Nonlinear Dynamics and Systems Theory, **2**(2) (2002) 185–202

Stability of Dynamic Systems on the Time Scales

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Received: January 11, 2000; Revised: June 5, 2001

Abstract: The paper dwells on the problems of stability of dynamical systems on a time scale. The paper is divided into the following sections: local existence and uniqueness, dynamic inequalities, existence of extremal solutions, comparison results, linear variation of parameters, nonlinear variation of parameters, global existence and stability, comparison theorems, stability criteria, etc.

Keywords: *Dynamical systems on a time scale; stability.*

Mathematics Subject Classification (2000): 34B99, 39A99.

1 Introduction

In both natural and engineering systems the lowest level is usually characterized by continuous variable dynamics and the highest by a logical decision making mechanism. The interaction of these different levels, with their different types of information, leads to a hybrid system. Many complicated control systems today (e.g. those for flight control, manufacturing systems, and transportation) have vast amount of computer code at their highest level. More pervasively, programmable logic controllers are widely used industrial process control. Virtually all control systems today issue continuous variable controls and perform logical checks that determine the mode, and hence the control algorithms the continuous variable system is operating under at any given moment.

Hybrid control systems are control systems that involve both continuous systems that involve both continuous and discrete dynamics and continuous and discrete controls. The continuous dynamics of such a system is usually modeled by a controlled vector field or difference equation. Its hybrid nature is expressed by a dependence on some discrete phenomena, corresponding to discrete states, dynamics and controls. The prototypical hybrid systems are digital controllers, computers, and subsystems modeled as finite automata coupled with controllers and plants modeled by partial or ordinary differential equations or difference equations. Thus such systems arise whenever one mixes logical decision making with continuous control laws. More specifically, real world examples of hybrid systems include systems with relays, switches, and hysteresis; disk drivers, transmissions, step motors; constrained robots; automated transportation systems; and modern flexible manufacturing and flight control systems.

In control theory, there has certainly been a lot of related work in the past, including variable structure systems, jump linear systems, systems with impulse effect, impulse control, and piecewise deterministic processes.

The mathematical modeling of several important dynamic processes has been via difference equations or differential equations. Difference equations also appear in the study of discretization methods for differential equations. In recent years, however, the investigation of the theory of difference equations (discrete time dynamic systems) has assumed a greater importance as a well desired discipline. In spite of this tendency of independence, there is a striking similarity or even duality between the theories of continuous and discrete dynamic systems. Many results in the theory of difference equations have been obtained as more or less natural discrete analogs of corresponding results of differential equations. Nevertheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation exhibits the chaotic behavior which can only happen for higher order differential equations. Moreover, additional assumptions are often required in the discrete case in order to overcome the topological deficiency of lacking connectedness. From a modeling point of view, it is perhaps more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals called time scale. In this survey paper we discuss the stability of dynamics systems on time scale.

2 Preliminaries

In this paper we use the calculus obtained in [1, 2] for unifying discrete and continuous dynamic systems.

Let $\mathbb{T}$ be a time scale (closed nonempty subset of R) with $t_0 \geq 0$ as a minimal element and no maximal element.

The points $\{t\}$ of $\mathbb{T}$ are classified as:

right-dense (rd), if $\sigma(t) = t$;

left-dense (ld), if $\rho(t) = t$;

right-scattered (rs), if $\sigma(t) > t$;

left-scattered (ls), if $\rho(t) < t$, where $\sigma(t)$, $\rho(t)$ are jump operators defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} \colon s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}.$$

Set $\mu^*(t) = \sigma(t) - t$ (*called graininess*), so that

$$\mathbb{T} \equiv R \Rightarrow \mu^*(t) = 0, \qquad \mathbb{T} \equiv Z \Rightarrow \mu^*(t) = 1.$$

Definition 2.1 The mapping $u \colon \mathbb{T} \to R$ is *rd-continuous* if it is continuous at each right-dense point and $\lim\limits_{s \to t^-} f(s) = f(t^-)$ exist at each left-dense.

Definition 2.2 A mapping $u\colon \mathbb{T} \to R$ is said to be *differentiable at* $t \in \mathbb{T}$, if there exists an $\alpha \in R$ such that for any $\epsilon > 0$ there exists a neighborhood U of t satisfying

$$|u(\sigma(t)) - u(s) - \alpha(\sigma(t) - s)| \le \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

Derivative of u is denoted by $u^{\triangle}(t)$.
Note: $\mathbb{T} = R \Rightarrow u^{\triangle} = \alpha = \frac{du(t)}{dt}$,
$\mathbb{T} = Z \Rightarrow u^{\triangle} = \alpha = u(t+1) - u(t)$.
If u is differentiable at t, then it is continuous at t.
If u is continuous at t and t is right-scattered, then u is differentiable and

$$u^{\triangle}(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.3 For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+u^{\triangle}(t)$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu^*(t,s)} < D^+u^{\triangle}(t) + \epsilon \quad \text{for} \quad s \in N_\epsilon, \quad s > t,$$

where $\mu(t,s) = \sigma(t) - s$.

In case t is rs and u is continuous at t, we have, as in the case of the derivative,

$$D^+u^{\triangle}(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.4 Let h be a mapping from $\mathbb{T}$ to R. The mapping $g\colon \mathbb{T} \to R$ is called the *antiderivative* of h on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\triangle}(t) = h(t)$ for $t \in \mathbb{T}$.

The following known properties of the antiderivative are useful.

(a) If $h\colon \mathbb{T} \to R$ is rd-continuous, then h has the antiderivative $g\colon t \to \int_s^t h(s)\,ds$, $s, t \in \mathbb{T}$.

(b) If the sequence $\{h_n\}_{n\in N}$ of rd-continuous functions $\mathbb{T} \to R$ converge uniformly on $[r,s]$ to rd-continuous function h then

$$\left(\int_r^s h_n(t)\,dt\right)_{n\in N} \to \int_r^s h(t)\,dt, \quad \text{in} \quad R.$$

A basic tool employed in the proofs is the following induction principle, well suited for time scales.

Suppose that for any $t \in \mathbb{T}$, there is a statement $A(t)$ such that the following conditions are verified:

(I) $A(t_0)$ is true;
(II) If t right-scattered and $A(t)$ is true, then $A(\sigma(t))$ is also true;
(III) For each right-dense t there exists a neighborhood U such that whenever $A(t)$ is true, $A(s)$ is also true for all $s \in U$, $s \ge t$;
(IV) For left-dense t, $A(s)$ is true for all $s \in [t_0, t)$ implies $A(t)$ is true.

Then the statement $A(t)$ is true for all $t \in \mathbb{T}$.

3 Local Existence and Uniqueness

In this section, we shall consider the initial value problem for dynamic systems on time scales and prove local existence and uniqueness results corresponding to Peano's and Perron's theorems. Let $\mathbb{T}^k$ represent the set of all nondegenerate points of the time scale $\mathbb{T}$.

Consider the initial value problem (IVP)

$$x^{\Delta} = f(t,x), \quad t \in \mathbb{T}^k, \quad x(t_0) = x_0, \tag{3.1}$$

where $f\colon \mathbb{T}^k \times R^n \to R^n$ and f is rd-continuous on $\mathbb{T}^k \times R^n$.

A map $x\colon \mathbb{T}^k \to R^n$ is a solution of IVP (3.1) if $x(t)$ is an antiderivative of $f(t,x(t))$ on $\mathbb{T}^k$ and satisfies $x(t_0) = x_0$.

Theorem 3.1 *Let $f \in C_{rd}[R_0, R^n]$, where $R_0 = [t_0, t_0 + a] \times B$, $[t_0, t_0 + a]$ is understood as $[t_0, t_0 + a] \cap \mathbb{T}^k$ and $B = \{x \in R^n \colon |x - x_0| \le b\}$. Then the IVP (3.1) has at least one solution $x(t)$ on $[t_0, t_0 + \alpha]$, where $\alpha = \min\left(a, \frac{b}{M}\right)$, M being the bound of $f(t,x)$ on R_0.*

Proof See [3] and cf. [1, 2].

Next we shall consider Perron type uniqueness result.

Theorem 3.2 *Assume that*

(i) *$g \in C_{rd}\big[[t_0, t_0 + a] \times [0, 2b], R_+\big]$ and for every t_1, $t_0 \le t_1 \le t_0 + a$, $u(t) \equiv 0$ is the only solution of*

$$u^{\Delta} = g(t,u), \quad u(t_1) = 0, \quad \text{on} \quad [t_1, t_0 + a];$$

(ii) *$f \in C_{rd}[R_0, R^n]$ and for each $t \in [t_0, t_0 + a]$, there exists a compact neighborhood U_t such that f^t in $U_t \times B$ satisfies*

$$|f(t,x) - f(t,y)| \le g(t, |x - y|), \quad (t,x),\ (t,y) \in U_t \times B.$$

Then the IVP (3.1) has a unique solution $x(t)$ on $[t_0, t_0 + a]$.

Proof See [3].

4 Dynamic Inequalities

In this section, we shall prove basic results on dynamic inequalities, that are needed to prove existence of extremal solutions. We shall first prove a result relative to a system of strict dynamic inequalities and then consider a similar result for nonstrict inequalities which is needed for later discussion. All inequalities between vectors are to be understood componentwise hereafter.

We need the following definition before we proceed further

Definition 4.1 A *function* $f \in C[R^n, R^n]$ is said to be *quasimonotone nondecreasing* if $x \le y$ and $x_i = y_i$ for some $1 \le i \le n$ implies $f_i(x) \le f_i(y)$.

Theorem 4.1 *Let $\mathbb{T}$ be the time scale with $t_0 \ge 0$ minimal element and no maximal element, $v, w\colon \mathbb{T} \to R^n$ be the rd-continuous mappings that are differentiable for each $t \in \mathbb{T}$ and satisfy*

$$v^{\Delta}(t) \le f(t, v(t)), \quad w^{\Delta}(t) \ge f(t, w(t)), \quad t \in \mathbb{T}, \tag{4.1}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $f(t,x)$ is quasimonotone nondecreasing in x, and, for $1 \le i \le n$, $f_i(t,x)\mu^*(t) + x_i$ is nondecreasing in x_i for each $t \in \mathbb{T}$.
Then $v(t_0) < w(t_0)$ implies $v(t) < w(t)$, for $t \in \mathbb{T}$.

Proof See [4].

The next result deals with nonstrict dynamic inequalities

Theorem 4.2 *Let $\mathbb{T}$ be the time scale as before, $v, w\colon \mathbb{T} \to R^n$ be the rd-continuous mappings that are differentiable for each $t \in \mathbb{T}$ and satisfy*

$$v^{\Delta}(t) \le f(t, v(t)), \quad w^{\Delta}(t) \ge f(t, w(t)), \quad t \in \mathbb{T}, \tag{4.2}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $f(t,x)$ is quasimonotone nondecreasing in x and for each i, $1 \le i \le n$, $f_i(t,x)\mu^*(t) + x_i$ is nondecreasing in x_i for $t \in \mathbb{T}$. Then $v(t_0) \le w(t_0)$ implies $v(t) \le w(t)$, $t \in \mathbb{T}$, provided f satisfies

$$f_i(t,x) - f_i(t,y) \le L \sum_{i=1}^{n} (x_i - y_i), \quad x \ge y. \tag{4.3}$$

Proof See [3].

5 Existence of Extremal Solutions

Using the result on strict dynamic inequalities proved in Section 4, we shall discuss, in this section, the existence of extremal solutions for dynamic systems. For that consider the IVP

$$u^{\Delta} = g(t,u), \quad u(t_0) = u_0 > 0, \tag{5.1}$$

where $g \in C_{rd}[R_0, R^n]$, $R_0 = \{([t_0, t_0 + a] \cap \mathbb{T}) \times B\}$,

$$B = \{u \in R^n \colon |u - u_0| \le b\}, \quad \text{and} \quad |g(t,u)| \le M \quad \text{on} \quad R_0.$$

Theorem 5.1 *Assume that*

(i) *$g(t,u)$ is quasimonotone nondecreasing in u,*
(ii) *for each i, $1 \le i \le n$, $g_i(t,u)\mu^*(t) + u_i$ is nondecreasing in u_i for each $t \in \mathbb{T}$.*

Then there exist minimal and maximal solutions of (5.1) on $I \equiv [t_0, t_0 + \eta] \cap \mathbb{T}$, where $\eta = \min\left(a, \frac{b}{2M+b}\right)$.

Proof See [4].

6 Comparison Results

Having the existence results and the theory of dynamic inequalities at our disposal, it is now easy to prove comparison results.

Theorem 6.1 *Let the assumptions of Theorem 5.1 hold and let* $m\colon I \equiv [t_0, t_0 + a) \cap \mathbb{T} \to R^n$ *be a mapping that is differentiable for each* $t \in I$ *and that satisfies*

$$m^{\Delta}(t) \leq g(t, m(t)), \qquad t \in I.$$

Then $m(t_0) \leq u_0$ *implies that* $m(t) \leq r(t)$, $t \in I$, *where* $r(t)$ *is the maximal solution of (5.1) existing on* I.

Proof See [4].

Definition 6.1 Let $x \in C^1_{rd}[\mathbb{T}, R^n]$. Given an $\epsilon > 0$, if there exists a neighbourhood N_ϵ of $t \in \mathbb{T}$ satisfying

$$\frac{1}{\mu(t,s)}\left[|x(\sigma(t))| - |x(\sigma(t)) - \mu(t,s)x^{\triangle}(t)\|\right] < \left[x, x^{\triangle}\right]_+ + \epsilon$$

for each $s \in N_\epsilon$ and $s > t$, where $\mu(t,s) = \sigma(t) - s$, then we say that $\left[x, x^{\triangle}\right]_+$ is the *generalised derivative* of $x(t)$. In case, $t \in \mathbb{T}$ is rs, then we have

$$\left[x, x^{\triangle}\right]_+ = \frac{1}{\mu^*(t)}\left[|x(\sigma(t))| - |x(t)|\right],$$

where $\mu^*(t) = \mu(t,t)$.

We can now prove the following comparison result.

Theorem 6.2 *Suppose that*

$$\left[x, x^{\triangle}\right]_+ \leq g(t, |x|) \quad on \quad \mathbb{T} \times R,$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R]$ *and* $g(t,u)\mu^*(t) + u$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$, *where* $x\colon \mathbb{T} \to R^n$ *is any rd-continuously differentiable function such that* $|x_0| \leq u_0$. *Then* $|x(t)| \leq r(t)$, $t \in \mathbb{T}$, *where* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of (5.1) existing on* $\mathbb{T}$.

Proof See [5, p.86] and [3].

7 Linear Variation of Parameters

Let $(\mathbb{T}, \mu, X)$ a dynamical triple, and $B(X)$ be a Banach algebra with unity of the continuous endomorphims on a Banach space X.

A *mapping* $A\colon \mathbb{T}^k \to B(X)$ is called *regressive*, if for each $t \in \mathbb{T}^k$ the mapping $A(t)\mu^*(t) + id\colon X \to X$ is invertible. This is the case e.g if $|A(t)\mu^*(t)| < 1$ for all $t \in \mathbb{T}$. Obviously in case $\mathbb{T} = R$ any A is regressive (since $\mu^* = 0$) and in case $\mathbb{T} = Z$, A is regressive if $|A(t)| < 1$ (since $\mu^* \equiv 1$).

Suppose $A\colon \mathbb{T}^k \to B(X)$ is rd-continuous and regressive and $F\colon \mathbb{T}^k \times X \to X$ is rd-continuous, then a mapping $x\colon \mathbb{T}^k \to X$ is called a *solution of the dynamic equation*

$$x^{\triangle} = A(t)x + F(t,x) \tag{7.1}$$

if $x^{\triangle}(t) = A(t)x(t) + F(t, x(t))$ for all $t \in \mathbb{T}^k$.

If a solution $x(\cdot)$ of (7.1) in addition satisfies the condition $x(\tau) = \eta$ for a pair $(\tau, \eta) \in \mathbb{T}^k \times X$, it is called a *solution of the initial value problem* (IVP)

$$x^{\triangle} = A(t)x + F(t,x), \qquad x(\tau) = \eta. \tag{7.2}$$

Consider the IVP, in the Banach algebra $B(X)$,

$$x^{\triangle} = A(t)x, \qquad x(\tau) = I, \tag{7.3}$$

where I is the unity of $B(X)$. By Theorem 3.2 , it admits exactly one solution $\Phi_A(\tau) := x(\cdot;\tau, I)$. We call it *principal solution*. The corresponding transition function is defined to be $\Phi_A(t,\tau) := \Phi_A(\tau)(t)$. In the particular case, when $A\colon \mathbb{T}^k \to B(X)$ is constant, we call the transition function, exponential function $(e_L(t,\tau))$. If $X = R$ and $C\colon \mathbb{T}^k \to R_+$, then $C(t)\mu^*(t) + 1 > 0$ satisfies the regressive property we can set $e_C(t,\tau) := \Phi_A(t,\tau)$.

Theorem 7.1 *We consider the IVP (7.2) with rd-continuous and regressive right-hand side. Then the solution of (7.2) is given by*

$$x(t) = \Phi_A(t,\tau)\eta + \int_{\tau}^{t} \Phi_A(t,\sigma(s))F(s,x(s)) \triangle s.$$

Proof See [1, 2].

8 Nonlinear Variation of Parameters

Theorem 8.1 *Let $\mathbb{T} = [\tau, s]$ be some compact measure chain. Assume that $f \in C_{rd}[\mathbb{T}^k \times R^n, R^n]$, and possesses rd-continuous partial derivatives f_x on $\mathbb{T}^k \times R^n$. Let L be a nonnegative constant with $L\mu(s,\tau) < 1$ and $|f_x(t,x)| \le L$ on $\mathbb{T}^k \times R^n$. Let the solution $x_0(t) = x(t,\tau,\eta)$ of*

$$x^{\triangle} = f(t,x), \quad x(\tau) = \eta, \quad \textit{exists for} \quad t \ge \tau. \tag{8.1}$$

Then

(i) $\Phi(t,\tau,\eta) = x_\eta(t,\tau,\eta)$ *exists and is the solution of*

$$y^{\triangle} = H(t,\tau,\eta)y, \tag{8.2}$$

where $H(t,\tau,\eta) = \lim\limits_{h\to 0} \int_0^1 f_x(t, px(t,\tau,\eta) - (1-p)x(t,\tau,\eta+h)) \triangle p$ *such that* $\Phi(\tau,\tau,\eta)$ *is the unit matrix;*

(ii) $\Psi(t,\tau,\eta) = x_\tau^{\triangle}(t,\tau,\eta)$ *exists, is the solution of*

$$z^{\triangle} = H(t,\sigma(\tau),\tau,\eta)z \tag{8.3}$$

such that $\Psi(\sigma(\tau),\tau,\eta) = -f(\tau,\eta)$*, where*

$$H(t,\sigma(\tau),\tau,\eta) = \int_0^1 f_x(t, px(t,\sigma(\tau),\eta) - (1-p)x(t,\tau,\eta)) \triangle p;$$

(iii) *the function* $\Phi(t,\tau,\eta)$, $\Psi(t,\tau,\eta)$ *satisfy the relation*

$$\Psi(t,\tau,\eta) = -\Phi(t,\sigma(\tau),\eta)f(\tau,\eta) + \int_{\sigma(\tau)}^{t} \Phi(t,\sigma(s),\eta)\big[H(s,\sigma(\tau),\tau,\eta) - H(s,\sigma(\tau),\tau,\eta)\big]\Psi(s,\tau,\eta) \triangle s.$$

Proof See [3].

Theorem 8.2 *Let* $\mathbb{T} = [\tau, s]$ *be some compact measure chain. Assume that* f, $F \in C_{rd}[\mathbb{T}^k \times R^n, R^n]$, *and* f_x *exists and be rd-continuous on* $\mathbb{T}^k \times R^n$.

Let L *be a nonnegative constant with* $L\mu(s,\tau) < 1$ *and* $|f_x(t,x)| \le L$ *on* $\mathbb{T}^k \times R^n$.

If $x(t,\tau,\eta)$ *is the solution* $x^{\triangle} = f(t,x)$, $x(\tau) = \eta$, *exists for* $t \ge \tau$, *any solution* $y(t,\tau,\eta)$ *of* $y^{\triangle} = f(t,y) + F(t,y)$, *with* $y(\tau) = \eta$, *satisfies the integral equation*

$$y(t,\tau,\eta) = x(t,\tau,\eta) + \int_{\tau}^{t} \Phi(t,\sigma(s),y(s))\,F(s,y(s)) \triangle s + \int_{\tau}^{t}\int_{\sigma(s)}^{t} \Phi(t,\sigma(p),y(s))\big[H(p,\sigma(s),s,y(s)) - H(p,\sigma(s),y(s))\big]\,\Psi(p,s,y(s)) \triangle p \triangle s.$$

Proof See [3].

Remark 8.1 It is easy to see from the definitions of $H(p,\sigma(s),s,y(s))$ and $H(p,\sigma(s), y(s))$ that they are identical if the measure chain is R, and consequently, the foregoing variation of parameter formula reduces to the usual Alekseev's formula (see [6]).

9 Global Existence and Stability

As an application of comparison Theorem 6.2, we shall prove, in this section, a global existence result and a simple stability result.

Theorem 9.1 *Assume that*

(i) $f \in C_{rd}[\mathbb{T}\times R^n, R^n]$, $g \in C_{rd}[\mathbb{T}\times R_+, R_+]$, $g(t,u)$ *is non-decreasing in* u *for each* $t \in \mathbb{T}$, *where* $\mathbb{T}$ *is the time scale with* $t_0 \ge 0$ *as the minimal element and has no maximal element and*

$$|f(t,x)| \le g(t,|x|) \quad for \quad (t,x) \in \mathbb{T}\times R^n;$$

(ii) *the maximal solution* $r(t)$ *of the scalar IVP*

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{9.1}$$

exists on $\mathbb{T}$.

Then the largest interval of existence of any solution $x(t)$ of

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0, \tag{9.2}$$

with $|x_0| \le u_0$ *is* $\mathbb{T}$.

Proof See [3].

To prove a simple stability result, we need the following definition of stability.

Definition 9.1 The trivial *solution* of (9.2) is said to be

(i) *stable* if given an $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta > 0$ such that $|x_0| \le \delta$ implies $|x(t)| \le \epsilon$, $t \ge t_0$;
(ii) *asymptotically stable* if it is stable and $\lim\limits_{t\to\infty} |x(t)| = 0$.

We are now in a position to prove a typical result on stability in terms of comparison principle (cf. [7, p.13]).

Theorem 9.2 *Assume that*

(i) $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$ *and for* $(t,x) \in \mathbb{T} \times R^n$,

$$[x, f(t,x)]_+ \equiv \lim_{h\to 0^+} \frac{1}{h}\big[|x + hf(t,x)| - |x|\big] \le g(t,|x|); \tag{9.3}$$

(ii) $g(t,u)\mu^*(t)$ *is nondecreasing in u for each* $t \in \mathbb{T}$.

Then the stability properties of the trivial solution of the IVP

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{9.4}$$

imply the corresponding stability properties of the trivial solution of

$$x^{\Delta} = f(t,x), \quad x(t_0) = x_0, \quad t_0 \ge 0, \quad t \in \mathbb{T}. \tag{9.5}$$

Proof Let the trivial solution of (9.4) be stable. Then, given $\epsilon > 0$ and $t_0 \ge 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$u \le u_0 < \delta \quad \text{implies} \quad u(t) < \epsilon, \quad t \in \mathbb{T}. \tag{9.6}$$

It is easy to claim that with these ϵ and δ, the trivial solution of (9.5) is also stable. If this were false, there would exist a solution $x(t)$ of (9.5) with $|x_0| < \delta$ and a $t_1 \in \mathbb{T}$, $t_1 > t_0$ such that $\epsilon \le |x(t_1)|$ and $|x(t)| \le \epsilon$, $t \in [t_0,t_1)$.

For $t \in [t_0,t_1]$, using condition (9.3), we get

$$m^{\Delta}(t) \le g(t,m(t)), \quad t \in [t_0,t_1], \quad m(t_0) \le u_0,$$

where $m(t) = |x(t)|$. Consequently, Theorem 6.2 yields

$$|x(t)| \le r(t), \quad t \in [t_0,t_1].$$

At $t = t_1$, we arrive at the contradiction

$$\epsilon \le |x(t_1)| \le r(t_1) < \epsilon,$$

proving the claim. One can prove similarly other concepts of stability, and we omit the details.

10 Comparison Theorems

Consider the dynamic system

$$x^{\triangle} = f(t,x), \qquad x(t_0) = x_0, \tag{10.1}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, and $x^{\triangle}$ denotes the derivative of x with respect to $t \in \mathbb{T}$. We shall assume, for convenience, that the solutions $x(t) = x(t,t_0,x_0)$ of (10.1) exist and are unique for $t \ge t_0$.

Definition 10.1 Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$. Then we define the generalized derivative of $V(t,x)$ relative to (10.1) as follows: given $\epsilon > 0$, there exists a neighbourhood $N(\epsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\mu(t,s)}\big[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s) f(t,x(t))\big] < D^+V^{\triangle}(t,x(t)) + \epsilon$$

for each $s \in N(\epsilon)$ and $s > t$, where $\mu(t,s) = \sigma(t) - s$ and $x(t)$ is any solution of (10.1).

In case, $t \in \mathbb{T}$ is right scattered and $V(t,x(t))$ is continuous at t, we have

$$D^+V^{\triangle}(t,x(t)) = \frac{1}{\mu^*(t)}\big[V(\sigma(t), x(\sigma(t))) - V(t,x(t))\big],$$

where $\mu^*(t) = \mu(t,t)$.

We are now in a position to prove the following comparison theorem in terms of Lyapunov function $V(t,x)$.

Theorem 10.1 *Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t,x)$ be locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, and let*

$$D^+V^{\triangle}(t,x(t)) \le g(t, V(t,x)),$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R_+]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$ and $r(t) = r(t,t_0,u_0)$ is the maximal solution of $u^{\triangle} = g(t,u)$, $u(t_0) = u_0 \ge 0$, existing on $\mathbb{T}$. Then, $V(t_0,x_0) \le u_0$ implies that $V(t,x(t)) \le r(t,t_0,u_0)$, $t \in \mathbb{T}$, $t \ge t_0$.

Proof See [3] and cf. [8].

Remark 10.1 If the inequalities between vectors is understood as componentwise, then Theorem 10.1 is valid for $V \in C_{rd}[\mathbb{T} \times R^n, R_+^N]$, $g \in C_{rd}[\mathbb{T} \times R_+^N, R_+^N]$, provided $g(t,u)$ is quasimonotone nondecreasing in u and $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$. The proof requires slight modification since we now need to use comparison result for differential systems relative to Vector Lyapunov functions. We therefore omit proving such a result.

We shall next discuss another comparison result which connects the solutions to dynamic systems which can be employed in perturbation theory.

Consider another dynamic system

$$x^{\Delta} = F(t,x), \qquad x(t_0) = x_0, \tag{10.2}$$

where $F \in C_{rd}[\mathbb{T} \times R^n, R^n]$. Relative to the system (10.1), let us assume that the following assumption (H) holds:

(H) the solutions $y(t,t_0,x_0)$ of (10.2) exist for all $t \geq t_0$, unique and rd-continuous with respect to the initial data and $|y(t,t_0,x_0)|$ is locally Lipschitzian in x_0.

For any $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$ and any fixed $t \in \mathbb{T}$, we define $D^+V^{\Delta}(s, y(t,s,x(s)))$ as follows: given $\epsilon > 0$, there exists a neighbourhood $N(\epsilon)$ of $s \in \mathbb{T}$, $t_0 \leq s \leq t$ such that

$$\frac{1}{\mu(s,r)}\big[V(\sigma(s), y(t,\sigma(s),x(\sigma(s)))) - V(s, y(t,s,x(\sigma(s))) - \mu(s,r)F(s,x(s))\big]$$
$$< D^+V(s,y(t,s,x(s))) + \epsilon$$

for each $r \in N(\epsilon)$ and $r > s$. As before, if $s \in \mathbb{T}$ is right-scattered and $V(s,y(t,s,x(s)))$ is continuous at s, then

$$D^+V(s,y(t,s,x(s))) = \frac{1}{\mu^*(s)}\big[V(\sigma(s), y(t,\sigma(s),x(\sigma(s))) - V(s, y(t,s,x(\sigma(s))\big],$$

with $\mu^*(s) = \mu(s,s)$.

We then have the following general comparison result which includes Theorem 10.1 as a special case.

Theorem 10.2 *Assume that the assumption (H) holds. Suppose that*

(i) $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, *$V(s,x)$ is locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd and for $t_0 < s \leq t$, $x \in R^n$*

$$D^+V^{\Delta}(s,y(t,s,x)) \leq g(s, V(s,y(t,s,x)));$$

(ii) $g \in C_{rd}[\mathbb{T}^k \times R_+, R]$, *$g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and the maximal solution $r(t,t_0,u_0)$ of*

$$u^{\Delta} = g(t,u), \quad u(t_0) = u_0 \geq 0, \quad \textit{exists for} \quad t \in \mathbb{T}.$$

Then, if $x(t) = x(t,t_0,x_0)$ is any solution of (10.1) we have

$$V(t, x(t,t_0,x_0)) \leq r(t,t_0,u_0), \qquad t \in \mathbb{T}, \tag{10.3}$$

provided $V(t_0, y(t,t_0,x_0)) \leq u_0$.

Proof See [3] and [8].

11 Stability Criteria

In this section, we shall consider some simple stability results. We list a few definitions concerning the stability of the trivial solution of (10.1) which we assume to exit (for details see [5]).

Definition 11.1 The *trivial solution* $x = 0$ of (10.1) is said to be

($S1$) *equi-stable*, if for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is rd-continuous in t_0 for each ϵ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| < \epsilon$ for $t \geq t_0$;

($S2$) *uniformly stable*, if the δ in ($S1$) is independent of t_0;

($S3$) *quasi-equi asymptotically stable*, if for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exist positive $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that $|x_0| < \delta_0$ implies $|x(t, t_0, x_0)| < \epsilon$ for $t \geq t_0 + T$;

($S4$) *quasi-uniformly asymptotically stable*, if δ_0 and T in ($S3$) are independent of t_0;

($S5$) *equi-asymptotically stable*, if ($S1$) and ($S3$) hold simultaneously;

($S6$) *uniformly asymptotically stable*, if ($S2$) and ($S4$) hold simultaneously.

Corresponding to the definitions ($S1$) to ($S6$), we can define the stability notions of the trivial solution $u = 0$ of (10.1) below. For example, the trivial solution $u = 0$ of (10.1) is equi-stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a function $\delta_0 = \delta_0(t_0, \epsilon)$ that is rd-continuous in t_0 for each ϵ, such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon$, $t \geq t_0$.

We are now in a position to prove a general result which provides sufficient conditions for stability criteria.

Theorem 11.1 *Assume that*

(i) $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, *$V(t,x)$ is locally Lipschitzian in x for each right dense $t \in \mathbb{T}$;*

(ii) *$b(|x|) \leq V(t,x) \leq a(|x|)$ for $(t,x) \in \mathbb{T} \times R^n$, where a, $b \in \mathcal{K} = \big[\sigma \in C[R_+, R_+]$: $\sigma(0) = 0$ and $\sigma(u)$ is increasing in $u\big]$;*

(iii) *$f(t,0) = 0$, $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and*

$$D^+V^{\Delta}(t,x) \leq g(t, V(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n.$$

Then the stability properties of the trivial solution of

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \geq 0 \tag{11.1}$$

imply the corresponding stability properties of the trivial solution of (10.1).

Proof Let $\epsilon > 0$ and $t_0 \in \mathbb{T}$ be given. Suppose that the trivial solution of (11.1) is equi-stable. Then given $b(\epsilon) > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$u_0 < \delta_1 \;\;\Rightarrow\;\; u(t) < b(\epsilon), \quad t \in \mathbb{T} \tag{11.2}$$

where $u(t) = u(t, t_0, u_0)$ is any solution of (11.1). Choose $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(\delta) < \delta_1. \tag{11.3}$$

We claim that if $|x_0| < \delta$, then $|x(t)| < \epsilon$, $t \in \mathbb{T}$, where $x(t) = x(t, t_0, x_0)$ is any solution of (11.1). If this is not true, there would exist a $t_1 \in \mathbb{T}$, $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (11.1) satisfying

$$|x(t)| < \epsilon, \quad t_0 \leq t < t_1 \quad \text{and} \quad |x(t_1)| \geq \epsilon. \tag{11.4}$$

Setting $m(t) = V(t, x(t))$ for $t_0 \le t \le t_1$ and using condition (iii), we get by Theorem 11.1, the estimate

$$V(t, x(t)) \le r(t, t_0, u_0), \qquad t_0 \le t \le t_1, \tag{11.5}$$

where $r(t, t_0, u_0)$ is the maximal solution of (11.1) with $V(t_0, x_0) \le u_0$. Now the relations (11.2), (11.4), (11.5) and the assumption (ii) yield

$$b(\epsilon) \le b(|x(t_1)|) \le V(t, x(t_1)) \le r(t_1, t_0, u_0) < b(\epsilon),$$

since $u_0 = V(t_0, x_0) \le a(|x_0|) < a(\delta) < \delta_1$ by (11.3). This contradiction proves the claim.

Other stability properties may be proved in a similar manner and hence the proof is complete.

We can obtain from Theorem 11.1, a result analogous to Lyapunov's first theorem for continuous and discrete cases immediately. This we state as a corollary.

Corollary 11.1 *The function $g(t, u) \equiv 0$ is admissible in Theorem 11.1 to yield uniform stability of the trivial solution of (11.1).*

A result analogous to Lyapunov's second theorem cannot be obtained directly from Theorem 11.1, since when we choose $g(t, u) = -c(u)$, $c \in \mathcal{K}$, $g(t, u)\mu^*(t) + u$ does not satisfy the monotone condition. However, a minor change in the argument proves such a result.

Corollary 11.2 *The choice of the function $g(t, u) = -c(u)$, $c \in \mathcal{K}$, in Theorem 11.1 implies uniform asymptotic stability of the trivial solution of (11.1).*

Proof See [3].

Usually, Lyapunov's second theorem has the assumption $D^+V^\Delta(t, x(t)) \le -c_0(|x|)$, $c_0 \in \mathcal{K}$. Since $V(t, x)$ has the upper estimate in (ii), which means $V(t, x)$ is decrescent, it is easy to show that $D^+V^\Delta(t, x(t)) \le -c(|x|)$, where $c(u) = c_0^{-1}(a(u))$. Of course, one can prove directly with the assumption $D^+V^\Delta(t, x(t)) \le -c_0(|x|)$ following the proof of Corollary 11.2.

When $V(t, x)$ is not assumed to be decrescent, that $V(t, 0) \equiv 0$, we have Marachkov's result for differential equations. We can extend such a result in the present set up.

Theorem 11.2 *Assume that $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t, x)$ is locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, $b(|x|) \le V(t, x)$ and $V(t, 0) \equiv 0$. Suppose further that $D^+V^\Delta(t, x(t)) \le -c(|x|)$, and $|f(t, x)| \le M$ for $(t, x) \in \mathbb{T} \times R^n$. Then the trivial solution is equi-asymptotically stable.*

Proof Let $\epsilon > 0$, and $t_0 \in \mathbb{T}$ be given. Since $V(t, x)$ is rd continuous and $V(t, 0) \equiv 0$, it is possible to find a $\delta = \delta(t_0, \epsilon) > 0$ satisfying $V(t_0, x_0) < b(\epsilon)$ iff $|x_0| < \delta$.

We claim that $|x_0| < \delta$ implies $|x(t)| < \epsilon$, $t \ge t_0$, the proof of which follows from Theorem 10.1 . Now set $\epsilon = \rho$, for some $\rho > 0$ and designate by $\delta_0 = \delta(t_0, \rho)$ so that we have $|x_0| < \delta_0$ implies $|x(t)| < \rho$, $t \ge t_0$. To prove the theorem, we need to show that $\liminf\limits_{t\to\infty} |x(t)| \ne 0$, then there exists a $T > 0$ such that for a given $\eta > 0$, we have $|x(t)| > \eta$, $t \ge T$. As a result, we get arguing as in Corollary 11.2, the estimate

$V(t,x(t)) \le V(t_0,x_0) - \int_T^t c(|x(s)|)\Delta s, t \ge T$, which yields a contradiction, for large t, $0 \le V(t_0,x_0) - c(\eta)(t-T)$.

Hence $\liminf_{t\to\infty} |x(t)| = 0$. Suppose then $\limsup_{t\to\infty} |x(t)| \ne 0$. Then, given an $\epsilon > 0$, there exist a divergent sequence $\{t_k\}$ such that $|x(t_k)| > \epsilon$. Each $t_k \in \mathbb{T}$ may belong to one of the following cases:

(i) t_k is rs and ls;
(ii) t_k is rs and ld;
(iii) t_k is rd and ls;
(iv) t_k is rd and ld;

without loss of generality, we can assume that there is a divergent subsequence $\{t_i\}$ of $\{t_k\}$ such that all t_i belong to one of the above four cases. In case (i), we have

$$V(\sigma(t_i), x(\sigma(t_i))) \le V(t_i, x(t_i)) - \mu^*(t_i)c(|x(t_i)|),$$

which yields, by successive application,

$$\begin{aligned} 0 \le V(\sigma(t_i), x(\sigma(t_i))) &\le V(t_0,x_0) - \sum_{j=1}^{i} \mu^*(t_j)c(|x(t_j)|) \\ &\le V(t_0,x_0) - c(\epsilon)\eta i. \end{aligned}$$

This leads to a contradiction as $i \to \infty$ since $\mu^*(t_i)$ is constant for each i, say η.

In cases (ii) to (iv), we can find another divergent sequence $\{t_i^*\}$ such that $t_i < t_i^*$ or $t_i^* < t_{i+1}$ satisfying

$$\begin{aligned} &|x(t_i)| = \epsilon, \quad &&|x(t_i^*)| = \tfrac{1}{2}\epsilon, \quad &&\tfrac{1}{2}\epsilon < |x(t)| < \epsilon, \quad &&t \in (t_i, t_i^*) \quad \textit{or} \\ &|x(t_i^*)| = \tfrac{1}{2}\epsilon, \quad &&|x(t_{i+1})| = \epsilon, \quad &&\tfrac{1}{2}\epsilon < |x(t)| < \epsilon, \quad &&t \in (t_i^*, t_{i+1}). \end{aligned}$$

Since $|f(t,x)| \le M$, it is easy to find $t_i - t_i^* > \frac{\epsilon}{2M}$ and $t_{i+1} - t_i^* > \frac{\epsilon}{2M}$.

It therefore follows that

$$0 \le V(t_i^*, x(t_i^*)) \le V(t_0,x_0) - ic\big(\tfrac{1}{2}\epsilon\big)\,\frac{\epsilon}{2M} < 0, \quad \text{for large} \quad i.$$

This is a contradiction. Similarly, we get contradiction for other case. Hence we have $\lim_{t\to\infty} |x(t)| = 0$ and the proof is complete.

12 A Technique in Stability Theory

As an application of Theorem 10.2, we shall consider a typical result on stability and asymptotic behavior of solutions of (10.2).

Theorem 12.1 *Assume that* (H) *holds and* (i) *of Theorem 10.2 is verified. Suppose that* $g \in C_{rd}[\mathbb{T}\times R_+, R]$, $g(t,u)\mu^*(t)+u$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$, $g(t,0) \equiv 0$, $f(t,0) \equiv 0$, $F(t,0) \equiv 0$ *and for* $(t,x) \in \mathbb{T} \times R^n$,

$$b(|x|) \le V(t,x) \le a(|x|), \qquad a,\, b \in \mathcal{K}. \tag{12.1}$$

Furthermore, suppose that the trivial solution of (10.2) is uniformly stable and $u = 0$ *of*

$$u^\Delta = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{12.2}$$

is uniformly asymptotically stable. Then, the trivial solution of (10.2) is uniformly asymptotically stable.

Proof Let $\epsilon > 0$ and $t_0 \in \mathbb{T}$ be given. The uniform stability of $u = 0$ of (12.2) implies that given $b(\epsilon) > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(\epsilon) > 0$ such that if $u_0 \le \delta_1$, then

$$u(t,t_0,u_0) < b(\epsilon), \qquad t \ge t_0. \tag{12.3}$$

Let $\delta_2 = a^{-1}(\delta_1)$. Since $y = 0$ of (10.2) is uniformly stable, given $\delta_2 > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(\epsilon)$ such that

$$|y(t,t_0,x_0)| < \delta_2, \quad t \ge t_0 \quad \text{if} \quad |x_0| < \delta. \tag{12.4}$$

We claim that $|x_0| < \delta$ also implies that $|x(t,t_0,x_0)| < \epsilon$, $t \in \mathbb{T}$, where $x(t,t_0,x_0)$ is any solution of (10.2). If this is not true, there would exist a solution $x(t,t_0,x_0)$ of (10.2) with $|x_0| < \delta$ and a $t_1 > t_0$ such that $|x(t_1,t_0,x_0)| \ge \epsilon$, $t_0 \le t \le t_1$. Then, by Theorem 10.2, we have

$$V(t, x(t,t_0,x_0)) \le r(t,t_0,V(t_0, y(t,t_0,x_0))), \qquad t_0 \le t \le t_1.$$

Consequently, by (12.1), (12.3) and (12.4), we get

$$\begin{aligned} b(\epsilon) &\le V(t_1, x(t_1,t_0,x_0)) \le r(t_1,t_0,a(|y(t_1,t_0,x_0)|)) \\ &\le r(t_1,t_0,a(\delta_2)) \le r(t_1,t_0,\delta_1) < b(\epsilon). \end{aligned}$$

This contradiction proves that $x = 0$ of (10.2) is uniformly stable.

To show uniform asymptotic stability, we set $\delta = \delta_0$. Then, from the foregoing argument, we have

$$b(|x(t,t_0,x_0)|) \le V(t,x(t,t_0,x_0)) \le r(t,t_0,V(t_0,y(t,t_0,x_0)))$$

for all $t \in \mathbb{T}$, if $|x_0| \le \delta_0$. From this it follows that

$$b(|x(t,t_0,x_0)|) \le r(t,t_0,\delta_1), \qquad t \in \mathbb{T}$$

which implies the uniform asymptotic stability of $x = 0$ because of the assumption that $u = 0$ of (12.2) is uniformly asymptotically stable. Hence the proof is complete.

Setting $F(t,x) = f(t,x) + R(t,x)$ in Theorem 12.1, we see that although the unperturbed system (10.2) is only uniformly stable, the perturbed system (10.2) is uniformly asymptotically stable, an improvement caused by the perturbing term.

13 Strict Stability

Consider the initial value problem

$$x^\Delta = f(t,x), \quad t \in \mathbb{T}, \quad x(t_0) = x_0, \tag{13.1}$$

where $f\colon \mathbb{T} \times R^n \to R^n$ and f is rd-continuous on $\mathbb{T} \times R^n$.

Let $K = \big\{a \in C_{rd}[\mathbb{T}, R_+]\colon a(u)$ is strictly increasing in u, $a(0) = 0$ and $a(u) \to \infty$ as $u \to \infty\big\}$.

Definition 13.1 The *trivial solution* of (13.1) is said to be

(S1) *strictly stable*, if given $\epsilon_1 > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ such that $|x_0| < \delta_1$ implies $|x(t)| < \epsilon_1$, $t \geq t_0$, and for every $0 < \delta_2 \leq \delta_1$, there exist an $0 < \epsilon_2 < \delta_2$ such that

$$\delta_2 < |x_0| \quad \text{implies} \quad \epsilon_2 < |x(t)|, \quad t \geq t_0; \tag{13.2}$$

(S2) *strictly uniformly stable*, if δ_1, δ_2 and ϵ_2 are independent of t_0;

(S3) *strictly attractive*, if given $\alpha_1 > 0$, $\epsilon_1 > 0$ and $t_0 \in \mathbb{T}$ for every $\alpha_2 \leq \alpha_1$ there exists $\epsilon_2 < \epsilon_1$ and $T_1 = T_1(t_0, \epsilon_1)$, $T_2 = T_2(t_0, \epsilon_1)$ such that

$$\alpha_2 \leq |x_0| \leq \alpha_1 \quad \text{implies} \quad \epsilon_2 < |x(t)| < \epsilon_1, \quad \text{for} \quad t_0 + T_1 \leq t \leq t_0 + T_2;$$

(S4) *strictly uniformly attractive*, if T_1, T_2 in (S3) are independent of t_0;

(S5) strictly asymptotically stable if (S3) holds and the trivial solution is stable;

(S6) strictly uniformly asymptotically stable if (S4) holds and the trivial solution is uniformly stable;

Remark 13.1 It is important to note that (S1) and (S3), or (S2) and (S4) cannot hold at the same time. If in (S1) it is not possible to find an ϵ_2 satisfying (13.2), we shall say that the trivial solution is stable. This can happen when $|x(t)| \to 0$ as $t \to \infty$ or $\liminf |x(t)| = 0$ and $\limsup |x(t)| \neq 0$.

Theorem 13.1 *Assume that*

(H_1) *for each* $0 < \eta < \rho$, $V_\eta \in C_{rd}[\mathbb{T} \times S_\rho, R_+]$, V_η *is locally Lipschitzian in* x *and for* $(t, x) \in \mathbb{T} \times S_\rho$ *and* $|x| \geq \eta$,

$$b_1(|x|) \leq V_\eta(t, x) \leq a_1(|x|), \quad a_1, b_1 \in K, \quad \textit{and} \quad D^+V_\eta^\Delta(t, x) \leq 0; \tag{13.3}$$

(H_2) *for each* σ, $0 < \sigma < \rho$, $V_\sigma \in C_{rd}[\mathbb{T} \times S_\rho, R_+]$, V_σ *is locally Lipschitzian in* x *and for* $(t, x) \in \mathbb{T} \times S_\rho$ *and* $|x| \leq \sigma$,

$$b_2(|x|) \leq V_\eta(t, x) \leq a_2(|x|), \quad a_2, b_2 \in K, \quad \textit{and} \quad D^+V_\eta^\Delta(t, x) \geq 0. \tag{13.4}$$

Then the trivial solution is strictly uniformly stable.

Proof See [15].

Theorem 13.2 *Let the assumptions of Theorem 13.1 hold except that the conditions (13.3) and (13.4) are replaced by*

$$D^+V_\eta^\Delta(t, x) \leq -c_1(|x|),$$

and

$$D^+V_\eta^\Delta(t, x) \geq -c_2(|x|),$$

where $c_1, c_2 \in K$.

Then the trivial solution of (13.1) is uniformly strictly asymptotically stable.

Proof See [15].

Before proving the general result in terms of the comparison principle, we need to consider the comparison differential system

$$u_1^{\Delta} = g_1(t, u_1), \qquad u_1(t_0) = u_0 \geq 0, \tag{13.5a}$$

$$u_2^{\Delta} = g_2(t, u_2), \qquad u_2(t_0) = u_0 \geq 0, \tag{13.5b}$$

where $g_1,\, g_2 \in C_{rd}[\mathbb{T} \times R_+, R]$.

We shall say that the comparison system (13.5) is strictly stable, if given $\epsilon_1 > 0$ and $t_0 \in \mathbb{T}$, there exist a $\delta_1 > 0$ such that $u_0 < \delta_1$ implies $u_1(t) < \epsilon_1,\ t \geq t_0$, and for every $\delta_2 \leq \delta_1$, there exists an $\epsilon_2,\ 0 < \epsilon_2 < \delta_2$ such that $\delta_2 < u_0$ implies that $\epsilon_2 < u_2(t)$, $t \geq t_0$. Where, $u_1(t)$, $u_2(t)$ are any solutions of (13.5a) and (13.5b) respectively.

Based on these definition, we can formulate other strict stability notions. Next result is formulated in terms of comparison principles.

Theorem 13.3 *Let the assumptions of Theorem 13.3 hold except that the conditions (13.3) and (13.4) are replaced by*

$$D^+V_{\eta}^{\Delta}(t,x) \leq g_1(t, V_{\eta}(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n.$$

and

$$D^+V_{\sigma}^{\Delta}(t,x) \geq g_2(t, V_{\sigma}(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n,$$

where $g_2(t,u) \leq g_1(t,u)$, $g_1,\, g_2 \in C_{rd}[\mathbb{T} \times R_+, R]$, $g_1(t,0) \equiv 0$, $g_2(t,0) \equiv 0$.

Then any strict stability concept of the comparison system implies the corresponding strict stability concept of the trivial solution of (13.1) respectively.

Proof See [15].

For several allied results, see [9 – 15].

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