor to certain assumptions of the Lyapunov theory. This allows us to generalize the approach in different directions.

The generalization for use in control is as follows.

(1) Equation (5.1) is regarded as a nominal equation and not as a perturbation one. The condition $f(0, t) = 0$ is dropped.

(2) The sets U, Δ, Δ_1 are not assumed to be small, on the contrary:

$$d(\Delta) \geq d(\Delta_1) = \sup \|x_1 - x_2\| \geq l > 0, \tag{5.5}$$

where $\|\cdot\|$ is a norm in R^n.

(3) The aim is to determine whether or not the motion $x(t)$ tends to a certain given domain $M \subset \Delta$ which is not a neighbourhood of the origin. In control applications the function $u(x,t)$ is to be chosen so as to make $x(t)$ enter M in finite time and remain there. We shall concentrate on sufficient conditions for the convergence $x(t) \to M$, and not on how to choose $u(\cdot)$. Consequently, the control function is assumed to have been chosen, so that we start with (5.4). More on how to choose $u(\cdot)$ can be found in [32, 35].

(4) Regarding the Lyapunov second method, the conditions $V(x) > 0$, $x \neq 0$, $V(0) = 0$ are dropped, the condition $dV/dt \leq 0$ modified, and certain other conditions are imposed. The functions $V(x)$ thus constructed are no longer Lyapunov functions and, to avoid confusion, they are called simply V-functions. We demonstrate, however, that stationary Lyapunov functions represent a subset in the set of general stationary V-functions.

(5) The sets Δ, Δ_1, M are explicitly introduced into the method, allowing us to obtain quantitative results.

Such are the major changes that aim at the two-fold objective:

(a) to facilitate direct control applications of Lyapunov's approach;

(b) to provide the means for investigation of nominal equations of a system and a tool for quantitative design of desired motions.

5.1 Geometry of V-functions

5.1.1 V-surfaces. We consider real C^1-functions $V(x)\colon R^N \to R$ such that for each constant $\nu_0 \in B \subset R$, B open, satisfy the following conditions:

(1^*) There exists a surface $V(x) = \nu_0$ which is unique (single-sheeted) and of a finite measure.

(2^*) There exist x_0 such that $V(x_0) < \nu_0$ and x_1 such that $V(x_1) > \nu_0$.

(3^*) The set

$$\Omega(\nu_0) = \{x \mid V(x) < \nu_0\} \tag{5.6}$$

is bounded in R^N.

We consider the closure of Ω, or the level set

$$\operatorname{cl}\Omega(\nu_0) = \{x \mid V(x) \leq \nu_0\}, \tag{5.7}$$

its boundary

$$\partial\Omega(\nu_0) = \{x \mid V(x) = \nu_0\} \tag{5.8}$$

and the open complement or the exterior of Ω:

$$C\operatorname{cl}\Omega(\nu_0) = \{x \mid V(x) > \nu_0\} = \operatorname{ext}\operatorname{cl}\Omega. \tag{5.9}$$

The condition (2^*) means that the interior and exterior of $\operatorname{cl}\Omega$ are not empty. If $V(x)$ is defined everywhere in R^N, then by (5.6), (5.8) (5.9) we have $\Omega + \partial\Omega + C\operatorname{cl}\Omega = R^N$. Also $\operatorname{ext}\Omega = \partial\Omega + C\operatorname{cl}\Omega \supset \operatorname{ext}\operatorname{cl}\Omega = C\operatorname{cl}\Omega$.

Lemma 5.1 *The boundary $\partial\Omega$ separates R^N into disjoint open sets:*

$$\Omega = \operatorname{int}\operatorname{cl}\Omega \quad \textit{and} \quad C\operatorname{cl}\Omega = \operatorname{ext}\operatorname{cl}\Omega, \quad \Omega \cap C\operatorname{cl}\Omega = \emptyset.$$

Lemma 5.2 *Any continuous curve L in R^N, joining $x_0 \in \Omega$ and $x_1 \in \operatorname{ext}\Omega$, intersects the boundary $\partial\Omega = \{x \mid V(x) = \nu_0\}$.*

Lemma 5.3 *If $v_0' < v_0$, then for the same $V(x)$ the surfaces $\partial\Omega(v_0')$ and $\partial\Omega(v_0)$ are strictly enclosed:*

$$\operatorname{cl}\Omega(\nu_0') \subset \Omega = \Omega(\nu_0). \tag{5.10}$$

Remark 5.1 The requirements of uniqueness and a finite measure of a V-surface are imposed in (1^*) to avoid unnecessary complications. Such pathological cases do exist, for example, the function $V = (x_1^2 + x_2^2)\sin^2(x_1^2 + x_2^2)$ with nice properties: $V(x) = 0$ for $\|x\|^2 = x_1^2 + x_2^2 = \pi n$, $n = 0, 1, \dots$, otherwise $V(x) > 0$, presents for each $\nu_0 > 0$ a countable (denumerable) set of surfaces $V = \nu_0$ in R^2 which can be constructed by the equation $\sin^2(x_1^2 + x_2^2) = \nu_0/(x_1^2 + x_2^2)$. Such functions are not allowed by the condition (1^*).

The set of C^1-functions satisfying $(1^*)-(2^*)-(3^*)$ is not empty. Any real ellipsoid centered at the origin

$$V = \sum a_i x_i^2 = \nu_0, \quad a_i > 0, \quad i = 1 \div n$$

presents such a V-function for $v_0 > 0$, that is, $v_0 \in B = R_+$, thereby with additional properties: $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$, that are not required in this research. The property $V(0) = 0$ disappears for ellipsoids centered not at the origin.

Non-sign-definite functions of the type:

$$V_k = \sum a_i (x_i - \alpha_i)^{2k} + \beta, \quad k = 1, 2, \dots, a_i > 0, \quad i = 1 \div n,$$

where β, α_i are real constants, are also allowed. Planes, cylinders, cones, paraboloids are not allowed since Ω is unbounded. Functions of the type

$$V_k = \sum a_i |x_i - \alpha_i|^{2k} + \beta, \quad k = 1, 2, \dots, a_i > 0, \quad i = 1 \div n$$

satisfy (1^*), (2^*), (3^*) for an appropriate interval $B \subset R$ but are not differentiable at $x_i = \alpha_i$. If however special care is taken at those corners, such functions can be allowed and were actually used for nonlinear stabilization in [28–30].

In some problems one might be interested in a bounded open region $\Delta \subset R^N$ only. In this case one can consider those $B \subset R$ and $\nu_0 \in B = B(\Delta)$ for which the conditions (1^*), (2^*), with x, x_0, x_1 all in $\Delta \subset R^N$ are satisfied and define the sets Ω_Δ, $C\operatorname{cl}\Omega_\Delta$ by the relations:

$$\Omega_\Delta = \{x \mid V(x) < \nu_0,\ x \in \Delta\} = \Omega(\nu_0, \Delta),$$
$$C\operatorname{cl}\Omega_\Delta = \{x \mid V(x) > \nu_0,\ x \in \Delta\}.$$

Clearly, $\Omega_\Delta \subset \Delta$ and is, therefore, always bounded so that (3^*) is automatically satisfied. To preserve the separation property in this case, we have to introduce it either directly by the condition:

(4^*a) The sets Ω_Δ and $C\Omega_\Delta$, nonempty by (2^*), are disjoint, that is

$$\Omega_\Delta \cap C\operatorname{cl}\Omega_\Delta = \emptyset;$$

or indirectly, by the condition:

(4*b) There exists $x_1 \in C\mathrm{cl}\,\Omega_\Delta$ such that $x_1 \notin \mathrm{cl}\,\Omega_\Delta$.

Condition (4*a) replaces Lemma 5.1 and it follows from (4*b) by Lemma 5.2. Geometrically it is clear that one of these conditions is necessary to exclude spiral and other surfaces that do not partition Δ into two disjoint subsets. Now planes, cylinders, paraboloids are allowed. We shall see, however, that this vast collection of V-functions is restricted by further considerations.

5.1.2 Moving V-surfaces. Suppose $x = x(t)$ is a C^1-function of time on $[t_0, \infty)$. Using one and the same $V(x)$, we can define the level function

$$\nu_0(t) = V(x(t)). \tag{5.11}$$

If this function is considered in (5.8) instead of a constant v_0, then we obtain a moving boundary

$$\partial\Omega(t) = \{x \mid V(x) = \nu_0(t)\} \tag{5.12}$$

and so in (5.12) $x \in R^N$ is any point on the surface and not the same as $x(t)$ in (5.11).

Take any $t_1 \in [t_0, t_f]$ and let the total derivative be negative:

$$\frac{d\nu_0}{dt} = \frac{dV}{dt} = \nabla V x' < 0, \qquad t = t_1. \tag{5.13}$$

Since $V(x) \in C^1$, then by continuity there exists $\delta > 0$ such that

$$t_2 = t_1 + \delta < t_f \quad \text{and} \quad \frac{d\nu_0}{dt} < 0 \quad \text{for all} \quad t \in [t_1, t_1 + \delta]. \tag{5.14}$$

The continuous function dv_0/dt is uniformly continuous on a closed segment $[t_1, t_2]$ and attains there its maximum:

$$\max \frac{d\nu_0}{dt} = c, \quad c < 0, \quad t \in [t_1, t_2]. \tag{5.15}$$

Now, (5.14) can be strengthened:

$$\frac{d\nu_0}{dt} \le -|c| < 0 \quad \text{for all} \quad t \in [t_1, t_2]. \tag{5.16}$$

Integrating (5.16) over $[t_1, t_2]$ yields

$$\nu_0(t_2) \le \nu_0(t_1) - |c|(t_2 - t_1) < \nu_0(t_1). \tag{5.17}$$

Thus, the new (moved) boundary $\delta\Omega(t_2)$ lies entirely in the interior of the old $\Omega(t_1)$, cf. Lemma 5.3:

$$\delta\Omega(t_2) \in \Omega(t_1) \tag{5.18}$$

and is separated from $\delta\Omega(t_1)$ by a band of the width (in terms of V-levels)

$$\Delta\nu_0 = \nu_0(t_1) - \nu_0(t_2) \ge |c|(t_2 - t_1). \tag{5.19}$$

Of course, here $c = c(\delta)$. Suppose now that (5.16) holds over the entire closed segment $[t_0, t_f]$. Then $c = \text{const} < 0$ and by the same argument we obtain that the curve $x(t)$ in finite time $\Delta t = t_f - t_0$ crosses the band between $\delta\Omega(t_0)$ and $\delta\Omega(t_f)$ at the moment $t = t_f + 0$ and stays there for a sufficiently small interval $(t_f, t_f + \varepsilon)$, $\varepsilon > 0$. If in addition

$$\frac{d\nu_0}{dt} = \nabla V x' < 0 \quad \text{for} \quad x(t) \in \delta\Omega(t_f), \;\; t \geq t_f, \tag{5.20}$$

where (5.20) is understood to hold every moment $t \geq t_f$ when the curve touches the boundary $\partial\Omega(t_f)$, then the curve $x(t)$ is not leaving the closure $\Omega(t_f)$, $\forall\, t \geq t_f$.

5.1.3 Carrying V-surfaces (V-carriages). Suppose that a family of trajectories $x(x_0, t_0, \cdot)$ is given by a differential equation

$$x' = \frac{dx}{dt} = f(x,t), \quad x_0 = x(t_0), \quad t \geq t_0. \tag{5.21}$$

Then (5.20) takes the simple form

$$\nabla V f(x,t) = \sigma(x,t) < 0 \tag{5.22}$$

and can be evaluated at every point of a region in space and time, in our case in $\Omega(t_0) \times [t_0, t_f]$, $\Omega(t_0) \subset R^N$, without integration of the equation (5.21). If $x_0 \in \Omega(t_0)$ and (5.22) holds for the closed region:

$$x \in \Omega(t_0) - \Omega(t_f), \quad \Omega(t_f) \subset \Omega(t_0) \;\; \text{(closed band in } R^N\text{)}$$
$$t \in [t_0, t_\alpha], \quad t_\alpha \geq t_f \;\; \text{(closed segment in time)}$$

then the same argument holds and the entire family of solutions of (5.21) once trapped in $\Omega(t_0)$ crosses the band $\Omega(t_0) - \Omega(t_f)$ in finite times (depending on x_0)

$$\Delta t(x_0) \leq \frac{1}{|c|}\,[\nu_0(t_0) - \nu_0(t_f)], \tag{5.23}$$

where

$$c = \max \sigma(x,t) = \text{const} < 0, \quad x \in \Omega(t_0) - \Omega(t_f), \quad t_0 \leq t \leq t_f$$

and every solution stays in $\Omega(t_f)$ at least until $t = t_\alpha$.

The construction resembles the well-known Lyapunov design. However, we do not require that $V(x)$ be sign-definite, nor that $V(0) = 0$.

5.2 The control theorem

Consider the set of all V-functions. Given $\Delta \subset R^N$, $M \subset \Delta$ and a function $V(x)$, define the following constants and sets ($\partial\Delta$, ∂M denote the boundaries of Δ, M):

$$\nu^+ = \sup V(x) \mid x \in \partial\Delta, \tag{5.24}$$

$$\Omega^+ = \{x \mid V(x) < \nu^+\}, \tag{5.25}$$

$$\nu^- = \inf V(x) \mid x \in \partial M, \tag{5.26}$$

$$\Omega^- = \{x \mid V(x) < \nu^-\}. \tag{5.27}$$

Unless otherwise stated, Ω^- is assumed to be non-empty. We assume $f(x,t)$ of (5.4) to be defined and solutions to exist in the closure Ω^+. Suppose that Ω^+ and Ω^- are simply connected. Discard all V-functions for which either $\nu^- \geq \nu^+$ or $\Delta \not\subseteq \Omega^+$, or $\Omega^- \not\subseteq M$. The remaining subset Π which is assumed to be non-empty contains only those $V(x)$ for which the following inclusions hold:

$$\Omega^- \subseteq M \subset \Delta \subseteq \Omega^+. \tag{5.28}$$

Denote the closed complement

$$CM = \Omega^+ - \Omega^-, \tag{5.29}$$

non-empty since $M \neq \Delta$.

Theorem 5.1 *Given $M \subset \Delta$, $x_0 = x(t_0) \in \Delta - M$ and a constant T, $t_f - t_0 > T > 0$, the motion $x(x_0, t_0, t)$ enters M not later than at the moment $t^* = t_0 + T$ and stays there, if there is a function $V \in \Pi$ such that for all $(x,t) \in CM \times [t_0, t_f)$ we have*

$$\nabla V f(x,t) \leq -c, \tag{5.30}$$

where

$$c = \frac{\nu^+ - \nu^-}{T} = \text{const} > 0. \tag{5.31}$$

Proof follows from the above considerations, see [33].

Remark 5.2 One cannot substitute M for Ω^- in (5.29).

It is apparent that the above theorem is well in the spirit of Lyapunov, with the difference that it presents sufficient conditions for guaranteed transfer from a given point into a given domain in finite time specified beforehand. This theorem can be specified to include the limit operation as $t \to \infty$ for the case of the perturbation equation in (5.4) with $t_f = \infty$, $f(0,t) = 0$, and to deduce the well known classical results of Lyapunov [1] in stability theory, see [34]. This makes clear that the set Π of V-functions is non-empty and contains positive definite functions used by Lyapunov. It also opens a way to apply known methods of constructing Lyapunov functions to more general functions $V \in \Pi$.

In [32] this approach is applied for differential games, cf. Theorem 4.2 above where θ is the target set. In [35] it is applied for asymptotic observer design in differential games with incomplete information.

In control applications, usually a part of coordinates of the state vector $x \in R^N$ are directly measured, or a function thereof that constitute the information vector $y = g(x,t) + \gamma$, $y \in R^k$, $k < n$, containing measurement noise $\gamma(t)$. In this case, a controller is taken either in the form $u = u(y,t)$ for the output feedback control, or in the form $u = u(z,t)$, where $z(t)$ is the observer, that is, an approximation to $x(t)$ computed from a model

$$\frac{dz}{dt} = h(y,u,t), \quad z(t_0) = z_0, \quad t \geq t_0 \tag{5.32}$$

constructed in such a way that the error

$$\epsilon(t) = z(t) - x(t) \tag{5.33}$$

does not leave some neighborhood of the origin and is attracted to the origin sufficiently fast. This way of obtaining an acceptable estimate of $x(t)$ for use in control is called asymptotical observation or adaptive identification [35, 37−44].

For a linear stationary control system

$$\frac{dx}{dt} = Ax + Bu, \quad y(t) = Cx \tag{5.34}$$

the construction of a model (5.32) is very simple [38, 39]:

$$\frac{dz}{dt} = Hz + Qy + Bu, \quad H = A - QC. \tag{5.35}$$

Subtracting (5.34) from (5.35), we get the error equation, cf. (5.33):

$$\frac{d\epsilon}{dt} = Hz - Ax + Qy = H(z - x) = H\epsilon(t). \tag{5.36}$$

The matrices A, B, C are known, and it remains to provide appropriate eigenvalues for the matrix H in (5.35), (5.36) by the choice of the matrix Q, see [40, 44].

For a nonlinear control system, the construction of the model (5.32) is not so simple and Lyapunov's approach should be used for a proper asymptotic observer design [35, 37, 41 – 43].

6 Stability by Time-Space Mosaic with Discontinuous Lyapunov Function

By a theorem of Massera [46], if the trivial solution $x = 0$ of a perturbation equation with Lipschitzian right-hand side is uniformly asymptotically stable in the large, then there exists a Lyapunov function $V(x, t)$ that guarantees this type of stability.

In practical cases, a particular solution may be uniformly asymptotically stable but not in the large. Too, stability in the large as well as uniform stability, though comfortable, are not usually required in practice.

Even if the existence of a Lyapunov function is established, there is no universal method for constructing Lyapunov functions, and its construction is difficult in almost all nontrivial cases. These difficulties led to the development of vector [22, 23] and matrix [24] Lyapunov functions which act on regions of the subdivided state space through which trajectories are passing.

The generalized perturbation equation described in Section 2 opens a way to use different contracting Lyapunov functions for different periods of time. The surfaces defined by such Lyapunov functions form a time-space mosaic, or in other words, a discontinuous Lyapunov function, which is easier to construct and which can serve for establishing stability of motion. This approach was developed in the joint work [25] with V.V.Rumyantsev.

In stability analysis, deviations $w(t)$ are studied in a neighborhood H of the origin and one is interested to determine whether or not for every $\eta > 0$ there exists $\delta(\eta) > 0$ such that if

$$\|w_0\| \le \delta(\eta), \tag{6.1}$$

then

$$\|w(t)\| < \eta \quad \text{for all} \quad t > t_0, \tag{6.2}$$

where $\|\cdot\|$ is the Euclidean norm. If the answer to this question is in the affirmative, then the motion $w(t) = 0$ is called stable, otherwise, unstable. It means that if there exists

$\eta_0 > 0$ such that, whatever small $\delta > 0$ may be, there is a moment $t_* > t_0$ at which $\|w(t_*)\| = \eta_0$, then the motion is unstable. If at some moments $t_i^* > t_0$, perturbations grow to a fraction of the magnitude of a nominal coordinate, $|w_j(t_i^*)| = \alpha_j |x_j(t_i^*)|$, $\alpha_j = \text{const} \geq 1,\ 1 \leq j \leq n$, then the motion is unstable.

A stable motion with the additional property

$$\lim \|w(t)\| = 0, \quad t \to \infty \tag{6.3}$$

is called asymptotically stable. These are the classical definitions of stability given by Lyapunov [1]. With the notation (2.3), it refers, of course, to the stability of the solution $x^0(t)$. Let us not fix the initial condition $x^0(t_0) = x_0 \in \Delta_0$, considering instead a collection of nominal solutions $\{x^0(t)\} = x^0(\{x_0\}, t_0, t)$ corresponding to a set $\{x_0\} \subseteq \Delta_0$ of initial conditions; the notation $\{x_0\}$ may mean a finite collection or a set, a continuum.

To study and solve the problem by Lyapunov's second (direct) method, C^1-functions $V(w,t)$, $W(w)$, $W^1(w)$, $W^*(w)$ are considered that vanish if $w = 0$,

$$V(0,t) = W(0) = W^1(0) = W^*(0) = 0, \quad t \geq t_0, \tag{6.4}$$

and have some additional properties.

Recall the basic theorems of Lyapunov's second method.

Theorem 6.1 (Lyapunov [1]) *If there exists a function $V(w,t)$ satisfying the conditions*

$$\text{(a)} \quad V(w,t) \geq W(w) > 0, \quad w \in H, \quad w \neq 0, \quad t \geq t_0; \tag{6.5}$$

$$\text{(b)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) \leq 0, \quad w \in H, \quad t \geq t_0 \tag{6.6}$$

on the trajectories of the perturbation equation, then the solution $w(t) = 0$ is stable.

Theorem 6.2 (Lyapunov [1]) *If there is a function $V(w,t)$ satisfying condition (a) and the strengthened (cf. (b)) conditions:*

$$\text{(c)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) \leq -W^1(w) < 0, \quad w \in H,\ w \neq 0,\ t \geq t_0; \tag{6.7}$$

$$\text{(d)} \quad W^*(w) \geq V(w,t), \quad w \in H, \quad t \geq t_0, \tag{6.8}$$

then the solution $w(t) = 0$ is asymptotically stable.

Theorem 6.3 (Chetaev [2]) *If there exists a function $V(w,t)$ satisfying the conditions:*

$$\text{(e)} \quad \textit{the set} \quad \Sigma_\eta^t = \{w \in H \mid V(w,t) > 0,\ t \geq t_0\} \cap \{\|w\| < \eta,\ \eta > 0\} \neq \emptyset \tag{6.9}$$

is nonempty for all $t \geq t_0$ and any small $\eta > 0$;

$$\text{(f)} \quad V(w,t) \quad \textit{is bounded within } \Sigma_\eta^t; \tag{6.10}$$

$$\text{(g)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) > 0, \quad w \in \Sigma_\eta^t, \quad w \neq 0, \tag{6.11}$$

on the trajectories of the perturbation equation, meaning that dV/dt *is positive definite in* Σ_η^t*, in the sense that for every small* $\varepsilon > 0$ *there is* $\gamma > 0$ *such that if* $V(w,t) \geq \varepsilon$*, then*

$$\frac{dV}{dt} \geq \gamma \quad \textit{for all} \quad t \geq t_0, \tag{6.12}$$

then the motion $w(t) = 0$ *is unstable.*

Geometrically, condition (6.11) together with (6.12) mean that if $w(t) \in \Sigma_\eta^t$ is uniformly separated from the boundary $\partial\Sigma_\eta^t$ for all $t \geq t_0$, then $dV/dt \geq \gamma > 0$ is uniformly separated from zero for all $t \geq t_0$, see [2, Section 13]. As distinct from (6.5), a function $V(w,t)$ in (6.9) need not be positive definite.

Of course, stability, asymptotic stability or instability of the solution $w(t) = 0$ implied by Theorems 6.1 – 6.3 means the same property of all nominal solutions $\{x^0(t)\}$ for which (6.6), or (6.7), or (6.11) – (6.12), respectively, are fulfilled.

Consider x^0 in (2.6), (2.7) and x in (2.8) not as a particular solution, but as a parameter. Then inequalities (6.6), (6.7), (6.11) become characteristics of a domain (simply connected open set)

$$E = \mathcal{D} \times (t', t' + T), \quad \mathcal{D} \subseteq \Delta \subseteq R^n, \quad t' \geq t_0 \text{ fixed}, \quad T > 0, \tag{6.13}$$

where $\mathcal{D}$ may vary with $t \in (t', t' + T)$.

With x, t considered as independent variables, the left-hand side of (6.6), (6.7), (6.11) becomes a function $F\colon R^n \times R^n \times R \to R$ of three arguments

$$F(w,x,t) = \frac{\partial V}{\partial t} + \nabla V \cdot q(w,x,t), \tag{6.14}$$

which coincides with the total derivative $V' = dV/dt$ of a chosen function $V(w,t)$ on trajectories $w(t)$ of the perturbation equation (2.6).

Consideration of such functions (6.14) and domains (6.13) is motivated by the need to evaluate the rate of attraction of perturbed motions to a nominal solution of (2.1) within a finite time interval, and for all nominal trajectories passing through domain E of (6.13). For processes evolving in a finite space-time region, such information may be useful irrespective of stability properties on $[t_0, \infty)$. In such considerations, perturbations w do not have to be small.

Definition 6.1 If for a chosen $V(w,t)$ satisfying (6.5) on an interval $(t', t' + T)$, the condition (6.6) or (6.7) holds for $(x,t) \in E$, then *domain* E is called *neutral* or *contractive*, respectively.

Definition 6.2 If for a chosen $V(w,t)$ satisfying (6.9), (6.10) on an interval $(t', t'+T)$, the condition (6.11) holds for $(x,t) \in E$, then *domain* E contains a repulsive sector Σ_η^t; such domain E is called *repulsive.*

The statement that a certain domain E is contractive, neutral or repulsive means that there is a function $V(w,t)$ mentioned in Definitions 6.1, 6.2 which renders the corresponding property of E. The availability of such a function defines the corresponding domains. For example, if $V(w,t)$ satisfies (6.5), (6.7) for all $t \geq t_0$, then our domain becomes a contractive band $E = \mathcal{D} \times [t_0, \infty)$ with one sole Lyapunov function which is the classical case.

Remark 6.1 The names *contractive* or *repulsive domain* relating to the (x,t)-space should not be confused with the names *domain of attraction* or *repulsion* relating to the w-space, as in Example 2.1.

To illustrate the geometry corresponding to Definitions 6.1, 6.2, we can use the standard argument of the Lyapunov stability theory [1, 2]. Consider, for example, a neutral domain $E_1 = \mathcal{D}_1 \times [t_0, t_1)$. For a given $\eta > 0$, let

$$\gamma_1 = \inf W_1, \quad \|w\| = \eta; \quad \text{due to (6.5)}, \;\; \gamma_1 > 0. \tag{6.15}$$

Since $V_1(w, t_0)$ does not depend on t, so due to (6.4) and to the continuity of V_1 there is $\delta > 0$ such that for $\|w\| \le \delta$ we have $V_1(w_0, t_0) < \gamma_1$. Choosing such initial conditions and due to the relation

$$V_1 - V_1(w_0, t_0) = \int_{t_0}^{t} V_1' \, dt, \quad V_1' \le 0 \quad \text{as of (6.6)}, \;\; t \in [t_0, t_1], \tag{6.16}$$

we obtain that $w(t)$ is such that the following conditions are satisfied

$$W_1 \le V_1(w,t) \le V_1(w_0, t_0) < \gamma_1, \qquad t \in [t_0, t_1] \tag{6.17}$$

implying $\|w(t)\| < \eta$ for $t \in [t_0, t_1]$.

It means that, over a neutral domain, perturbations within a ball $\|w\| < \eta$, where (6.4) – (6.6) are satisfied cannot escape this ball whatever $(x,t) \in E_1 = \mathcal{D}_1 \times [t_0, t_1)$. If $t_1 = \infty$, stability follows.

If we have strict inequality $V_1' < 0$ in (6.16), compare with (6.7), then domain E_1 is contractive. If $t_1 = \infty$ and we use the additional condition (6.8), then asymptotic stability follows by the standard argument [1, 2].

However, if we consider two adjacent domains with different functions V_1, V_2 (with one common function it would be one single domain), then neutrality or contractivity of the union does not follow from the same property for component domains. Indeed, continuing the argument (6.13) – (6.17) for $E_2 = \mathcal{D}_2 \times [t_1, t_2]$, we denote $\eta_1 = \|w(t_1)\|$. Clearly, $\eta \ge \eta_1 > 0$ since, otherwise, the value $w(t_1) = 0$ of the solution $w(t) \not\equiv 0$ would contradict the uniqueness of a solution emanating from the point $(t_1, 0)$ due to the existence of the trivial solution $w(t) \equiv 0$. Let

$$\gamma_2 = \inf_{\|w\| = \eta_1} W_2. \tag{6.18}$$

Since $V_2(w, t_1)$ does not depend on t so due to (6.4) and to the continuity of V_2, there is $\delta_2 > 0$ such that for $\|w_1\| \le \delta_2$ we have $V_2(w_1, t_1) < \gamma_2$. However, $w_1 = w(t_1) = w(w_0, t_0, t_1)$ comes from E_1 and cannot be chosen so as $\|w_1\| \le \delta_2$ for appropriate $\delta_2 > 0$. Hence, to continue the argument and to assure that finite or countable union of adjacent neutral (contractive) domains be also neutral (contractive), we have to impose the following condition.

Consistency condition. A sequence of adjacent or overlapping neutral (contractive) domains $E_1, E_2, \ldots$ with functions $V_1, V_2, \ldots$, acting on $[t_0, t_1)$, $[t_1, t_2), \ldots$, and satisfying (6.4) – (6.6), or (6.7) for contractive domains, is called consistent if the functions $V_1, V_2, \ldots$ are such that, with the initial condition $\|w\| \le \delta(\gamma_1)$ for a given $\eta > 0$

in (6.15), we have $V_2(w_1,t_1) < \gamma_2$ for $w_1 = w(w_0,t_0,t_1)$ and any $x \in \mathcal{D}_2(t_1)$, then $V_3(w_2,t_2) < \gamma_3$ for $w_2 = w(w_1,t_1,t_2) = w(w_0,t_0,t_2)$ and any $x \in \mathcal{D}_3(t_2)$, etc., for all V_n, $n = 2,3,4,\dots$ in the sequence. It simply means that the solution $w(w_0,t_0,t)$ at times $t = t_1,t_2,\dots,t_n,\dots$ is picked by the next function with the same properties as previous functions plus the property of no escape from the sphere (ball) already attained. Consistent domains do exist, for example, if $V_n = c_n\|w\|^2$ or if V_n are considered as pieces on $[t_{n-1},t_n)$ of one single Lyapunov function $V(w,t)$, $t \in [t_0,\infty)$, existing under certain conditions [4, 46].

Definition 6.3 If there is a band $E_0 = \mathcal{D}_0 \times (t_0,\infty)$, $\mathcal{D}_0 \subseteq \Delta$, that can be covered by a finite or countable chain of consistent neutral (respectively, contractive) domains, such a *band* is called *neutral* (*respectively, contractive*).

Theorem 6.4 *Every solution which is entirely in a neutral band is stable.*

Proof There is a sequence of functions $V_1,V_2,\dots$ acting on $[t_0,t_1),[t_1,t_2),\dots$ and satisfying (6.4) – (6.6) that corresponds to a cover by a finite or countable chain of consistent neutral domains. If the chain is finite, we prove the theorem after a number of repetitions of the above argument (6.13) – (6.17) since the last $t_k = \infty$. If the chain is countable, then $t_n \to \infty$, thus, for every $t \in [t_0,\infty)$ there is a subsegment to which it belongs, yielding $\|w(t)\| < \eta$ for all $t \geq t_0$.

A solution which is entirely in a contractive band may not be asymptotically stable though its stability follows from Theorem 6.4 since (6.6) is implied by (6.7). If the chain is finite and for the last function $V_k(w,t)$ acting on $[t_k,\infty)$ the condition (6.8) is satisfied, then asymptotic stability follows from the classical Lyapunov Theorem [1].

For a countable chain of consistent contractive domains, consider a sequence of corresponding functions

$$V_i(w,t), \quad t \in [t_{i-1},t_i), \quad t_i \to \infty \text{ as } i \to \infty, \quad i = 1,2,\dots, \tag{6.19}$$

each acting over corresponding domain E_i of finite time length $\Delta t_i = t_i - t_{i-1} \geq \tau > 0$. Functions (6.19) may be regarded as components of a piecewise continuous function $V(w,t)$ acting on $[t_0,\infty)$, which components should satisfy the consistency condition stated above.

Now, condition (6.8) can be extended onto the sequence (6.19) as follows. From (6.5), (6.8) we have

$$W^*(w) \geq V(w,t) \geq W(w) > 0, \quad w \in H, \quad w \neq 0, \tag{6.20}$$

where $V(w,t)$ represents $V_i(w,t)$ over each $[t_{i-1},t_i)$ of (6.19). Since $W^*(w) \to 0$ as $\|w\| \to 0$, so for appropriate $\eta > \eta_* > 0$ the surface $V(w,t) = \gamma$ is enclosed in the spherical ring

$$\eta \geq \|w\| \geq \eta_*, \tag{6.21}$$

provided that $\eta > \gamma > \eta_*$ and the ring (6.21) is in the region H. Indeed, it is sufficient to take such η, η_* that the sphere $\|w\| = \eta$ is circumscribed around $W^*(w) = \eta_1 \leq \eta$, and the sphere $\|w\| = \eta_*$ is inscribed in $W(w) = \eta_2 \geq \eta_*$, $\eta_1 > \eta_2$. Since $\eta_1 = W^*(w) \to 0$, as $\|w\| \to 0$, we can take $\eta \to 0$. Vice versa, if (6.8) holds, then for any spherical ring (6.21) in the region H, by virtue of (6.20), (6.8), there exist functions of (6.19) acting over this ring (we say in such case that ring (6.21) is covered by consistent contractive domains).

Take a decreasing sequence $\eta = \eta_1 > \dots > \eta_k > \eta_{k+1} > \dots$, $\lim \eta_k = 0$, and consider rings $R_k = \{w \in H \mid \eta_k \geq \|w\| \geq \eta_{k+1}\}$, $k = 1, 2, \dots$. Consider all functions $V_i(w,t)$ from (6.19) acting over the ring R_k. By (6.7) every $V_i' < 0$ which means that there exists $W_i^1(w)$ such that over the segment of definition of $V_i(w,t)$ we have definite negative and bounded from zero total derivatives

$$-V_i'(w,x,t) \geq W_i^1(w) > 0, \quad w \in H, \quad w \neq 0, \\ (x,t) \in E_i = \mathcal{D}_i \times [t_{i-1}, t_i). \tag{6.22}$$

Let

$$\gamma_{ik} = \inf W_i^1(w) \geq \gamma_k > 0, \quad w \in R_k. \tag{6.23}$$

The uniform bound $\gamma_k > 0$ exists for all V_i acting over R_k since otherwise $W_i^1(w)$ would not be separated from zero within closed R_k, not containing zero, in contradiction with definition of a positive definite function.

Now, integrating the piecewise continuous function $V(w,t)$ with components (6.19) along a trajectory (or a part thereof) lying entirely within R_k, we obtain by (6.22), (6.23)

$$V - V(w_*, t_*) = \int_{t^*}^{t} V' \, dt \leq -\sum_i \gamma_{ik}(t_i - t_{i-1}) \leq -\gamma_k(t - t_*), \tag{6.24}$$

where the sum covers all components $V_i(w,t)$ acting over R_k and t_* is the starting time of a perturbed trajectory. From (6.5), (6.24), we get

$$0 < V(w,t) \leq V(w_*, t_*) - \gamma_k(t - t_*), \quad \gamma_k > 0, \tag{6.25}$$

meaning that there is only finite time $(t - t_*) \leq T_k < \infty$ during which a trajectory can stay within R_k. Since the band is contractive, the perturbed trajectory $w(t)$ will leave R_k, approaching zero, so that for $t > t_* + T_k$ we have $\|w(t)\| < \eta_{k+1}$. By (6.8), for any ring R_k, $k = 1, 2, \dots$, there are V_i from (6.19) that act over that ring, hence $\lim_{t\to\infty} \|w(t)\| = \lim_{k\to\infty} \eta_k = 0$. This proves the following theorem.

Theorem 6.5 *If a contractive band is such that for any $\eta > 0$ there is $N(\eta)$ such that for all $i \geq N(\eta)$ functions $V_i(w,t)$ of (6.19) satisfy the condition $\eta \geq V_i(w,t) > 0$, $w \in H$, $w \neq 0$, $t \in [t_{i-1}, t_i)$, $t_i \to \infty$, as $i \to \infty$, then every solution passing entirely within such a band is asymptotically stable.*

Remark 6.2 The above arguments resemble the analysis based on property (A) or (B) in [4, Sections 4, 5], under which there exists a Lyapunov function $V(w,t)$ acting on $[t_0, \infty)$ with sign definite derivative that renders asymptotic stability of certain nominal solution $x^0(t)$. However, it may be difficult to find such a function and, if found, it serves one particular solution only. Functions (6.19) may be easier to construct, and they serve all solutions passing through corresponding domains E_i. If considered as components of one function $V(w,t)$, this function, though generally discontinuous, renders, under certain conditions, the same conclusions about stability or asymptotic stability as a classical Lyapunov function.

Remark 6.3 In contrast and similarity with vector Lyapunov functions introduced, e.g. in [22, 23], that create a space mosaic based on the idea that each subsequent function (all acting on $[t_0, \infty)$) covers a manifold (or a part thereof) where preceding functions are inconclusive (e.g. where $V' = 0$), the functions (6.19) correspond to a time-space mosaic of consistent domains which domains, if forming a band extending over $[t_0, \infty)$, deliver the same stability properties as a conventional Lyapunov function.

Functions V_i of (6.19) corresponding to a chain of consistent contractive domains can be used to obtain quantitative results concerning the measure of contraction within every domain E_i, see [25].

7 Conclusions

Developments presented in this survey complement the classical stability theory in different directions. First, it seems important that investigation of stability should be possible without integration of equations of motion. This possibility is provided by the generalized perturbation equation which implicitly contains trajectories of the nominal equation passing through the x-space included as parameter-space in the generalized perturbation equation acting in the w-space of perturbations. As a by-product, such relaxation of a fixed particular solution around which the classical perturbation equation is constructed allows us to investigate stability of all nominal solutions passing through the x-space. Thus, the explicit integration of the nominal equation which is difficult if not impossible in many practical cases becomes unnecessary. This also opens the avenue for numerical investigation of stability.

Second, Lyapunov functions usually constructed as smooth functions do not have to be differentiable. They can be even discontinuous, if certain consistency condition is respected. This expansion of the class of possible Lyapunov functions is of much interest in view of difficulties encountered in attempts to construct a Lyapunov function for a more complicated practical system.

Further, the extension of the Barbashin-Krasovskii theorem onto nonperiodic systems has been long overdue. Indeed, it was puzzling that this important and much used theorem should be valid only for systems with such easy-to-see fashionable property as being stationary or with a periodic right-hand side. The result presented in Section 4 extends the validity of this theorem to systems of class A whose solutions satisfy a condition that resembles the Cauchy compactness criterion.

Another generalization was to apply the idea of decomposition of motion (embodied in Lyapunov's approach) to the controller and observer design for nominal systems. This development required the relaxation or modification of classical Lyapunov conditions, leading, in fact, to new functions and to a different framework. Well in the spirit of Lyapunov, this approach can be used for new classes of problems such as motion control, dynamic games and asymptotic observer design. Quite naturally, in application to stability and stabilization it brings us back to the classical Lyapunov results.

Using this framework and the generalized perturbation equation, it became possible to develop a time-space mosaic method, a sort of Lyapunov-like assembly line along the time axis, that allows us to substitute a single continuous Lyapunov function acting on $[t_0, \infty)$ by separate independent functions easier to construct, provided the consistency condition is satisfied. Apart from analytical advantages in stability analysis, it opens a way to "practical stability" evaluations (on a finite interval of time, cf. [17, 27]) through on-line computations of the rate of attraction. If combined with the space-splitting furnished by vector and matrix Lyapunov functions, see [22 – 24], this presents a complete time-space mosaic in $R^n \times R$ which could provide a powerful tool for solution of complicated practical problems.

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Mathematical Analysis in a Model of Obligate Mutualism with Food Chain Populations

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Abstract: This paper is concerned with a three-species food chain whose populations interact with a mutualist. The mutualism is obligate for one of the predators, and is modeled by a system of autonomous ordinary differential equations. Persistence and extinction criteria are developed in the cases of trivial, periodic and almost periodic dynamics.

Keywords: *Food chain; obligate mutualism; persistence; extinction; stability; periodic solutions; almost periodic solutions.*

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1 Introduction

The main thrust of this paper is to model obligate mutualism with the middle and top predators of a three-species food chain. The cases of facultative mutualism with the prey and middle predator populations have been considered in [24].

Previously, models of mutualism with predator-prey systems have been considered in [2, 12, 16, 24, 27, 34]. Models of obligate mutualism have been discussed in [7, 12, 13, 14]. For general discussions of mutualism the reader is referred to [1, 7, 11, 32].

Most models of mutualism are two dimensional. There has been a fair amount of work recently on three dimensional models, where the mutualism occurs between prey

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(see eg. [2, 15, 27]); predators (see [2, 13, 14, 15, 27]), perhaps both [30], or competitors (see [2, 9, 27, 29, 33]), etc. However, to date the only results dealing with mutualism in food chains are contained in [24].

Our main concern in this paper will be to develop criteria for the persistence or extinction of populations considered in our model. Persistence and extinction criteria for food chains and/or mutualism models have been discussed in [13, 15, 16, 17, 18, 21, 24].

At this time we give definitions of extinction, persistence and nonpersistence. First we define extinction. We say that $N(t) > 0$ exhibits *extinction* if $\lim_{t\to\infty} N(t) = 0$. We note that nonpersistence (defined below) does not necessarily imply extinction for all initial values $N(0)$. If $\lim_{t\to\infty} N(t) = 0$ for all $N(0) > 0$, we say that our system exhibits *total extinction* with respect to the $N(t)$ population. We will employ the notation R_v^+ to denote the positive v-axis and $\bar{R}_v^+$ for its closure, for any variable v. R_{vw}^+ denotes the positive $v - w$ plane and $\bar{R}_{vw}^+$ its closure etc.

Further if populations $N_1, \ldots, N_k$ exhibit total extinction in the space $R^+_{v_1,\ldots,v_\ell}$, we denote this by $\mathcal{E}_{N_1,\ldots,N_k} \to 0$. Here $N_1, \ldots, N_k$ and $v_1, \ldots, v_\ell$ are subsets of the set $\{u, x, y, z\}$.

We now define persistence with respect to the positive orthant in R^n (see [4, 5] for more general definitions). We say that $N(t)$, $N(0) > 0$, *persists* if $N(t) > 0$ for all $t > 0$ and $\liminf_{t\to\infty} N(t) > 0$. We say that $N(t)$ *uniformly persists* if, further, $\liminf_{t\to\infty} N(t) \geq \delta > 0$ for all $N(0) \in \mathring{R}_+$, where $\mathring{R}_+$ is the interior of R_+^n. Finally, we say that a vector $(N_1(t), \ldots, N_n(t)) \in R_+^n$ (uniformly) persists if each component (uniformly) persists. If any component fails to persist, we say that *nonpersistence* occurs.

In Section 2, we discuss our model. Section 3 contains an equilibrium analysis and a review of known persistence criteria. Section 4, gives persistence and extinction criteria for the total model including reversal of outcome. In particular, criteria are developed for the first time to the best of our knowledge for the case of almost periodic dynamics. Included in this are examples to illustrate our results. Section 5 contains a brief discussion.

2 The Models

In this section we describe a general model of interactions between a mutualist population and populations of a food chain. The mathematical formulation of obligate relationships between the mutualist and two different trophic levels of the food chain are also described. Finally, we estimate the region of attraction in each case, showing that the models are well-behaved.

We consider the autonomous system,

$$\begin{aligned}
\frac{du}{dt} &= uh(u, x, y, z),\\
\frac{dx}{dt} &= \alpha x g(u, x) - y p_1(u, x) - z p_2(u, x),\\
\frac{dy}{dt} &= y[-s_1(u, y) + c_1(u) p_1(u, x)] - z q(u, y),\\
\frac{dz}{dt} &= z[-s_2(u, z) + c_2(u) p_2(u, x) + c_3(u) q(u, y)],\\
u(0) &= u_0 \geq 0, \quad x(0) = x_0 \geq 0, \quad y(0) = y_0 \geq 0, \quad z(0) = z_0 \geq 0,
\end{aligned} \tag{2.1}$$

as a model of a mutualist-food chain interaction with continuous birth and death processes. The variable $u(t)$ represents the density of the mutualist at time t and $x(t)$, $y(t)$, $z(t)$ denote the prey, predator, and superpredator densities respectively.

The function $h(u,x,y,z)$ represents the specific growth rate of the mutualist population. We assume that $h(u,x,y,z)$ possesses the following properties.

(H1) $h(0,x,y,z) > 0$, $\frac{\partial h}{\partial u}(u,x,y,z) \le 0$.
(H2) There exists a unique function $L(x,y,z) > 0$, such that $h(L(x,y,z),x,y,z) = 0$.

The function $g(u,x)$ is the specific growth rate of the prey x in the absence of any predation. We assume that

(G1) $g(u,0) > 0$, $\frac{\partial g}{\partial x}(u,x) \le 0$.
(G2) There exists a unique $K(u) > 0$ such that $g(u,K(u)) = 0$.
(G3) $\frac{\partial g}{\partial u}(u,x) \ge 0$.

Next, the functions $p_i(u,x)$, $i = 1,2$ and $q(u,y)$ denote the predator's functional response to the prey and mutualist densities. We assume that,

(P1) $p_i(u,0) = 0$, $\frac{\partial p_i}{\partial x}(u,x) > 0$, $i = 1,2$, $q(u,0) = 0$, $\frac{\partial q}{\partial y}(u,y) > 0$.

The functions $s_1(u,y)$ and $s_2(u,z)$ are the specific death rates of the predators y and z, in the absence of predation. We assume that

(S1) $\frac{\partial s_1(u,y)}{\partial y} > 0$, $\frac{\partial s_2(u,z)}{\partial z} > 0$.
(S2) $\frac{\partial s_1(u,y)}{\partial u} \le 0$, $\frac{\partial s_2(u,z)}{\partial u} \ge 0$, $c_1'(u) \ge 0$, $c_i'(u) \le 0$, $i = 2,3$.

The non-negative functions $c_i(u)$, $i = 1,2,3$ are the conversion rates of prey biomass to the predator biomass. The implications of the above conditions are described in detail in [24]. Finally, we assume that all the functions are smooth enough so that existence and uniqueness of initial value problems hold and any required analysis can be carried out.

In model (2.1), we will think of α as a bifurcation parameter.

2.1 Obligate mutualism with the bottom-predator

In this section we consider the case of obligate mutualism between the mutualist u and the predator y. In addition to H(1-2), we assume the following for the specific growth rate $h(u,x,y,z)$ of the mutualist:

(H3) $\frac{\partial h}{\partial x}(u,x,y,z) \le 0$, $\frac{\partial h}{\partial y}(u,x,y,z) > 0$, $\frac{\partial h}{\partial z}(u,x,y,z) \le 0$.
(H4) $\lim\limits_{y\to\infty} L(0,y,0) = \widetilde{L} < \infty$.

The condition (H3) implies that u derives benefit from the predator population and that there might be a cost to the mutualist due to its interactions with the predators. The condition (H4) implies that u has a finite carrying capacity, no matter how much benefit it derives.

Further we assume

(P2) $\frac{\partial p_1(u,x)}{\partial u} \ge 0$, $\frac{\partial p_2(u,x)}{\partial u} \le 0$, $\frac{\partial q(u,y)}{\partial u} \le 0$.

This condition implies that the mutualist can benefit the bottom-predator by increasing its predator's response and/or by decreasing the response of the superpredator.

In order for system (2.1) to exhibit obligate mutualism between u and y, the food chain must collapse in the absence of the mutualist and the predator y must become extinct. Thus we require that the subsystem:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x g(0,x) - y p_1(0,x) - z p_2(0,x),\\ \frac{dy}{dt} &= y[-s_1(0,y) + c_1(0) p_1(0,x)] - z q(0,y), \\ \frac{dz}{dt} &= z[-s_2(0,z) + c_2(0) p_2(0,x) + c_3(0) q(0,y)],\end{aligned} \tag{2.2}$$

with $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$, exhibits extinction and $\lim_{t\to\infty} y(t) = 0$. As observed in [11] this happens when either

(S3a) $\lim_{x\to\infty} p_1(0,x) \le \frac{s_1(0,0)}{c_1(0)}$

or

(S3b) $p_1(0,\hat{x}) = \frac{s_1(0,0)}{c_1(0)}$ and $\hat{x} \ge K(0)$.

In conclusion whenever hypotheses H(1-4), G(1-3), P(1,2) and S(1-3) hold, mutualism occurs between u and y and is obligate for the predator y.

The following result establishes that under the above hypotheses, system (2.1) possesses a region of attraction. The proof is similar to one given in [24].

Theorem 2.1 *Let the hypotheses* H(1-4), G(1-3), P(1,2), S(1-3) *hold. Then the set*

$$\begin{aligned}\mathcal{C} = \{(u,x,y,z)\colon & 0 \le u \le \widetilde{L},\ \ 0 \le x \le \widetilde{K},\ \ 0 \le \widetilde{c}_1 x + y \le \widetilde{M},\\ & 0 \le c_2(\widetilde{L})x + c_3(\widetilde{L})y + z \le \widetilde{N},\ \ 0 \le c_2(0)x + c_3(0)y + z \le \widetilde{N}\},\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\widetilde{K} &= \max_{0\le u\le \widetilde{L}} K(u), \qquad \widetilde{c}_1 = \max_{0\le u\le \widetilde{L}} c_1(u),\\ \widetilde{M} &= \frac{c_1(\widetilde{L})\widetilde{K}}{s_1(\widetilde{L},0)}\,[\alpha g(\widetilde{L},0) + s_1(\widetilde{L},0)],\end{aligned} \tag{2.4}$$

$$\widetilde{N} = \frac{1}{s_2(0,0)}\left[c_2(0)\widetilde{K}\big(\alpha g(\widetilde{L},0) + s_2(0,0) + c_3(0)\widetilde{M}\Big(\frac{c_1(\widetilde{L})}{\widetilde{L}\,\widetilde{K} + s_2(0,0)}\Big)\right]$$

and

$$\widetilde{p}_1 = \max_{0\le u\le \widetilde{L}} p_1(u,\widetilde{K}),$$

is positively invariant and attracts all solutions starting with nonnegative initial-values.

2.2 Obligate mutualism with the top-predator

The system (2.1) exhibits mutualism between u and z, which is obligate for the top-predator z, whenever in addition to H(1-2), G(1-3), P1, S(1,3), the following assumptions hold:

(H3*) $\frac{\partial h(u,x,y,z)}{\partial x} \le 0,\ \frac{\partial h(u,x,y,z)}{\partial y} \le 0,\ \frac{\partial h(u,x,y,z)}{\partial z} > 0.$

(H4*) $\lim_{z\to\infty} L(0,0,z) = \widetilde{L} < \infty.$

(P2*) $\frac{\partial p_2(u,x)}{\partial u} \ge 0,\ \frac{\partial q(u,y)}{\partial u} \ge 0.$

(S2*) $\frac{\partial s_2(u,y)}{\partial u} \le 0,\ c_2'(u) \ge 0,\ c_3'(u) \ge 0.$

The following condition ensures that in the absence of u, z will become extinct.

(S4*a) $c_2(0) \lim_{x\to\infty} p_2(0,x) + c_3(0) \lim_{y\to\infty} q(0,y) \le s_2(0,0)$.

or

(S4*b) $c_2(0)p_2(0,\underline{x}) + c_3(0)q(0,\underline{y}) = s_2(0,0)$, for some $\underline{x}$ and $\underline{y}$, where $\underline{x} \ge K(0)$.

Finally the mutualist can indirectly benefit the predator z, by affecting the death rate, the predator response function or the conversion rate of prey biomass to the predator biomass of the predator y.

Under the above stated hypotheses by similar arguments as for Theorem 2.1, we can prove the following by using standard techniques (see e.g. [17]).

Theorem 2.2 *Let the hypotheses* H(1,2,3*, 4*), G(1-3), P(1,2*), S(1,2*,4*) *hold. Then the set*

$$\mathcal{D} = \{(u,x,y,z)\colon 0 \le u \le \widetilde{L},\ \ 0 \le x \le \widetilde{K},\ \ 0 \le \widetilde{c}_1 x + y \le \widetilde{M},\\ 0 \le c_2(\widetilde{L})x + c_3(\widetilde{L})y + z \le \widetilde{N}\}, \tag{2.5}$$

where the constants are given in (2.4), and

$$\widetilde{p}_1 = \max_{0\le u\le \widetilde{L}} p_1(u,\widetilde{K}),$$

is positively invariant and attracts all solutions starting with nonnegative initial-values.

3 The Equilibria

The question of existence and non-existence of various equilibria of system (2.1) and their stabilities are discussed in detail in [24]. Below we describe the information needed to study the question of reversal of outcome in our system for the two cases under consideration.

Case I: Obligate mutualism between u and y

The system (2.1) possesses the equilibrium $E_0(0,0,0,0)$ and one dimensional equilibria $E_1(L_0,0,0,0)$, $E_2(0,K_0,0,0)$, where $L_0 = L(0,0,0)$ and $K_0 = K(0)$. The two dimensional equilibrium $E_3(\widetilde{u},\widetilde{x},0,0)$ always exists. The equilibrium $E_5(0,x_2,0,z_2)$ in the $x{-}z$ plane may or may not exist. The three dimensional equilibria, if they exist are of the form $E_6(u_3,x_3,y_3,0)$ and $E_7(u_4,x_4,0,z_4)$. We note that a three dimensional submodel has an equilibrium if it is uniformly persistent (see [4]).

Case II: Obligate mutualism between u and z

In this case the equilibria $E_0(0,0,0,0)$, $E_1(L_0,0,0,0)$, $E_2(0,K_0,0,0)$, $E_3(\widetilde{u},\widetilde{x},0,0)$ always exist. The equilibrium $E_4(0,x_1,y_1,0)$ in the $x-y$ plane may or may not exist. The three dimensional equilibria if they exist are of the form $E_6(u_3,x_3,y_3,0)$ and $E_7(u_4,x_4,0,z_4)$.

Next, we list information regarding the eigenvalues of the variational matrix, computed at the various equilibria so that their stabilities may be discussed.

The eigenvalues of E_2 in the y and z-directions are

$$\alpha_i \triangleq -s_i(0,0) + c_i(0)p_i(0,K_0), \quad i = 1,2. \tag{3.1}$$

The eigenvalues of E_3 in the y and z directions are

$$\beta_i \triangleq -s_i(\widetilde{u},0) + c_i(\widetilde{u})p_i(\widetilde{u},\widetilde{x}), \quad i = 1,2. \tag{3.2}$$

The eigenvalues of E_4 in the z-direction and of E_5 in the y-direction are

$$\gamma \triangleq -s_2(0,0) + c_2 p_2(0,x_1) + c_3(0)q(0,y_1), \tag{3.3}$$

and

$$\delta \triangleq -s_1(0,0) + c_1(0)p_1(0,x_2) - z_2 q_y(0,0), \tag{3.4}$$

respectively.

The eigenvalues of E_6 and E_7 in the z and y directions are

$$\xi \triangleq -s_2(u_3,0) + c_2(u_3)p_2(u_3,x_3) + c_3(u_3)q(u_3,y_3) \tag{3.5}$$

and

$$\eta \triangleq -s_1(u_4,0) + c_1(u_4)p_1(u_4,x_4) - z_4 q_y(u_4,0), \tag{3.6}$$

respectively.

The above values are computed in a straightforward manner using standard techniques of ordinary differential equations.

4 Reversal of Outcome

Case I: Obligate mutualism between u and y

Suppose that for the system (2.1) the hypotheses H(1-4), G(1-3), P(1-2), S(1-3) hold. The obligate relationship between u and y implies that $\mathcal{E}_{y\to 0}$ in R^+_{xy} and R^+_{xyz}, that is, in the absence of mutualism, the predator y becomes extinct. However, we will show that with mutualism present, system (2.1) can exhibit uniform persistence resulting in a reversal of the outcome exhibited by the food chain submodel. The following result specifies a set of conditions leading to such a reversal. The proof follows using techniques similar to those used in [24] and is thus omitted. First we assume the following additional hypotheses for technical mathematical reasons.

(H5) Let E_5 (if it exists) be globally asymptotically stable with respect to solutions initiating in $\overset{\circ}{R}{}^+_{xz}$.

(H6) Let the equilibria E_6 and E_7 be globally asymptotically stable in $\overset{\circ}{R}{}^+_{uxy}$ and $\overset{\circ}{R}{}^+_{uxz}$, respectively.

Theorem 4.1 *Let the hypotheses* H(1-6), G(1-3), P(1,2) *and* S(1-3) *hold. Then the system (2.1) is uniformly persistent whenever* $\xi > 0$ *and* $\eta > 0$*, where* ξ *and* η *are given by (3.5) and (3.6), respectively.*

The above theorem can be interpreted as follows. If the predator y is unable to survive on its own, then the mutualist could help the predator population to survive. As observed in [13], the mutualist can benefit the mutualist predator in several ways: by increasing the prey growth rate, by increasing the rate of predation of its prey x, by providing an alternate food source for the mutualist-predator and by enhancing the efficiency of utilization of the prey by the mutualist-predator. Below we illustrate each of these cases with an example. All examples considered are of the form

$$\begin{aligned}
\frac{du}{dt} &= u\left(1-\frac{u}{L+\ell y}\right),\\
\frac{dx}{dt} &= \alpha x\left(1-\frac{x}{K+ku}\right)-(\gamma_0+\gamma_1 u)xy-\frac{\delta_0}{1+\delta_1 u}\,xz,\\
\frac{dy}{dt} &= y\big[-s_{10}+s_{11}u-s_{12}y+(c_{10}+c_{11}u)(\gamma_0+\gamma_1 u)x-\xi_0 z\big],\\
\frac{dz}{dt} &= z\left[-s_{20}-s_{21}u-s_{22}z+\frac{c_{20}}{1+c_{21}u}\frac{\delta_0}{1+\delta_1 u}x+\frac{c_{30}\xi_0}{1+c_{31}u}y\right],
\end{aligned} \tag{4.1}$$

where all the constants are assumed to be nonnegative.

In the absence of the mutualist u, there will be an equilibrium in R^+_{xy},

$$(x,y)=\left(\frac{K(s_{10}\gamma_0+\alpha s_{12})}{\alpha s_{12}+Kc_{10}\gamma_0^2},\frac{\alpha(Kc_{10}\gamma_0-s_{10})}{\alpha s_{12}+Kc_{10}\gamma_0^2}\right),$$

unless $Kc_{10}\gamma_0 \le s_{10}$. Thus for obligate mutualism we require

$$Kc_{10}\gamma_0 \le s_{10}. \tag{4.2}$$

Example 4.1 When $\gamma_1=\delta_1=s_{11}=c_{11}=s_{21}=c_{21}=c_{31}=0$ and $k>0$, mutualism occurs by means of the mutualist enhancing the prey growth rate.

The region of attraction for the system is contained in the set

$$\begin{aligned}
\mathcal{B}=\{(u,x,y,z)\colon 0\le u\le L+\ell\widetilde{M},\ 0\le x\le K+k(L+\ell\widetilde{M}),\\
0\le y\le \widetilde{M},\ 0\le z\le \widetilde{N}\},
\end{aligned} \tag{4.3}$$

where

$$\widetilde{M}=\frac{-s_{10}+c_{10}\gamma_0(K+kL)}{s_{12}-c_{10}\gamma_0 k\ell},$$

$$\widetilde{N}=\frac{1}{s_{22}}\big(-s_{20}+Kc_{20}\delta_0+K(L+\ell\widetilde{M})+c_{30}\xi_0\widetilde{M}\big).$$

We assume that $\widetilde{M}$ and $\widetilde{N}$ are positive, otherwise the system will always exhibit extinction. The equilibria in $\overline{R}^+_{ux}$ are $E_0(0,0,0,0)$, $E_1(L,0,0,0)$, $E_2(0,K,0,0)$, $E_3(L,K+kL,0,0,0)$. The equilibrium $E_5\left(0,\frac{K(\delta_0 s_{20}+\alpha s_{22})}{\alpha s_{22}+Kc_{20}\delta_0^2},0,\frac{\alpha(-s_{20}+Kc_{20}\delta_0)}{\alpha s_{22}+Kc_{20}\delta_0^2}\right)$ exists provided $Kc_{20}\delta_0>s_{20}$.

The subsystem in R^+_{uxy} is uniformly persistent whenever

$$\beta_1=-s_{10}+c_{10}\gamma_0(K+kL)>0, \tag{4.4}$$

in which case the equilibrium in R^+_{uxy} is given by

$$E_6(u_3,x_3,y_3,0)=(L+\ell y_3,\frac{s_{10}+s_{12}y_3}{c_{10}\gamma_0},y_3,0),$$

where

$$y_3 = \frac{b + \sqrt{b^2 + 4k\ell c_{10}\gamma_0^2\beta_1}}{2k\ell c_{10}\gamma_0^2}$$

and

$$b = \alpha k\ell c_{10}\gamma_0 - \alpha s_{12} - c_{10}\gamma_0^2(K + kL).$$

The subsystem in R^+_{uxz} is uniformly persistent provided

$$\beta_2 = -s_{20} + c_{20}\delta_0(K + kL) > 0, \tag{4.5}$$

and then the equilibrium in R^+_{uxz} is given by

$$E_7(u_4, x_4, 0, z_4) = \left(L, \frac{(\alpha s_{22} + \delta_0 s_{20})(K + kL)}{\alpha s_{22} + (K + kL)c_{20}\delta_0^2}, 0, \frac{\alpha\beta_2}{\alpha s_{22} + (K + kL)c_{20}\delta_0^2}\right).$$

The symmetric matrix $\mathcal{B}(u, x, y)$ corresponding to E_6 is given by

$$\begin{aligned} b_{11} &= \frac{1}{(L + \ell y_3)}, \\ b_{12} &= \frac{-\alpha k c_{10} x}{2(K + ku)(K + k(L + \ell y_3))}, \\ b_{13} &= -\frac{\ell u}{2(L + \ell y)(L + \ell y_3)}, \\ b_{22} &= \frac{\beta c_{10}}{K + k(L + \ell y_3)}, \\ b_{23} &= 0 \quad \text{and} \quad b_{33} = s_{12}. \end{aligned}$$

It is positive definite in the region of attraction of the subsystem in R^+_{uxy}, whenever

$$s_{12}\left(\frac{\alpha c_{10}}{L + \ell y_3} - \frac{\alpha^2 k^2 c_{10}^2 (K + k(L + \ell\widetilde{M}))^2}{4K^2(K + k(L + \ell y_3))}\right) - \frac{\alpha c_{10}\ell^2(L + \ell\widetilde{M})^2}{4L^2(L + \ell y_3)^2} > 0. \tag{4.6}$$

The matrix $\mathcal{D}(u, x, z)$ corresponding to E_7 is given by

$$d_{11} = \frac{1}{L}, \quad d_{12} = \frac{\alpha k c_{20} x}{2(K + kL)(K + ku)}, \quad d_{13} = 0,$$

$$d_{22} = \frac{\alpha c_{20}}{K + kL}, \quad d_{23} = 0, \quad d_{33} = s_{22}.$$

It is positive definite whenever

$$\frac{1}{L} - \frac{\alpha k^2 c_{20}(K + kL)^2}{4(K + kL)K^2} > 0. \tag{4.7}$$

Thus the system (4.1) will be uniformly persistent whenever (4.4)–(4.7) hold and

$$\xi = -s_{20} + c_{20}\delta_0 x_3 + c_{30}\xi_0 y_3 > 0, \tag{4.8}$$

and

$$\eta = -s_{10} + c_{10}\gamma_0 x_4 - \xi_0 z_4 > 0. \tag{4.9}$$

Example 4.2 When $k = \delta_1 = s_{11} = c_{11} = s_{21} = c_{21} = c_{31} = 0$ and $\gamma_1 > 0$, the mutualist enhances the rate of predation of the mutualist-predator y. Here the region of attraction is contained in the set

$$\begin{aligned}\mathcal{B} = \{(u,x,y,z)\colon 0 \le u \le L + \ell\widetilde{M},\ \ 0 \le x \le K,\ \ 0 \le y \le \widetilde{M},\\ 0 \le z \le \frac{1}{s_{22}}\,(-s_{20} + Kc_{20}\delta_0 + c_{30}\xi_0\widetilde{M})\},\end{aligned} \tag{4.10}$$

where

$$\widetilde{M} = -\frac{s_{10} + Kc_{10}(\gamma_0 + \gamma_1 L)}{s_{12} - Kc_{10}\gamma_1\ell}.$$

The equilibria in $\overline{R}^+_{ux}$ are given by $E_0(0,0,0,0)$, $E_1(L,0,0,0)$, $E_2(0,K,0,0)$, $E_3(L,K,0,0)$.

The subsystem in R^+_{uxy} is uniformly persistent whenever

$$\beta_1 = -s_{10} + Kc_{10}(\gamma_0 + \gamma_1 L) > 0, \tag{4.11}$$

in which case the equilibrium $E_6(u_3,x_3,y_3,0) = (L+\ell y_3, \frac{1}{\alpha}[\alpha - (\gamma_0 + \gamma_1(L+\ell y_3))y_3], y_3, 0)$, where from Descartes' rule of signs y_3 is the unique positive root of the equation

$$y^3 + \frac{2}{\gamma_1\ell}\,(\gamma_0 + \gamma_1 L)y^2 + \big(Kc_{10}(\gamma_0 + \gamma_1 L)^2 + \alpha s_{12} - \alpha K c_{10}\gamma_1\ell\big)y - \alpha\beta_1 = 0. \tag{4.12}$$

The subsystem in R^+_{uxz} is uniformly persistent provided

$$\beta_2 = -s_{20} + c_{20}\delta_0 K > 0, \tag{4.13}$$

in which case $E_7(u_4,x_4,0,z_4) = \Big(L, \frac{K(\alpha s_{22} + \delta_0 s_{20})}{\alpha s_{22} + Kc_{20}\delta_0^2}, 0, \frac{\alpha\beta_2}{\alpha s_{22} + Kc_{20}\delta_0^2}\Big)$. Now we consider the global asymptotic stability of E_6 and E_7 in R^+_{uxy} and R^+_{uxz}, respectively.

The symmetric matrix $\mathcal{B}(u,x,y)$ corresponding to $E_6(u_3,x_3,y_3,0)$ is given by

$$b_{11} = \frac{1}{u_3}, \quad b_{12} = \frac{\gamma_1 y}{2}, \quad b_{13} = -\frac{1}{2}\bigg(\frac{\ell u}{u_3(L+\ell y)} + c_{10}\gamma_1 x - s_{11}\bigg),$$

$$b_{22} = \frac{c_{10}}{K}, \quad b_{23} = 0, \quad b_{33} = s_{12}.$$

It is positive definite in its region of attraction whenever

$$\frac{4}{s_{12}}\left(\frac{1}{Ku_3} - \frac{\gamma_1^2 c_{10}\widetilde{M}^2}{4}\right) - \frac{1}{K}\bigg(\frac{\ell(L+\ell\widetilde{M})}{u_3 L} + Kc_{10}\gamma_1 - s_{11}\bigg) > 0, \tag{4.14}$$

where $\widetilde{M}$ is given by (4.10). The symmetric matrix $\mathcal{D}(u,x,z)$ corresponding to E_7 is given by

$$d_{11} = \frac{1}{u_4}, \quad d_{12} = 0, \quad d_{13} = \frac{1}{2}s_{21},$$

$$d_{22} = \frac{\alpha c_{20}}{K}, \quad d_{23} = 0, \quad d_{33} = s_{22}.$$

It is positive definite in its region of attraction whenever

$$4s_{22} - Ls_{21}^2 > 0. \tag{4.15}$$

Thus whenever (4.11), (4.13)–(4.15) hold and

$$\xi = -s_{20} + c_{20}\delta_0 x_3 + c_3\xi_0 y_3 > 0, \tag{4.16}$$

and

$$\eta = -s_{10} + c_{10}(\gamma_0 + \gamma_1 u_4)x_4 - \xi_0 z_4 > 0, \tag{4.17}$$

the system (4.1) is uniformly persistent.

Example 4.3 When $k = \gamma_1 = \delta_1 = c_{11} = s_{21} = c_{21} = c_{31} = 0$ and $s_{11} > 0$, the mutualist provides the mutualist-predator with an alternate food source.

The region of attraction is contained in the set

$$\mathcal{B} = \{(u,x,y,z) \colon 0 \le u \le L + \ell\widetilde{M},\ \ 0 \le x \le K,\ \ 0 \le y \le \widetilde{M},\ \ 0 \le z \le \widetilde{N}\}, \tag{4.18}$$

where

$$\begin{aligned} \widetilde{M} &= -\frac{s_{10} + s_{11}L + c_{10}\gamma_0 K}{s_{12} - s_{11}\ell}, \\ \widetilde{N} &= \frac{1}{s_{22}}(-s_{20} + Kc_{20}\delta_0 + c_3\xi_0\widetilde{M}). \end{aligned} \tag{4.19}$$

The equilibria in $\overline{R}^+_{ux}$ are $E_0(0,0,0,0)$, $E_1(L,0,0,0)$, $E_2(0,K,0,0)$ and $E_3(L,K,0,0)$. The subsystem in R^+_{uxy} is uniformly persistent whenever

$$\beta_1 = -s_{10} + s_{11}L + Kc_{10}\gamma_0 > 0. \tag{4.20}$$

The subsystem in R^+_{uxz} is uniformly persistent whenever

$$\beta_2 = -s_{20} + Kc_{20}\delta_0 > 0. \tag{4.21}$$

Whenever the inequalities (4.20) and (4.21) hold the equilibria in R^+_{uxy} and R^+_{uxz} are given by

$$E_6(u_3,x_3,y_3,0) = \left(L + \ell y_3, \frac{K(s_{10}\gamma_0 + \alpha s_{12} - \alpha s_{11}\ell + s_{11}L\gamma_0)}{Kc_{10}\gamma_0^2 - \alpha s_{11}\ell + \alpha s_{12}}, \frac{\alpha\beta_1}{Kc_{10}\gamma_0^2 - \alpha s_{11}\ell + \alpha s_{12}}, 0\right)$$

and

$$E_7(u_4,x_4,0,z_4) = \left(L, \frac{s_{20} + s_{22}z_4}{c_{20}\delta_0}, 0, \frac{\alpha\beta_2}{Kc_{20}\delta_0^2 + \alpha s_{22}}\right).$$

The symmetric matrix $\mathcal{B}(u,x,y)$ corresponding to E_6 is given by

$$\begin{aligned} &b_{11} = \frac{1}{u_3}, \quad b_{12} = 0, \quad b_{13} = -\frac{\ell u}{2u_3}, \\ &b_{22} = \frac{\alpha c_{10}}{K}, \quad b_{23} = 0, \quad b_{33} = s_{12}. \end{aligned}$$

It is positive definite in the region of attraction whenever

$$s_{12} - \frac{\ell^2}{4u_3}(L + \ell\widetilde{M})^2 > 0. \tag{4.22}$$

The matrix corresponding to E_7, $\mathcal{D}(u,x,z) = \operatorname{diag}\left(\frac{1}{u_4}, \frac{\alpha c_{20}}{K}, s_{22}\right)$ is always positive definite.

Thus the system (4.1) will be uniformly persistent whenever inequalities (4.20) – (4.22) hold and

$$\xi = -s_{20} + c_{20}\delta_0 x_3 + c_{30}\xi_0 y_3 > 0, \tag{4.23}$$

and

$$\eta = -s_{10} + s_{11}u_4 + c_{10}\gamma_0 x_4 - \xi_0 z_4 > 0. \tag{4.24}$$

Example 4.4 When $k = \gamma_1 = \delta_1 = s_{11} = s_{21} = c_{21} = c_{31} = 0$ and $c_{11} > 0$, the mutualist enhances the efficiency of the utilization of the prey by the mutualist-predator. Here the region of attraction is contained in the set

$$\mathcal{B} = \{(u,x,y,z)\colon 0 \le u \le L + \ell\widetilde{M},\ \ 0 \le x \le K,\ \ 0 \le y \le \widetilde{M},\ \ 0 \le z \le \widetilde{N}\}, \tag{4.25}$$

where

$$\widetilde{M} = \frac{-s_{10} + K\gamma_0(c_{10} + c_{11}L)}{s_{12} - K\gamma_0 c_{11}\ell},$$

$$\widetilde{N} = \frac{-s_{20} + Kc_{20}\delta_0 + c_{30}\xi_0\widetilde{M}}{s_{22}}.$$

The equilibrium in R^+_{ux} is $E_3(L,K,0,0)$. The subsystems in R^+_{uxy} and R^+_{uxz} are uniformly persistent provided

$$\beta_1 = -s_{10} + (c_{10} + c_{11}L)\gamma_0 K > 0, \tag{4.26}$$

and

$$\beta_2 = -s_{20} + Kc_{20}\delta_0 > 0. \tag{4.27}$$

The equilibrium

$$E_6(u_3,x_3,y_3,0) = \left(L + \ell y_3,\ \frac{K}{\alpha}(\alpha - \gamma_0 y_3),\ \frac{b + \sqrt{b^2 + 4\alpha\beta_1}}{2Kc_{11}\gamma_0^2\ell},\ 0\right),$$

where

$$b = \alpha(\ell K c_{11}\gamma_0 - s_{12}) - K\gamma_0^2(c_{10} + c_{11}L).$$

The equilibrium

$$E_7(u_4,x_4,0,z_4) = \left(L,\ \frac{K(\alpha s_{22} + \delta_0 s_{21})}{\alpha s_{22} + Kc_{20}\delta_0^2},\ 0,\ \alpha\beta_2\right).$$

The symmetric matrix $\mathcal{B}(u,x,y)$ corresponding to E_6 is given by

$$b_{11} = \frac{1}{u_3},\quad b_{12} = 0,\quad b_{13} = -\frac{\ell u}{2u_3(L + \ell y)} - \frac{1}{2}c_{11}\gamma_0 x,$$

$$b_{22} = \frac{\alpha}{K}(c_{10} + c_{11}u_3),\quad b_{23} = 0,\quad b_{33} = s_{12}.$$

It is positive definite in its region of attraction whenever

$$\frac{4s_{12}}{u_3} - \left(\frac{\ell(L + \ell\widetilde{M})}{u_3 L} + Kc_{11}\gamma_0\right)^2 > 0. \tag{4.28}$$

The symmetric matrix corresponding to E_7 is given by

$$\mathcal{D}(u,x,z) = \operatorname{diag}\left(\frac{1}{L},\ \frac{\alpha c_{20}}{K},\ s_{22}\right).$$

Thus whenever inequalities (4.26) – (4.28) hold and

$$\xi = -s_{20} - s_{21}u_3 + c_{20}\delta_0 x_3 + c_{30}\xi_0 y_3 > 0, \tag{4.29}$$

and

$$\eta = -s_{10} + (c_{10} + c_{11}u_4)\gamma_0 x_4 - \xi_0 z_4 > 0, \tag{4.30}$$

the given system is uniformly persistent.

In the above examples, all boundary equilibria of predator-prey type were globally asymptotically stable in their respective predator-prey planes, i.e. we assumed that hypotheses (H5) and (H6) hold.

We now allow for the possibility that (H5) and/or (H6) be violated, in which case there could be periodic solutions in $\overset{\circ}{R}{}^{+}_{xz}$ and periodic, almost periodic, or recurrent motions in $\overset{\circ}{R}{}^{+}_{uxz}$ and $\overset{\circ}{R}{}^{+}_{uxy}$.

Persistence criteria have been obtained in three dimensional systems when periodic solutions occur in the predator-prey planes. To the best of our knowledge the almost periodic case for four dimensions has not yet been considered.

Hence we next demonstrate that uniform persistence can occur even when one or more of the three-dimensional subsystems have almost periodic solutions. We note that the closure Σ of an almost periodic orbit is a compact, minimal set and every solution in Σ is almost periodic (see [26]).

We state and prove a theorem for persistence in the case where almost periodic solutions occur in R^+_{uxy}, but that E_7 is globally stable with respect to $\overset{\circ}{R}{}^{+}_{uxz}$. Let there be k nontrivial almost periodic solutions in R^+_{uxy}, denoted $(\phi_i(t), \psi_i(t), \xi_i(t), 0)$, with disjoint closures $\sum\limits_i$, $i = 1, \ldots, k$.

Theorem 4.2 *Let the hypotheses* H(1-5), G(1-3), P(1,2) *and* S(1-3) *hold, and* E_7 *be globally stable with respect to* $\overset{\circ}{R}_{+uxz}$. *Also let the omega limit sets of all solutions initiating in* R^+_{uxy} *lie in the acyclic set* $\left\{ \bigcup\limits_{i=1}^{k} \Sigma_i \cup E_6 \right\}$. *Then the system (2.1) is uniformly persistent whenever* $\xi > 0$, $\eta > 0$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t [-s_2(\phi_i(r), 0) + c_2(\phi_i(r))p_2(\phi_i(r), \psi_i(r)) + c_3(\phi_i(r))q(\phi_i(r), \xi_i(r))]\, dr > 0, \quad i = 1, \ldots, k. \tag{4.31}$$

Proof First we observe that the limit in the inequality (4.31) exists. Also as each Σ_i is a compact minimal set, it lies in R^+_{uxy} and the subsystem in R^+_{uxy} is uniformly persistent. The uniform persistence of (2.1) will follow (see [5]) if we can show that the stable sets, $W^s(\Sigma_i)$ and $W^s(E_j)$ do not intersect R^+_{uxyz} and each of them is isolated in $\overline{R}^+_{uxyz}$.

First, we show that $W^s(\Sigma_i) \cap R^+_{uxyz} = \phi$, $1 \le i \le k$. Let $\Phi(t) = (\phi(t), \psi(t), \xi(t), 0)^T$ be any almost periodic solution in Σ_{i_0} and $X(t) = (u(t), x(t), y(t), z(t))^T$, $X(0) = X_0 \in$

R^+_{uxyz} be any solution starting sufficiently close to $\Phi(t)$. Linearizing $X(t)$ about $\Phi(t)$ we obtain

$$Y'(t) = A(t)Y(t), \tag{4.32}$$

where $Y(t) = (u_1(t), x_1(t), y_1(t), z_1(t))^T$ is the linearized vector variable, and

$$A(t) = \begin{pmatrix} \phi h_u + h & \phi h_x & \phi h_y & \phi h_z \\ \alpha\psi g - \psi p_1 & \alpha g + \alpha g_x - \xi p_{1_x} & -p_1 & -p_2 \\ -\psi(-s_{1u} + c_1' p_{1u}) & \psi c_1 p_{1x} & -s_1 + c_1 p_1 - \xi s_{1y} & -q \\ 0 & 0 & 0 & -s_2 + c_2 p_2 + c_3 q \end{pmatrix},$$

where all the functions are evaluated at $\Phi(t)$. Solving the last equation in (4.32) we obtain

$$z_1(t) = z_1(0) \exp \int_0^t [-s_2(\phi(r), 0) + c_2(\phi(r))p_2(\phi(r), \psi(r)) + c_3(\phi(r))q(\phi(r), \xi(r))]\, dr.$$

Now since $\Phi(t)$ lies in Σ_{i_0} and the solutions through Σ_{i_0} are uniformly stable in both directions in Σ_{i_0}, the inequality (4.31) (with $i = i_0$) implies that $z_1(t) > 0$ for $t \geq 0$ and is an increasing function for sufficiently large t. Thus any solution in R^+_{uxyz}, starting sufficiently close to Σ_{i_0} eventually gets away from it.

Hence, $\Omega(X_0) \not\subset \Sigma_{i_0}$. Thus $W^s(\Sigma_{i_0}) \cap R^+_{uxyz} = \phi$, $1 \leq i \leq k$.

Since all boundary equilibria are hyperbolic we conclude as in [24] that $W^s(E_i) \cap R^+_{uxyz} = \emptyset$, $1 \leq i \leq 7$.

Now suppose that for some i_0, Σ_{i_0} is not an isolated invariant set in R^+_{uxyz}. Then there must exist closed invariant sets in arbitrarily close neighbourhoods of Σ_{i_0}. Let $M \supset \Sigma_{i_0}$ be such a closed invariant set. Then by repeating the arguments, given above we conclude that Σ_{i_0} repels the solutions starting in M/Σ_{i_0} and hence they must leave M. However this contradicts the fact that M is invariant. Hence the proof.

Remark 4.3 The acyclic condition of the above theorem is always satisfied when each Σ_i is either asymptotically stable or completely unstable in R^+_{uxy} and there do not exist any homoclinic orbits in R^+_{uxy}.

Remark 4.4 In the event that almost periodic solutions exist for the subsystems in R^+_{uxz} a criterion similar to the one given by the above theorem can be obtained.

Case II: Obligate mutualism between u and z

The system (2.1) exhibits obligate mutualism between the mutualist u and the top-predator z, whenever the hypotheses H(1,2,3*,4*), G(1-3), P(1,2*) and S(1,2*,4*) hold.

Also from the hypothesis S(4*) the mutualism is obligate for the predator. Hence $\mathcal{E}_{z\to 0}$ in R^+_{xz} and R^+_{xyz} and the equilibrium E_5 does not exist. To obtain the persistence criteria in this case, we need to introduce the following additional hypothesis:

(H5*) Let the equilibrium E_4 (if it exists) be globally asymptotically stable with respect to solutions initiating in $\overset{\circ}{R}{}^+_{xy}$.

The following result holds for system (2.1).

Theorem 4.3 *Let the hypotheses* H(1,2,3*,4*,5*,6), G(1-3), P(1,2*), S(1,2*,4*) *hold. Then system (2.1) is uniformly persistent whenever* $\xi > 0$ *and* $\eta > 0$.

Persistence in system (2.1) can result in any of the following ways.

The mutualist u can directly benefit the mutualist-predator z, by enhancing the growth rate of the prey x, by providing an alternate food supply, by increasing its rate of predation or by enhancing the efficiency of utilization of the prey(s). Below we illustrate each of these cases with an example. We also note that the mutualist's interaction with the predator y can also lead to a beneficial effect for the top-predator.

Consider the system

$$\begin{aligned} u' &= u\left(1 - \frac{u}{L + \ell z}\right), \\ x' &= \alpha x\left(1 - \frac{x}{K + ku}\right) - \frac{\gamma_0}{1 + \gamma_1 u} xy - (\delta_0 + \delta_1 u)xz, \\ y' &= y\left[-s_{10} + s_{12}y + \frac{c_1\gamma_0}{1 + \gamma_1 u} xy\right] - (\xi_0 + \xi_1 u)yz, \\ z' &= z[-s_{20} - s_{21}u - s_{22}z + (c_{20} + c_{21}u)(\delta_0 + \delta_1 u)x + c_3(\xi_0 + \xi_1 u)y], \end{aligned} \tag{4.33}$$

where all the constants are assumed to be nonnegative.

It is easily seen that in the absence of the mutualist, $y(t) \le \frac{-s_{10} + Kc_1\gamma_0}{s_{12}}$. Hence assume that

$$s_{10} < Kc_1\gamma_0, \tag{4.34}$$

otherwise $\mathcal{E}_{y\to 0}$ in R^+_{uxyz}. Furthermore for obligate mutualism to occur we require that

$$Kc_{20}\delta_0 + c_3\xi_0 \frac{(-s_{10} + Kc_1\gamma_0)}{s_{12}} \le s_{20}. \tag{4.35}$$

Example 4.5 When $\gamma_1 = \delta_1 = \xi_1 = s_{21} = c_{21} = 0$ and $k > 0$, mutualism occurs by means of mutualist enhancing the rate of growth of the prey x.

The region of attraction is contained in the set

$$\begin{aligned} \mathcal{C} = \{(u, x, y, z) \colon 0 \le u \le L + \ell\widetilde{N},\ \ 0 \le x \le K + k(L + \ell\widetilde{N}), \\ 0 \le y \le \widetilde{M},\ \ 0 \le z \le \widetilde{N}\}, \end{aligned} \tag{4.36}$$

where

$$\widetilde{M} = \frac{-s_{10} + c_1\gamma_0(K + k(L + \ell\widetilde{N}))}{s_{11}}$$

and

$$\widetilde{N} = \frac{-s_{20} + c_{20}\delta_0(K + kL) + c_3\xi_0\widetilde{M}}{s_{22} - c_{20}\delta_0 + K\ell}.$$

The equilibrium in R^+_{ux} is $(L, K + kL)$. The subsystems in R^+_{uxy} and R^+_{uxz} are uniformly persistent whenever

$$\beta_1 = -s_{10} + c_1\gamma_0(K + kL) > 0, \tag{4.37}$$

and

$$\beta_2 = -s_{20} + c_{20}\delta_0(K + kL) > 0. \tag{4.38}$$

The equilibrium

$$E_6(u_3, x_3, y_3, 0) = \left(L, \frac{(K+kL)(\gamma_0 s_{10} + \alpha s_{11})}{\alpha s_{11} + (K+kL)\gamma_0^2 c_1}, \frac{\alpha\beta_1}{\alpha s_{11} + (K+kL)\gamma_0^2 c_1}, 0\right),$$

$$E_7(u_4, x_4, 0, z_4) = (L + \ell z_4, x_4, 0, z_4),$$

where

$$x_4 = \frac{1}{\alpha}(\alpha - \delta_0 z_4)(K + kL + k\ell z_4),$$

$$z_4 = \frac{b_4 + \sqrt{b_0^2 + 4c_{20}\delta_0^2\alpha k\ell\beta_2}}{2c_{20}\delta_0^2 k\ell},$$

and $b_0 = \alpha(-s_{22} + c_{20}\delta_0 k\ell) - c_{20}\delta_0^2(K + kL)$. The symmetric matrix $\mathcal{B}(u, x, y)$ corresponding to E_8 is given by

$$b_{11} = \frac{1}{u_3}, \quad b_{12} = \frac{-c_1\alpha kx}{2(K + ku_3)(K + ku)}, \quad b_{13} = 0,$$

$$b_{22} = \frac{\alpha c_1}{K + ku_3}, \quad b_{23} = 0, \quad b_{33} = s_{11}.$$

Thus $\mathcal{B}(u, x, y)$ is positive definite in its region of attraction whenever

$$4K^2(K + ku_3) - \alpha u_3 c_1 k^2 (K + kL)^2 > 0. \tag{4.39}$$

The symmetric matrix $\mathcal{D}(u, x, z)$ corresponding to E_7 is given by

$$d_{11} = \frac{1}{u_4}, \quad d_{12} = \frac{-\alpha k c_{20} x}{2(K + ku_4)(K + ku)}, \quad d_{13} = -\frac{1}{2}\frac{\ell u}{u_4(L + \ell z)},$$

$$d_{22} = \frac{\alpha c_{20}}{(K + ku_4)}, \quad d_{23} = 0, \quad b_{33} = s_{22}.$$

The region of attraction of the subsystem in R^+_{uxz} is contained in the set

$$\mathcal{C}_1 = \{(u, x, z)\colon 0 \le u \le L + \ell N_1,\ 0 \le x \le K + k(L + \ell N_1),\ 0 \le z \le N_1\},$$

where $N_1 = \frac{\beta_2}{s_{22} - c_{20}\delta_0 k\ell}$. The matrix $\mathcal{D}(u, x, z)$ is positive definite in $\mathcal{B}_1$ whenever

$$4c_{20}s_{22} - \frac{\alpha u_4 c_{20}^2 s_{22} k^2}{(K + ku_4)K^2}(K + kL + k\ell N_1)^2 - \frac{\ell^2}{L^2 u_4}(L + \ell N_1)^2 > 0. \tag{4.40}$$

Thus the system (4.31) is uniformly persistent whenever the inequalities (4.35)–(4.38) hold and

$$\xi = -s_{20} + c_{20}\delta_0 x_3 + c_3\xi_0 y_3 > 0 \tag{4.41}$$

and

$$\eta = -s_{10} + c_1\gamma_0 x_4 - \xi_0 z_4 > 0. \tag{4.42}$$

Example 4.6 When $k = \gamma_1 = \delta_1 = \xi_1 = c_{21} = 0$ and $s_{21} > 0$, mutualism occurs by means of providing an alternate food source to the top-predator. The region of attraction is contained in the set

$$\mathcal{C} = \{(u,x,z)\colon 0 \le u \le L + \ell\widetilde{N},\ \ 0 \le x \le K,\ \ 0 \le y \le \widetilde{M},\ \ 0 \le z \le \widetilde{N}\}, \tag{4.43}$$

where

$$\widetilde{M} = \frac{-s_{10} + Kc_1\gamma_0}{s_{11}} \quad \text{and} \quad \widetilde{N} = \frac{-s_{20} + s_{21}L + Kc_{20}\delta_0 + c_3\xi_0\widetilde{M}}{s_{22} - \ell s_{21}}.$$

The equilibrium in R^+_{ux} is $(L, K, 0, 0)$. The subsystems in R^+_{uxy} and R^+_{uxz} are uniformly persistent whenever

$$\beta_1 = -s_{10} + c_1\gamma_0 K > 0, \tag{4.44}$$

and

$$\beta_2 = -s_{20} + s_{21}L + c_{20}\delta_0 K > 0, \tag{4.45}$$

respectively, in which case the equilibria are

$$E_6(u_3, x_3, y_3, 0) = \left(L,\ \frac{K(\gamma_0 s_{10} + \alpha s_{12})}{Kc_1\gamma_0^2 + \alpha s_{12}},\ \frac{\alpha(-s_{10} + Kc_1\gamma_0)}{Kc_1\gamma_0^2 + \alpha s_{12}},\ 0\right)$$

and

$$E_7(u_4, x_4, 0, z_4) = \left(L + \ell z_4,\ \frac{K((s_{20} - s_{21}L)\delta_0 + (s_{22} - s_{21}\ell)\alpha)}{Kc_{20}\delta_0^2 + \alpha(s_{22} - s_{21}\ell)},\ 0,\ \frac{\alpha\beta_2}{Kc_{20}\delta_0^2 + \alpha(s_{22} - s_{21}\ell)}\right).$$

The symmetric matrix corresponding to E_6 is $\mathcal{B}(u,x,y) = \operatorname{diag}\left(\frac{1}{u_3}, \frac{\alpha c_1}{K}, s_{12}\right)$. The symmetric matrix $\mathcal{D}(u,x,z)$, corresponding to E_7 is given by

$$b_{11} = \frac{1}{u_4}, \quad b_{12} = 0, \quad b_{13} = \left(\frac{\ell u}{u_4(L + \ell z)} + s_{21}\right),$$
$$b_{22} = \frac{\alpha c_{20}}{K}, \quad b_{23} = 0, \quad b_{33} = s_{22}.$$

The region of attraction of the subsystem in R^+_{uxz} is contained in the set

$$\mathcal{C}_1 = \left\{(u,x,z)\colon 0 \le u \le L + \frac{\ell\beta_2}{s_{22} - s_{21}\ell},\ \ 0 \le x \le K,\ \ 0 \le z \le \frac{\beta_2}{s_{22} - s_{21}\ell}\right\}.$$

The matrix $\mathcal{D}(u,x,z)$ is positive definite in $\mathcal{C}_1$ whenever

$$s_{22} - u_4\left(s_{21} + \frac{\ell}{Lu_4}\left(L + \frac{\ell\beta_2}{s_{22} - s_{21}\ell}\right)\right)^2 > 0. \tag{4.46}$$

Therefore the system (4.31) will be uniformly persistent whenever inequalities (4.42)–(4.44) hold and

$$\xi = -s_{20} + s_{21}u_3 + c_{20}\delta_0 x_3 + c_3\xi_0 y_3 > 0, \tag{4.47}$$

and

$$\eta = -s_{10} + c_1\gamma_0 x_4 - \xi_0 z_4 > 0. \tag{4.48}$$

Example 4.7 When $k = \gamma_1 = \delta_1 = \xi_1 = s_{21} = 0$ and $c_{21} > 0$, mutualism occurs by mutualist enhancing the utilization of the prey by the top-predator. Here the region of attraction is contained in

$$\mathcal{C} = \{(u,x,y,z)\colon 0 \le u \le L + \ell\widetilde{N},\ \ 0 \le x \le K,\ \ 0 \le y \le \widetilde{M},\ \ 0 \le z \le \widetilde{N}\}, \tag{4.49}$$

where

$$\widetilde{M} = \frac{-s_{10} + Kc_1\gamma_0}{s_{11}} \quad \text{and} \quad \widetilde{N} = \frac{-s_{20} + (c_{20} + c_{21}L)\delta_0 K + c_3\xi_0\widetilde{M}}{s_{22} - Kc_{21}\ell\delta_0}.$$

The equilibrium in R^+_{ux} is $E_3(L,K,0,0)$. The subsystems in R^+_{uxy} and R^+_{uxz} are uniformly persistent whenever

$$\beta_1 = -s_{20} + Kc_1\gamma_0 > 0, \tag{4.50}$$

and

$$\beta_2 = -s_{20} + (c_{20} + c_{21}L)K\delta_0 > 0. \tag{4.51}$$

The equilibrium $E_6(u_3,x_3,y_3,0)$ in R^+_{uxy} is the same as in Example 4.6 and the corresponding matrix $\mathcal{B}(u,x,y) = \operatorname{diag}\Big(\frac{1}{u_3}, \frac{\alpha c_1}{K}, s_{12}\Big)$. The equilibrium

$$E_7(u_4,x_4,0,z_4) = \left(L + \ell z_4, \frac{s_{20} + s_{22}z_4}{\delta_0(c_{20} + c_{21}u_4)}, 0, \frac{-b_0 + \sqrt{b_0^2 + 4K\ell\alpha\beta_2\delta_0^2 c_{21}}}{2\delta_0^2 K\ell c_{21}}\right),$$

where $b_0 = \alpha s_{22} - K\ell\alpha\delta_0 c_{21} + K\delta_0^2(c_{20} + c_{21}L)$. The symmetric matrix $\mathcal{D}(u,x,z)$ corresponding to E_7 is given by

$$d_{11} = \frac{1}{u_4}, \quad d_{12} = 0, \quad d_{13} = -\frac{1}{2}\left(\frac{\ell u}{(L + \ell z)u_4} + c_{21}\delta_0 x\right),$$
$$d_{22} = \frac{\alpha}{K}(c_{20} + c_{21}u_4), \quad d_{23} = 0, \quad d_{33} = s_{22}.$$

The region of attraction for the subsystem in R^+_{uxz} is contained in the set

$$\mathcal{C}_1 = \left\{(u,x,z)\colon 0 \le u \le L + \ell N_1,\ 0 \le x \le K,\ 0 \le z \le \frac{\beta_2}{s_{22} - K\ell\delta_0 c_{21}} = N_1\right\}.$$

The matrix $\mathcal{D}(u,x,z)$ is positive definite in $\mathcal{A}_1$ whenever

$$4s_{22} - u_4\left(\frac{\ell(L + \ell N_1)}{Lu_4} + c_{21}\delta_0 K\right)^2 > 0. \tag{4.52}$$

Thus if inequalities (4.48) – (4.50) hold and

$$\xi = -s_{20} + (c_{20} + c_{21}u_3)\delta_0 x_3 + c_3\xi_0 y_3 > 0 \tag{4.53}$$

and

$$\eta = -s_{10} + c_1\gamma_0 x_4 - \xi_0 z_4 > 0, \tag{4.54}$$

then the system will be uniformly persistent.

Example 4.8 When $k = \gamma_1 = \xi_1 = s_{21} = c_{21} = 0$ and $\delta_1 > 0$, mutualism occurs by means of the mutualist increasing the rate of predation by the predator z on the prey x.

$$\mathcal{C} = \{(u,x,y,z)\colon 0 \le u \le L + \ell\widetilde{N},\ 0 \le x \le K,\ 0 \le y \le \widetilde{M},\ 0 \le z \le \widetilde{N}\}, \tag{4.55}$$

where

$$\widetilde{M} = \frac{-s_{10} + Kc_1\gamma_0}{s_{11}} \quad \text{and} \quad \widetilde{N} = \frac{-s_{20} + c_{20}(\delta_0 + \delta_1 L)K + c_3\xi_0\widetilde{M}}{s_{22} - Kc_{20}\ell\delta_1}.$$

The equilibrium in R^+_{ux} is $E_3(L,K,0,0)$. The subsystem in R^+_{uxy} is uniformly persistent if

$$\beta_1 = -s_{10} + c_1\gamma_0 K > 0. \tag{4.56}$$

The equilibrium E_6 in R^+_{uxy} is the same as in Example 4.7 and is always globally asymptotically stable with respect to solutions initiating in R^+_{uxy}.

The subsystem in R^+_{uxz} will be uniformly persistent whenever

$$\beta_2 = -s_{20} + c_{20}(\delta_0 + \delta_1 L)K > 0. \tag{4.57}$$

The equilibrium in R^+_{uxz} is

$$E_7(u_4, x_4, 0, z_4) = \left(L + \ell z_4,\ \frac{K}{\alpha}(\alpha - (\delta_0 + \delta_1 u_4)z_4), 0, z_4\right),$$

where z_4 is the unique positive root of the cubic equation

$$k\ell^2 c_{20}\delta_1^2 z^3 + 2K\ell c_{20}\delta_1(\delta_0 + \delta_1 L)z^2 + (\alpha s_{22} - \alpha K\ell c_{21}\delta_1 + Kc_{20}(\delta_0 + \delta_1 L)^2)z - \alpha\beta_2 = 0.$$

The symmetric matrix $\mathcal{D}(u,x,z)$ corresponding to E_7 is given by

$$d_{11} = \frac{1}{u_4}, \quad d_{12} = \frac{1}{2}c_{20}\delta_1 z, \quad d_{13} = -\frac{1}{2}\left(\frac{\ell u}{u_4(L + \ell z)} + c_{20}\delta_1 x\right),$$
$$d_{22} = \frac{\alpha}{K}c_{20}, \quad d_{23} = 0, \quad d_{33} = s_{22}.$$

The region of attraction for the subsystem in R^+_{uxz} is contained in

$$\mathcal{C}_1 = \left\{(u,x,z)\colon 0 \le u \le L + \ell N_1,\ 0 \le x \le K,\ 0 \le z \le \frac{\beta_2}{s_{22} - K\ell c_{20}\delta_1} = N_1\right\}.$$

The matrix $\mathcal{D}(u,x,z)$ is positive definite in $\mathcal{A}_1$ provided

$$s_{22}\left(\frac{4\alpha}{u_4} - Kc_{20}\delta_1^2 N_1\right) - \alpha\left(\frac{\ell}{u_4 L}(L + \ell N_1) + Kc_{20}\delta_1\right)^2 > 0. \tag{4.58}$$

Thus whenever the inequalities (4.54)−(4.56) hold and

$$\xi = -s_{20} + c_{20}(\delta_0 + \delta_1 u_3)x_3 + c_3\xi_0 y_3 > 0, \tag{4.59}$$

and

$$\eta = -s_{10} + c_1\gamma_0 x_4 - \xi_0 z_4 > 0, \tag{4.60}$$

the given system is uniformly persistent.

5 Discussion

The main focus in this paper is to examine the possible effects of an obligate mutualist on the middle and top predator in a food chain. In particular, it was shown how a mutualist could reverse the outcome of extinction in the case of no mutualism to persistence in the case of mutualism.

Such mutualisms occur in nature. Examples are cleaner mutualists. The large iguanas of the Galapagos Islands may be thought of as either middle predators or top predators depending on whether or not their eggs are subject to predation [10]. Similarly for the giant tortoises [8]. Both have evolved a mutualism with finches which act as cleaner mutualists by removing ticks and other pests from the iguanas and tortoises. Such cleaner mutualism has been shown to be obligate in the Carribean [28]. in that if the cleaning is not performed, the individuals (in this case certain fish) will soon die.

A remaining problem to be analyzed is the case where the mutualism is obligate on both mutualists. This is left to future work.

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Robust Stability: Three Approaches for Discrete-Time Systems

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Abstract: The paper presents the results of estimating the robust stability bounds for discrete system in terms of three approaches based on scalar, vector and hierarchical Lyapunov functions. It is shown that the hierarchical Lyapunov function allows one to obtain the most wide bounds for the uncertain matrix in the investigation of discrete system. A numerical example is cited which illustrates the application of the proposed approach.

Keywords: *Discrete-time system; robust bounds; scalar; vector and hierarchical Lyapunov functions.*

Mathematics Subject Classification (2000): 93D09, 93D30.

1 Introduction

In the present paper we consider an uncertain discrete-time system

$$x(\tau+1) = A\,x(\tau) + f(x(\tau),\alpha), \tag{1.1}$$

where $x \in R^n$, $\tau \in \mathcal{N} = \{t_0 + k, k = 0,1,2,\dots\}$, $t_0 \in R$, A is a constant $n \times n$ matrix, $f\colon R^n \times S \to R^n$, $\alpha \in S \subseteq R^d$, $d \geqslant 1$ is a compact set. Under specific conditions (we don't cite them here) dynamics of the system (1.1) are topologically equivalent with dynamics of the system

$$x(\tau+1) = (A+E)\,x(\tau), \tag{1.2}$$

where A is the same matrix, as in system (1.1), and E is an uncertain $n \times n$ matrix, about which it is known that it lies in some compact set $S_1 \subset R^{n\times n}$. Further we will investigate the system (1.2).

Our purpose is to compare the results of estimating the robust bounds of discrete system obtained in terms of three approaches involving scalar, vector and hierarchical

Lyapunov function. In the paper it is shown that the hierarchical Lyapunov function provides more wide bounds for estimation of the uncertain matrix.

2 Scalar Approach

We assume that for the matrix A the condition $|\sigma_i(A)| < 1$ is realized for all $i = 1, 2, \dots, n$. In this case the Lyapunov equation

$$A^{\mathrm{T}}PA - P = -G \tag{2.1}$$

has a unique solution $P \in R^{n\times n}$ for arbitrary symmetric and positive definite matrix $G \in R^{n\times n}$. Moreover the matrix P is symmetric and positive definite. According to the results of paper [6], we apply the function

$$v(x) = (x^{\mathrm{T}}Px)^{\frac{1}{2}}. \tag{2.2}$$

in robustness analysis of the system (1.2). Let us denote by $\sigma_m(P)$, $\sigma_M(P)$ the maximum and minimum eigenvalues of the matrix P.

Following the paper [6] we get the assertion.

Theorem 2.1 *Let the nominal system*

$$x(\tau+1) = A\,x(\tau)$$

be asymptotically stable. If

$$\|E\| < \mu(G), \tag{2.3}$$

where

$$\mu(G) = \frac{\sigma_m(G)}{\sigma_M^{\frac{1}{2}}(P-G)\sigma_M^{\frac{1}{2}}(P) + \sigma_M(P)},$$

then the uncertain system (1.2) is asymptotically stable.

Here $\|E\| = \sup\limits_{\|x\|\leqslant 1} \|Ex\|$, $\|x\| = (x^{\mathrm{T}}x)^{\frac{1}{2}}$ is the Euclidean norm of vector x.

It is known [6], that $\mu(G)$ takes the largest value, if $G = I$ in (2.1). The expression (2.3) is a robust bound for the system (1.2), obtained in the framework of scalar approach with the function (2.2).

3 Vector Approach

We decompose system (1.2) into two interconnected subsystems

$$\hat{S}_i\colon \quad x_i(\tau+1) = (A_i + E_i)\,x_i(\tau) + (B_j + U_j)\,x_j(\tau), \quad i,\, j = 1, 2 \quad \text{and} \quad i \neq j. \tag{3.1}$$

Here $x_i \in R^{n_i}$, A_i and B_i are submatrices of the known matrix

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}, \tag{3.2}$$

E_i and U_i are submatrices of the uncertain matrix

$$E = \begin{pmatrix} E_1 & U_1 \\ U_2 & E_2 \end{pmatrix}, \tag{3.3}$$

where $B_1, U_1 \in R^{n_1\times n_2}$, $B_2, U_2 \in R^{n_2\times n_1}$, and $A_i, E_i \in R^{n_i\times n_i}$, $i = 1, 2$.

Assumption 3.1 *We assume that:*

(1) *the nominal subsystems*

$$x_i(\tau+1) = A_i x_i(\tau) \tag{3.4}$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices $P_i \in R^{n_i \times n_i}$, *which satisfy the Lyapunov matrix equations*

$$A_i^{\mathrm{T}} P_i A_i - P_i = -G_i, \quad i = 1,2, \tag{3.5}$$

where G_i *are arbitrary symmetric and positive definite matrices;*

(2) *there exists a constant* $\gamma \in (0,1)$ *such that*

$$\|B_1\|\,\|B_2\| < \gamma^2 \mu_1 \mu_2$$

where $\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i)\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i))^{-1}$, P_i *are solutions of the Lyapunov matrix equations (3.5) for the matrices* $G_i = I_{n_i}$, I_{n_i} *are* $n_i \times n_i$ *identity matrices,* $i = 1,2$.

We define the constants

$$\begin{aligned}
&a = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2), \quad b = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)(\|B_1\| + \|B_2\|),\\
&\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i)\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i))^{-1}, \quad i = 1,2,\\
&\alpha_i = \sigma_M^{\frac{1}{2}}(P_i)\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i) + \sigma_M^{\frac{1}{2}}(P_i))^{-1}, \quad i = 1,2,\\
&c = \gamma^2 \alpha_1 \alpha_2 - \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)\|B_1\|\,\|B_2\|,\\
&\epsilon = \frac{1}{2a}((b^2 + 4ac)^{\frac{1}{2}} - b),
\end{aligned}$$

where P_i are solutions of the Lyapunov matrix equations (3.5) for the matrices $G_i = I_{n_i}$, $i = 1,2$.

Theorem 3.1 *Assume that for the uncertain system (1.2) the decomposition (3.1) – (3.3) takes place and all conditions of Assumption 3.1 are satisfied. If the submatrices* E_i *and* U_i *satisfy the inequalities*

$$\|E_i\| \leqslant (1-\gamma)\mu_i, \quad \|U_i\| < \epsilon, \quad i = 1,2, \tag{3.6}$$

then the equilibrium $x = 0$ *of (1.2) is asymptotically stable.*

Proof For the nominal subsystems (3.4) by (3.5) we construct the normlike functions

$$v_i(x_i) = (x_i^{\mathrm{T}} P_i x_i)^{\frac{1}{2}}, \quad i = 1,2, \tag{3.7}$$

and the scalar function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2), \tag{3.8}$$

where d_1, d_2 are some positive constants.

For the first forward differences $\Delta v_i(x_i)$ of the functions (3.7) with respect to τ along the solutions of (3.1) we have the estimates:

$$\Delta v_i(x_i)\big|_{\hat S_i} = v_i(A_i x_i) - v_i(x_i) + v_i((A_i+E_i)x_i) - v_i(A_i x_i) + v_i((A_i+E_i)x_i$$
$$+ (B_i+U_i)x_j) - v_i((A_i+E_i)x_i) \leqslant (x_i^{\mathrm T} A_i^{\mathrm T} P_i A_i x_i)^{\frac12} - (x_i^{\mathrm T} P_i x_i)^{\frac12} + \sigma_M^{\frac12}(P_i)\|E_i x_i\|$$
$$+ \sigma_M^{\frac12}(P_i)\|(B_i+U_i)x_j\| \leqslant \frac{x_i^{\mathrm T} A_i^{\mathrm T} P_i A_i x_i - x_i^{\mathrm T} P_i x_i}{(x_i^{\mathrm T} A_i^{\mathrm T} P_i A_i x_i)^{\frac12} + (x_i^{\mathrm T} P_i x_i)^{\frac12}} + \sigma_M^{\frac12}(P_i)\|E_i\|\,\|x_i\|$$
$$+ \sigma_M^{\frac12}(P_i)(\|B_i\|+\|U_i\|)\|x_j\| \leqslant -(\alpha_i - \sigma_M^{\frac12}(P_i)\|E_i\|)\|x_i\| + \sigma_M^{\frac12}(P_i)(\|B_i\|+\|U_i\|)\|x_j\|,$$

$i,\ j = 1,2,\ i \neq j$. Here we use the known inequality [6]

$$(p^{\mathrm T} P p)^{\frac12} - (q^{\mathrm T} P q)^{\frac12} \leqslant \sigma_M^{\frac12}(P)\|p-q\|$$

for all $p,\, q \in R^n,\ P \in R^{n\times n}$ is a symmetric and positive definite matrix. From here we arrive to the following inequality

$$\Delta v(x)\big|_{(\hat S_1,\hat S_2)} \leqslant d_1 \Delta v_1(x_1)\big|_{\hat S_1} + d_2 \Delta v_2(x_2)\big|_{\hat S_2} \leqslant -\tilde d^{\mathrm T} W z, \tag{3.9}$$

where $\tilde d = (d_1, d_2)^{\mathrm T},\ z = (\|x_1\|, \|x_2\|)^{\mathrm T},\ W = (w_{ij})$ is a 2×2 matrix with the elements

$$w_{ij} = \begin{cases} \alpha_i - \sigma_M^{\frac12}(P_i)\,\|E_i\| & \text{if} \quad i = j, \\ -\sigma_M^{\frac12}(P_i)(\|B_i\|+\|U_i\|) & \text{if} \quad i \neq j. \end{cases}$$

As all conditions of Theorem 3.1 are satisfied, it is not difficult to verify that the matrix W is the M-matrix [8]. Really

$$w_{11}w_{22} - w_{12}w_{21} = [\alpha_1 - \sigma_M^{\frac12}(P_1)\|E_1\|][\alpha_2 - \sigma_M^{\frac12}(P_2)\|E_2\|] - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)$$
$$\times (\|B_1\|+\|U_1\|)(\|B_2\|+\|U_2\|) > [\alpha_1 - \sigma_M^{\frac12}(P_1)(1-\gamma)\mu_1][\alpha_2 - \sigma_M^{\frac12}(P_2)(1-\gamma)\mu_2]$$
$$- \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\|+\epsilon)(\|B_2\|+\epsilon)$$
$$= \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\|+\epsilon)(\|B_2\|+\epsilon)$$
$$= -\sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\epsilon^2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\|+\|B_2\|)\epsilon + \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}$$
$$\times (P_2)\|B_1\|\,\|B_2\| = -a\epsilon^2 - b\epsilon + c.$$

By condition (2) of Assumption 2.1

$$c = \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\|B_1\|\,\|B_2\| = \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\big[\gamma^2\mu_1\mu_2 - \|B_1\|\,\|B_2\|\big] > 0$$

and therefore $-a\epsilon^2 - b\epsilon + c = 0$, and $w_{11}w_{22} - w_{12}w_{21} > 0$.

It is clear that the function (3.8) is positive definite and it's first forward difference (3.9) is negative definite. These conditions are sufficient [9] for the asymptotic stability of the equilibrium $x = 0$ of (1.2).

The proof of Theorem 3.1 is complete.

Thus the inequalities (3.6) are the robust bounds for the system (1.2), obtained in terms of the vector approach.

4 Hierarchical Approach

As is known [7], the essence of this method is as follows: beginning from the constructing an auxiliary Lyapunov function, we take into account a hierarchical structure of the system (1.2) or realize a multilevel decomposition of the initial system. Further the second approach is applied precisely.

We decompose each subsystems (3.1) into two interconnected components

$$\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = (A_{ij}+E_{ij})\,x_{ij}(\tau)+(B_{ij}+U_{ij})\,x_{ik}(\tau), \quad i,\,j,\,k=1,2, \quad j\neq k, \tag{4.1}$$

where $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $A_{ij}, E_{ij} \in R^{n_{ij}\times n_{ij}}$, $B_{i1}, U_{i1} \in R^{n_{i1}\times n_{i2}}$, and $B_{i2}, U_{i2} \in R^{n_{i2}\times n_{i1}}$,

$$A_i = \begin{pmatrix} A_{i1} & B_{i1} \\ B_{i2} & A_{i2} \end{pmatrix}, \qquad E_i = \begin{pmatrix} E_{i1} & U_{i1} \\ U_{i2} & E_{i2} \end{pmatrix}.$$

Assume that the matrices B_i and U_i have a block structure:

$$B_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)} \end{pmatrix}, \qquad U_i = \begin{pmatrix} F_{11}^{(i)} & F_{12}^{(i)} \\ F_{12}^{(i)} & F_{22}^{(i)} \end{pmatrix},$$

where $M_{jk}^{(i)}, F_{jk}^{(i)} \in R^{n_{ij}\times n_{lk}}$, $i,\,j,\,k,\,l=1,2$, $i\neq l$.

We extract from (4.1) the independent components

$$C_{ij}: \quad x_{ij}(\tau+1) = (A_{ij}+E_{ij})\,x_{ij}(\tau), \quad i,\,j=1,2,$$

with the same designations of variables as in system (4.1).

In order to state the robust bounds we require the following assumptions.

Assumption 4.1 *The nominal components*

$$x_{ij}(\tau+1) = A_{ij}x_{ij}(\tau), \quad i,\,j=1,2,$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices P_{ij}*, which satisfy the Lyapunov matrix equations*

$$A_{ij}^{\mathrm{T}}P_{ij}A_{ij} - P_{ij} = -G_{ij}, \quad i,\,j=1,2, \tag{4.2}$$

where G_{ij} *are arbitrary symmetric and positive definite matrices.*

Let P_{ij} be solutions of the Lyapunov matrix equations (4.2) for the identity matrices $G_{ij} = I_{ij}$. We define the constants

$$\begin{aligned}
&\alpha_{ij} = \sigma_M^{\frac12}(P_{ij})\mu_{ij} = (\sigma_M^{\frac12}(P_{ij}-I_{ij}) + \sigma_M^{\frac12}(P_{ij}))^{-1},\\
&\mu_{ij} = (\sigma_M^{\frac12}(P_{ij}-I_{ij})\sigma_M^{\frac12}(P_{ij}) + \sigma_M(P_{ij}))^{-1},\\
&\epsilon_i = \frac{1}{2a_i}((b_i^2+4a_ic_i)^{\frac12} - b_i),\\
&a_i = \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2}),\\
&b_i = \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2})(\|B_{i1}\| + \|B_{i2}\|),\\
&c_i = \gamma_i^2\alpha_{i1}\alpha_{i2} - \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2})\|B_{i1}\|\,\|B_{i2}\|, \quad i,\,j=1,2.
\end{aligned}$$

Assumption 4.2 *There exist constants* $\gamma_i \in (0,1)$ *such that*

$$\|B_{i1}\|\,\|B_{i2}\| < \gamma_i^2 \mu_{i1}\mu_{i2}, \quad i = 1,2.$$

Let us construct an auxiliary function on the base of the functions

$$v_{ij}(x_{ij}) = (x_{ij}^{\mathrm{T}} P_{ij} x_{ij})^{\frac{1}{2}},$$

by formular

$$v_i(x_i) = d_{i1}v_{i1}(x_{i1}) + d_{i2}v_{i2}(x_{i2}), \quad i = 1,2,$$

where d_{ij} are some positive constants. We introduce 2×2 matrices $W_i = (w_{jk}^{(i)})$ with the elements

$$w_{jk}^{(i)} = \begin{cases} \gamma_i \alpha_{ij} & \text{if} \quad j = k, \\ -\sigma_M^{\frac{1}{2}}(P_{ij})(\|B_{ij}\| + \overline{\epsilon}_i) & \text{if} \quad j \neq k. \end{cases}$$

Here $0 < \overline{\epsilon}_i < \epsilon_i$.

Further we need the following proposition.

Lemma 4.1 *We assume that*

(1) *discrete system (1.2) is decomposed on the first level to the system (3.1) and on the second level to the systems (4.1);*
(2) *all conditions of Assumptions 4.1 and 4.2 are satisfied;*
(3) *for the submatrices* E_{ij}, U_{ij} *of the matrices* E_i, $i = 1,2$, *the estimates*

$$\|E_{ij}\| \leqslant (1-\gamma_i)\mu_{ij}, \quad \|U_{ij}\| \leqslant \overline{\epsilon}_i, \quad i,\, j = 1,2.$$

are realized.

Then there exist vectors $\hat{d}_1, \hat{d}_2 \in R^2$ *with positive components such that the first forward differences* $\Delta v_i(x_i)\big|_{C_{ij}}$ *for the functions* $v_i(x_i)$ *satisfy the estimates*

$$\Delta v_i(x_i)\big|_{C_{ij}} \leqslant -\hat{d}_i^{\mathrm{T}} W_i z_i, \quad i = 1,2 \tag{4.3}$$

and the matrices W_i *are the* M*-matrices.*

Here $\hat{d}_i = (d_{i1}, d_{i2})^{\mathrm{T}}$ and $z_i = (\|x_{i1}\|, \|x_{i2}\|)^{\mathrm{T}}$.

The proof of Lemma 4.1 is analogous to that of Theorem 3.1.

Under the hypotheses of Lemma 4.1 the matrices W_i are the M-matrices and, according to [8], the vectors $\hat{d}_i^{\mathrm{T}} W_i = (d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)}, d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)})$ have positive components.

Let us denote

$$\pi_i = \min\{d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)}; d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)}\}, \quad i = 1,2,$$

$$m = \frac{1}{2}\left(\frac{\pi_1\pi_2}{(d_{11}\sigma_M^{\frac{1}{2}}(P_{11}) + d_{12}\sigma_M^{\frac{1}{2}}(P_{12}))\,(d_{21}\sigma_M^{\frac{1}{2}}(P_{21}) + d_{22}\sigma_M^{\frac{1}{2}}(P_{22}))}\right)^{\frac{1}{2}} \tag{4.4}$$

and give a method of optimal choice of the constants d_{i1}, d_{i2}, $i = 1,2$.

Lemma 4.2 *Let the matrices W_1 and W_2 be the M-matrices and $w_{12}^{(i)}, w_{21}^{(i)} < 0$, then*

$$\sup_{d\in D} m(d) = m(d_1^*, 1, d_2^*, 1) = \frac{1}{2}\left(\frac{w_{11}^{(1)}w_{22}^{(1)} - w_{12}^{(1)}w_{21}^{(1)}}{\sigma_M^{\frac{1}{2}}(P_{11})(w_{22}^{(1)} - w_{21}^{(1)}) + \sigma_M^{\frac{1}{2}}(P_{12})(w_{11}^{(1)} - w_{12}^{(1)})}\times\right.$$
$$\left.\times\,\frac{w_{11}^{(2)}w_{22}^{(2)} - w_{12}^{(2)}w_{21}^{(2)}}{\sigma_M^{\frac{1}{2}}(P_{21})(w_{22}^{(2)} - w_{21}^{(2)}) + \sigma_M^{\frac{1}{2}}(P_{22})(w_{11}^{(2)} - w_{12}^{(2)})}\right)^{\frac{1}{2}}, \tag{4.5}$$

where

$$D = \left\{d = (d_{11}, d_{12}, d_{21}, d_{22})^{\mathrm{T}} \in R^4 : -\frac{w_{21}^{(1)}}{w_{11}^{(1)}} < \frac{d_{11}}{d_{12}} < -\frac{w_{22}^{(1)}}{w_{12}^{(1)}},\ -\frac{w_{21}^{(2)}}{w_{11}^{(2)}} < \frac{d_{21}}{d_{22}} < -\frac{w_{22}^{(2)}}{w_{12}^{(2)}}\right\},$$
$$d_1^* = \frac{w_{22}^{(1)} - w_{21}^{(1)}}{w_{11}^{(1)} - w_{12}^{(1)}}, \qquad d_2^* = \frac{w_{22}^{(2)} - w_{21}^{(2)}}{w_{11}^{(2)} - w_{12}^{(2)}}.$$

Proof As the matrices W_1 and W_2 are the M-matrices, then $w_{11}^{(i)}, w_{22}^{(i)} > 0$, $w_{12}^{(i)}, w_{21}^{(i)} < 0$ and consequently, $-\frac{w_{22}^{(i)}}{w_{12}^{(i)}} > -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} > 0$. On computing of the constant π_i and m we can set $d_{12} = d_{22} = 1$, $d_{11} = d_1$, $d_{21} = d_2$ and $d_i \in D_i = \left\{d_i \in R\colon -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < -\frac{w_{22}^{(i)}}{w_{12}^{(i)}}\right\}$, $i = 1, 2$. Let us denote

$$m_i(d_i) = \frac{\pi_i}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})} \quad i = 1, 2,$$

and note that

$$\sup_{d\in D} m(d) = \frac{1}{2}\Big(\sup_{d_1\in D_1} m_1(d_1) \sup_{d_2\in D_2} m_2(d_2)\Big). \tag{4.6}$$

By (4.4) for the function $m_i(d_i)$ we get the expressions

$$m_i(d_i) = \begin{cases} \dfrac{d_iw_{11}^{(i)} + w_{21}^{(i)}}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})}, & \text{if} \quad -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i \leqslant d_i^*, \\ \dfrac{d_iw_{12}^{(i)} + w_{22}^{(i)}}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})}, & \text{if} \quad d_i^* \leqslant d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}. \end{cases}$$

For the first derivatives $m_i'(d_i)$ we have

$$m_i'(d_i) = \begin{cases} \dfrac{w_{11}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i2}) - w_{21}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i1})}{\left(d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})\right)^2}, & \text{if} \quad -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*, \\ \dfrac{w_{12}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i2}) - w_{22}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i1})}{\left(d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})\right)^2}, & \text{if} \quad d_i^* < d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}, \end{cases}$$

therefore $m_i'(d_i) > 0$ for $-\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*$ and $m_i'(d_i) < 0$ for $d_i^* < d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}$. From here it follows that

$$\sup_{d_i\in D_i} m_i(d_i) = m_i(d_i^*) = \frac{w_{11}^{(i)}w_{22}^{(i)} - w_{12}^{(i)}w_{21}^{(i)}}{\sigma_M^{\frac{1}{2}}(P_{i1})(w_{22}^{(i)} - w_{21}^{(i)}) + \sigma_M^{\frac{1}{2}}(P_{i2})(w_{11}^{(i)} - w_{12}^{(i)})}.$$

Substituting by the values of $m_i(d_i^*)$ into (4.6), we get the identity (4.5). Lemma 4.2 is proved.

Assumption 4.3 *Let for the submatrices $M_{jk}^{(i)}$ of the matrices B_i the inequalities*

$$\overline{m} = \max \|M_{jk}^{(i)}\| < m$$

be realized for all $i,\ j,\ k = 1,2$.

The following proposition is basic in the method of hierarchical Lyapunov functions in the robust stability problem of the system (1.2).

Theorem 4.1 *We assume that for the uncertain system (1.2) the two-level decomposition (3.1), (4.1) is realized and all conditions of Assumptions 4.1 – 4.3 are satisfied. If the inequalities*

$$\|E_{ij}\| \leqslant (1-\gamma_i)\mu_{ij}, \quad \|U_{ij}\| \leqslant \overline{\epsilon}_i, \quad \|F_{jk}^{(i)}\| < m - \overline{m}$$

are fulfilled for all $i,\ j,\ k = 1,2$, then the equilibrium $x = 0$ of the system (1.2) is asymptotically stable.

Proof Under the hypotheses of Lemma 4.1 there exist constants $d_{ij} > 0$ for which $\hat{d}_i^{\mathrm{T}} W_i z_i > 0$. In view of designations (4.4), we get from estimate (4.3)

$$\Delta v_i(x_i)\big|_{S_i} \leqslant -\pi_i\left(\|x_{i1}\|^2 + \|x_{i2}\|^2\right)^{\frac{1}{2}} = -\pi_i\|x_i\|, \quad i = 1,2.$$

Since for $i \neq k$ the estimates

$$\Delta v_{i1}(x_{i1})\big|_{\widehat{S}_i} \leqslant \Delta v_{i1}(x_{i1})\big|_{S_i} + \sigma_M^{\frac{1}{2}}(P_{i1})(2\overline{m} + \|F_{11}^{(i)}\| + \|F_{12}^{(i)}\|)\|x_k\|,$$
$$\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \leqslant \Delta v_{i2}(x_{i2})\big|_{S_i} + \sigma_M^{\frac{1}{2}}(P_{i2})(2\overline{m} + \|F_{21}^{(i)}\| + \|F_{22}^{(i)}\|)\|x_k\|,$$

are true, then

$$\begin{aligned}\Delta v_i(x_i)\big|_{\widehat{S}_i} &= d_{i1}\Delta v_{i1}(x_{i1})\big|_{S_i} + d_{i2}\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \leqslant -\pi_i\|x_i\| + \big[d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1})\big(2\overline{m} + \\ &+ \|F_{11}^{(i)}\| + \|F_{12}^{(i)}\|\big) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\big(2\overline{m} + \|F_{21}^{(i)}\| + \|F_{22}^{(i)}\|\big)\big]\|x_k\|.\end{aligned} \tag{4.7}$$

For the function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2)$$

in view of estimates (4.7) we get

$$\Delta v(x)\big|_S = d_1\Delta v_1(x_1)\big|_{\widehat{S}_1} + d_2\Delta v_2(x_2)\big|_{\widehat{S}_2} \leqslant -\hat{d}^{\mathrm{T}} W\, z, \tag{4.8}$$

where $\hat{d} = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$ and W is a 2×2-matrix with the elements

$$w_{jk} = \begin{cases} \pi_j & \text{for} \quad j = k, \\ -d_{j1}\sigma_M^{\frac{1}{2}}(P_{j1})(2\overline{m} + \|F_{11}^{(j)}\|) + \|F_{12}^{(j)}\|) - & \\ \qquad -d_{j2}\sigma_M^{\frac{1}{2}}(P_{j2})(2\overline{m} + \|F_{21}^{(j)}\|) + \|F_{22}^{(j)}\|) & \text{for} \quad j \neq k. \end{cases}$$

Under the hypotheses of Theorem 4.1 the matrix W in the estimate (4.8) is the M-matrix. Thus the matrices W_1, W_2, W are the M-matrices and it is sufficient [3] for asymptotic stability of the system (1.2).

5 Discussion and Some Applications

The hierarchical approach in robust stability problem permits a more complete allowance for the dynamic characteristics of the nominal system on each hierarchical level and thus a more exact definition of robust bounds for the system (1.2). We illustrate efficiency of the approach proposed in the paper by a simple example.

Let us assume that in the system (1.2) the matrix A has the form

$$A = \begin{pmatrix} 0.5 & 0.01 & 0.03 & 0 \\ 0.01 & 0.125 & 0 & 0.03 \\ 0.03 & 0 & 0.25 & 0.005 \\ 0 & 0.03 & 0.005 & 0.125 \end{pmatrix}. \tag{5.1}$$

5.1 Scalar approach

Let us compute the matrices and constants occurring in the framework of the scalar approach (see Theorem 2.1):

$$P = \begin{pmatrix} 1.336149 & 0.008512 & 0.032104 & 0.000737 \\ 0.008512 & 1.017019 & 0.000708 & 0.007761 \\ 0.032104 & 0.000708 & 1.068495 & 0.002057 \\ 0.000737 & 0.007761 & 0.002057 & 1.016891 \end{pmatrix};$$

$$\sigma(P) \approx 1.340176; \quad \sigma_M(P - I) \approx 0.340176; \quad \mu \approx 0.496185.$$

Here I is a 4×4-unit matrix. From here the robust bound for the system (1.2) with the matrix (5.1) is determined by the inequality

$$\|E\| < 0.496185 \tag{5.2}$$

for all matrices $E \in S_1$.

5.2 Vector approach

According to this approach we decomposed the matrix (5.1) and denote

$$A_1 = \begin{pmatrix} 0.5 & 0.01 \\ 0.01 & 0.125 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.25 & 0.005 \\ 0.005 & 0.125 \end{pmatrix}, \qquad B_1 = B_2 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.03 \end{pmatrix}.$$

The uncertain matrix E is represented in the form (3.3). The matrices and constants occurring in the framework of vector the approach are:

$$P_1 \approx \begin{pmatrix} 1.333581 & 0.008469 \\ 0.008469 & 1.016029 \end{pmatrix}, \qquad P_2 \approx \begin{pmatrix} 1.066699 & 0.002031, \\ 0.002031 & 1.015902 \end{pmatrix},$$
$$\sigma_M(P_1) \approx 1.333807, \quad \sigma_M(P_2) \approx 1.066780, \quad \mu_1 \approx 0.449733, \quad \mu_2 \approx 0.749800.$$

Hence we have the estimates of submatrices norms in the form

$$\|E_1\| \leqslant 0.499733(1-\gamma), \quad \|E_2\| \leqslant 0.749800(1-\gamma), \quad \gamma \in (0,1). \tag{5.3}$$

Let $\gamma = 0.25$. Besides $\epsilon \approx 0.012303$.

Finally, for the matrix E represented in the form (3.3), we get the estimates:

$$\|E_1\| \leqslant 0.374800, \quad \|E_2\| \leqslant 0.562350, \quad \|U_i\| < 0.012303, \quad i = 1,2. \tag{5.4}$$

For example the matrix

$$\widetilde{E} = \operatorname{diag}\{0.37,\, 0.37,\, 0.56,\, 0.56\}$$

satisfies the inequalities (5.4). But $\|\widetilde{E}\| = 0.56$, and consequently, the norm of uncertain matrix $\widetilde{E}$ does not satisfy the inequality (5.2).

5.3 Hierarchical approach

According to the proposed algorithm we accomplish the two-level decomposition of system (1.2) with the matrix (5.1) and as a result we get:

$$A_{11} = 0.5, \quad A_{12} = 0.125, \quad A_{21} = 0.25, \quad A_{22} = 0.125.$$

Let

$$\gamma_1 = 0.5, \qquad \gamma_2 = 0.125.$$

Numerical values of corresponding constants are:

$$\begin{aligned} &\sigma_M(P_{11}) \approx 1.333333, \quad \sigma_M(P_{12}) \approx 1.015873, \quad \mu_{11} = 0.5, \quad \mu_{12} = 0.875, \\ &\sigma_M(P_{21}) \approx 1.066666, \quad \sigma_M(P_{22}) \approx 1.015873, \quad \mu_{21} = 0.75, \quad \mu_{22} = 0.875, \\ &\qquad \epsilon_1 \approx 0.320718, \qquad \epsilon_2 \approx 0.096261. \end{aligned}$$

We shall set $\bar{\epsilon}_1 = 0.05$, and $\bar{\epsilon}_2 = 0.006$. In this case for the matrices W_1 and W_2 we get the expressions

$$W_1 \approx \begin{pmatrix} 0.288675 & -0.069282 \\ -0.060474 & 0.440958 \end{pmatrix}, \qquad W_2 \approx \begin{pmatrix} 0.096824 & -0.011360 \\ -0.011086 & 0.110239 \end{pmatrix}.$$

The matrices W_1 and W_2 are the M-matrices as their non-diagonal elements are negative and their principal minors are positive.

The constant m is computed by the formular (4.5): $m \approx 0.038392$. Thus, the following restrictions are imposed on submatrices of E:

$$
\begin{gathered}
\|E_{11}\| \leqslant 0.25, \quad \|E_{12}\| \leqslant 0.4375, \quad \|E_{21}\| \leqslant 0.65625, \quad \|E_{22}\| \leqslant 0.765625, \\
\|U_{1j}\| \leqslant 0.05, \quad \|U_{2j}\| \leqslant 0.006, \quad \|F^{(i)}_{jk}\| < 0.008392.
\end{gathered} \tag{5.5}
$$

For example the matrix

$$
\overline{E} = \operatorname{diag}\{0.25,\, 0.43,\, 0.65,\, 0.76\}
$$

satisfies the inequalities (5.5). Since $\|\overline{E}\| = 0.76$, the matrix $\overline{E}$ does not satisfy condition (5.2). Moreover $\|\operatorname{diag}\{0.65,\, 0.76\}\| = 0.76 > 0.75$ and it means that for the matrix $\overline{E}$ conditions (5.3) are not satisfied for any $\gamma \in (0,1)$.

Thus, the general conclusion from this example is: the hierarchical Lyapunov function allows a more complete use of the potential possibilities of direct Lyapunov method in robustness analysis of discrete system (1.2).

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Asymptotic Behaviour of Feedback Controlled Systems and the Ubiquity of the Brockett-Krasnosel'skiĭ-Zabreĭko Property

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Abstract: A well-known topological barrier – the Brockett-Krasnosel'skiĭ-Zabreĭko necessary condition on the underlying vector field – to stability of equilibria (or stabilizability of equilibria by regular feedback) of ordinary differential equations (or controlled differential equations) is shown to persist in a wider context of differential inclusions (encompassing controlled differential equations with nonsmooth feedback) that exhibit attracting compacta.

Keywords: *Brockett-Krasnosel'skiĭ-Zabreĭko condition; feedback controlled system.*

Mathematics Subject Classification (2000): 34D05, 93D15.

1 Introduction

Let $f\colon \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz and consider the system

$$\dot{x} = f(x). \tag{1}$$

By [1, Theorem 52.1], if (1) has an asymptotically stable (that is, Lyapunov stable and attractive) equilibrium ξ, then the (isolated) zero ξ of $-f$ has index $\mathrm{ind}(-f,\xi) = 1$ and so, for all $\epsilon > 0$ sufficiently small, $\deg_{\mathrm{B}}(-f, \mathbb{B}_\epsilon(\xi), 0) = 1$, where $\deg_{\mathrm{B}}$ denotes Brouwer degree and $\mathbb{B}_\epsilon(\xi)$ denotes the open ball of radius ϵ centred at ξ. Therefore,

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by properties of Brouwer degree, $f(\mathbb{R}^N)$ contains an open neighbourhood of 0. Now let $f\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be locally Lipschitz and consider the controlled system

$$\dot{x} = f(x, u). \tag{2}$$

If (2) is stabilizable in the sense that there exists a time-invariant locally Lipschitz feedback $u = k(x)$ that renders some point of $\mathbb{R}^N$ an asymptotically stable equilibrium of the feedback system $\dot{x} = f(x, k(x))$, then, by the above result, the image of f contains an open neighbourhood of 0. This is Brockett's necessary condition for stabilizability, originally proved in [2, Theorem 1]; for discussions on variants and ramifications of Brockett's condition, see, for example, [3−11]. In either case of an uncontrolled (1) or controlled (2) system, if $f\colon D \to \mathbb{R}^N$ is such that $f(D)$ contains an open neighbourhood of 0, we say that f has the BKZ (Brockett-Krasnosel'skiĭ-Zabreĭko) property.

In this paper, the necessity of the BKZ property is investigated in a wider context of differential inclusions under hypotheses weaker than asymptotic stability/stabilizability of equilibria. For example, amongst other consequences for (1), the results of the paper imply that, if any of the following hold, then f has the BKZ property:

(a) some compact set C is *globally attractive* for solutions of (1);
(b) some closed ball is a *locally asymptotically stable* (Lyapunov stable and locally attractive) set for (1);
(c) (1) is L^p*-stable* for some $1 \le p < \infty$ (in the sense that every maximal solution has interval of existence $\mathbb{R}_+$ and is of class L^p).

Within the control framework of (2), these observations have natural counterparts: f has the BKZ property if there exists a (possibly discontinuous) feedback k such that the feedback-controlled system (a) has a globally attractive compact set, or (b) has a locally asymptotically stable closed ball, or (c) is L^p-stable (in the above sense).

2 Notation and Terminology

For a Banach space X and non-empty $C \subset X$, d_C denotes the distance function given by

$$d_C(x) := \inf_{c \in C} \|x - c\| \quad \forall\, x \in X.$$

For non-empty $B, C \subset X$,

$$d(B, C) := \sup_{b \in B} d_C(b).$$

The open ball of radius $r \ge 0$ centred at $z \in \mathbb{R}^N$ is denoted $\mathbb{B}_r(z)$ (with closure $\overline{\mathbb{B}}_r(z)$), to which the conventions $\mathbb{B}_0(z) := \emptyset$ and $\overline{\mathbb{B}}_0(z) := \{z\}$ apply; if $z = 0$, then we simply write $\mathbb{B}_r$ (respectively, $\overline{\mathbb{B}}_r$) in place of $\mathbb{B}_r(0)$ (respectively, $\overline{\mathbb{B}}_r(0)$). The boundary of a set Ω is denoted $\partial\Omega$. We write $\mathbb{R}_+ := [0, \infty)$.

Throughout, a sequence (x_n) is regarded as synonymous with a map $n \mapsto x_n$ with domain $\mathbb{N}$. We shall frequently extract subsequences of sequences. In order to avoid proliferation of subscripts, the notation $(x_{\sigma(n)})$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing map, is adopted to indicate a subsequence of (x_n). If $\big((x_{\sigma_k(n)})\big)_{k \in \mathbb{N}}$ is a sequence of subsequences of (x_n) nested in the following sense

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots ,$$

then σ_k is to be interpreted as a k-fold composition of strictly increasing maps $\mathbb{N} \to \mathbb{N}$, with $\sigma_k = \hat{\sigma}_k \circ \sigma_{k-1}$ for all $k \geq 2$: the sequence $(x_{\sigma_n(n)}) \subset (x_n)$ will be referred to as the diagonal sequence.

$AC(I;\mathbb{R}^N)$ denotes the space of functions $I \to \mathbb{R}^N$ defined on an interval I and absolutely continuous on compact subintervals thereof.

$\mathcal{U}(D)$ denotes the space of upper semicontinuous maps $x \mapsto F(x) \subset \mathbb{R}^N$, defined on $D \subset \mathbb{R}^N$, with non-empty convex compact values: if $D = \mathbb{R}^N$, then we simply write $\mathcal{U}$.

We record the following well-known facts (see, for example, [12]):

Proposition 2.1 *Let* $F \in \mathcal{U}(D)$.

(i) *If* $K \subset D$ *is compact, then* $F(K)$ *is compact.*

(ii) *For each* $\epsilon > 0$, *there exists locally Lipschitz* $f_\epsilon \colon D \to \mathbb{R}^N$ *such that*

$$d(\mathrm{graph}\,(f_\epsilon), \mathrm{graph}\,(F)) < \epsilon$$

(*any such* f_ϵ *is said to be an* ϵ*-approximate selection for* F).

3 Set-Valued Maps: Degree and the BKZ Property

If $F \in \mathcal{U}(D)$ is such that $F(D)$ contains an open neighbourhood of 0, then F is said to have the BKZ property.

Let $\mathcal{M} := \{(F,\Omega,p) \mid F \in \mathcal{U}(D),\ \Omega \text{ an open bounded subset of } D,\ p \in \mathbb{R}^N \backslash F(\partial\Omega)\}$. As discussed in [8] within the framework of [13] (see, also, [14–16]), there exists a map $\deg \colon \mathcal{M} \to \mathbb{Z}$ with the properties:

P1. $\deg(F,\Omega,p) = \deg_{\mathrm{B}}(f_\epsilon,\Omega,p)$ for all $\epsilon > 0$ sufficiently small, where $\deg_{\mathrm{B}}$ denotes Brouwer degree and $f_\epsilon \colon \overline{\Omega} \to \mathbb{R}^N$ is any ϵ-approximate selection for $F\big|_{\overline{\Omega}}$;

P2. if $q \colon [0,1] \to \mathbb{R}^N \backslash F(\partial\Omega)$ is continuous, then $\deg(F,\Omega,q(t))$ is independent of t;

P3. if $\deg(F,\Omega,p) \neq 0$, then $p \in F(x)$ for some $x \in \Omega$.

Lemma 3.1 *Let* $(F,\Omega,0) \in \mathcal{M}$. *If* $\deg(F,\Omega,0) \neq 0$, *then* F *has the BKZ property.*

Proof Since $0 \notin F(\partial\Omega)$, $d_{F(x)}(0) > 0$ for all $x \in \partial\Omega$. Let $(x_n) \subset \partial\Omega$ be a convergent sequence with limit $x \in \partial\Omega$. Let $(x_{\sigma(n)})$ be a subsequence with

$$\lim_{n\to\infty} d_{F(x_{\sigma(n)})}(0) = \liminf_{n\to\infty} d_{F(x_n)}(0).$$

For each n, let z_n be a minimizer of $\|\cdot\|$ over compact $F(x_{\sigma(n)})$ (and so $\|z_n\| = d_{F(x_{\sigma(n)})}(0)$). By upper semicontinuity of F, for each $\epsilon > 0$,

$$z_n \in F(x_{\sigma(n)}) \subset F(x) + \mathbb{B}_\epsilon.$$

By compactness of $F(x)$ and since $\epsilon > 0$ is arbitrary, we may conclude that (z_n) has a convergent subsequence (which we do not reliabel) with limit $z \in F(x)$. Therefore,

$$d_{F(x)}(0) \leq \|z\| = \lim_{n\to\infty} \|z_n\| = \liminf_{n\to\infty} d_{F(x_n)}(0)$$

and so $x \mapsto d_{F(x)}(0)$ is lower semicontinuous and positive-valued on compact $\partial\Omega$. It follows that there exists $\mu > 0$ such that $p \notin F(\partial\Omega)$ for all $p \in \mathbb{B}_\mu$. By properties P2 and P3,

$$p \in \mathbb{B}_\mu \quad \Longrightarrow \quad p \in F(x) \quad \text{for some} \quad x \in \Omega.$$

Therefore, F has the BKZ property.

4 Differential Inclusions

Let $F \in \mathcal{U}$ and consider the differential inclusion (subsuming (1))

$$\dot{x}(t) \in F(x(t)). \tag{3}$$

By an F-arc, we mean a function $x \in AC(I;\mathbb{R}^N)$ that satisfies (3) for almost all $t \in I$.

The following is a particular case of [17, Theorem 3.1.7].

Proposition 4.1 *Let $F \in \mathcal{U}$, let $K \subset \mathbb{R}^N$ be compact, let $I := [a,b]$, let $(\epsilon_n) \subset (0,\infty)$ be a decreasing sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$ and, for each $n \in \mathbb{N}$, define $F_n \colon x \mapsto F(x) + \mathbb{B}_{\epsilon_n}$.*

Let sequence $(x_n) \subset AC(I;\mathbb{R}^N)$ be such that, for each $n \in \mathbb{N}$, x_n is an F_n-arc with $x_n(I) \subset K$. Then (x_n) has a subsequence that converges uniformly to an F-arc $x \in AC(I;\mathbb{R}^N)$.

Next, we prove (by arguments similar to those used in establishing [18, Lemma 5 (p.8)], see also remarks on page 78 therein) a variant of the above, tailored to our later purposes.

Proposition 4.2 *Let $F \in \mathcal{U}$ and let $(s_n) \subset [a,b]$ be a convergent sequence with limit $s \in (a,b]$. If $(x_n) \subset AC([a,b];\mathbb{R}^N)$ is a sequence of F-arcs and there exists $r > 0$ such that, for all $n \in \mathbb{N}$, $\|x_n(t)\| \le r$ for all $t \in [a,s_n]$, then (x_n) has a subsequence $(x_{\sigma(n)})$ such that $(x_{\sigma(n)}|_{[a,s]})$ converges to an F-arc $x \in AC([a,s];\mathbb{R}^N)$.*

Proof Let $(\delta_k) \subset (0, s-a)$ be a decreasing sequence with $\delta_k \downarrow 0$ as $k \to \infty$. Write $I_k := [a, s-\delta_k]$. By Proposition 4.1, the sequence (x_n) has a subsequence, which we label $(x_{\sigma_1(n)})$, such that $(x_{\sigma_1(n)}|_{I_1})$ converges uniformly to an F-arc $x^1 \in AC(I_1;\mathbb{R}^N)$. Again by Proposition 4.1, the sequence $(x_{\sigma_1(n)})$ has a subsequence, which we label $(x_{\sigma_2(n)})$, such that $(x_{\sigma_2(n)}|_{I_2})$ converges uniformly to an F-arc $x^2 \in AC(I_2;\mathbb{R}^N)$ (with $x^2|_{I_1} = x^1$). By induction, we generate a sequence of subsequences of (x_n),

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots$$

such that, for all k, $\big(x_{\sigma_k(n)}|_{I_k}\big)$ converges to an F-arc $x^k \in AC(I_k;\mathbb{R}^N)$ with $x^k|_{I_{k-1}} = x^{k-1}$ for all $k \ge 2$. Therefore, the diagonal sequence of restrictions to $[a,s)$, that is, the sequence $(x_{\sigma_n(n)}|_{[a,s)})$, converges to the F-arc $x \colon [a,s) \to \overline{\mathbb{B}}_r$ defined by the property:

$$\forall\, k \in \mathbb{N} \quad x(t) = x^k(t) \quad \forall\, t \in I_k = [a, s-\delta_k].$$

By compactness of $F(\overline{\mathbb{B}}_r)$, it follows that the bounded F-arc x is uniformly continuous and so extends to an F-arc on the closed interval $[a,s]$ by defining $x(s) := \lim_{t\uparrow s} x(t)$.

4.1 The initial-value problem

Let $F \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^N$, the initial-value problem

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = x^0 \tag{4}$$

has a solution and every solution can be extended to a maximal solution. By a solution, we mean an F-arc $x \in AC([0,\omega);\mathbb{R}^N)$, with $0 < \omega \le \infty$ and $x(0) = x^0$; by a maximal solution, we mean a solution having no proper right extension which is also a solution. Moreover, if $x \colon [0,\omega) \to \mathbb{R}^N$ is maximal and $\omega < \infty$, then $\limsup_{t\uparrow\omega} \|x(t)\| = +\infty$.

Proposition 4.3 *Let non-empty $K \subset \mathbb{R}^N$ be compact. Assume that, for each $x^0 \in K$, every maximal solution of (4) has interval of existence $\mathbb{R}_+$. For $T > 0$, define*

$$\Sigma_T(K) := \bigcup_{t\in[0,T]} \big\{x(t) \mid x \in AC([0,T];\mathbb{R}^N) \text{ is an } F\text{-arc with } x(0) \in K\big\} \subset \mathbb{R}^N$$

and write $\Sigma_\infty(K) := \bigcup_{T>0} \Sigma_T(K)$.

(a) *For all $T > 0$, the set $\Sigma_T(K)$ is compact.*

(b) *Let non-empty C_1, $C_2 \subset \mathbb{R}^N$ be compact, with $C_1 \subset C_2 \subset K$ and $C_1 \cap \partial C_2 = \emptyset = K \cap \partial C_2$. Assume that, for every maximal solution x of (4) with $x^0 \in K$, $d_{C_1}(x(t)) \to 0$ as $t \to \infty$. Then there exists $T > 0$ such that $\Sigma_T(K) = \Sigma_\infty(K)$ and, for all $x^0 \in \Sigma_\infty(K)$, every maximal solution x of (4) has interval of existence $\mathbb{R}_+$ and has the properties:*

  (i) *$x(\mathbb{R}_+) \subset \Sigma_\infty(K)$;*

  (ii) *$x(t) \in C_2$ for some $t \in [0,T]$.*

Proof (a) Let $T > 0$ be arbitrary. Seeking a contradiction, suppose that $\Sigma_T(K)$ is unbounded. Then there exist a constant $\delta > 0$, a sequence $(t_n) \subset [0,T]$ and a sequence (x_n) of maximal solutions of (4) such that

$$x_n(0) \in K \quad \text{and} \quad \|x_n(t_n)\| > (n+1)\delta \quad \forall\, n \in \mathbb{N}.$$

By continuity of the solutions, it follows that, for each $n \in \mathbb{N}$, there exist s_n^k, $k = 1, \dots, n$, such that

$$\|x_n(s_n^k)\| = (k+1)\delta \quad \text{and} \quad \|x_n(t)\| < (k+1)\delta \quad \forall\, t \in [0, s_n^k) \tag{5}$$

and $s_n^1 < s_n^2 < \dots < s_n^n$ for all $n \geq 2$.

From (s_n^1), extract a convergent subsequence $(s^1_{\sigma_1(n)})$ with limit $s^1 \in [0,T]$. By compactness of $F(\overline{\mathbb{B}}_{2\delta}(0))$, $s^1 > 0$. Write $I_1 := [0, s^1]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_1(n)}|_{I_1})$ converges uniformly to an F-arc $x^1 \in AC(I_1;\mathbb{R}^N)$; moreover, by (5), $\|x^1(s^1)\| = 2\delta$. From $(s^2_{\sigma_1(n)})$, extract a subsequence $(s^2_{\sigma_2(n)})$ with limit $s^2 \in [0,T]$. By compactness of $F(\overline{\mathbb{B}}_{3\delta}(0))$, $s^2 > s^1$. Write $I_2 := [0, s^2]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_2(n)}|_{I_2})$ converges uniformly to an F-arc $x^2 \in AC(I_2;\mathbb{R}^N)$ with $x^2|_{I_1} = x^1$; moreover, by (5), $\|x^2(s^2)\| = 3\delta$. By induction, we generate a strictly increasing sequence $(s^k) \subset [0,T]$, with limit $s \in [0,T]$, and a sequence of subsequences of (x_n),

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \dots \supset (x_{\sigma_k(n)}) \supset \cdots$$

such that the diagonal sequence of restricted functions $(x_{\sigma_n(n)}|_I)$, where $I := [0,s)$, converges to the F-arc $x \in AC(I;\mathbb{R}^N)$ defined by the property that, for each $k \in \mathbb{N}$,

$$x(t) = x^k(t) \quad \forall\, t \in I_k := [0, s^k].$$

Clearly, $x(0) \in K$. Furthermore, $\|x(s^k)\| = (k+1)\delta$ for all $k \in \mathbb{N}$ and so x has no proper right extension that is also an F-arc. This contradicts the hypothesis that all maximal solutions of (4), with $x^0 \in K$, have interval of existence $\mathbb{R}_+$. Therefore, $\Sigma_T(K)$ is bounded.

Let $(y_n) \subset \Sigma_T(K)$ be a convergent sequence with limit y. Then $y_n = x_n(t_n)$ for some sequence $(t_n) \subset [0,T]$ and some sequence of F-arcs $(x_n) \subset AC([0,T];\mathbb{R}^N)$ with $x_n(0) \in K$ for all n. Without loss of generality, we may assume that (t_n) is convergent, with limit $t \in [0,T]$. By boundedness of $\Sigma_T(K)$, there exists compact C such that $x_n([0,T]) \subset C$ for all n. By Proposition 4.1, passing to a subsequence if necessary, we may assume that (x_n) converges uniformly to an F-arc $x \in AC([0,T];\mathbb{R}^N)$, with $x(0) \in K$. Therefore,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n(t_n) = x(t) \in \Sigma_T(K),$$

and so $\Sigma_T(K)$ is closed.

(b) It suffices to show that there exists $T > 0$ such that, for every maximal solution x of (4), with $x^0 \in K$, $x(t) \in C_2$ for some $t \in [0,T]$ (in which case, $\Sigma_T(K) = \Sigma_\infty(K)$). Seeking a contradiction, suppose that no such T exists. Then there is a sequence $(x_n) \subset AC(\mathbb{R}_+;\mathbb{R}^N)$ such that, for each $n \in \mathbb{N}$, $x_n(0) \in K$ and $d_{C_2}(x_n(t) > 0$ for all $t \in I_n := [0,n]$. By part (a) above, for each $k \in \mathbb{N}$, the sequence $(x_n|_{I_k})$ is bounded. Therefore, repeated application of Proposition 4.1 yields a sequence of subsequences $(x_n) \supset (x_{\sigma_1(n)}) \supset (x_{\sigma_2(n)}) \cdots$ such that, for each $k \in \mathbb{N}$, the sequence $(x_{\sigma_k(n)}|_{I_k})$ converges uniformly to an F-arc $x^k \in AC(I_k;\mathbb{R}^N)$ with $d_{C_2}(x^k(t)) \geq 0$ for all $t \in I_k$. It follows that the diagonal sequence $(x_{\sigma_n(n)})$ converges to the F-arc $x \in AC(\mathbb{R}_+;\mathbb{R}^N)$ defined by the property that, for each $k \in \mathbb{N}$, $x(t) = x^k(t)$ for all $t \in I_k$. Therefore, $d_{C_2}(x(t)) \geq 0$ for all $t \in \mathbb{R}_+$, which contradicts the hypothesis that every maximal solution approaches $C_1 \subset C_2$ (recall that $C_1 \cap \partial C_2 = \emptyset$).

Remark 4.1 Proposition 4.3(a) is closely akin to [18, Theorem 3 (p.79)]. Proposition 4.3(b-i) is essentially an assertion that $\Sigma_\infty(K)$ is compact and is an invariant set for (4) in the sense that, for each $x^0 \in \Sigma_\infty(K)$, every maximal solution of (4) has trajectory in $\Sigma_\infty(K)$. A similar observation occurs in the proof of [7, Theorem 11].

4.2 Persistence of the BKZ property

The following is essentially Theorem 1 of [8].

Theorem 4.1 *Let $F \in \mathcal{U}$. If there exist $0 < \tau < \delta < \rho$ and $T > 0$ such that*

$$\|x^0\| \leq \delta \quad \Longrightarrow \quad \begin{cases} \|x(t)\| \leq \rho & \forall\, t \in [0,T] \\ \|x(t)\| \leq \tau & \forall\, t \in [T,2T] \end{cases}$$

for every maximal solution x of (4)*, then F has the BKZ property.*

In view of Lemma 3.1, to prove this result it suffices to show that $\deg(F,\mathbb{B}_\delta,0) \neq 0$. In the Appendix, we provide a proof which incorporates minor corrections to the proof in [8].

In what follows, several specific consequences of the above result are highlighted: simply stated, the first of these (Theorem 4.2) asserts that, if there exists a compact set that attracts all maximal solutions of (4), then F has the BKZ property.

A non-empty set $C \subset \mathbb{R}^N$ is said to be attractive for (4) if there exists an open neighbourhood $\mathcal{N}$ of C (that is, an open set containing the closure of C) with the property that, for each $x^0 \in \mathcal{N}$, every maximal solution $x\colon [0,\omega) \to \mathbb{R}^N$ of (4) is such that $d_C(x(t)) \to 0$ as $t \uparrow \omega$ (if C is compact, then $\omega = \infty$): C is globally attractive if the latter property holds with $\mathcal{N} = \mathbb{R}^N$. Non-empty C is said to be stable for (4) if, for each open neighbourhood $\mathcal{N}_1$ of C, there is an open neighbourhood $\mathcal{N}_2$ of C such that, for each $x^0 \in \mathcal{N}_2$, every maximal solution of (4) has trajectory in $\mathcal{N}_1$.

Theorem 4.2 *Let $F \in \mathcal{U}$. Let $C \subset \mathbb{R}^N$ be non-empty and compact. If C is globally attractive for* (4)*, then F has the BKZ property.*

Proof By global attractivity of compact C, every maximal solution of (4) has interval of existence $\mathbb{R}_+$. Fix $r > 0$ such that $\overline{\mathbb{B}}_r \supset C$. By Proposition 4.3, the set $\Sigma_\infty(\overline{\mathbb{B}}_{3r})$ is compact and positively invariant.

Let $\tau > 3r$ be sufficiently large so that $\Sigma_\infty(\overline{\mathbb{B}}_{3r}) \subset \overline{\mathbb{B}}_\tau$ and choose $\delta > \tau$. By Proposition 4.3(b), there exists $T > 0$ such that, for every F-arc $x \in AC(\mathbb{R}_+;\mathbb{R}^N)$ with $\|x(0)\| \le \delta$, $\|x(t)\| \le 3r$ for some $t \in [0,T]$. Since $\overline{\mathbb{B}}_{3r} \subset \Sigma_\infty(\overline{\mathbb{B}}_{3r})$, it follows that, for each x^0,

$$\|x^0\| \le \delta \quad \Longrightarrow \quad x(t) \in \Sigma_\infty(\overline{\mathbb{B}}_{3r}) \quad \text{for some} \quad t \in [0,T]$$

for every maximal solution of (4). Therefore, by (positive) invariance of $\Sigma_\infty(\overline{\mathbb{B}}_{3r}) \subset \overline{\mathbb{B}}_\tau$,

$$\|x^0\| \le \delta \quad \Longrightarrow \quad \|x(t)\| \le \tau \quad \forall\, t \in [T,\infty)$$

for every maximal solution of (4).

By Proposition 4.3(a), there exists $\rho > \delta$ such that

$$\|x^0\| \le \delta \quad \Longrightarrow \quad \|x(t)\| \le \rho \quad \forall\, t \in [0,T].$$

Therefore, the hypotheses of Theorem 4.1 hold and so the result follows.

Next, we highlight a further consequence of the above theorem which, for example, implies that, if (1) generates a global semiflow and is L^p stable in the sense that all solutions are of class L^p for some $1 \le p < \infty$, then f has the BKZ property.

Corollary 4.1 *Let $F \in \mathcal{U}$. Let $g\colon \mathbb{R}^N \to \mathbb{R}_+$ be lower semicontinuous with the properties:*

(a) $C := g^{-1}(0)$ *is compact;*
(b) $\inf_{z\in K} g(z) > 0$ *for any closed set $K \subset \mathbb{R}^N$ with $K \cap C = \emptyset$.*

If, for each $x^0 \in \mathbb{R}^N$, every maximal solution of (4) *has interval of existence $\mathbb{R}_+$ and $\int_0^\infty g(x(t))\,dt < \infty$, then F has the BKZ property.*

Proof By [19, Theorem 10 (i)], the compact set $C = g^{-1}(0)$ is globally attractive for (4) and the result follows by Theorem 4.2.

In Theorem 4.2, in order to conclude that F has the BKZ property, hypotheses of a global nature were imposed (global in the sense that, for each $x^0 \in \mathbb{R}^N$, every maximal solution was posited to approach C). The following theorem imposes hypotheses of a local nature under which the BKZ property again persists: in particular, if there exists a closed ball that is locally asymptotically stable for (4), then F has the BKZ property.

Theorem 4.3 *If there exists a closed ball* $\overline{\mathbb{B}}_r(z) =: B$ *which is both stable and attractive for* (4)*, then* F *has the BKZ property.*

Proof Without loss of generality, we may assume $z = 0$ and so $B = \overline{\mathbb{B}}_r \equiv \overline{\mathbb{B}}_r(0)$. By stability and attractivity of compact B, there exist $\alpha, \beta \in \mathbb{R}_+$ such that, for all $x^0 \in \mathbb{R}^N$,

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad \begin{cases} d_B(x(t)) \le \beta & \forall\, t \in \mathbb{R}_+ \\ d_B(x(t)) \to 0 & \text{as} \quad t \to \infty \end{cases}$$

for every maximal solution of (4). Let $\gamma \in (0, \alpha)$ be arbitrary. By stability of B, there exists $\mu \in (0, \gamma)$ such that, for all x^0,

$$d_B(x^0) \le \mu \quad \Longrightarrow \quad d_B(x(t)) \le \gamma \quad \forall\, t \in \mathbb{R}_+ \tag{6}$$

for every maximal solution of (4). By Proposition 4.3(b), there exists $T > 0$ such that, for all x^0,

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad d_B(x(t)) \le \mu \quad \text{for some} \quad t \in [0, T]$$

which, together with (6), yields

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad d_B(x(t)) \le \gamma \quad \forall\, t \ge T$$

for every maximal solution x of (4). We may now conclude that the hypotheses of Theorem 4.1 hold (with $\tau = \gamma + r$, $\delta = \alpha + r$ and $\rho = \beta + r$) and the proof is complete.

5 Feedback Control

We now turn to the main concern of the paper, namely, the consequences of the above results in a context of feedback control systems.

Let $f\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be continuous and consider the controlled system

$$\dot{x} = f(x, u). \tag{7}$$

Henceforth, we assume that f has the property that, for every non-empty convex set $C \subset \mathbb{R}^M$, the set $f(x, C) \subset \mathbb{R}^N$ is convex for all $x \in \mathbb{R}^N$.

As *admissible feedback controls* for (7), we take the class $\mathcal{K}$ of upper semicontinuous maps $x \mapsto k(x) \subset \mathbb{R}^M$ on $\mathbb{R}^N$, with non-empty convex and compact values. Therefore, for every feedback $k \in \mathcal{K}$, the map $F_k\colon x \mapsto f(x, k(x))$ is of class $\mathcal{U}$.

5.1 Persistence of the BKZ property in feedback systems

For system (7), a feedback $k \in \mathcal{K}$ is said to render a compact set $C \subset \mathbb{R}^N$ stable (respectively, attractive) if C is stable (respectively, attractive) for (4) with $F = F_k$.

The following theorem and corollary are immediate consequences of Theorem 4.2 and Corollary 4.1.

Theorem 5.1 *Let* $k \in \mathcal{K}$ *and let* $C \subset \mathbb{R}^N$ *be non-empty and compact. If either of the following holds, then* f *has the BKZ property:*

(i) k *renders* C *globally attractive for* (4)*;*

(ii) k *renders some closed ball* B *stable and attractive for* (4).

Corollary 5.1 *Let $k \in \mathcal{K}$ and let $g\colon \mathbb{R}^N \to \mathbb{R}_+$ be as in Corollary 4.1. If, for each $x^0 \in \mathbb{R}^N$, every maximal solution of (4) with $F = F_k$ has interval of existence $\mathbb{R}_+$ and $g \circ x \in L^1(\mathbb{R}_+)$, then f has the BKZ property.*

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Appendix: Proof of Theorem 4.1

Let $D := \overline{\mathbb{B}}_\rho$ and let $\hat{F} \in \mathcal{U}(D)$ denote the restriction of $F \in \mathcal{U}$ to D.

Observe that $0 \notin \hat{F}(\partial\mathbb{B}_\delta)$ (otherwise, there exists a constant solution $t \mapsto x^0$ of (4) with $\|x^0\| = \delta$, contradicting the hypotheses). Therefore $\deg(\hat{F}, \mathbb{B}_\delta, 0)$ is well-defined and, in view of Lemma 3.1, to complete the proof it suffices to show that $\deg(\hat{F}, \mathbb{B}_\delta, 0) \neq 0$.

By Proposition 2.1(ii) and property P1 of degree, there exists a sequence (f_n) of locally Lipschitz functions $D \to \mathbb{R}^N$ such that:

$$\begin{aligned} \deg(\hat{F}, \mathbb{B}_\delta, 0) &= \deg_{\mathrm{B}}(f_n, \mathbb{B}_\delta, 0) \quad \forall\, n\,; \\ d(\mathrm{graph}\,(f_n),\, \mathrm{graph}\,(\hat{F})) &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{8}$$

By compactness of $\hat{F}(D)$, the functions f_n are bounded and so, for each n, the equation $\dot{x} = f_n(x)$ generates a semiflow $\varphi_n \colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Write $I := [0, 2T]$ and $X := C(I; \mathbb{R}^N)$ (with the uniform norm). On $\overline{\mathbb{B}}_\delta$ define

$$\mathcal{F} \colon x^0 \mapsto \big\{ x \in X \mid x \text{ an } \hat{F}\text{-arc with } x(0) = x^0 \big\}$$

with $\mathrm{graph}\,(\mathcal{F}) := \{(x^0, x) \mid x^0 \in \overline{\mathbb{B}}_\delta,\ x \in \mathcal{F}(x^0)\}$. For each n, define $\phi_n \colon \overline{\mathbb{B}}_\delta \to X$ by

$$(\phi_n(x^0))(t) := \varphi_n(t, x^0) \quad \forall\, t \in I.$$

Fix ϵ such that $0 < \epsilon < \delta - \tau$. We claim that

$$d(\mathrm{graph}\,(\phi_m),\, \mathrm{graph}\,(\mathcal{F})) < \epsilon \quad \text{for some} \quad m \in \mathbb{N}. \tag{9}$$

Suppose otherwise. Then there exists a sequence $(x_n^0) \subset \overline{\mathbb{B}}_\delta$ such that

$$d_{\mathrm{graph}\,(\mathcal{F})}((x_n^0, \phi_n(x_n^0))) \geq \epsilon \quad \forall\, n. \tag{10}$$

By Proposition 4.1, we may assume (without loss of generality) that $(\phi(x_n^0)) \subset X$ converges uniformly to an $\hat{F}$-arc $x \in AC(I; \mathbb{R}^N)$ with $x(0) \in \overline{\mathbb{B}}_\delta$ (and so $(x(0), x) \in \mathrm{graph}\,(\mathcal{F})$), which contradicts (10). Therefore, (9) is true.

Let $x^0 \in \overline{\mathbb{B}}_\delta$ be arbitrary. By (9), there exists $y^0 \in \overline{\mathbb{B}}_\delta$, with $\|x^0 - y^0\| < \epsilon$, and $y \in \mathcal{F}(y^0)$ such that $\|\varphi_m(t, x^0) - y(t)\| < \epsilon$ for all $t \in I$. Since the set $\{y(t) \mid y \in \mathcal{F}(\overline{\mathbb{B}}_\delta)\}$ lies in the ball $\mathbb{B}_\tau$ for all $t \in [T, 2T]$, we may conclude:

$$\text{for all} \quad x^0 \in \overline{\mathbb{B}}_\delta, \quad \varphi_m(t, x^0) \in \mathbb{B}_\delta \quad \text{for all} \quad t \in [T, 2T]. \tag{11}$$

Define continuous $h \colon [0,1] \times \overline{\mathbb{B}}_\delta \to \mathbb{R}^N$ by

$$h(s, x^0) := \begin{cases} f_m(x^0), & s = 0 \\ \frac{1}{sT}\big[(\phi_m(x^0))(sT) - x^0\big], & 0 < s \leq 1. \end{cases}$$

We conclude that $h(s, x^0) \neq 0$ for all $(s, x^0) \in [0,1] \times \partial\mathbb{B}_\delta$ by the following argument. Suppose $h(0, x^0) = f_m(x^0) = 0$ for some $x^0 \in \partial\mathbb{B}_\delta$. Then, $\varphi_m(t, x^0) = x^0 \in \partial\mathbb{B}_\delta$ for all $t \in I$, which contradicts (11). Now suppose $h(s, x^0) = 0$ for some $(s, x^0) \in (0,1] \times \partial\mathbb{B}_\delta$. Then $\varphi_m(nsT, x^0) = x^0 \in \partial\mathbb{B}_\delta$ for all $n \in \mathbb{N}$ with $ns \leq 2$. In particular, there exists $n \in \mathbb{N}$ such that $1 \leq ns \leq 2$ and $\varphi_m(nsT, x^0) = x^0 \in \partial\mathbb{B}_\delta$. This contradicts (11).

Therefore, by (8) and the homotopic invariance property of the Brouwer degree,

$$\begin{aligned} \deg(\hat{F}, \mathbb{B}_\delta, 0) &= \deg_{\mathrm{B}}(f_m, \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(h(0,\cdot), \mathbb{B}_\delta, 0) \\ &= \deg_{\mathrm{B}}(h(1,\cdot), \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(g_m, \mathbb{B}_\delta, 0), \end{aligned}$$

where, for notational convenience, g_m denotes the function

$$g_m\colon x^0 \mapsto [(\phi_m(x^0))(T) - x^0]/T.$$

Now consider the continuous map

$$h_0\colon [0,1] \times \overline{\mathbb{B}}_\delta, \qquad (s, x^0) \mapsto (1-s)g_m(x^0) - sx^0.$$

Noting that h_0 is a homotopic connection of the function g_m and the odd map $o : x^0 \mapsto -x^0$ and $h_0(s, x^0) \neq 0$ for all $(s, x^0) \in [0,1] \times \partial\mathbb{B}_\delta$ by properties of the Brouwer degree, we may now conclude that

$$\deg(\hat{F}, \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(o, \mathbb{B}_\delta, 0) \neq 0.$$

This completes the proof.

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Hamilton's Action Function in Stability Problem of Conservative Systems

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Abstract: In the paper the equilibrium stability of conservative systems both holonomic and nonholonomic in case when the appropriate force function of a system has not a local maximum in the equilibrium state is considered. For the investigation of stability the Hamilton action is used as a function of phase variables and time.

Keywords: *Hamilton's action function; stability.*

Mathematics Subject Classification (2000): 70H05, 70H14.

0 Introduction

The most vulnerable area of researches on stability of systems based on the application of the Liapunov second method is the problem of finding a Liapunov function (including its analogues and modifications), especially if the problem is not solved in the framework of linear approximation. For this reason relevant stability (instability) theorems involving auxiliary functions, whose construction remains a problem, are actually inefficient or even useless.

In view of the above, the approach has become of theoretical and practical significance when beforehand the class of considered systems (for example conservative, reversible, with constant phase volume, etc.) is defined, for which the construction of a Liapunov function or its analogue is possible. Thus, the solution of the problem about stability (instability) passes into constructive direction at once for the whole class of systems. Just such idea can be realized concerning the class of conservative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{0.1}$$

whose Lagrangian L can retain its sign at least on some set of motions.

Within the framework of the proposed approach the Liapunov second method should be interpreted somewhat wider in comparison with the classical approach. In particular, the ideas incorporated in the second method are put in the forefront instead of the specific theorems covered by it. Also allowing for the peculiarities of the examined systems takes an important place.

1 About Hamilton Action as a Function of Phase Variables

In the construction of a Liapunov function analogue for conservative systems it is proposed to use the Hamilton action S as a function of phase coordinates and time. The possibility to obtain the equations of conservative systems from the condition for the Hamilton action S being stationary

$$\delta S = \delta \int_0^{t_1} L(q,\dot q)\, d\tau = 0, \tag{1.1}$$

enables the action S to be recognized as a carrier of information on conservative systems. In view of this fact, we replace the fixed value t_1 by the current value t in the expression for the action S and consider S as a magnitude which characterizes the true motion of the system, i.e. as the action function

$$S = \int_0^{t} L(q,\dot q)\, d\tau. \tag{1.2}$$

It means that the values

$$\begin{aligned} q = q(t, q_0, \dot q_0), &\qquad \dot q = \dot q(t, q_0, \dot q_0), \\ q_0 = q(t=0), &\qquad \dot q_0 = \dot q(t=0) \end{aligned} \tag{1.3}$$

in the integrand of equality (1.2) satisfy the equations (0.1). Let us assume further that the Lagrangian $L(q,\dot q) \in C^2(D_q \times R^n_{\dot q})$ and

$$\begin{aligned} L(q,\dot q) &= L_2(q,\dot q) + L_1(q,\dot q) + L_0(q) \\ &= \frac{1}{2}\,\dot q^T A(q)\dot q + f(q)^T \dot q + L_0(q), \end{aligned} \tag{1.4}$$

where the quadratic form $L_2(0,\dot q)$ is positive definite, the point $q = \dot q = 0$ corresponds to the equilibrium state of system (0.1), (1.4), $f(0) = 0$, $L_0(0) = 0$. Besides, let the solution (1.3) satisfy the definition of a flow [1]. This does not limit generality of the consideration, since the instability of the equilibrium state is dealt with below.

Replacing t by τ in (1.3) and carrying out integration in equality (1.2), we obtain

$$S = \tilde S(\tau, q_0, \dot q_0)\big|_0^t \in C^{(1,1,1)}_{tq_0\dot q_0}(R \times s_\delta), \tag{1.5}$$

where the vector $(q_0, \dot q_0)$ belongs to the neighborhood $s_\delta = \{(q_0, \dot q_0) \in D_q \times R^n_{\dot q},$ $\|q_0 \oplus \dot q_0\| < \delta\}$ of the point $q = \dot q = 0$. Taking into account that the solution (1.3) defines a flow and thus

$$q_0 = q(-t, q, \dot q), \qquad \dot q_0 = \dot q(-t, q, \dot q), \tag{1.6}$$

we have from (1.5)

$$S = S^*(\tau, q(\tau), \dot q(\tau))|_0^t \in C^{(1,1,1)}_{tq\dot q}(R \times D_q \times R^n_{\dot q}). \tag{1.7}$$

The use of the Hamilton action function S in the form of (1.7) as an analogue of the Liapunov function provides greater possibility for more complete representation of the internal properties of the system in question than the standard approach within the Liapunov second method. Actually, this is confirmed by the investigations of the inversion of the Lagrange-Dirichlet and Routh theorems [2−8]. Sufficient conditions of instability obtained in these investigations are more general in comparison with the ones known earlier (see the reviews [7, 9−11]. In particular, the following result is true

Theorem 1.1 [6] *Let a number* $\varepsilon > 0$ $(D \supset \overline{s^*_\varepsilon})$ *exist such that:*

(1) $\omega = \{q \in s^*_\varepsilon \colon L_0(q) > 0\} \neq \emptyset,\ 0 \in \partial\omega$;
(2) $\partial L_0/\partial q \neq 0\ \ \forall q \in \omega$;
(3) $L_0 - \frac{1}{2} f^T A^{-1} f \geq 0\ \ \forall q \in \omega$.

Then the equilibrium state $q = \dot q = 0$ *of system (0.1), (1.4) is unstable.*

Corollary 1.1 *Let the system be natural* ($L_2 = T,\ L_1 \equiv 0,\ L_0 = -\Pi$, *where the functions* T *and* Π *are kinetic energy and potential energy of system respectively) and let a number* $\varepsilon > 0$ $(D \supset \overline{s^*_\varepsilon})$ *exist such that:*

(1) $\omega = \{q \in s^*_\varepsilon \colon \Pi(q) < 0\} \neq \emptyset,\ 0 \in \partial\omega$;
(2) $\partial\Pi/\partial q \neq 0\ \ \forall q \in \omega$.

Then the equilibrium state $q = \dot q = 0$ *of the system is unstable.*

Corollary 1.2 [2] *The isolated equilibrium state* $q = \dot q = 0$ *of a natural system is unstable if in this state the potential energy* $\Pi(q)$ *has not a local minimum.*

Corollary 1.3 [3] *Let the Lagrangian* L *in the neighborhood of the point* $q = \dot q = 0$ *be analytical function. Then the equilibrium state* $q = \dot q = 0$ *of a natural system is unstable if the potential energy* $\Pi(q)$ *at the point* $q = 0$ *has not a local minimum.*

Remark 1.1 In special case when the expression $L_0 - \frac{1}{2} f^T A^{-1} f$ has the local minimum (not necessarily strict) at the point $q = 0$, the restriction (2) in Theorem 1.1 (and the restriction (2)) in Corollary 1.1 respectively) can be omitted.

Remark 1.2 The statement of the problem about the equilibrium instability of a natural system under the assumptions of Corollary 1.3 is due to Liapunov [12].

2 The Application of the Hamilton's Action Function S to the Investigation of Stability of Conservative Nonholonomic Systems

The use of the Hamilton's action function S can also appear to be useful in analysing the equilibrium stability of nonholonomic systems. As it is well known [11, 13], the equilibria set of a nonholonomic system is larger than a set of critical points of the appropriate Lagrangian L. Restricting the investigation to stability analysis of the equilibria of nonholonomic systems which are critical points of the Lagrangian L, Whittaker [14] somewhat narrowed the class of considered nonholonomic systems. However this more narrow class is of interest first of all because many of the approaches characteristics of stability investigation of holonomic systems [15−19] are applicable to it. It turns out that for the systems in question, the application of the Hamilton's action function S is efficient as well as for the holonomic ones.

So, we shall consider a nonholonomic system written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q}-\frac{\partial L}{\partial q}=B^T(q)\lambda, \qquad \lambda=(\lambda_1,\dots,\lambda_l)^T, \tag{2.1}$$

$$B(q)\,\frac{dq}{dt}=0, \tag{2.2}$$

where $B(q)=(b_{ij}(q))$ is a $l\times n$ matrix $(i=1,\dots,l,\ j=1,\dots,n,\ l<n)$, λ is the l-vector of the Lagrange multipliers, $L(q,\dot q)$, $B(q)\in C^2(D_q\times R^n_{\dot q})$ and the Lagrangian L is defined by expression (1.4).

Let the quadratic form $\dot q^T A(0)\dot q$ be positive definite as before, the point $q=\dot q=0$ correspond to the equilibrium state of system (2.1), (2.2), (1.4) and $f(0)=0$, $L_0(0)=0$.

Nonintegrable relations (2.2) which restrict the generalized velocities of the system $(\operatorname{rank} B(q)=l)$ are nonholonomic constraints.

It is well known [20, 21] that the equations of motion of nonholonomic system can be obtained on the basis of the Hamilton principle in the Hölder form

$$\int_0^{t_1}\delta L(q,\dot q)\,d\tau=0. \tag{2.3}$$

In contrast to the holonomic systems, the Hamilton principle in the form of (2.3) is no longer the principle of stationary action when equality (1.1) is valid. Apparently, it was Hertz [22], who first called his attention to the fact that equality (1.1) ceases to be true for nonholonomic systems. Nevertheless the indisputable fact is that in the case of nonholonomic systems the Lagrangian $L(q,\dot q)$ is still the key characteristics of the system. Taking this into account, we shall consider the action function of the form of (1.2) for nonholonomic system too.

By analogy with the case of holonomic systems it is assumed that the solution of the considered nonholonomic system is extendable on the whole axis $t\in R$ and so in view of the assumptions on smoothness of $L(q,\dot q)$ and $B(q)$ satisfies the definition of a flow. This fact, as above, does not restrict generality of the consideration and enables the action function S to be represented in the form of (1.7).

In what follows system (2.1), (2.2), (1.4) will be written as

$$\begin{aligned}\dot q&=\frac{\partial H}{\partial p},\\ \dot p&=-\frac{\partial H}{\partial q}+B^T(q)\lambda,\\ B(q)\,&\frac{\partial H}{\partial p}=0,\end{aligned} \tag{2.4}$$

where

$$H(q,p) = \frac{1}{2}\,p^T A^{-1} p - p^T A^{-1} f - L_0 + \frac{1}{2}\,f^T A^{-1} f = h = \text{const}\,. \tag{2.5}$$

Let us consider in R^n the set π which is defined by the equations

$$B(q)q = 0. \tag{2.6}$$

Since $\operatorname{rank} B(q) = l$, the equations (2.6) can always be solved with respect to any l components of the vector of generalized coordinates q. Let us designate by $\widehat{\Psi}(q)$ the restriction of an arbitrary function $\Psi(q)$ to the set π.

Alongside the set π we shall also define the sets

$$\Omega = \bigl\{(q,p) \in s_\varepsilon = \{(q,p) \in D_q \times R^n_p,\ \|q \oplus p\| < \varepsilon\}\colon\ H = h = 0\bigr\},$$

$$\Omega^+ = \bigl\{(q,p) \in s_\varepsilon\colon\ H = h > 0\bigr\},$$

$$\Omega_1^+ = \left\{(q,p) \in s_\varepsilon\colon\ H = h > 0,\ H - q\,\frac{\partial H}{\partial q} + (B(q)q)\lambda > 0\right\}.$$

Theorem 2.1 *Let the function* $L_0 - \frac{1}{2}\,f^T A^{-1} f$ *have a local minimum (not necessarily strict) at the point* $q = 0$ *and besides:*

$$\omega^* = \bigl\{q \in s^*_\varepsilon = \{q \in D_q,\ \|q\| < \varepsilon\}\colon\ \widehat{L}_0(q) > 0\bigr\} \neq \emptyset.$$

Then the equilibrium state $q = \dot q = 0$ *of system (2.1), (2.2), (1.4) is unstable.*

Before proving this theorem we need the following lemma.

Lemma 2.1 *Under the assumptions of Theorem 2.1 the set*

$$\Omega^0 = \left\{(q,p) \in s_\varepsilon\colon\ H = 0, -q\,\frac{\partial H}{\partial q} > 0,\ B(q)q = 0\right\}$$

is not empty.

Proof On the basis of the theorem about the mean [23] we have the equality

$$H(q,p) - H(0,0) = q\,\frac{\partial H}{\partial q}\,(\theta q, \theta p) + p\,\frac{\partial H}{\partial p}\,(\theta q, \theta p), \qquad \theta \in (0,1). \tag{2.7}$$

First let us assume that $L_0 - \frac{1}{2}\,f^T A^{-1} f > 0$. Then, taking into account the relation

$$L = p\dot q - H = \frac{1}{2}\,p^T A^{-1} p + L_0 - \frac{1}{2}\,f^T A^{-1} f = p\,\frac{\partial H}{\partial p}$$

$$\forall\,(q,p) \in \Omega, \quad q \in \omega^*,$$

we conclude that the term $q\,\partial H/\partial q$ on the right-hand side of equality (2.7) is negative $\forall\,(q,p) \in \Omega,\ q \in \omega^*$. Since the vectors q and θq are collinear, on the basis of (2.7) and the definition of ω^* we conclude that $\Omega^0 \neq \emptyset$.

If $L_0 - \frac{1}{2}\,f^T A^{-1} f \equiv 0$, then representing the Hamiltonian H as

$$H = \frac{1}{2}\,p^T A^{-1}(p - 2f) - (L_0 - \frac{1}{2}\,f^T A^{-1} f)$$

and assuming $p = 2f$ $(\|f\| \neq 0)$, we come to the similar conclusion.

Corollary 2.1 *Under the assumptions of lemma the set Ω_1^+ is not empty.*

Proof Let us fix a point $(q^0, p^0) \in \Omega^0$. Taking into account the fact that Ω is the boundary for set Ω^+, we carry out small perturbation of the point (q^0, p^0):

$$\|(q^* - q^0) \oplus (p^* - p^0)\| < \eta, \quad 0 < \eta = \text{const}$$

such that the point (q^*, p^*) becomes the element of the set Ω^+. Then owing to the continuity of the product $q\,\partial H/\partial q$ and the equality (2.6) $\forall (q,p) \in \Omega^0$ the number η (and consequently the appropriate perturbation) can be chosen small so that the inequalities

$$H(q^*, p^*) - q\,\frac{\partial H}{\partial q}\bigg|_{q=q^*,\, p=p^*} + (B(q^*)q^*)\lambda > 0, \qquad H(q^*, p^*) > 0$$

are satisfied. This implies the validity of the corollary.

Proof of Theorem 2.1 Let us assume that the equilibrium state $q = \dot q = 0$ of the initial system (2.1), (2.2), (1.4) is stable.

Following [6], we consider the function

$$V = \frac{qp}{S_1^2 + 1},$$

where

$$S_1 = S^*\Big(t, q, \tfrac{\partial H}{\partial p}\Big) = S_1^*(t, q, p) \in C_{tqp}^{(1,1,1)}(R \times D_q \times R_p^n).$$

Its time derivative along the vector field defined by the equations (2.4) is of the form

$$\begin{gathered} \frac{dV}{dt} = \frac{L}{S_1^2 + 1}\,(1 - \mu) + \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2 + 1}, \\ \mu = 2qp\,\frac{S_1}{S_1^2 + 1}. \end{gathered} \tag{2.8}$$

According to the assumption about equilibrium stability, there always exists the positive semitrajectory $\gamma_1^+ \subset s_\varepsilon$ of the considered system passing through the point $(q^*, p^*) \in \Omega_1^+$. Let us integrate equality (2.8) over a segment of the semitrajectory γ_1^+ which corresponds to the interval $[t_1, t_2]$, where the numbers t_1 and t_2 are such that

$$\gamma_1^+\big|_{t_1}^{t_2} \subset \Omega_1^+. \tag{2.9}$$

We notice that as $\overline{\gamma_1^+}$ is a compact set, the absolute value of the velocity of the appropriate representing point $(q(t, q^*, p^*),\, p(t, q^*, p^*))$ moving along γ_1^+ is uniformly bounded. Consequently, it is possible to specify a number $a > 0$ such that $t_2 - t_1 \geq a$ irrespective of how large the values $t_1,\, t_2 \in R$ are. In result of integration of equality (2.8) we have

$$\frac{qp}{S_1^2 + 1}\bigg|_{t_1}^{t_2} = \arctan\, S_1\big|_{t_1}^{t_2} + o\big(\arctan\, S_1\big|_{t_1}^{t_2}\big) + \int\limits_{t_1}^{t_2} \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2 + 1}\,dt. \tag{2.10}$$

Function $\arctan S_1$ appearing in the right-hand side of equality (2.10) is a multifunction with branch points $S_1 = \pm\infty$. At the same time the intersection of the level set of the Hamiltonian $H = h > 0$ and the small neighborhood of the equilibrium state is the manifold, in any point of which the function S_1 does not become infinite. Further, without loss of generality, it is possible to consider that the inequality $S_1|_{t=t_1} > 1$ is valid.

Under condition (2.9) we shall choose the length of the interval $[t_1, t_2]$ small enough to deal with the domain of principal values of the function $\arctan S_1$, using, for example, the representation of the latter as

$$\arctan S_1 = \frac{\pi}{2} - \frac{1}{S_1} + \frac{1}{3S_1^3} - \dots .$$

Then from equality (2.10) when $\varepsilon > 0$ is sufficiently small we obtain

$$\frac{1}{S_1}\bigg|_{t_1}^{t_2} + O\bigg(\frac{1}{3S_1^3}\bigg|_{t_1}^{t_2}\bigg) + o\bigg(\frac{1}{S_1}\bigg|_{t_1}^{t_2}\bigg) = \int_{t_1}^{t_2} \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2+1}\,dt. \qquad (2.11)$$

Noticing that by (2.9) the right-hand side of equality (2.11) is positive, we come to the contradiction, because according to structure (1.5) of the Lagrangian L:

$$\frac{dS_1}{dt} = p\dot{q} - H = \frac{1}{2}\,p^TA^{-1}p + L_0 - \frac{1}{2}\,f^TA^{-1}f > 0 \quad \forall\,(q,p) \in \gamma_1^+\big|_{t_1}^{t_2}$$

the expression in its left-hand side is negative. Thus the assumption about stability of the examined equilibrium state is false. Theorem 2.1 is proved.

Corollary 2.2 *Let* $L_0 - \frac{1}{2}\,f^TA^{-1}f \ge 0\ \forall\,q \in s_\varepsilon^*$ *and besides the function* $L_0(q)$ *have a strict local minimum at the point* $q = 0$.

Then the equilibrium state $q = \dot{q} = 0$ *of system (2.1), (2.2), (1.4) is unstable.*

Corollary 2.3 *Under the assumptions of Corollary 2.2 the manifold* M *of the equilibrium states of system (2.1), (2.2), (1.4) defined by the equations*

$$\frac{\partial L_0(q)}{\partial q} + B^T(q)\lambda = 0, \qquad \dot{q} = 0$$

is unstable.

Proof According to Corollary 2.2 the equilibrium state $q = \dot{q} = 0$ of system (2.1), (2.2), (1.4) is unstable. Let us show that the instability holds true relating to both variables q and variables $\dot{q}$. For this purpose we shall assume on the contrary that instability of the equilibrium state under consideration motivates only the leave of q-vector. Then, irrespective of the smallness of the initial perturbation, there is an orbit of the system, whose representing point in a small enough neighborhood of the point $q = 0$ reaches some sphere $\|q\|^2 = \eta^2$, $\eta = \text{const}$. Let integer ξ $(0 < \xi = \text{const})$ correspond to the minimum of function $L_0(q)$ on this sphere. It is clear that ξ does not depend on the smallness of perturbation.

On the basis of equality

$$L_2(q,\dot{q}) - L_0(q) = h = \text{const} \qquad (2.12)$$

we have

$$\frac{1}{2}\,\dot{q}^T A(q)\dot{q} \geq h + \xi. \tag{2.13}$$

Since according to the proof of Theorem 2.1 the equilibrium state $q = \dot{q} = 0$ of the examined system is unstable when $h > 0$, it follows from (2.13) that $\|\dot{q}\| \geq \varepsilon > 0$. In this case the number ε does not depend on the smallness of the initial perturbation and thus the instability of the equilibrium state is accompanied by leaving of $\dot{q}$-vector. Taking into account that $\dot{q} = 0$ on the manifold M of the equilibrium states of the system, according to definition [24, p.34] we make conclusion about the validity of Corollary 2.3.

The sense of Corollary 2.3 is that it establishes a relationship between instability of the fixed equilibrium state of nonholonomic system and that of the whole manifold of the equilibrium states, the existence of which is a distinguishing property of nonholonomic systems [25].

Corollary 2.4 *In special case when the nonholonomic constraints are absent:* $B(q) = 0$*, it is possible to omit the condition* $\omega^* \neq \emptyset$ *in the Theorem 2.1 (cf. [6, 26, 27]).*

As we see, if a system is nonholonomic then this fact is embodied in the character of instability conditions, however it should be remembered that the latter are only sufficient. Therefore, unfortunately, the question about the real influence of nonholonomic constraints on the equilibrium stability remains still open.

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Dynamics of Bidirectional Associative Memory Networks with Processing Delays

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Abstract: A mathematical model describing the dynamical interactions of the bidirectional associative memory networks, incorporating among other things processing time delays, has been proposed in this paper. The existence and stability characteristics of the equilibrium patterns have been discussed. Results on local asymptotic stability of the equilibrium patterns have been presented. Three sets of easily verifiable sufficient conditions describing the global stability of the equilibrium patterns of these networks are obtained.

Keywords: *Bidirectional associative memory networks; global stability.*

Mathematics Subject Classification (2000): 34K20, 34K15, 92B20, 94CXX.

1 Introduction

Mathematical models describing the dynamical interactions of the bidirectional associative memory (BAM for short) networks have been a subject of numerous investigations, Kosko [14−16], Simpson ([23] and the references there in). In particular, the following BAM network model, known as Hopfield network is expressed by the following system of equations:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j) + I_i, \tag{1.1}$$

for $i = 1, 2, \dots, n$ (see, e.g. [11, 12, 16]). As may be seen this model describes the activation dynamics among the various neurons in one single neuronal field. In (1.1), a_i for $i = 1, 2, \dots, n$ represent the passive decay rates, b_{ij} denotes the synaptic connection weights between i-th and j-th neurons, $f_j(x_j)$, for $j = 1, 2, \dots, n$ denote signal

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propagation functions and I_i for $i = 1, 2, \ldots, n$ denote the exogenous inputs to the i-th neuron.

The Hopfield model illustrates an autoassociative BAM. Autoassociativity means that the network topology reduces to only one field, F_X of neurons. The synaptic connection matrix M symmetrically intraconnects the n neurons in Hopfield network. Thus $M = M^T$ and hence it is termed as a BAM model according to [7, 11, 16, 23]. Hopfield network is governed by feedback law. As an important generalization of the Hopfield equation, the following system of equations [7, 14–16, 23]:

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(y_j) + I_i, \\ y_j'(t) &= -c_j y_j(t) + \sum_{i=1}^{m} d_{ji} g_i(x_i) + J_j, \end{aligned} \tag{1.2}$$

in which $b_{ij} = d_{ji}$, have been proposed to describe the BAM network in the neuronal fields F_X and F_Y. Kohonen [13, 16] described these two layer networks as Heteroassociative networks. In (1.2), a_i for $i = 1, 2, \ldots, m$ and c_j for $j = 1, 2, \ldots, n$ denote the passive decay rates, b_{ij}, d_{ji} for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, are synaptic connection strengths, f_j for $j = 1, 2, \ldots, n$ and g_i for $i = 1, 2, \ldots, m$ denote propagational signal functions and I_i for $i = 1, 2, \ldots, m$ and J_j for $j = 1, 2, \ldots, n$ are exogenous inputs.

Evidently if $F_X = F_Y$, the system (1.2) includes (1.1) and hence the notion of BAM described by (1.2) reduces to that expressed by Hopfield network. In their studies, Cohen-Grossberg [2, 7, 16] assumed that the synaptic connection matrices are symmetric, as in the case of Hopfield networks. Kosko [16], expressed that when b_{ij} and d_{ji} differ, fixed point equilibrium tends not to occur, instead equilibrium behaviour may be oscillatory or a periodic. We disagree with this view by presenting various global stability criteria under the circumstances that b_{ij} and d_{ji} can differ.

We now consider the networks, in which the synaptic connection matrices B and D need not satisfy $B = D^T$ or vice versa and $B = B^T$ and regard these networks as BAM networks. However, if the matrices B and D satisfy $B = D$, $B = B^T$ our definition of BAM networks reduces to the earlier known definition. Clearly the above definition of BAM is more general and allows us to consider arbitrary connection matrices.

It is generally known that in the biological and artificial neural networks as well, time delays arise due to the propagation of information. More specifically, in the electronic implementation of analog neural networks, time delays occur in the communication and response of neurons due to the finite switching speed of amplifiers. Usually constant fixed time delays in models of delayed feedback systems serve as good approximations in simple circuits having a small number of cells. Due to the spatial nature of neural networks, resulting in the parallel pathways of a variety of axon sizes and lengths, in [24, 26] distributed time delays representing transmission of information have been considered.

In the present paper, we propose a BAM model incorporating a fixed discrete delay to represent the processing delays. It is known that in the mammals, the processing of information at the neuronal level is rather slow implying that the neuron is not very efficient, but when these neurons are connected in a network, their efficiency increases, see an der Heiden [1] and Kosko [16, p. 45]. The following model in a more general form

$$
\begin{aligned}
x_i' &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t)) + I_i, \\
y_i' &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j(t)) + J_i,
\end{aligned}
\tag{1.3}
$$

where $i = 1, 2, \dots, n$, has been suggested in [14–16] to describe the activation dynamics of the neurons in the absence of time delays. These equations lead us to the understanding that the neurons process the information instantaneously, contrary to what has been observed in the references [1, 16]. Thus the activational dynamics described by the system (1.3) seems unrealistic. It is also natural to think that any system (whether biological or man made), which responds instantaneously accumulates certain amount of strain over a period of time, which may result in its break down. From these considerations, it appears that a certain amount of delay (time-lag) in its performance is necessary for its well being. Thus, we modify the network equations (1.3) to include a discrete time delay in the signal response functions and accordingly, our model equations assume the form

$$
\begin{aligned}
x_i' &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t-\tau)) + I_i, \\
y_i' &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j(t-\tau)) + J_i,
\end{aligned}
\tag{1.4}
$$

for $i = 1, 2, \dots, n$ (see [22]). Our model (1.4) includes the earlier proposed models involving discrete time delays [7, 18–20]. It is important to mention here that in biological/man made systems increasing time delays always render the system attain instability, see Cushing [3], Mac Donald [17]. However, it has been established by Freedman and Sree Hari Rao [5] that a proper interplay between the time delay and various other parameters of the system, may help stabilize the otherwise unstable systems. Thus from this discussion, we certainly cannot neglect the time delays but rather like to study their influence on the stability behaviour of the system. To be more precise, we shall be interested mainly in examining the effect of time delays on the maintenance and preservation of stability/instability of the equilibrium. It is worth pointing out that the delay parameter τ, may be regarded as a mechanism to limit the strain in the performance of the network, particularly when it processes information instantaneously.

In (1.4) the parameter τ corresponds to the time delay arising due to the processing of information at neuronal level. In artificial neural networks, this time delay arises due to the finite switching speed of amplifiers. The passive decay rates a_i, c_i for $i = 1, 2, \dots, n$ are assumed to be positive constants. The numbers b_{ij}, d_{ij} for $i, j = 1, 2, \dots, n$ are synaptic connection strengths between the i-th and j-th neurons in the neuronal fields F_X and F_Y. I_i and J_i for $i = 1, 2, \dots, n$ are exogenous inputs. The functions f_i and g_i for $i = 1, 2, \dots, n$ are signal response functions. The initial functions associated with the system (1.4) are given by

$$
x_i(s) = p_i(s), \qquad y_i(s) = q_i(s) \tag{1.5}
$$

for $s \in [-\tau, 0]$ and each $i = 1, 2, \ldots, n$, where p_i and q_i, for each $i = 1, 2, \ldots, n$ are assumed to be continuous functions on $[-\tau, 0]$.

This paper is organized as follows. In Section 2, results on the existence of a unique equilibrium pattern are presented. The influence of processing delays on the stability behaviour of the network has been discussed in Section 3. Conditions for the length of delay for which stability has been maintained have been presented. A result on the preservation of stability/instability of the equilibrium has been presented in Section 3. Stability of bifurcating periodic solutions is discussed in Section 4. Three independent sets of sufficient conditions for the global asymptotic stability of the equilibrium patterns have been presented in Section 5. Examples illustrating the merit of our results have been presented in Section 6. Finally a discussion follows in Section 7.

2 Existence and Uniqueness of Equilibrium Pattern

It is easy to see that the equilibria of the system (1.4) are solutions of the following system of equations:

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j^*) + I_i, \\ c_i y_i^* &= \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j^*) + J_i, \end{aligned} \tag{2.1}$$

for $i = 1, 2, \ldots, n$.

We shall state the following conditions on the signal functions f_i and g_i for $i = 1, 2, \ldots, n$, which will be utilized in this work. There exist positive constants $\alpha_i(\lambda_i)$ and $\beta_i(\mu_i)$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} \|f_j(\lambda_j, u_j(t)) - f_j(\lambda_j, v_j(t))\| &\le \alpha_j(\lambda_j)\|u_j - v_j\|, \\ \|g_j(\mu_j, u_j(t)) - g_j(\mu_j, v_j(t))\| &\le \beta_j(\mu_j)\|u_j - v_j\| \end{aligned} \tag{2.2}$$

for λ, μ, u, $v \in R^n$ and $t \in [0, \infty)$ and $\|\cdot\|$ denotes any appropriate norm on R^n.

We now, present our first result on the existence of a unique equilibrium pattern (x^*, y^*).

Theorem 2.1 *Assume that (1.5), (2.2) are satisfied. In addition assume that the decay rates a_i, c_i, the synaptic weights b_{ij}, d_{ij} and the parameters α_i, β_i satisfy the following inequalities:*

$$\frac{\alpha_i \sum\limits_{j=1}^{n} |b_{ji}|}{a_i} < 1 \quad \text{and} \quad \frac{\beta_i \sum\limits_{j=1}^{n} |d_{ji}|}{c_i} < 1 \tag{2.3}$$

for each $i = 1, 2, \ldots, n$.

Then for any pair of input vectors (I, J) the system (1.4) has a unique positive equilibrium pattern (x^*, y^*) satisfying the equations (2.1).

In the following, we present a result guaranteeing the existence of a unique equilibrium pattern (x^*, y^*) to our model equations (1.4), at the expense of dropping the Lipschitzian hypotheses (2.2). This allows us to include non-Lipschitzian signal response functions in our equations. Here and henceforth the term "*Non-Lipschitz*" should be understood as *not necessarily Lipschitz*. Further, the decay rates a_i, c_i, synaptic connections b_{ij}, d_{ij} are accorded more freedom in the following result, than the restrictions placed on them in Theorem 2.1.

We now rewrite the system of equations (1.4) as:

$$X'(t) = F(X(t)) \tag{2.4}$$

in which

$$X(t) = (x_1(t), \dots, x_n(t),\, y_1(t), \dots, y_n(t))^T$$

and

$$F(X(t)) = \begin{pmatrix} -a_1 x_1(t) + \sum\limits_{j=1}^{n} b_{1j} f_j(\lambda_j, y_j(t-\tau)) + I_1 \\ \vdots \\ -a_n x_n(t) + \sum\limits_{j=1}^{n} b_{nj} f_j(\lambda_j, y_j(t-\tau)) + I_n \\ -c_1 y_1(t) + \sum\limits_{j=1}^{n} d_{1j} g_j(\mu_j, x_j(t-\tau)) + J_1 \\ \vdots \\ -c_n y_n(t) + \sum\limits_{j=1}^{n} d_{nj} g_j(\mu_j, x_j(t-\tau)) + J_n \end{pmatrix}. \tag{2.5}$$

We consider the initial value problem associated with the autonomous system (2.4), in which the initial functions are given by

$$x_i(s) = p_i(s), \qquad y_i(s) = q_i(s), \tag{2.6}$$

for $s \in (-\tau, 0]$ and for $i = 1, 2 \dots n$, where p_i and q_i are assumed to be continuous functions of bounded variation on $(-\tau, 0]$. Let S be an open subset of R^{2n}. For any $\xi \in R^{2n}$, we define $\|\xi\| = \sum\limits_{i=1}^{2n} |\xi_i|$.

We now present a lemma which is an application of the theorem in [21] and is useful in proving the next theorem.

Lemma 2.1 *Let $F\colon S \to R^{2n}$ be continuous and satisfy the following condition: Corresponding to each point $\xi \in S$, its neighbourhood U, there exists a constant $k > 0$, and functions h_j and Φ_l for $j = 1, 2, \dots, n$ and $l = 1, 2, \dots, n, n+1, \dots, 2n$ such that*

$$\|F(\xi) - F(\eta)\| \le k\|\xi - \eta\| + k \sum_{l=1}^{2n} |\Phi_l(h_j(\xi)) - \Phi_l(h_j(\eta))| \tag{2.7}$$

on U, where each $h_j\colon U \to R$ is a continuously differentiable function in ξ satisfying

$$\sum_{i=1}^{2n} \frac{\partial h_j(\xi)}{\partial \xi_i} F_i(\xi) \ne 0 \quad \text{on} \quad U \tag{2.8}$$

and each $\Phi_l\colon R \to R$, $l = 1, 2, \dots, n, n+1, \dots, 2n$ is continuous and of bounded variation on bounded sub intervals. Then there exists a unique solution for the initial value problem (2.4)–(2.5) on any interval containing the initial functions (2.6).

Theorem 2.2 *Assume that the hypotheses of Lemma 2.1 hold for the functions f_i and g_i for each $i = 1, 2, \ldots, n$. Then the system of equations (2.1) admits a unique solution, yielding a unique equilibrium pattern for our model equations (1.4).*

It is easy to modify the results on existence and uniqueness of equilibrium in [24, 26] in proving these results.

3 Delay Dependent Stability Results

In this section, we examine the influence of the time lags on the stability of the equilibrium pattern of (1.4). The linearized system associated with (1.4) is given by

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij}\alpha_j y_j(t-\tau), \\ y_i'(t) &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij}\beta_j x_j(t-\tau) \end{aligned} \tag{3.1}$$

for $i = 1, 2, \ldots, n$. It is convenient to represent the linearized system of $2n$ delay equations (3.1) as amplitudes u_i and v_i for $i = 1, 2, \ldots, n$, along the $2n$ eigen values of the connection matrices respectively. Following the study in [18], we let r_i and s_i be the connection eigen values of b_{ij} and d_{ij} respectively for $i,\, j = 1, 2, \ldots, n$ and accordingly, we have

$$\begin{aligned} u_i'(t) &= -a_i u_i(t) + r_i\alpha_i v_i(t-\tau), \\ v_i'(t) &= -c_i v_i(t) + s_i\beta_i u_i(t-\tau) \end{aligned} \tag{3.2}$$

for $i = 1, 2, \ldots, n$. Since the neural gains α_i and β_i are positive for each $i = 1, 2, \ldots, n$, then $r_i\alpha_i$ and $s_i\beta_i$ have the same sign as those of r_i and s_i respectively for $i = 1, 2, \ldots, n$. We now introduce that the complex characteristic exponent λ_i and define $u_i(t) = u_i(0)e^{\lambda_i t}$, $v_i(t) = v_i(0)e^{\lambda_i t}$ for each $i = 1, 2, \ldots, n$. Substituting this form into (3.2) yields

$$\lambda_i^2 + (a_i + c_i)\lambda_i + a_i c_i - \alpha_i\beta_i r_i s_i e^{-2\lambda_i\tau} = 0 \tag{3.3}$$

for $i = 1, 2, \ldots, n$.

We now let

$$a_i + c_i = a, \quad a_i c_i = b, \quad -\alpha_i\beta_i r_i s_i = c, \quad \lambda_i = \lambda \tag{3.4}$$

for $i = 1, 2, \ldots, n$. Clearly, both a and b are positive. Then, (3.3) yields,

$$\lambda^2 + a\lambda + b + ce^{-2\lambda\tau} = 0. \tag{3.5}$$

The equation (3.5) has roots with negative real parts, if

$$F(\lambda, \tau) = \lambda^2 + a\lambda + b + ce^{-2\lambda\tau} \neq 0 \quad \text{for} \quad \operatorname{Re}\lambda \geq 0. \tag{3.6}$$

In this section, we examine the following aspects related to the stability/instability of the equilibrium patterns of the system (1.4). Throughout this section we shall use the term *stability* to mean and imply *asymptotic stability* of the equilibrium patterns for the model equations.

(i) If the equation (3.5) is stable for $\tau = 0$, then for what other values of $\tau > 0$, it is stable? This amounts to determining an interval for τ, say, $0 \le \tau \le \tau_*$, such that for all values of τ in this interval, the equation (3.5.) is stable or (3.6) holds.

(ii) For $\tau > 0$ large, what type of stability for equation (3.5) prevails? More specifically, if the equation (3.5) is stable (or unstable) for $\tau = 0$, can it continue to be stable(or unstable) for all values of $\tau > 0$?

Clearly, for stability of the equations (3.5), one has to see that it has no pure imaginary zeros or zeros with positive real parts. We note that $\lambda \neq 0$ is a root of (3.5) if and only if $b + c \neq 0$.

Our first result provides us an estimate on the length of the delay parameter τ, say τ_* which ensures that the equilibrium pattern is asymptotically stable for all values of τ satisfying the inequalities, $0 \le \tau \le \tau_*$. For some biological models, methods to estimate τ_0, has been presented in Erbe, Freedman and Sree Hari Rao [4] and Freedman, Sree Hari Rao and Jayalakshmi [6]. We utilize these techniques to establish the following result.

Theorem 3.1 *Assume that the following inequality*

$$b + c > 0 \tag{3.7}$$

is satisfied. Then the equilibrium pattern is asymptotically stable for all values of τ *satisfying*

$$0 \le \tau < \tau_* = \frac{\sqrt{c^2(b + |c|) + 2a^2|c|(b + c)} - |c|\sqrt{b + |c|}}{2a|c|\sqrt{b + |c|}}. \tag{3.8}$$

Proof It is easy to see that the inequality (3.7) is a consequence of stability for $\tau = 0$, which follows from the application of Routh Hurwitz method. Since λ is a continuous function of the parameter τ, all eigen values will continue to have negative real parts for sufficiently small $\tau > 0$.

If $\lambda = \mu + i\nu$ satisfies (3.5), then μ and ν are real solutions of

$$\mu^2 - \nu^2 + a\mu + b + ce^{-2\mu\tau} \cos 2\tau\nu = 0, \tag{3.9}$$

$$2\mu\nu + a\nu - ce^{-2\mu\tau} \sin 2\tau\nu = 0. \tag{3.10}$$

Let $\hat{\tau}$ be such that $\mu(\hat{\tau}) = 0$, then from (3.9) and (3.10), we have

$$-\hat{\nu}^2 + b + c \cos 2\hat{\tau}\hat{\nu} = 0, \tag{3.11}$$

$$a\hat{\nu} - c \sin 2\hat{\tau}\hat{\nu} = 0. \tag{3.12}$$

We see that the conditions for the asymptotic stability of the equilibrium, following the procedure suggested in Freedman and Sree Hari Rao [5], are given by

$$\operatorname{Im} F(i\nu_0) > 0, \tag{3.13}$$

$$\operatorname{Re} F(i\nu_0) = 0, \tag{3.14}$$

where $F(s) = s^2 + as + b + ce^{-2\tau s}$ and ν_0 is the smallest positive root of (3.14). The conditions in our case become

$$-\nu_0^2 + b + c\,\cos 2\tau\nu_0 = 0, \tag{3.15}$$

$$a\nu_0 > c\,\sin 2\tau\nu_0. \tag{3.16}$$

To get our estimate on the length of delay we shall utilize the inequality (3.16) and the equation (3.15), which if simultaneously satisfied, are sufficient to guarantee stability. Rewriting the same, we get

$$\nu^2 - b = c\,\cos 2\tau\nu, \tag{3.17}$$

$$a\nu > c\,\sin 2\tau\nu. \tag{3.18}$$

We recall that the equilibrium will be stable if the inequality (3.18) holds at $\nu = \nu_0$, where ν_0 is the first positive root of the equation (3.17). Our technique is to find an upper bound ν_+ on ν_0, independent of τ, and then to estimate τ so that (3.18) holds for all values of ν, $0 \le \nu \le \nu_+$ and hence in particular at $\nu = \nu_0$.

Since the right hand side of (3.17) is less than or equal to $|c|$, the unique positive solution of

$$\nu^2 - b = |c|, \tag{3.19}$$

denoted by ν_+, is always greater than or equal to ν_0. Clearly,

$$\nu_+ = \sqrt{b + |c|}. \tag{3.20}$$

Note that ν_+ is independent of τ. We need an estimate on τ so that (3.18) holds for all $0 \le \nu \le \nu_+$.

Note that at $\tau = 0$, this inequality becomes $a\nu > 0$. However at $\tau = 0$, the solution of (3.17) is $\nu_0 = \sqrt{b + c}$. Hence (3.18) is valid at $\tau = 0$, $\nu = \nu_0$. So by continuity it will continue to hold for small enough $\tau > 0$ and $\nu = \nu_0$.

From (3.18), we get

$$a\nu^2 > c\nu\,\sin 2\tau\nu. \tag{3.21}$$

From (3.17), we get

$$c\nu\,\sin 2\tau\nu + 2ac\,\sin^2\tau\nu < a(b + c). \tag{3.22}$$

Denote the left hand side of (3.22) by $\phi(\tau,\nu)$. We now use the inequalities $\sin 2\tau\nu \le 2\tau\nu$ and $\sin^2\tau\nu \le \tau^2\nu^2$. Then

$$\phi(\tau,\nu) \le \psi(\tau,\nu) \equiv 2|c|\tau\nu^2 + 2a|c|\tau^2\nu^2.$$

We note that for $0 \le \nu \le \nu_+$, we have $\phi(\tau,\nu) \le \psi(\tau,\nu) \le \psi(\tau,\nu_+)$. Hence if $\psi(\tau,\nu_+) < a(b + c)$, then $\phi(\tau,\nu_0) < a(b + c)$.

Let τ_* denote the unique positive root of $\psi(\tau,\nu_+) = a(b + c)$. Then

$$2a|c|\tau^2\nu_+^2 + 2|c|\nu_+^2\tau = a(b + c).$$

Thus

$$\tau_* = \frac{\sqrt{c^2(b + |c|) + 2a^2|c|(b + c)} - |c|\sqrt{b + |c|}}{2a|c|\sqrt{b + |c|}}. \tag{3.23}$$

Then for $\tau < \tau_*$, the Nyquist criterion holds, and τ_* is the estimate for the length of delay for which stability is preserved.

Remark 3.1 Utilizing the relation (3.4) and evaluating the estimate for τ in the special case $b = \frac{a^2}{4}$ and $c = \frac{a^2}{2}$, we get $\tau_* = \frac{\sqrt{5}-1}{2(a_i+c_i)}$. Clearly, τ_* decreases with increasing a_i or c_i or *vice versa.* This special situation is interesting in the sense that there is an explicit relation between the delay τ and the passive decay rates a_i and c_i.

In the next result we shall improve the estimate on τ_* in this special case by employing a different technique which has been presented in detail in Sree Hari Rao and Phaneendra [25].

In the following, we utilize a method as suggested in [10], to find the stability interval, in which the equilibrium is asymptotically stable.

Theorem 3.2 *Assume that the hypothesis (3.7) is satisfied. In addition, let* $b = \frac{a^2}{4}$, $c = \frac{a^2}{2}$ *in (3.5) are satisfied. Then the equilibrium pattern is asymptotically stable for all* τ *satisfying*

$$0 \le \tau < \tau_* = \frac{\pi}{2a} = \frac{\pi}{2(a_i + c_i)}. \tag{3.24}$$

Proof Let $F(\lambda, \tau) \equiv 4\lambda^2 + 4a\lambda + a^2 + 2a^2 e^{-2\lambda\tau}$.

Now, $F_1(\lambda) = F(\lambda, 0) = 4\lambda^2 + 4a\lambda + 3a^2 = 0$ has roots with negative real parts since $a > 0$.

Now let $F_2(\lambda) \equiv 4\lambda^2 + 4a\lambda + 3a^2$, which is obtained by replacing $e^{-\lambda\tau}$ with -1.

Clearly, $F_2(\lambda) \neq 0$. Now, substituting, $e^{-\lambda\tau} = \frac{1-T\lambda}{1+T\lambda}$ for $T > 0$, in the equation

$$4\lambda^2 + 4a\lambda + a^2 + 2a^2 e^{-2\lambda\tau} = 0, \tag{3.25}$$

we obtain

$$\begin{aligned} F_3(\lambda) \equiv 4T^2\lambda^4 &+ 4T(2 + aT)\lambda^3 + (4 + 8aT + 3a^2T^2)\lambda^2 \\ &+ 2a(2 - aT)\lambda + 3a^2 = 0. \end{aligned} \tag{3.26}$$

Equation (3.26) has pure imaginary roots $\lambda = i\nu$, $\nu > 0$ if and only if

$$\tau = \frac{2}{\omega}\left[\tan^{-1}(\omega T) - k\pi\right], \qquad k = 0, \pm 1, \dots . \tag{3.27}$$

In order to obtain the pure imaginary roots for the equation (3.26), we set the Routh-Hurwitz determinant to zero. Then we obtain,

$$3a^4T^4 + 16a^3T^3 + 8a^2T^2 - 16 = 0. \tag{3.28}$$

Differentiating (3.25) with respect to τ and simplifying, we get

$$\frac{d\lambda}{d\tau} = \frac{a^2\lambda e^{-2\lambda\tau}}{2\lambda + a - a^2\tau e^{-2\lambda\tau}}. \tag{3.29}$$

Evaluating $\frac{d\lambda}{d\tau}$, $\lambda = i\nu$, we obtain

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda = i\nu} = \frac{a^3\nu \sin 2\tau\nu + 2a^2\nu^2 \cos 2\tau\nu}{\Delta}, \tag{3.30}$$

where

$$\Delta = a^2 + 4\nu^2 + a^4\tau^2 - 2a^3\tau\,\cos 2\tau\nu + 4a^2\nu\tau\,\sin 2\tau\nu.$$

Now, solving the equations (3.9) and (3.10) at $\mu = 0$ and $\tau = \tau^*$ for $\sin 2\tau_*\nu$ and $\cos 2\tau_*\nu$, we have

$$\sin 2\tau_*\nu = \frac{2\nu}{a}, \quad \cos 2\tau_*\nu = \frac{4\nu^2 - a^2}{2a^2}. \tag{3.31}$$

Thus from (3.30) and (3.31), we have at $\tau = \tau_*$,

$$\mathrm{Re}\,\frac{d\lambda}{d\tau}\bigg|_{\lambda=i\nu} = \frac{4\nu^4 + a^2\nu^2}{\Delta}, \tag{3.32}$$

$$\Delta = a^2 + 4\nu^2 + a^4\tau^2 + 4a\tau\nu^2 + a^3\tau.$$

From (3.26) and (3.28), it follows that for $T = \frac{2\sqrt{2}-2}{a}$ and $\nu = \frac{a}{2}$, we have $F_3(i\nu) = 0$. Further, at $\tau = \tau_* = \frac{\pi}{2a}$,

$$\mathrm{Re}\,\frac{d\lambda}{d\tau}\bigg|_{\lambda=i\nu} = \frac{2a^2}{(8 + 4\pi + \pi^2)}, \tag{3.33}$$

which proves the theorem. Further, from (3.33), it is clear that the transversality condition of the Hopf bifurcation is satisfied.

Remark 3.2 Following the method used in the above theorem, when $b = c$, the inequality $a^2 - 2b < 0$ implies that (3.5) has a pair of pure imaginary zeros which upon substitution of the values of a and b lead us to the contradiction that $a_i^2 + c_i^2 < 0$. Thus, this situation (though mathematically acceptable for the characteristic equation (3.5) in its most general form involving the coefficients a, b, c) can not arise for this specific model under consideration.

Remark 3.3 Following the discussion in Section 1, we understand that the strain on the network arising out of its instantaneous processing of the information (the case where the time delay $\tau = 0$) may result in the break down in course of time. To avoid the unwanted breakdown in the nervous system, we have proposed the introduction of processing delays. Our Theorems 3.1 and 3.2 clearly ensure the stability of the network in the presence of processing delays so long as they lie in the interval $[0, \tau_*]$. Notice that the expression for τ_* in terms of the relations (3.4) may be written as

$$\tau_* = \frac{\sqrt{\alpha_i^2\beta_i^2 r_i^2 s_i^2 (a_i c_i + \alpha_i\beta_i|r_i s_i|) + 2\alpha_i\beta_i|r_i s_i|(a_i + c_i)^2(a_i c_i - \alpha_i\beta_i r_i s_i)}}{2\alpha_i\beta_i|r_i s_i|(a_i + c_i)\sqrt{a_i c_i + \alpha_i\beta_i|r_i s_i|}} - \frac{1}{2(a_i + c_i)}. \tag{3.34}$$

Clearly, τ_* is expressed among others mainly in terms of the passive decay rates a_i, c_i of the network. Also in the Remark 3.1, we have noted that in the special case where $b = \frac{a^2}{4}$ and $c = \frac{a^2}{2}$, the increase/decrease in τ_* is related to the passive decay rates being smaller/greater (respectively). But it is evident from biological considerations that if the decay rates are smaller then the network takes longer time to return to equilibrium and to process the subsequent inputs. On the other hand larger decay rates make τ_* very small, which according to our view strains the network and may result in an eventual breakdown. Further, one may understand, the vital decay rates, as the decay rates of the membrane potentials. Also the expression (3.34) for τ_* involves apart from decay rates, other network parameters as well. Thus a proper interplay between the processing delays and the various network parameters is essential for the network to have stability.

The next result describes a situation in which, instability at $\tau = 0$ will be preserved for all $\tau > 0$. That is no matter how large the processing delays be, the network continues to have instability if it starts with instability at $\tau = 0$. Thus the inequality (3.7) for maintenance of stability can not be relaxed.

Theorem 3.3 *Assume the conditions* $a^2+2c = 0$, $a^2 = 4b$ *hold. Then for all* $\tau \geq 0$, *the equilibrium is unstable. Further, at* $\tau = \tau_* = \frac{3\pi}{2a}$, *(3.5) has a pair of pure imaginary roots and*

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\nu} = \frac{a^2}{53.2146} > 0.$$

Proof The conditions $a^2+2c = 0$, $a^2 = 4b$ imply that the condition (3.7) is violated. Now, following the lines of argument in Theorem 3.2, we get

$$3a^4T^4 + 56a^2T^2 - 64aT - 16 = 0. \tag{3.35}$$

A positive value of T is given by $\frac{2\sqrt{2}+2}{a}$ and $\nu = \frac{a}{2}$. Now

$$\tau = \frac{2}{\omega}\tan^{-1}(\omega T) = \frac{3\pi}{2a}. \tag{3.36}$$

Now, at $\tau = \tau^* = \frac{3\pi}{2a} = \frac{3\pi}{2(a_i+c_i)}$

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\nu} = \frac{a^2}{53.2146} > 0,$$

which proves the theorem.

Observe that Theorem 3.3 presents conditions under which the instability of the equilibrium pattern would be maintained in the special case, in which $c = \frac{-a^2}{2}$, $b = \frac{a^2}{4}$. Now, our next result is a general result which explains the circumstances under which the network does not change its stability. More specifically, if the network is stable (unstable) in the absence of processing delays, it remains stable(unstable) in the presence of processing delays.

Theorem 3.4 *Assume condition* (H_1) *holds. Then if the equilibrium is stable (unstable) at* $\tau = 0$, *then the equilibrium remains stable (unstable) for all* $\tau > 0$.

Assume condition (H_2) *holds. Then if the equilibrium is unstable for any* $\tau = \tau_* \geq 0$, *then it will be unstable for all* $\tau \geq \tau_*$.

(H_1) (i) $b^2 - c^2 \geq 0$ *or*

(ii) $a^4 - 4a^2b + 4c^2 < 0$ *holds*

(H_2) $b^2 - c^2 < 0$

Proof Here we analyze the question of stability by examining the sign of the derivative of the real part of the eigen values with respect to τ, as the real part crosses 0. That is, we analyze $\frac{d\mu}{d\tau}(\hat{\tau})$, where $\mu(\hat{\tau}) = 0$. If this derivative is positive (negative), then clearly, a stabilization (destabilization) can not take place at that value of $\hat{\tau}$.

We first note that the imaginary part $\hat{\nu} = \nu(\hat{\tau})$ must satisfy the equation

$$\nu^4 + (a^2 - 2b)\nu^2 + b^2 - c^2 = 0. \tag{3.37}$$

We begin with equations (3.9) and (3.10) and differentiate with respect to τ. Then setting $\tau = \hat{\tau}$, $\mu = 0$ and $\nu = \hat{\nu}$ gives us the two equations in $\frac{d\mu}{d\tau}(\hat{\tau})$ and $\frac{d\nu}{d\tau}(\hat{\tau})$:

$$\begin{aligned} \xi \frac{d\mu}{d\tau}(\hat{\tau}) - \eta \frac{d\nu}{d\tau}(\hat{\tau}) &= C, \\ \eta \frac{d\mu}{d\tau}(\hat{\tau}) + \xi \frac{d\nu}{d\tau}(\hat{\tau}) &= D, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} \xi &= a - 2c\hat{\tau}\,\cos 2\hat{\tau}\hat{\nu}, \qquad & \eta &= (2\hat{\nu} + 2c\hat{\tau}\,\sin 2\hat{\tau}\hat{\nu}), \\ C &= 2c\hat{\nu}\,\sin \hat{\tau}\hat{\nu}, & D &= 2c\hat{\nu}\,\cos 2\hat{\tau}\hat{\nu}. \end{aligned} \tag{3.39}$$

Solving (3.38) gives

$$\frac{d\mu}{d\tau} = \frac{\xi C + \eta D}{\xi^2 + \eta^2} \tag{3.40}$$

and clearly, $\frac{d\mu}{d\tau}(\hat{\tau})$ has the same sign as that of $\xi C + \eta D$. From (3.39), after some simplifications we get

$$\xi C + \eta D = 2\nu^2[2\nu^2 + (a^2 - 2b)]. \tag{3.41}$$

Let

$$F(z) = z^2 + (a^2 - 2b)z + (b^2 - c^2) \tag{3.42}$$

[which is the left hand side of (3.37) with $\hat{\nu}^2 = z$]; then $F(\hat{\nu}^2) = 0$ and we note that

$$\frac{dF}{dz}(\hat{\nu}^2) = \frac{\xi^2 + \eta^2}{2\hat{\nu}^2}\frac{d\mu}{d\tau}(\hat{\tau}). \tag{3.43}$$

Hence, we can describe the criteria for preservation of instability (stability) as follows:

(1) If the polynomial $w = F(z)$ has no positive roots, there can be no change of stability.

(2) If $w = F(z)$ is increasing(decreasing) at all of its positive roots, instability (stability) is preserved.

We now proceed to analyze $F(z)$. Since, $a^2 - 2b > 0$ and if $F(0) = b^2 - c^2 < 0$, then by Decarte's rule of signs, $F(z)$ has at most one positive root. If $F(0) = b^2 - c^2 \geq 0$, $F(z)$ has no positive roots. Similarly, when $a^4 - 4a^2b + 4c^2 < 0$, then $F(z)$ has no real roots.

To summarize, we state the following conditions: Condition (H_1) implies that $w = F(z)$ has no positive roots and condition (H_2) implies that $w = F(z)$ has at most one positive root.

Corollary 3.1 *If $b^2 < c^2$, and if the equilibrium is stable at $\tau = 0$, there exists $\hat{\tau} > 0$ for which this equilibrium is unstable for $\tau > \hat{\tau}$.*

We now present a bifurcation result. This theorem gives conditions under which the equilibrium that is asymptotically stable for $0 \le \tau < \tau_*$ bifurcates at $\tau = \tau_*$ into small amplitude periodic solutions.

Theorem 3.5 *Let $b + c > 0$ and $b - c < 0$ are satisfied. Then there exists a τ_*, the smallest value of τ for which the equations (3.9) and (3.10) have a solution such that $\mu = 0$. For $\tau < \tau_*$ the equilibrium is asymptotically stable. For $\tau > \tau_*$ the equilibrium is unstable. Further, as τ increases through τ_* the equilibrium bifurcates into small amplitude periodic solutions.*

Proof Suppose $\mu = 0$, $\nu = \nu_0$ at $\tau = \tau_*$. Then (3.9) and (3.10) yield

$$-\nu_0^2 + b + c\cos 2\tau_0\nu_0 = 0, \tag{3.44}$$

$$a\nu_0 - c\sin 2\tau_0\nu_0 = 0. \tag{3.45}$$

Eliminating τ_* by squaring and adding (3.44) and (3.45)

$$\nu_0^4 + (a^2 - 2b)\nu_0^2 + b^2 - c^2 = 0. \tag{3.46}$$

Since $a^2 - 2b > 0$ only positive root of (3.46) is given when $b - c < 0$. Accordingly,

$$\nu_0 = \pm\frac{\sqrt{2}}{2}\big\{ -(a^2 - 2b) + \{a^4 - 4a^2b + 4c^2\}^{\frac{1}{2}}\big\}^{\frac{1}{2}}. \tag{3.47}$$

We now solve (3.44) and (3.45) for τ_*. Accordingly, we get

$$c^2\cos^2 2\tau_*\nu_0 - a^2c\cos 2\tau_*\nu_0 - (a^2b + c^2) = 0. \tag{3.48}$$

Let $f(z) = c^2z^2 - a^2cz - (a^2b + c^2)$. Clearly, (3.48) has a real solution of the form $\cos 2\tau_*\nu_0 = k$, where $|k| < 1$. From (3.45) this solution in τ_* is of the form

$$\tau_* = \frac{1}{2\nu_0}\sin^{-1}\Big(\frac{a\nu_0}{c}\Big) + \frac{n\pi}{\nu_0}, \qquad n = 0, 1, 2, \dots, \tag{3.49}$$

where the positive value of ν_0 is given by (3.47). Hence the τ_* required by the theorem is obtained by choosing $n = 0$.

Clearly, for $\tau = 0$, the equilibrium is stable. Hence by continuity it remains to be stable for $\tau < \tau_*$. We now show that $\frac{d\mu}{d\tau}\Big|_{\tau=\tau_*} > 0$ when $\nu = \nu_0$ and $n = 0, 1, 2, \dots$. This will imply that there is at least one eigenvalue with positive real part for $\tau > \tau_*$, $n = 0$ and hence the equilibrium is unstable for $\tau > \tau_*$. From (3.44) and (3.45) differentiating with respect to τ and after some simplifications, we get for $\nu = \nu_0$, $\tau = \tau_*$, $\mu = 0$,

$$\frac{d\mu}{d\tau}\Big|_{\tau=\tau_*} = \frac{1}{\Delta}\big[4\nu_0^4 + 2\nu_0^2(a^2 - 2b)\big] > 0.$$

This completes the proof.

4 Stability of Bifurcating Periodic Solutions

In this section, we discuss the stability behaviour of solutions for (1.4) in the neighbourhood of $\tau = \tau_*$ as given in Section 3. We scale the time so as to fix the delay equal to 1 and accordingly set $s = t\tau$, $\tilde{x}_i(s) = x_i(\tau s)$, $\tilde{y}_i(s) = y_i(\tau s)$. Equations (1.4) become after replacing $\tilde{x}_i$ by x_i, $\tilde{y}_i$ by y_i and s by t again,

$$\begin{aligned} x_i'(t) &= \tau\Big[-a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t-1))\Big], \\ y_i'(t) &= \tau\Big[-c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\lambda_j, x_j(t-1))\Big] \end{aligned} \tag{4.1}$$

for $i = 1, 2, \dots, n$.

Since r_i and s_i are eigen values of the matrices b_{ij} and d_{ij} respectively expressing along the amplitudes, the system (4.1) become

$$\begin{aligned} x_i'(t) &= \tau\big[-a_i x_i(t) + r_i f_i(\lambda_i, y_i(t-1))\big], \\ y_i'(t) &= \tau\big[-c_i y_i(t) + s_i g_i(\lambda_i, x_i(t-1))\big] \end{aligned} \tag{4.2}$$

for $i = 1, 2, \dots, n$.

Expanding the above system around the equilibrium using Taylor's series and simplifying we get

$$\begin{aligned} x_i'(t) &= \tau\big[-a_i x_i(t) + r_i\alpha_i y_i(t-1) + r_i\gamma_i y_i^2(t-1) + o(3)\big], \\ y_i'(t) &= \tau\big[-c_i y_i(t) + s_i\beta_i x_i(t-1) + s_i\delta_i x_i^2(t-1) + o(3)\big] \end{aligned} \tag{4.3}$$

for $i = 1, 2, \dots, n$, where $\alpha_i = f'(y^*)$, $\gamma_i = \frac{f''}{2}(y^*)$, $\beta_i = g'(x^*)$, $\delta_i = \frac{g''}{2}(x^*)$ and $o(3)$ denotes the terms of third order and above.

Now we present our theorem following the method suggested in [9] on stability of bifurcating periodic solutions.

Theorem 4.1 *The system (4.2) has a Hopf bifurcation near $\tau = \tau_*$, obtained in Section 3. The direction of bifurcation is determined by*

$$\mu_2 = \frac{-\operatorname{Re} C_1(0)}{\alpha'(0)} \tag{4.4}$$

and the stability of bifurcating periodic solutions is determined by $\beta_2 = 2\operatorname{Re} C_1(0)$, where

$$\alpha'(0) = \operatorname{Re} \frac{d\lambda}{d\tau}\Big|_{\lambda = i\omega_0,\, \tau=\tau_0}$$

and

$$C_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}. \tag{4.5}$$

The quantities in the above expression are given by the following:

$$
\begin{aligned}
\frac{g_{20}}{2} &= \bar{D}\tau_* r_i \gamma_i B^2 q^2(-1) + \bar{D}\bar{C}\tau_* s_i \delta_i q^2(-1),\\
\frac{g_{02}}{2} &= \bar{D}\tau_* r_i \gamma_i \bar{B}^2 \bar{q}^2(-1) + \bar{D}\bar{C}\tau_* s_i \delta_i \bar{q}^2(-1),\\
g_{11} &= 2\bar{D}\tau_* r_i \gamma_i B\bar{B} q(-1)\bar{q}(-1) + 2\tau_* \bar{D}\bar{C} s_i \delta_i q(-1)\bar{q}(-1)\\
\frac{g_{21}}{2} &= \bar{D}\tau_* r_i \gamma_i (W_{20}^{(2)}(-1)\bar{B}q(-1) + 2W_{20}^{(2)}(-1)Bq(-1),\\
&\quad + \bar{D}\bar{C}\tau_* s_i \delta_i (W_{20}^{(1)}(-1)\bar{q}(-1) + 2W_{20}^{(1)}(-1)q(-1)
\end{aligned}
$$

and

$$
\begin{aligned}
W_{20}^{(1)}(\theta) &= \sigma_1\ e^{i\omega_0\theta} + \sigma_2\ e^{-i\omega_0\theta} + \sigma_f\ e^{2i\omega_0\theta},\\
W_{20}^{(2)}(\theta) &= \mu_1\ e^{i\omega_0\theta} + \mu_2\ e^{-i\omega_0\theta} + \mu_f\ e^{2i\omega_0\theta},\\
W_{11}^{(1)}(\theta) &= \rho_1\ e^{i\omega_0\theta} + \rho_2\ e^{-i\omega_0\theta} + \rho_f,\\
W_{11}^{(2)}(\theta) &= \chi_1\ e^{i\omega_0\theta} + \chi_2\ e^{-i\omega_0\theta} + \chi_f
\end{aligned}
$$

in which

$$
\begin{aligned}
\sigma_1 &= \frac{2\tau_*\bar{D}T_1 i}{\omega_0}, \quad \sigma_2 = \frac{2\tau_*\bar{D}T_2 i}{3\omega_0}, \quad \mu_1 = \sigma_1 B, \quad \mu_2 = \sigma_2 \bar{B},\\
\rho_1 &= \frac{-\tau_*\bar{D}T_3 i}{\omega_0}, \quad \rho_2 = \frac{\tau_*\bar{D}T_4 i}{\omega_0}, \quad \chi_1 = \rho_1 B, \quad \chi_2 = \rho_2 \bar{B},\\
\sigma_f &= \frac{C_{20}^{(1)}(2i\omega_0 + \tau_* c_i) + \tau_* r_i \alpha_i C_{20}^{(2)}}{(2i\omega_0 + \tau_* a_i)(2i\omega_0 + \tau_* c_i) - \tau_*^2 r_i s_i \alpha_i \beta_i e^{-2i\omega_0}},\\
\mu_f &= \frac{C_{20}^{(1)} s_i \beta_i\ e^{-2i\omega_0} + C_{20}^{(2)}(2i\omega_0 + \tau_* a_i)}{(2i\omega_0 + \tau_* a_i)(2i\omega_0 + \tau_* c_i) - \tau_*^2 r_i s_i \alpha_i \beta_i e^{-2i\omega_0}},\\
\rho_f &= \frac{r_i \alpha_i C_{11}^{(2)} + C_{11}^{(1)}}{\tau_*(a_i c_i - r_i \alpha_i s_i \beta_i)},\\
\chi_f &= \frac{s_i \beta_i C_{11}^{(1)} + a_i C_{11}^{(2)}}{\tau_*(a_i c_i - r_i \alpha_i s_i \beta_i)}.
\end{aligned}
$$

Further

$$
\begin{aligned}
C_{20}^{(1)} &= H_{20}^{(0)} + \tau_* r_i \alpha_i (\mu_1 + \mu_2) - (2i\omega_0 + \tau_* a_i)(\sigma_1 + \sigma_2),\\
C_{20}^{(2)} &= H_{20}^{(0)} + \tau_* s_i \beta_i (\sigma_1 e^{-i\omega_0}) - (2i\omega_0 + \tau_* c_i)(\mu_1 + \mu_2),\\
C_{11}^{(1)} &= H_{11}^{(0)} - \tau_* a_i (\rho_1 + \rho_2) + \tau_* r_i \alpha_i (\lambda_1 + \lambda_2),\\
C_{11}^{(2)} &= H_{11}^{(0)} - \tau_* c_i (\lambda_1 + \lambda_2) + \tau_* s_i \beta_i (\rho_1\ e^{-i\omega_0} + \rho_2 e^{i\omega_0}),
\end{aligned}
$$

$$H_{20}(\theta) = -2\tau_* \left[\bar{D}T_1 q(\theta) + DT_1 \bar{q}(\theta)\right] + 2\tau_* \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1 < \theta < 0, \\ \begin{bmatrix} r_i\gamma_i q^2(-1)B^2 \\ s_i\delta_i q^2(-1) \end{bmatrix} \theta = 0, \end{cases}$$

$$H_{11}(\theta) = -\tau_* \left[\bar{D}T_3 q(\theta) + DT_4 \bar{q}(\theta)\right] + 2\tau_* \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1 < \theta < 0, \\ \begin{bmatrix} r_i\gamma_i B\bar{B} q(-1)\bar{q}(-1) \\ s_i\delta_i q(-1)\bar{q}(-1) \end{bmatrix} \theta = 0. \end{cases}$$

The expressions T_1, T_2, T_3 and T_4 are given by the following.

$$\begin{aligned} T_1 &= r_i\gamma_i q^2(-1)B^2 + \bar{C}s_i\delta_i q^2(-1), \\ T_2 &= r_i\gamma_i q^2(-1)B^2 + Cs_i\delta_i q^2(-1), \\ T_3 &= 2r_i\gamma_i q(-1)\bar{q}B\bar{B} + 2\bar{C}s_i\delta_i q(-1)\bar{q}(-1), \\ T_4 &= 2r_i\gamma_i q(-1)\bar{q}B\bar{B} + 2Cs_i\delta_i q(-1)\bar{q}(-1). \end{aligned}$$

The terms B, C and D are components in the following

$$q(\theta) = \begin{bmatrix} 1 \\ B \end{bmatrix} e^{i\omega_0\theta}, \quad -1 < \theta \leq 0,$$

$$q^*(\theta) = D \begin{bmatrix} 1 \\ C \end{bmatrix} e^{i\omega_0\theta}, \quad 0 \leq \theta < 1,$$

where

$$B = \frac{-s_i\beta_i\tau_*}{c_i\tau_* + i\omega_0}, \qquad C = \frac{-r_i\alpha_i\tau_*}{c_i\tau_* - i\omega_0},$$

$$\bar{D} = \frac{1}{(1 - e^{-i\omega_0})(1 + \bar{C}B)}.$$

5 Global Stability Results

In this section, we present results dealing with the circumstances under which the equilibrium pattern (x^*, y^*) of (1.4) relative to a given input pair (I, J), is globally asymptotically stable. Our results are analogous to those obtained in [24].

Theorem 5.1 *Assume that the hypotheses (2.2) are satisfied. Then the equilibrium pattern of (1.4) is globally asymptotically stable, provided the following inequalities hold:*

$$\frac{\beta_i \sum_{j=1}^{n} |d_{ji}|}{a_i} \leq 1 \quad \text{and} \quad \frac{\alpha_i \sum_{j=1}^{n} |b_{ji}|}{c_i} \leq 1. \tag{5.1}$$

Proof The following change of variables,

$$u_i(t) = x_i(t) - x_i^*, \qquad v_i(t) = y_i(t) - y_i^*$$

transform the system (1.4) to

$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} b_{ij}\bigl\{f_j(\lambda_j, v_j + y_j^*) - f_j(\lambda_j, y_j^*)\bigr\},$$

$$\dot{v}_i(t) = -c_i v_i(t) + \sum_{j=1}^{n} d_{ij}\bigl\{g_j(\mu_j, u_j + x_j^*) - g_j(\mu_j, x_j^*)\bigr\}$$

for $i = 1, 2, \dots, n$. In view of (2.2) this may be written as

$$\begin{aligned} \dot{u}_i(t) &\le -a_i u_i(t) + \sum_{j=1}^{n} |b_{ij}|\alpha_j |v_j(t-s)|, \\ \dot{v}_i(t) &\le -c_i v_i(t) + \sum_{j=1}^{n} |d_{ij}|\beta_j |u_j(t-s)| \end{aligned} \tag{5.2}$$

for $i = 1, 2, \dots, n$.

Employing the Lyapunov functional,

$$\begin{aligned} V(u(t), v(t)) = \sum_{i=1}^{n} \Biggl[|u_i(t)| + |v_i(t)| &+ \sum_{j=1}^{n} |b_{ij}|\alpha_j \int_{t-\tau}^{t} |v_j(z)|\,dz \\ &+ \sum_{j=1}^{n} |d_{ij}|\beta_j \int_{t-\tau}^{t} |u_j(z)|\,dz \Biggr] \end{aligned} \tag{5.3}$$

and proceeding along the lines of the proof of Theorem 3.1 (see [24]) the remaining proof of this theorem may be completed.

We now present our next result on the global asymptotic stability.

Theorem 5.2 *Assume that the inequalities*

$$a_i > \frac{1}{2}\sum_{j=1}^{n} |b_{ij}|, \qquad c_i > \frac{1}{2}\sum_{j=1}^{n} |d_{ij}| \tag{5.4}$$

for $i = 1, 2, \dots, n$ *are satisfied. Further, assume that there exists constants* $\gamma_i > 0$ *and* $\delta_i > 0$ *satisfying*

$$\frac{\beta_i^2 \sum\limits_{j=1}^{n} |d_{ji}|}{a_i - \frac{1}{2}\sum\limits_{j=1}^{n} |b_{ij}|} \le \frac{\gamma_i}{\delta_i} \le \frac{c_i - \frac{1}{2}\sum\limits_{j=1}^{n} |d_{ij}|}{\alpha_i^2 \sum\limits_{j=1}^{n} |b_{ji}|} \tag{5.5}$$

for $i = 1, 2, \dots, n$.

Then the equilibrium pattern (x^*, y^*) of (1.4) is globally asymptotically stable.

Proof We consider the following functional

$$V(x(t), y(t)) = \sum_{i=1}^{n} \Big[\gamma_i \, \frac{(x_i(t) - x_i^*)^2}{2} + \delta_i \, \frac{(y_i(t) - y_i^*)^2}{2} + \gamma_i \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau}^{t} \big\{ f_j(\lambda_j, y_j(u)) - f_j(\lambda_j, y_j^*) \big\}^2 \, du + \delta_i \sum_{j=1}^{n} |d_{ij}| \int_{t-\tau}^{t} \big\{ g_j(\mu_j, x_j(u)) - g_j(\mu_j, x_j^*) \big\}^2 \, du \Big]. \tag{5.6}$$

It is easy to see that $V(x^*, y^*) = 0$ and $V(x, y) \geq \omega(\|z\|)$, where

$$\omega(\|z\|) = \sum_{i=1}^{n} \frac{\epsilon_i}{2} \big[(x_i - x_i^*)^2 + (y_i - y_i^*)^2 \big]$$

for $z = (x - x^*,\, y - y^*)$ with $\epsilon_i = \min\{\gamma_i, \delta_i\}$. Clearly $\omega(\|z\|)$ is positive definite.

Now, the derivative of V along the solutions of system (1.4) may be written as

$$\begin{aligned} V'(x(t), y(t)) = \sum_{i=1}^{n} \Big\{ \gamma_i \Big[(x_i(t) - x_i^*) \Big(- a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t-\tau)) + I_i \Big) + \sum_{j=1}^{n} |b_{ij}| \big\{ f_j(\lambda_j, y_j(t)) - f_j(\lambda_j, y_j^*) \big\}^2 \\ - \sum_{j=1}^{n} |b_{ij}| \big\{ f_j(\lambda_j, y_j(t-\tau)) - f_j(\lambda_j, y_j^*) \big\}^2 \Big] \\ + \delta_i \Big[(y_i(t) - y_i^*) \Big(- c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j(t-\tau)) + J_i \Big) \\ + \sum_{j=1}^{n} |d_{ij}| \big\{ g_j(\mu_j, x_j(t)) - g_j(\mu_j, x_j^*) \big\}^2 \\ - \sum_{j=1}^{n} |d_{ij}| \big\{ g_j(\mu_j, x_j(t-\tau)) - g_j(\mu_j, x_j^*) \big\}^2 \Big] \Big\}. \end{aligned} \tag{5.7}$$

Proceeding along the lines of argument of Theorem 3.3 (see [24]), we obtain,

$$V'(x, y) < -\Phi(\|z\|),$$

where

$$\Phi(\|z\|) = \sum_{i=1}^{n} k_i \big[(x_i - x_i^*)^2 + (y_i - y_i^*)^2 \big],$$

$k_i = \min\{\xi_i, \eta_i\}$, for $i = 1, 2, \dots, n$. The numbers ξ_i, η_i for $i = 1, 2, \dots, n$ are given by

$$\xi_i = \gamma_i\left(a_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|\right) - \delta_i\beta_i^2\sum_{j=1}^{n}|d_{ji}|$$

and

$$\eta_i = \delta_i\left(c_i - \frac{1}{2}\sum_{j=1}^{n}|d_{ij}|\right) - \gamma_i\alpha_i^2\sum_{j=1}^{n}|b_{ji}|.$$

Clearly, one may see that ξ_i, η_i are non-negative for each $i = 1, 2, \dots, n$ and so is k_i.

Now the conclusion follows from [8].

Theorem 5.3 *Assume that the neuronal gains α_i, β_i, the synaptic connection weights b_{ij}, d_{ij} and the decay rates a_i, c_i of the neuronal fields F_X and F_Y respectively, satisfy the inequality*

$$\sum_{j=1}^{n}(\alpha_j|b_{ij}| + \beta_j|d_{ij}|) < k_i = \min\{a_i, c_i\} \tag{5.8}$$

for $i = 1, 2, \dots, n$.

Then the equilibrium solution (x^*, y^*) of (1.4) is globally asymptotically stable.

Proof For each $i = 1, 2, \dots, n$, define

$$Q_i(t) = \|(x_i(t) - x_i^*,\, y_i(t) - y_i^*)\| = |x_i(t) - x_i^*| + |y_i(t) - y_i^*| \quad \text{for} \quad t \in [-\tau, \infty). \tag{5.9}$$

Then using (1.4), (2.2) we get

$$D^+Q_i(t) \le \Big[-a_i|x_i(t) - x_i^*| + \sum_{j=1}^{n}\alpha_j|b_{ij}|\,|y_j(t-\tau) - y_j^*| - c_i|y_i(t) - y_i^*| + \sum_{j=1}^{n}\beta_j|d_{ij}|\,|x_j(t-\tau) - x_j^*|\Big]$$

which in turn yields,

$$D^+Q_i(t) \le \Big[-k_iQ_i(t) + \sum_{j=1}^{n}(\alpha_j|b_{ij}| + \beta_j|d_{ij}|)Q_j(t-\tau)\Big] \tag{5.10}$$

for each $i = 1, 2, \dots, n$.

Since, $Q_i(t) \le M$ for $t \in [-\tau, 0]$ and for each $i = 1, 2, \dots, n$, we now claim that

$$Q_i(t) \le M = \max_{1\le i\le n}\Big[\sup_{-\tau\le s\le 0}\big(|\phi_i(s) - x_i^*| + |\psi_i(s) - y_i^*|\big)\Big] \tag{5.11}$$

for $t \ge 0$.

If the inequality (5.11) does not hold for all $t \geq 0$, then there must exist a $t_1 > 0$ and some i such that

$$Q_i(t_1) = M, \quad Q_j(t) \begin{cases} < M, & \text{for} \quad i = j, \ -\tau \leq t < t_1, \\ \leq M, & \text{for} \quad i \neq j, \ -\tau \leq t \leq t_1. \end{cases}$$

It is easy to see that

$$D^+ Q_i(t_1) \geq 0. \tag{5.12}$$

But, from (5.8) and (5.10), we have

$$\begin{aligned} D^+ Q_i(t_1) &\leq \Big[- k_i M + \sum_{j=1}^{n} \big(\alpha_j |b_{ij}| + \beta_j |d_{ij}| \big) M \Big] \\ &= - \Big[k_i - \sum_{j=1}^{n} \big(\alpha_j |b_{ij}| + \beta_j |d_{ij}| \big) \Big] M < 0 \end{aligned}$$

which contradicts (5.12) and thus (5.11) holds for all $t \geq 0$.

Now, for each $i = 1, 2, \ldots, n$, let

$$\lim_{t \to \infty} \sup Q_i(t) = \bar{\sigma}_i \quad \text{and} \quad \lim_{t \to \infty} \inf Q_i(t) = \underline{\sigma_i}.$$

Clearly $0 \leq \underline{\sigma_i} \leq \bar{\sigma}_i < \infty$, for $i = 1, 2, \ldots, n$. Without loss of generality, assume that $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_n$. We shall prove that $\bar{\sigma}_1 = 0$. Suppose that $\bar{\sigma}_1 > 0$.

Now, choose $\epsilon > 0$ in such a way that the inequality

$$0 < \epsilon \leq \frac{k_1 - \sum\limits_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)}{2[k_1 + \sum\limits_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)(1 + M)]} \bar{\sigma}_1 \tag{5.13}$$

is satisfied.

Since $\lim\limits_{t \to \infty} \sup Q_i(t) = \bar{\sigma}_i$ by definition (for this ϵ) there exists a $t_2 > 0$ such that for $t \geq t_2$, we have

$$Q_i(t - \tau) \leq \bar{\sigma}_i + \epsilon \leq \bar{\sigma}_1 + \epsilon,$$

for $i = 1, 2, \ldots, n$ and $\tau > 0$.

Then, from (5.10), for $t \geq t_2$, it follows that

$$D^+ Q_1(t) \leq \Big[- k_1 Q_1(t) + \sum_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)(\bar{\sigma}_1 + \epsilon) \Big]. \tag{5.14}$$

We first prove $\bar{\sigma}_1 = \underline{\sigma}_1$. If $\bar{\sigma}_1 > \underline{\sigma}_1$, then there are infinite number of intervals on which $Q_1(t)$ is non decreasing. We can choose $t_4 > t_3 \geq t_2$ such that $Q_1(t)$ is non decreasing on (t_3, t_4) and

$$Q_1(t) > \bar{\sigma}_1 - \epsilon \quad \text{for} \quad t \in (t_3, t_4).$$

From (5.14), for $t \in (t_3, t_4)$, we have

$$
\begin{aligned}
D^+Q_1(t) &\le -\left[k_1(\bar\sigma_1 - \epsilon) - (\bar\sigma_1 + \epsilon)\sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right] \\
&= -\left[\left\{k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right\}\bar\sigma_1 \right. \\
&\quad \left. - \left\{k_1 + \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|\right\}\epsilon\right]
\end{aligned}
$$

and using (5.13) one can see that

$$
D^+Q_1(t) \le -\frac{\bar\sigma_1}{2}\left[k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right] < 0 \tag{5.15}
$$

which is a contradiction to the statement that $Q_1(t)$ is non-decreasing over (t_3, t_4). Accordingly we must have $\bar\sigma_1 = \underline{\sigma}_1 = \sigma$ (say).

Since $\bar\sigma_1 = \underline{\sigma}_1 = \sigma > 0$, there must exist a $t_5 \ge t_2$ such that for $t \ge t_5$ we have

$$
\sigma - \epsilon < Q_1(t) < \sigma + \epsilon
$$

and

$$
Q_i(t) \le \sigma + \epsilon \quad \text{for} \quad i = 2, \dots, n.
$$

For $t \ge t_5$, from (5.15) we have

$$
0 \le Q_1(t) \le Q_1(t_5) - \frac{\sigma}{2}\left[k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right](t - t_5)
$$

which is a contradiction. Hence $\sigma = 0$ and thus

$$
\lim_{t\to\infty} \|(x_i(t) - x_i^*,\, y_i(t) - y_i^*)\| = 0
$$

for $i = 1, 2, \dots, n$, implying that (x^*, y^*) is globally asymptotically stable.

6 Examples

In this section, we present several examples illustrating our results. Further, we establish that the various stability criteria are independent.

Example 6.1 [24] Consider the network described by the system

$$
\begin{aligned}
\dot x_i &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(\lambda_j, y_j(t-\tau)) + I_i, \\
\dot y_i &= -c_i y_i(t) + \sum_{j=1}^{2} d_{ij} g_j(\mu_j, x_j(t-\tau)) + J_i.
\end{aligned} \tag{6.1}
$$

where $i = 1, 2$.

Now choose

$$a_1 = 3, \quad a_2 = 3, \quad c_1 = 3, \quad c_2 = 4,$$
$$\lambda_1 = 5/6, \quad \lambda_2 = 4/5, \quad \mu_1 = 1/2, \quad \mu_2 = 3/4,$$
$$I_1 = -1/2, \quad I_2 = 1, \quad J_1 = 1/4, \quad J_2 = -2,$$
$$[b_{ij}] = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}.$$

Further, choose the signal functions f_i and g_i for $i = 1, 2$ as follows:

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tanh(\lambda_1 y_1) \\ \tanh(\lambda_2 y_2) \end{pmatrix},$$
$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tanh(\mu_1 x_1) \\ \tanh(\mu_2 x_2) \end{pmatrix}.$$

Observe that all the hypotheses of Theorem 5.1 are satisfied while the hypotheses (5.5) of Theorem 5.2 and (5.8) of Theorem 5.3 are violated. From this it is clear that the equilibrium pattern is globally asymptotically stable by virtue of Theorem 5.1.

Example 6.2 [24] Consider the system (6.1) in which

$$a_1 = 3/2, \quad a_2 = 6, \quad c_1 = 7, \quad c_2 = 2.3,$$
$$\lambda_1 = 8, \quad \lambda_2 = 16, \quad \mu_1 = 11/6, \quad \mu_2 = 3/2,$$
$$I_1 = 3, \quad I_2 = 1, \quad J_1 = 2, \quad J_2 = 4,$$
$$[b_{ij}] = \begin{bmatrix} 1/5 & 1/3 \\ 1/2 & 1/4 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} -1/2 & 1/4 \\ -1/3 & -1/5 \end{bmatrix}$$

and the signal functions are given by

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+e^{-\lambda_1 y_1}} \\ \frac{1}{1+e^{-\lambda_2 y_2}} \end{pmatrix},$$
$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tanh(\mu_1 x_1) \\ \tanh(\mu_2 x_2) \end{pmatrix}$$

for $i = 1, 2$.

Notice that all conditions of Theorem 5.2 are satisfied while some of the inequalities (5.1) of Theorem 5.1 and inequalities (5.8) of Theorem 5.3 are violated.

Example 6.3 [24] Again consider the system (6.1) and choose

$$a_1 = 1.4, \quad a_2 = 1.3, \quad c_1 = 1.25, \quad c_2 = 1.35,$$
$$\lambda_1 = 1/3, \quad \lambda_2 = 1/9, \quad \mu_1 = 1/2, \quad \mu_2 = 1,$$
$$I_1 = 1/2, \quad I_2 = 1/4, \quad J_1 = -1/3, \quad J_2 = 1/5,$$

and

$$[b_{ij}] = \begin{bmatrix} 5/2 & 1/2 \\ 3/2 & 1/2 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 5/2 \end{bmatrix}$$

and the signal functions are

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tanh(\lambda_1 y_1) \\ \tanh(\lambda_2 y_2) \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+e^{-\mu_1 x_1}} \\ \frac{1}{1+e^{-\mu_2 x_2}} \end{pmatrix}.$$

It follows easily that all the requirements of Theorem 5.3 are satisfied while some of the inequalities of the hypotheses (5.1) of Theorem 5.1 and those of (5.4) of Theorem 5.2 are not satisfied.

Remark 6.1 Note that from Examples 6.1, 6.2 and 6.3, it follows that the global asymptotic stability criteria guaranteed by Theorems 5.1, 5.2 and 5.3 are independent.

7 Discussion

In this paper, a model describing the activation dynamics of neurons in a bidirectional associative memory network involving processing delays has been presented. The importance of the necessity of introducing processing delays in the model has been highlighted. Results on maintenance and preservation of stability together with circumstances leading to instability have been presented. We have established that a proper interplay between the processing delays and the various system parameters is highly essential to have global stability behaviour of the network.

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Asymptotic Methods for Stability Analysis of Markov Impulse Dynamical Systems

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Abstract: The paper deals with n-dimensional dynamical system of impulse type whose dynamical characteristics are dependent on the step Markov process with rapid switchings. The phase motion has small jumps at the moments of switchings and satisfies the ordinary differential equation in the intervals of constancy of the Markov process. The intensity of switchings, the quantities of jumps and the vector field of the differential equation are dependent on the phase coordinates and Markov process. Under some assumptions the limit averaged ordinary differential equation, the limit differential equation switched by the merged Markov process, the diffusion approximation and the limit stochastic differential equation of Ornstein-Uhlenbeck type for normalized deviations are constructed. It is proved that one can use the limit equations for stability analysis of an initial impulse dynamical system.

Keywords: *Stability analysis; impulsive dynamical systems; Markov process.*

Mathematics Subject Classification (2000): 34D20, 34A37, 34F05.

1 Introduction

The problem of asymptotic analysis of dynamical systems under small random perturbations has been discussed in many mathematical and engineering papers. Apparently, R.Z. Khasminsky was the first mathematician to have proved that the probabilistic limit theorems may be successfully used for differential equations with random right parts. The approach proposed in [12] makes it possible to apply for asymptotic analysis of real stochastic structural dynamical systems not only the Krylov-Bogolyubov averaging procedure but also diffusion approximation (see, for example, [6] and review there). It should be mentioned that in spite of the fact that the above result has been developed in [12] for

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the analysis of differential equations on a finite time interval, the diffusion approximation procedure has been applied in many engineering papers for Lyapunov stability analysis, that is, for analysis of differential equations as $t \to \infty$. To prove the validity of this approach the authors of papers [3, 5, 14, 15] had to use not only a special type of limit theorem in Skorokhod space [16] but also martingale techniques and a stochastic version of the Second Lyapunov method developed for stochastic Ito differential equations in [13]. These asymptotic methods of stochastic stability analysis have been applied in the above-mentioned papers to differential equations with continuous trajectories. But some dynamical systems of the recent Economics (see, for example, [1, 4, 9, 10] and review there) require an extension of "smooth" models to allow the phase motion to have a jump type discontinuity. A possible approach to this problem developed in [11, 17−19, 21] is discussed in the present paper.

To formulate the problems one needs first of all to describe the switching step process $\{y(t),\, t \geq 0\}$ with values in the set $\mathbf{Y}$. We suppose for simplicity that $\mathbf{Y}$ is discrete at most countable space but all our results easily can be reformulated for any metric topological space. We will assume that the above switching process is a right continuous homogeneous Markov process [8] with a weak infinitesimal operator defined by the equality

$$Qv(y) := a(y) \sum_{z \in \mathbf{Y}} [v(z) - v(y)] p(y,\, z)$$

for any bounded mapping $v\colon \mathbf{Y} \to \mathbb{R}$, where $p(y, z)$ is the transition probability of the embedded Markov chain and $a(y)$ is the intensity of switchings which satisfies the inequality $0 < \hat{a}_1 \leq a(y) \leq \hat{a}_2 < \infty$ for any $y \in \mathbf{Y}$. It is well known [8, 16] that the above $\{y(t)\}$ is a piecewise constant process with the switching moments $\{\tau_j,\, j \in \mathbb{N}\}$ which have the conditional exponential distributions defined by the equalities: $\tau_0 = 0$,

$$\mathbb{P}\left\{\tau_j - \tau_{j-1} > t \mid y(\tau_{j-1}) = y\right\} = \exp\{-a(y)t\}, \qquad j \in \mathbb{N}.$$

Now one can describe the Impulse Dynamical System (IDS) in $\mathbb{R}^n$ with small parameter $\varepsilon \in (0, 1)$ this paper deals with. The phase motion $x(t)$ of this system satisfies:

– *the initial condition*

$$x(0) = x; \tag{1}$$

– *the differential equation*

$$\frac{dx}{dt} = \varepsilon\, f(x,\, y(t), \varepsilon) \tag{2}$$

for all $t \in (\tau_{j-1}, \tau_j)$, $j \in \mathbb{N}$;

– *the condition of jump*

$$x(t) = x(t-0) + \varepsilon g(x(t-0),\, y(t-0), \varepsilon) \tag{3}$$

for all $t \in \{\tau_j,\ j \in \mathbb{N}\}$, where

$$f(x,\, y, \varepsilon) = f_1(x,\, y) + \varepsilon\, f_2(x,\, y), \qquad g(x,\, y, \varepsilon) = g_1(x,\, y) + \varepsilon\, g_2(x,\, y) \tag{4}$$

and $f_j(x, y)$, $g_j(x, y)$, $j = 1, 2$ are twice boundedly continuously differentiable on x functions.

Under the above assumptions it is easy to prove [21] that *the pair* $\{x(t),\, y(t)\}$ *is a homogeneous Markov process with the weak infinitesimal operator*

$$(\mathcal{L})v(x,\, y) := \varepsilon(f(x,\, y,\, \varepsilon),\, \bigtriangledown)\, v(x,\, y) + Qv(x,\, y) + \varepsilon G^{\varepsilon} v(x,\, y), \tag{5}$$

where

$$G^{\varepsilon} v(x,\, y) = \frac{a(y)}{\varepsilon} \sum_{z \in \mathbf{Y}} [v(x + \varepsilon g(x,\, y, \varepsilon),\, z) - v(x,\, z)] p(y,\, z), \tag{6}$$

$(.,.)$ is scalar product and $\bigtriangledown$ is an operator-gradient in $\mathbb{R}^n$.

In this paper we will discuss the problem of asymptotic analysis of the IDS (2) – (3) for sufficiently small positive ε. Under the condition of ergodicity of the Markov process $\{y(t)\}$ with limit distribution $\{\mu(y),\, y \in \mathbf{Y}\}$ we shall do this starting with the limit *averaged ordinary differential equation* for (2) – (3)

$$\frac{d\bar{x}}{dt} = \bar{F}_1\,(\bar{x}), \tag{7}$$

where

$$\bar{F}_1(x) := \sum_{y \in \mathbf{Y}} F_1(x,\, y)\mu(y), \qquad F_1(x,\, y) := f_1(x,y) + a(y) g_1(x,y). \tag{8}$$

It will be proven that this deterministic approximation may be successfully used not only on a finite interval but also for asymptotic stability analysis of the initial system. If $F_1(x) \equiv 0$ one will then be able to do the next step in asymptotic analysis of (2) – (3) using the limit theorem in Skorokhod space [16]. This approach leads us in the second section to the limit Ito stochastic differential equation which also can be successfully used for stability analysis of (2) – (3). The third section contains a derivation of a merger procedure and stability theorem based on the merged differential equation.

A word should be said about tools. To prove the limit theorems for (2) – (3) the methods and results of paper [2] can be successfully applied. But in the above paper the author uses specially constructed recurrent equations in the moments of switchings and does not use any infinitesimal characteristics of the Markov process $\{x(t),\, y(t)\}$. This approach is poorly consistent with the Second Lyapunov method, which is mainly used for stability analysis of stochastic dynamical systems [3, 5, 6, 13, 14] and is the main tool of our paper. To prove the classical averaging or merger theorems, unlike the martingale approach of [14], this paper applies the Lyapunov method with specially constructed Lyapunov functions, reflecting the distance between corresponding solutions of the system (2) – (3) and averaged or merged differential equations.

2 Averaging and Stability

Let us assume that the spectrum $\sigma(Q)$ of the weak infinitesimal operator Q has the simple spectrum point 0, $\sigma(Q) \setminus \{0\} \subset \{z \in \mathbf{C} : \Re z < -\rho < 0\}$ and let the distribution $\{\mu(y)\}$ be the solution of the equation $Q^* \mu = 0$, where Q^* is a conjugate operator. Under these conditions one can extend [8] the potential of the above Markov process and define the linear continuous operator $\Pi\colon \mathbb{B}\,(\mathbf{Y}) \to \mathbb{B}\,(\mathbf{Y})$ by equality

$$(\Pi v)(y) := \int_0^{\infty} \sum_{z \in \mathbf{Y}} v(z)[P(t, y, z) - \mu(z)]\, dt, \tag{9}$$

where $\mathbb{B}(\mathbf{Y})$ is the space of bounded mappings $\{v(y),\, y \in \mathbf{Y}\}$ of $\mathbf{Y}$ to $\mathbb{R}$ and $P(t, y, z)$ is the transition probability.

It is easy to prove that any considerable variations of any solution of (2)–(3) can happen only on a sufficiently large time interval of order ε^{-1}. Therefore it is convenient to pass to the slow time $s = \varepsilon t$ and to analyze the process with rapid switchings defined by the equality $x^\varepsilon(s) := x(s/\varepsilon)$.

Theorem 2.1 (Averaging principle) *Under the above assumptions the processes* $\{x^\varepsilon(s)\}$ *for any* $r > 0$, $T > 0$ *uniformly on* $y \in \mathbf{Y}$, $x \in U_r := \{|x| \le r\}$, $t \in [0, T]$ *converge on probability as* $\varepsilon \to 0$ *to the solution of (7) with initial condition* $\bar{x}(0) = x$, *that is, for any* $\delta > 0$

$$\lim_{\varepsilon \to 0} \sup_{y, |x| < r} \mathbb{P}_{x,y}\Big(\sup_{0 \le t \le T} |x^\varepsilon(t) - \bar{x}(t, x)| > \delta \Big) = 0.$$

Proof Under the assumptions of twice continuously boundedly differentiability on x of the functions $f_1(x, y)$ and $g_1(x, y)$, the function $\bar{F}_1(x)$ from (8) also has the continuous bounded derivative $D\bar{F}_1(x)$ and therefore the Cauchy problem $\bar{x}(0) = \bar{x}$ for (7) has a unique solution $\bar{x}(s, \bar{x})$ for any $\bar{x} \in \mathbb{R}^n$. It is easy to prove that the joined process $\{x^\varepsilon(s),\, y(s/\varepsilon),\, \bar{x}(s)\}$ one can consider as the Markov process with the weak infinitesimal operator [8]

$$\mathbb{L}(\varepsilon) := (f(x,\, y,\, \varepsilon),\, \bigtriangledown^{(x)}) + (\bar{F}_1(\bar{x}),\, \bigtriangledown^{(\bar{x})}) + \frac{1}{\varepsilon}\, Q + G^\varepsilon,$$

where the gradients are acting by indicated indices. Let us choose constant c so large that for all $\varepsilon \in (0, 1)$ and phase variables x, y, $\bar{x}$ the function

$$v_\varepsilon(x,\, y,\, \bar{x}) := |x - \bar{x}|^2 + \varepsilon[2(x - \bar{x},\, (\Pi F_1)(x,\, y)) + c\,(1 + |x|^2 + |\bar{x}|^2)]$$

satisfies the inequalities

$$|x - \bar{x}|^2 + \varepsilon\,(1 + |x|^2 + |\bar{x}|^2) \le v_\varepsilon(x,\, y,\, \bar{x}) \le |x - \bar{x}|^2 + \varepsilon\, c_1\,(1 + |x|^2 + |\bar{x}|^2)$$

with some positive constant c_1. Applying the equality

$$Q\,(\Pi F_1)(x,\, y) = -\,F_1(x,\, y) + \bar{F}_1(x) \tag{10}$$

and well-known Dynkin's formula [8]

$$\mathbb{E}_{x,y}\, v_\varepsilon(x^\varepsilon(t),\, y(t/\varepsilon),\, \bar{x}(t, \bar{x})) = v_\varepsilon(x,\, y,\, \bar{x}) + \int_0^t \mathbb{E}_{x,y}\, \mathbb{L}(\varepsilon)\, v_\varepsilon(x^\varepsilon(s),\, y(s/\varepsilon),\, \bar{x}(s, \bar{x}))\, ds$$

one can obtain the inequality $\mathbb{L}(\varepsilon) v_\varepsilon(x,\, y,\, \bar{x}) \le k\, v_\varepsilon(x,\, y,\, \bar{x})$ which guarantees the stochastic process

$$\zeta(t, x, y, \bar{x}) := v_\varepsilon(x^\varepsilon(t),\, y(t/\varepsilon),\, \bar{x}(t, \bar{x}))\, \exp\{-kt\}$$

be supermartingale [7, 13]. To complete the proof one can use the supermartingale properties and to write the inequalities

$$\mathbb{P}_{x,y}\Big(\sup_{0\le t\le T}|x^\varepsilon(t)-\bar{x}(t,x)|\ge\delta\Big)\le\mathbb{P}\Big(\sup_{0\le t\le T}\zeta(t,\,x,\,y,\,x)\ge\delta^2\,e^{-kT}\Big)$$
$$\le\delta^{-2}\,e^{kT}v_\varepsilon(x,\,y,\,x)\le\varepsilon\,c_1\,\delta^{-2}\,e^{kT}\,(1+2|x|^2)$$

for any $\delta>0,\ T>0$.

Let now $f(0,y,\varepsilon)\equiv g(0,y,\varepsilon)\equiv 0$. Then also $\bar{F}_1(0)=0$ and both systems (2) – (3) and (7) have the trivial solution. We will say that the trivial solution of (7) is exponentially stable if there exist positive constants $M,\ \gamma$ such that $|\bar{x}(t,\bar{x})|\le M|\bar{x}|\,\exp\{-\gamma t\}$ for any $t\ge 0$ and $\bar{x}\in\mathbb{R}^n$. For the IDS (2) – (3) we will use the following two definitions of stability [13]:

1) the trivial solution of (2) – (3) is called *asymptotically stochastic stable* if for any $\eta>0$ there exists a δ-neighborhood $B_\delta:=\{|x|<\delta\}$ such that any motion starting within B_δ remains within an η-neighborhood with probability not less than $1-\eta$ and tends to zero as $t\to\infty$;
2) the trivial solution of (2) – (3) is called *exponentially p-stable*, if there exist positive numbers K and β such that the inequality $\mathbb{E}|x(t)|^p\le K\,|x|^p\times\exp\{-\beta t\}$ is satisfied for all $t\ge 0$ and initial conditions $x\in\mathbb{R}^n,\ y\in\mathbf{Y}$.

Theorem 2.2 *Under the above assumptions if the trivial solution of (7) is exponentially stable then for any $p>0$ there exists $\varepsilon_p>0$ such that the trivial solution of IDS (2) – (3) is exponentially p-stable for any $\varepsilon\in(0,\varepsilon_p)$.*

Proof Owing to exponential decrease of the solutions of (7) and the boundedness of the derivative of $\bar{F}_1(x)$ one can define the Lyapunov function

$$v^{(p)}(x):=\int\limits_0^T|\bar{x}(t,x)|^p\,dt,$$

where $T=\frac{\ln M+\ln p}{\gamma}$ and the constants $M,\ \gamma$ are taken from the above definition of exponential stability. It is easy to verify that this function satisfies the inequalities

$$m_1\,|x|^p\le v^{(p)}(x)\le m_2\,|x|^p \tag{11}$$

with some positive constants m_1,m_2. By definition of the gradient and due to exponential stability of (7) one can write the inequalities

$$(\bar{F}_1(x),\,\triangledown)\,v^{(p)}(x)=|\bar{x}(T,\,x)|^p-|x|^p\le-\frac{1}{2}\,|x|^p\le-\frac{1}{2m_2}\,v^{(p)}(x) \tag{12}$$

for any $x\in\mathbb{R}^n$. To prove the theorem we will use the Lyapunov function

$$v_\varepsilon^{(p)}(x,y):=v^{(p)}(x)+\varepsilon((\Pi F_1)(x,y),\,\triangledown)\,v^{(p)}(x).$$

By definition (9) and due to equality

$$Q((\Pi F_1)(x,\,y),\,\triangledown)\,v^{(p)}(x)+(F_1(x,y),\,\triangledown)\,v^{(p)}(x)=(\bar{F}_1(x),\,\triangledown)\,v^{(p)}(x)$$

and inequalities (11) – (12) one can choose such a constant $\varepsilon_p > 0$ that the above Lyapunov function satisfies the inequalities

$$\hat{m}_1\,|x|^p \le v_\varepsilon^{(p)}(x,y) \le \hat{m}_2\,|x|^p, \qquad \mathbb{L}(\varepsilon)\,v_\varepsilon^{(p)}(x,y) \le -\frac{1}{4m_2}\,v_\varepsilon^{(p)}(x,y)$$

with some positive constants $\hat{m}_1$, $\hat{m}_2$ for any $\varepsilon \in (0, \varepsilon_p)$. Using Dynkin's formula for the stochastic process

$$\xi(s) := v_\varepsilon^{(p)}(x^\varepsilon(s),\, y(s/\varepsilon))\, e^{\frac{1}{4m_2}\,s}$$

one can get the inequalities

$$\hat{m}_1 e^{\frac{1}{4m_2}\,s}\,\mathbb{E}_{x,y}\,|x^\varepsilon(s)|^p \le v_\varepsilon^{(p)}(x,\,y) \le \hat{m}_2\,|x|^p,$$

for any $s \ge 0$ and the proof is complete.

By using the supermartingale property of the above defined stochastic process $\xi(s)$ one can make sure that *under the conditions of the Theorem 2.2 the trivial solution of the IDS (2) – (3) is asymptotically stochastic stable for all sufficiently small positive* ε.

3 Diffusion Approximation and Stability

In this Section we will assume that in addition to the condition of ergodicity of the Markov process and twice bounded differentiability of the functions (4) on x the average function satisfies the condition $\bar{F}_1(x) \equiv 0$. Thus, any solution of the averaged equation (7) is constant and we have no information on the behavior of the solutions of the IDS (2) – (3). Then we can go to the "very slow" time $\theta = \varepsilon s = \varepsilon^2 t$, where t is the initial time of the IDS (2) – (3). Let us denote $x_\varepsilon(\theta) := x(\theta/\varepsilon^2)$. The infinitesimal operator of the Markov process $\{x_\varepsilon(\theta),\, y(\theta/\varepsilon^2)\}$ has the form

$$\mathcal{L}_\varepsilon := \frac{1}{\varepsilon}(f(x,\,y,\,\varepsilon),\,\bigtriangledown) + \frac{1}{\varepsilon^2}\,Q + \frac{1}{\varepsilon}\,G^\varepsilon. \tag{13}$$

In spite of the fact that the operator (13) has a singular type as $\varepsilon \to 0$ under the above condition one can prove the following assertions.

Lemma 3.1 [21] *For any positive p there exist positive constants c_p, γ_p, ε_p such that*

$$\mathbb{E}_{x,y}\,|x_\varepsilon(\theta)|^p \le c_p\,(1+|x|)^p\,e^{\gamma_p\theta}$$

for all $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$, $\varepsilon \in (0, \varepsilon_p)$, $\theta > 0$.

Corollary 3.1 [21] *For any $T > 0$, $r > 0$ there exists $\varepsilon_T > 0$ such that*

$$\lim_{\rho\to\infty}\ \sup_{0\le\varepsilon\le\varepsilon_T}\ \mathbb{P}_{x,y}\Big(\sup_{0\le\theta\le T}\,|x_\varepsilon(\theta)| \ge \rho\Big) = 0$$

uniformly on $y \in \mathbf{Y}$ and $x \in U_r$.

The family of the stochastic processes $\{\{x_\varepsilon(\theta),\, 0 \le \theta \le T\},\, \varepsilon \in (0,\, \varepsilon_0)\}$ with initial condition $x_\varepsilon(0) = x$ we will consider as the family of random variables in Skorokhod space [16] $D([0,\, T],\, \mathbb{R}^n)$. The probability measures corresponding to these random variables we will denote $\mathbb{P}^\varepsilon$. Owing to Corollary 3.1 we may confirm that for any natural m and any moments of time $\theta_m > \theta_{m-1} > \dots > \theta_1 \geq 0$ the distribution family of the random vectors $\{x_\varepsilon(\theta_1),\, x_\varepsilon(\theta_2),\, \dots,\, x_\varepsilon(\theta_m)\}$ is weak compact. That is, the family $\{\mathbb{P}^\varepsilon\}$ is relatively compact (as $\varepsilon \to 0$) in the meaning of the weak convergence of finite-dimensional distributions. We will prove that there exist the weak limit of the family $\{\mathbb{P}^\varepsilon\}$ as $\varepsilon \to 0$. To describe the limit process let us introduce the vector

$$b(x) := \sum_{y \in \mathbf{Y}} [f_2(x,\, y) + a(y)\, g_2(x,\, y)]\mu(y) + \sum_{y \in \mathbf{Y}} [\Pi\, DF_1(x,\, y)]\, F_1(x,\, y)\, \mu(y) - \sum_{y \in \mathbf{Y}} [DF_1(x,\, y)]\, g_1(x,\, y)\, \mu(y)$$

and the positive symmetrical matrix $\sigma(x)$ defined by the equality

$$(\sigma(x)\, z,\, z) = 2 \sum_{y \in \mathbf{Y}} \big[(F_1(x,\, y),\, z)\, (\Pi\, F_1(x,\, y),\, z) - (g_1(x,\, y),\, z)\, \big(f_1(x,\, y) + \frac{1}{2}\, a(y)\, g_1(x,\, y),\, z\big)\big]\, \mu(y)$$

with an arbitrary vector $z \in \mathbb{R}^n$.

Theorem 3.1 (Diffusion approximation) *Under the above assumptions the family* $\{\mathbb{P}^\varepsilon\}$ *weak converges as* $\varepsilon \to 0$ *to the diffusion Markov process with weak infinitesimal operator*

$$\mathbb{L}_0 := (b(x),\, \triangledown) + \frac{1}{2}\, (\sigma(x)\, \triangledown,\, \triangledown). \tag{14}$$

Proof To prove this theorem it is sufficient to verify [16] that for any twice continuously differentiable function $v(x)$ with bounded support the equality

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{0<h<\varepsilon} \left| \mathbb{E}_{x,y} \left\{ v(x_\varepsilon(s+h)) - v(x_\varepsilon(s)) - \int_s^{s+h} \mathbb{L}_0\, v(x_\varepsilon(\tau))\, d\tau \right\} \right| = 0$$

can be written for all $0 \leq s < T$, $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$. This equality one can get using Dynkin's formula for $\mathbb{E}_{x,y}\, w(x_\varepsilon(s),\, y(s/\varepsilon^2),\, \varepsilon)$, where

$$w(x, y, \varepsilon) = v(x) + \varepsilon\, (\Pi\, F_1(x, y),\, \triangledown)\, v(x) + \varepsilon^2\, u(x, y),$$

and $u(x,\, y)$ is a solution of the equation

$$Q\, u(x,\, y) = -\Big\{ (f_2(x,\, y) + a(y)\, g_2(x,\, y) + [\Pi\, D\, F_1(x,\, y)]\, F_1(x,\, y) - [D\, F_1(x,\, y)]\, g_1(x,\, y) - b(x),\, \triangledown)\, v(x) + ([D \triangledown v(x)]\, F_1(x,\, y),\, \Pi\, F_1(x,\, y)) - \Big([D \triangledown v(x)]\, g_1(x,\, y),\, f_1(x,\, y) + \frac{1}{2}\, a(y)\, g_1(x,\, y) \Big) - \frac{1}{2}\, (\sigma(x) \triangledown,\, \triangledown)\, v(x) \Big\}.$$

Owing to the Fredholm alternative [11] and by construction the vector $b(x)$ and the matrix $\sigma(x)$ the above equation has solution and the proof is complete.

There exists [8] a Markov process $X(t)$ with infinitesimal operator (14) which satisfies the stochastic Ito differential equation

$$dX = b(X)\,dt + \sum_{k=1}^{n} \sigma_k(X)\,d\,w_k(t), \tag{15}$$

where $w_k(t),\ k = 1, 2, \dots, n$ are the coordinates of the standard Wiener process in $\mathbb{R}^n$ and the matrices $\sigma_k(X),\ k = 1, 2, \dots, n$ are defined such that the process $X(t)$ has the infinitesimal operator (14). Equation (15) is called [2] *the diffusion approximation of the process* $\{x_\varepsilon(t)\}$.

If $\bar{F}_1(x)$ is not identically equal to zero one can use the diffusion approximation for *the normalized deviations* $\xi_\varepsilon(t) := [x(t/\varepsilon) - \bar{x}(t)]/\sqrt{\varepsilon}$ as $\varepsilon \to 0$ applying Theorem 3.1 to the $2n$ dimensional process $\{\xi_\varepsilon(t),\, \bar{x}(t)\}$ with small parameter $\sqrt{\varepsilon}$.

Theorem 3.2 [19, 21] *Under the assumptions of this Section the probability measures* $\{\hat{\mathbb{P}}^\varepsilon\}$ *corresponding to the normalized deviations* $\{\xi_\varepsilon(t),\, 0 \le t \le T\}$ *weak converge as* $\varepsilon \to 0$ *to the measure* $\hat{\mathbb{P}}$ *corresponding to the solution* $\{\hat{X}(t),\, 0 \le t \le T\}$ *of the stochastic Ito equation*

$$d\hat{X} = D\bar{F}_1(\bar{x}(t))\hat{X}\,dt + \sum_{k=1}^{n} \sigma_k(\bar{x}(t))\,dw_k(t) \tag{16}$$

with initial condition $\hat{X}(0) = 0$*, where* $\bar{x}(t)$ *is the solution of (7) with the initial condition* $\bar{x}(0) = x$*.*

The diffusion approximation (15) in just the same way as for the Markov dynamical systems without jumps [3, 5] can be successfully used for stability analysis of the IDS (2) – (3).

Theorem 3.3 *Under the assumptions of this Section if the trivial solution of (15) is exponentially* p*-stable then the trivial solution of the IDS (2) – (3) is also exponentially* p*-stable for all sufficiently small* ε*.*

Proof It is shown in [13] that trivial solution of equation (12) is exponentially p-stable if and only if there exists such a sufficiently smooth Lyapunov function $V(x)$ that

$$h_1|x|^p \le V(x) \le h_2|x|^p, \quad \mathbb{L}_0 V(x) \le -h_3|x|^p, \quad \|D^l \bigtriangledown V(x)\| \le h_4|x|^{p-l-1}$$

for any $x \in R^n,\ l = 1, 2, 3$ and some positive constants $h_j,\ j = 1, 2, 3, 4$. To prove the theorem we will use the Lyapunov function

$$V_\varepsilon(x, y) = V(x) + \varepsilon(\Pi F_1(x, y),\, \bigtriangledown)V(x) + \varepsilon^2\, U_2(x, y)$$

where $U_2(x, y)$ satisfies the equation

$$Q\,U_2(x, y) = -\Big\{(F_1(x, y),\, \bigtriangledown)\,(\Pi F_1(x, y),\, \bigtriangledown)V(x) + (f_2(x, y) + a(y)g_2(x, y),\, \bigtriangledown)V(x) + \frac{1}{2}\,(g_1(x, y),\, \bigtriangledown)V(x) + \mathbb{L}_0\, V(x)\Big\}.$$

One can apply the infinitesimal operator $\mathcal{L}_\varepsilon$ to the function $V_\varepsilon(x,y)$ and to obtain the equality $\mathcal{L}_\varepsilon V_\varepsilon(x,y) = \mathbb{L}_0 V(x) + r(x,y,\varepsilon)$, where the last term satisfies the inequality $|r(x,y,\varepsilon)| \le \alpha(\varepsilon)|x|^p$ with some infinitesimal $\alpha(\varepsilon)$ as $\varepsilon \to 0$. It is easy to verify that there exist the positive constants ε_0, r_1, r_2, r_3 such that the function $V_\varepsilon(x,y)$ satisfies the inequalities

$$r_1\,|x|^p \le V_\varepsilon(x,y) \le r_2\,|x|^p, \qquad \mathcal{L}_\varepsilon V_\varepsilon(x,y) \le -r_3\,|x|^p \le -\frac{r_3}{r_2}\,V_\varepsilon(x,y)$$

for all $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$, $\varepsilon \in (0,\varepsilon_0)$. To complete the proof one can use the same calculations as in the end of the proof of Theorem 2.2.

4 Merger and Stability

To illustrate the asymptotic merger method of stability analysis proposed in [14] we will suppose that the infinitesimal operator of the step Markov process has the form $Q_\varepsilon = Q_0 + \varepsilon Q_1$, where

$$Q_j\,v(y) := \sum_{y\in\mathbf{Y}} [v(z) - v(y)]\,p_j(y,\,z), \quad j = 0,1$$

and $p_j(y,z)$ as functions of $z \in \mathbf{Y}$ are positive uniformly bounded on $y \in \mathbf{Y}$ discrete measures. Let $\{y_\varepsilon(t),\, t \ge 0\}$ be the Markov process corresponding to this infinitesimal operator. It is easy to see that this process is a step Markov process [8]. We will assume that the operator Q_0 has 0 as an isolated simple eigenvalue of multiplicity h, the eigenfunctions of this operator are defined by equalities

$$q_j\,(y) = \begin{cases} 1, & \text{for} \quad y \in \mathbf{Y}_j \\ 0, & \text{for} \quad y \in \mathbf{Y}_k,\ k \ne j \end{cases}$$

with nonintersecting supports $\mathbf{Y}_j$, $j = \overline{1,h}$ and the remaining part of the spectrum is situated in the half-plane $\{\lambda \in \mathbf{C} \colon \Re\lambda < -\rho\}$ for some positive ρ. The conjugate operator Q_0^* also [8, 11] has 0 as an isolated eigenvalue of multiplicity h and h invariant probabilistic measures $\mu_k(y)$ with the same supports $\mathbf{Y}_k$, $k = \overline{1,h}$.

In this section we will deal with stochastic process $\{x_\varepsilon(t),\, t \ge 0\}$ which satisfies the differential equation

$$\frac{dx_\varepsilon}{dt} = \varepsilon f(x_\varepsilon,\, y_\varepsilon(t),\, \varepsilon), \tag{17}$$

for all $t \in (\tau^\varepsilon_{j-1},\, \tau^\varepsilon_j)$, $j \in \mathbb{N}$, and the conditions of jump

$$x_\varepsilon(t) = x_\varepsilon(t-) + \varepsilon g(x_\varepsilon(t-),\, y_\varepsilon(t-),\, \varepsilon), \tag{18}$$

for all $t \in \{\tau^\varepsilon_j,\ j \in \mathbb{N}\}$, where $\{\tau^\varepsilon_j,\ j \in \mathbb{N}\}$ are switching time moments of the process $\{y_\varepsilon(t), t \ge 0\}$ and functions $f(x,y,\varepsilon)$, $g(x,y,\varepsilon)$ were defined in Section 2. The system (17)–(18) we will consider in slow time $s = \varepsilon t$ denoting $\tilde{x}_\varepsilon(s) = x_\varepsilon(s/\varepsilon)$.

To define the limit merged Markov process for the family $\{\tilde{x}_\varepsilon(s)\}$ as $\varepsilon \to 0$ [14] one needs the merged state space $\hat{\mathbf{Y}} := \{\mathbf{Y}_1,\, \mathbf{Y}_2,\, \ldots,\, \mathbf{Y}_h\}$ which for simplicity we will denote $\hat{\mathbf{Y}} := \{1, 2, \ldots, h\}$ and the infinitesimal matrix $\Gamma = \{\gamma_j^k\}$, where

$$\gamma_j^k := \begin{cases} \sum\limits_{y \in \mathbf{Y}_k} p_1(y, \mathbf{Y}_j)\, \mu_k(y), & \text{if} \quad j \neq k, \\ -\sum\limits_{\substack{l=1 \\ l \neq k}}^{h} \gamma_k^l, & \text{if} \quad j = k, \end{cases} \tag{19}$$

$k,\, j = \overline{1,\, h}$. Corresponding to this infinitesimal matrix process $\{\hat{y}(t),\ t \geq 0\}$ is called *a merged Markov process.* To use the merger method of [14] first of all one has to define the function

$$\tilde{F}_1(x,\, y) \equiv \sum_{z \in \mathbf{Y}_k} (f_1(x,\, z) + g_1(x,\, z) p_0(z, \mathbf{Y})) \mu_k(z), \qquad y \in \mathbf{Y}_k$$

for each $k = \overline{1, h}$ and differential equation

$$\frac{d\tilde{x}_\varepsilon}{ds} = \tilde{F}_1(\tilde{x}_\varepsilon(s),\, y_\varepsilon(s/\varepsilon)). \tag{20}$$

Substituting the above defined merged step Markov process $\{\hat{y}(s),\, s \geq 0\}$ instead of the initial Markov process $\{y_\varepsilon(s/\varepsilon),\, s \geq 0\}$ in (20) we will obtain *the limit merged differential equation for the system* (17)–(18)

$$\frac{d\hat{x}}{ds} = \hat{F}_1(\hat{x}(s), \hat{y}(s)), \tag{21}$$

where $\hat{F}_1(x, k) := \tilde{F}_1(x, y)$ for any $k = \overline{1, h}$ and $y \in \mathbf{Y}_k$.

Owing to assumption on spectrum structure of the operator Q_0 one can define [11] the projective operator $\mathcal{P}$ by the equalities

$$\forall\ y \in \mathbf{Y}_k, \ \ v \in \mathbb{B}(\mathbf{Y}) \colon \ (\mathcal{P}v)(y) \equiv \sum_{z \in \mathbf{Y}_k} v(z) \mu_k(z)$$

for each $k = \overline{1, h}$ and the linear continuous operator $\hat{\Pi} \colon \mathbb{B}\,(\mathbf{Y}) \to \mathbb{B}\,(\mathbf{Y})$ by equality

$$(\hat{\Pi} v)(y) := \int_0^\infty \sum_{z \in \mathbf{Y}} P_0(t, y, z)(v - \mathcal{P}v)(z)\, dt, \tag{22}$$

where $P_0(t, y, z)$ is the transition probability corresponding to infinitesimal operator Q_0. The operator (22) we will use in the same way as the potential Π in Section 2.

Theorem 4.1 (Merger principle) *Under the above assumptions the family of processes $\{x^\varepsilon(s)\}$ weak converges as $\varepsilon \to 0$ to the solution of (21) with corresponding initial condition.*

Proof It is easy to prove that the processes $\{x_\varepsilon(s),\, y_\varepsilon(s/\varepsilon),\, \tilde{x}_\varepsilon(s)\}$ one can consider jointly as the Markov process with the weak infinitesimal operator [8]

$$\tilde{\mathbb{L}}(\varepsilon) := (f(x, y, \varepsilon),\, \nabla^{(x)}) + (\tilde{F}_1(\tilde{x}, y),\, \nabla^{(\tilde{x})}) + \frac{1}{\varepsilon}\,(Q_0 + \varepsilon Q_1) + \tilde{G}^\varepsilon,$$

where the operator $\tilde{G}^\varepsilon$ is defined by the equality

$$\tilde{G}^\varepsilon v(x,y) = \frac{1}{\varepsilon}\sum_{y\in\mathbf{Y}}[v(x+\varepsilon g(x,y,\varepsilon),\,z) - v(x,z)](p_0(y,z)+\varepsilon p_1(y,z))$$

and the gradients are acting by indicated indices. As in Section 2 we will use the function

$$v_\varepsilon(x,y,\tilde{x}) := |x-\tilde{x}|^2 + \varepsilon[2(x-\tilde{x},\,(\tilde{\Pi}F_1)(x,y)) + c\,(1+|x|^2+|\tilde{x}|^2)],$$

which for sufficiently large c satisfies the inequalities

$$|x-\tilde{x}|^2 + \varepsilon\,(1+|x|^2+|\tilde{x}|^2) \le v_\varepsilon(x,y,\tilde{x}) \le |x-\tilde{x}|^2 + \varepsilon c_1\,(1+|x|^2+|\tilde{x}|^2)$$

with some positive constant c_1 for all $\varepsilon\in(0,1)$ and $x\in\mathbb{R}^n,\ y\in\mathbf{Y},\ \tilde{x}\in\mathbb{R}^n$. Applying the equality

$$Q_0\,(\tilde{\Pi}F_1)(x,y) = -F_1(x,y) + \tilde{F}_1(x,y)$$

one can obtain the inequality $\tilde{\mathbb{L}}(\varepsilon)v_\varepsilon(x,y,\tilde{x}) \le kv_\varepsilon(x,y,\tilde{x})$ with some positive constant k. Hence, for any $x\in\mathbb{R}^n,\ y\in\mathbf{Y},\ \tilde{x}\in\mathbb{R}^n$ the stochastic process $v_\varepsilon(x_\varepsilon(t),\,y_\varepsilon(t/\varepsilon),\,\tilde{x}_\varepsilon(t))$ $\times\exp\{-kt\}$ is a positive supermartingale [7]. Therefore under the initial conditions $x_\varepsilon(0)=x,\ \tilde{x}_\varepsilon(0)=x$ one can write the inequality

$$\mathbb{P}_{x,y}\Big(\sup_{0\le t\le T}|x_\varepsilon(t)-\tilde{x}_\varepsilon(t)|\ge\delta\Big) \le \varepsilon c_1\,\delta^{-2}\,e^{k_2T}\,(1+2|x|^2) \tag{23}$$

for any $\delta>0,\ T>0$.

Under the assumptions of twice continuously boundedly differentiability on x of the functions $f_1(x,y)$ and $g_1(x,y)$, the function $\tilde{F}_1(x,y)$ also has two continuous bounded derivatives and therefore one can use the merger method and results of paper [14]. According to the above paper for any $T>0$ and $\tilde{x}\in\mathbb{R}^n$ the solution $\{\tilde{x}_\varepsilon(t),\,t\in[0,\,T]\}$ of Cauchy problem $\tilde{x}_\varepsilon(0)=x$ for (20) defines on Skorokhod space $D([0,T],\,\mathbb{R}^n)$ the family of probability measures $\{\mathbb{P}_\varepsilon,\,\varepsilon\in(0,1)\}$ which weak converges as $\varepsilon\to0$ to the probability measure corresponding to the solution of the Cauchy problem $\hat{x}(0)=x$ for (21). This assertion and inequality (23) complete the proof.

Let now $f(0,y,\varepsilon)\equiv g(0,y,\varepsilon)\equiv0$. Then also $\tilde{F}_1(0,y)\equiv0$ and both systems (17)–(18) and (21) have the trivial solution.

Theorem 4.2 *Under the above assumptions if the trivial solution of (21) is exponentially p-stable for all sufficiently small ε and some $p>0$ then there exists $\varepsilon_p>0$ such that the trivial solution of IDS (17)–(18) is exponentially p-stable for any $\varepsilon\in(0,\varepsilon_p)$.*

Proof Owing to exponential decrease of the p-moments of the solutions of (21) and a boundedness of the x-derivative of $\tilde{F}_1(x,y)$ one can define function

$$y\in\mathbb{Y}_k\colon v^{(p)}(x,y)\equiv\hat{v}^{(p)}(x,k) := \int_0^T \mathbb{E}_{x,k}|\hat{x}(t)|^p\,dt,\quad k=\overline{1,h},$$

with so large a constant T that the above function satisfies the inequalities $m_1|x|^p\le v^{(p)}(x,y)\le m_2|x|^p$ with some positive constants $m_1,\ m_2$. Owing to exponential p-stability of (21), the inequality $(\hat{F}_1(x,k),\,\triangledown)\hat{v}^{(p)}(x,k)+\Gamma\hat{v}^{(p)}(x,k)\le -m_3\hat{v}^{(p)}(x,k)$ is held with some positive constant m_3 for any $k=\overline{1,h}$ and $x \in \mathbb{R}^n$. To prove the theorem we will use the Lyapunov function

$$v_\varepsilon^{(p)}(x,y) := v^{(p)}(x,y) + \varepsilon\tilde{\Pi}\{(F_1(x,y),\,\triangledown)\,v^{(p)}(x,y) + Q_1 v^{(p)}(x,y)\},$$

which satisfies the inequalities $\hat{m}_1|x|^p \le v_\varepsilon^{(p)}(x,y) \le \hat{m}_2|x|^p$ with some positive constants $\hat{m}_1,\ \hat{m}_2$ for any $\varepsilon \in (0,1)$. By definition of the operator $\tilde{\Pi}$ one can write the equality

$$\begin{aligned}
&(F_1(x,y),\,\triangledown)\,v^{(p)}(x,y) + Q_1 v^{(p)}(x,y) + Q_0 v_1^{(p)}(x,y)\\
&= (\tilde{F}_1(x,y),\,\triangledown)\,v^{(p)}(x,y) + \mathcal{P}Q_1 v^{(p)}(x,y) + \varepsilon r(x,y,\varepsilon)\\
&= (\hat{F}_1(x,k),\,\triangledown)\,\hat{v}^{(p)}(x,k) + \Gamma\hat{v}^{(p)}(x,k) + \varepsilon r(x,y,\varepsilon)\\
&\le -m_3\hat{v}^{(p)}(x,k) + \varepsilon\alpha(\varepsilon)|x|^p
\end{aligned}$$

and therefore

$$\begin{aligned}
&(F_1(x,y),\,\triangledown)\,v^{(p)}(x,y) + Q_1 v^{(p)}(x,y) + Q_0 v_1^{(p)}(x,y)\\
&\le -m_3\hat{v}^{(p)}(x,k) + \varepsilon\alpha(\varepsilon)|x|^p
\end{aligned}$$

for any $y \in \mathbf{Y}_k$ and $k=\overline{1,h}$, where $\alpha(\varepsilon)$ is infinitesimal as $\varepsilon \to 0$. Owing to the above inequalities there exist such positive constants ε_p that for any $\varepsilon \in (0,\varepsilon_p)$

$$\mathbb{L}(\varepsilon)v_\varepsilon^{(p)}(x,y) \le -\frac{m_3}{2}\,v_\varepsilon^{(p)}(x,y).$$

Now we can use Dynkin's formula for $\mathbb{E}_{x,y}\Big\{v_\varepsilon^{(p)}(x_\varepsilon(s),\ y_\varepsilon(s/\varepsilon))\exp(sm_3/2)\Big\}$ and complete the proof as it has been done in the proof of Theorem 2.2.

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