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Asymptotic Stability for a Conducting Electromagnetic Material with a Dissipative Boundary Condition

Giovambattista Amendola[†]

Dipartimento di Matematica Applicata "U.Dini", Facoltà di Ingegneria, via Diotisalvi 2, 56126-Pisa, Italy

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Abstract: In this work we study the existence and uniqueness of the solution for a linear electromagnetic material characterized by the memory effects due to a rate-type equation for the electric conduction when a general dissipative boundary condition is assumed on the boundary of the solid. We show the existence of a domain of dependence and we give some limitations of the values of the material constants which assure the asymptotic stability of the solution.

Keywords: *Linear electromagnetism; asymptotic stability.*

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1 Introduction

The purpose of this paper is to keep on studying memory effects in electromagnetic systems, which occur through a rate-type equation for the electric conduction. Its presence in the system of equations has been recently considered in another work [5], where we have supposed that the boundary of the solid is a perfect conductor.

In the present work a homogeneous, isotropic and conducting solid, characterized also by linear constitutive equations for the electric displacement and the magnetic induction, is considered on supposing that a general dissipative boundary condition holds on its boundary.

After introducing the field equations, the thermodynamic restrictions on the constitutive equations and the free energy in Section 2, we formulate the initial-boundary value problem. Thus, we show that a domain of dependence inequality exists for these bodies and we derive a useful energy estimate.

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In Section 4 we prove the existence and uniqueness theorems for the weak and the strong solution of this evolutive problem; then, we study the asymptotic stability, which holds under suitable hypotheses on the values of the material constants of the medium.

2 Basic Equations and Thermodynamic Restrictions

Let $\mathcal{B}$ be an electromagnetic solid, which occupies at time t a bounded and regular domain $\Omega \subset R^3$ with a smooth boundary $\partial\Omega$.

The electromagnetic phenomena of $\mathcal{B}$ are described by Maxwell's equations

$$\dot{\mathbf{D}}(x,t) = \nabla \times \mathbf{H}(x,t) - \mathbf{J}(x,t) - \mathbf{J}_f(x,t), \quad \dot{\mathbf{B}}(x,t) = -\nabla \times \mathbf{E}(x,t), \tag{2.1}$$

$$\nabla \cdot \mathbf{D}(x,t) = 0, \quad \nabla \cdot \mathbf{B}(x,t) = 0, \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic fields, $\mathbf{J}$ is the electric current density, $\mathbf{D}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction; moreover, $\mathbf{J}_f$ is a forced current density which must be considered as a given function of the position $x \in \Omega$ and $t \in R^+$. In $(2.2)_1$ we have supposed that the free charge density is zero.

Besides Maxwell's equations we must consider the thermodynamic principles [1−2].

The Dissipation Principle states that for any cyclic process the following inequality

$$\oint [\dot{\mathbf{D}}(x,t) \cdot \mathbf{E}(x,t) + \dot{\mathbf{B}}(x,t) \cdot \mathbf{H}(x,t) + \mathbf{J}(x,t) \cdot \mathbf{E}(x,t)]\, dt \geq 0 \tag{2.3}$$

holds, the equality sign referring to reversible processes.

The Second Law for smooth isothermal processes yields

$$\dot{\psi}(x,t) \leq \dot{\mathbf{D}}(x,t) \cdot \mathbf{E}(x,t) + \dot{\mathbf{B}}(x,t) \cdot \mathbf{H}(x,t) + \mathbf{E}(x,t) \cdot \mathbf{J}(x,t), \tag{2.4}$$

where ψ is the free energy.

Let us assume that $\mathcal{B}$ is a homogeneous and isotropic conductor, whose constitutive equations are linear and given by

$$\mathbf{D}(x,t) = \varepsilon \mathbf{E}(x,t), \quad \mathbf{B}(x,t) = \mu \mathbf{H}(x,t), \tag{2.5}$$

where both the dielectric constant ε and the permeability μ are positive constants. For the electric conduction we suppose that the following rate-type equation

$$\alpha \dot{\mathbf{J}}(x,t) + \mathbf{J}(x,t) = \sigma \mathbf{E}(x,t) \tag{2.6}$$

holds, where α is a positive parameter and σ denotes the conductivity, which is assumed constant too.

Using (2.5) and the relation derived from (2.6) for $\mathbf{E}$, inequality (2.4) becomes [5]

$$\oint \frac{d}{dt} \frac{1}{2} \left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2 + \frac{\alpha}{\sigma} \mathbf{J}^2 \right) dt + \oint \frac{1}{\sigma} \mathbf{J}^2\, dt \geq 0, \tag{2.7}$$

which, taking into account that the integration is made on cycles, yields

$$\sigma > 0. \tag{2.8}$$

Finally, we can introduce the free energy

$$\psi(x,t) = \frac{1}{2}\left[\varepsilon \mathbf{E}^2(x,t) + \mu \mathbf{H}^2(x,t) + \frac{\alpha}{\sigma}\,\mathbf{J}^2(x,t)\right], \tag{2.9}$$

which satisfies (2.4) on account of (2.6).

3 Formulation of the Problem and Domain of Dependence

Maxwell's equations (2.1), taking account of the constitutive equations (2.5), and (2.6) take the form

$$\nabla \times \mathbf{H}(x,t) - \varepsilon \dot{\mathbf{E}}(x,t) - \mathbf{J}(x,t) = \mathbf{f}(x,t), \tag{3.1}$$

$$\nabla \times \mathbf{E}(x,t) + \mu \dot{\mathbf{H}}(x,t) = \mathbf{g}(x,t), \tag{3.2}$$

$$\alpha \dot{\mathbf{J}}(x,t) + \mathbf{J}(x,t) - \sigma \mathbf{E}(x,t) = \mathbf{l}(x,t), \tag{3.3}$$

on introducing the source terms $\mathbf{g}$ and $\mathbf{l}$, which are two known functions of x and t as well as $\mathbf{f} \equiv \mathbf{J}_f$; the other two equations (2.2) reduce to

$$\nabla \cdot \mathbf{E}(x,t) = 0, \qquad \nabla \cdot \mathbf{H}(x,t) = 0, \tag{3.4}$$

in $\Omega \times R^+$.

The initial conditions are

$$\mathbf{E}(x,0) = \mathbf{E}_0(x), \quad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad \mathbf{J}(x,0) = \mathbf{J}_0(x) \quad \forall\, x \in \Omega, \tag{3.5}$$

while on $\partial\Omega$ we consider a linear dissipative boundary condition, characterized by the following definition [8].

We first denote by Σ the set of the states, to which $\mathbf{E}$ and $\mathbf{H}$ belong together $\mathbf{J}$, and introduce the function space

$$I(\Omega) = \left\{ \mathbf{E} \in L^2(\Omega)\colon \int\limits_{\Omega} \mathbf{E}(x,t) \cdot \nabla \phi(x,t)\, dx = 0 \quad \forall\, \phi \in C_0^\infty(\Omega, R) \right\},$$

which allows us to consider equations (3.4) automatically satisfied if both $\mathbf{E}$ and $\mathbf{H}$ belong to it.

Definition 3.1 A *linear* and *dissipative boundary condition* $\Sigma' \subset \Sigma$ is a *linear closed subset* of $I(\Omega) \times I(\Omega)$ such that $C_0^1(\Omega) \times C_0^1(\Omega) \subset \Sigma'$ and

$$\int\limits_{\partial\Omega} \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x)\, da \geq 0 \quad \forall\, (\mathbf{E}, \mathbf{H}) \in \Sigma' \tag{3.6}$$

with the static condition

$$\int\limits_{\partial\Omega} |\mathbf{H} \cdot \mathbf{n}|^2\, da = 0,$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

We shall denote by P the problem (3.1)–(3.6).

Lemma 3.1 *The electromagnetic fields* $\mathbf{E}$ *and* $\mathbf{H}$ *satisfy this inequality*

$$\chi\psi(x,t)+\mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\mathbf{u}(x)\geq 0, \qquad \chi=2(\varepsilon\mu)^{-1/2}, \tag{3.7}$$

for any unit vector $\mathbf{u}(x)$.

Proof From the definition of the free energy, given by (2.9), it follows that

$$|\mathbf{E}(x,t)|\leq[2\psi(x,t)/\varepsilon]^{1/2}, \qquad |\mathbf{H}(x,t)|\leq[2\psi(x,t)/\mu]^{1/2} \tag{3.8}$$

and hence we obtain (3.7) easily.

Let

$$E(A,t)=\int\limits_A \psi(x,t)\,dx \tag{3.9}$$

be the total energy for every domain $A\subset\Omega$, where ψ is given by (2.9), we have the following theorem.

Theorem 3.1 *If the triplet* $(\mathbf{E},\mathbf{H},\mathbf{J})$ *is a solution of the problem* P*, for every* (x_0,T) $\in\Omega\times R^+$ *the total energy satisfies*

$$\begin{aligned}E(B(x_0,\rho),T)\leq E(B(x_0,\rho+\chi T),0)+\int\limits_0^T\int\limits_{\Omega\cap B(x_0,\rho+\chi(T-t))}&[\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma\\ +\mathbf{g}(x,t)\cdot\mathbf{H}(x,t)-\mathbf{f}(x,t)\cdot\mathbf{E}(x,t)]\,dx\,dt,&\end{aligned} \tag{3.10}$$

where χ *is given by (3.7)*$_2$ *and* $B(x_0,\rho)=\{x\in\Omega\colon |x-x_0|\leq\rho\}$.

Proof We introduce the weighted energy

$$E_\phi(\Omega,t)=\int\limits_\Omega \psi(x,t)\phi(x,t)\,dx, \tag{3.11}$$

where ψ is expressed by (2.9) in terms of the solution $(\mathbf{E},\mathbf{H},\mathbf{J})$ of the problem P and $\phi(x,t)\in C_0^\infty(\Omega,R^+)$, and we consider its derivative with respect to time, where $\dot{\mathbf{E}}$, $\dot{\mathbf{H}}$ and $\dot{\mathbf{J}}$ can be eliminated by means of (3.1)−(3.3). Using the identity $\nabla\times\mathbf{E}\cdot\mathbf{H}-\nabla\times\mathbf{H}\cdot\mathbf{E}=\nabla\cdot(\mathbf{E}\times\mathbf{H})$, we get

$$\begin{aligned}\dot{E}_\phi(\Omega,t)=\int\limits_\Omega&[\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma+\mathbf{g}(x,t)\cdot\mathbf{H}(x,t)-\mathbf{f}(x,t)\cdot\mathbf{E}(x,t)\\ &-\mathbf{J}^2(x,t)/\sigma]\phi(x,t)\,dx+\int\limits_\Omega[\mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\nabla\phi(x,t)+\psi(x,t)\dot{\phi}(x,t)]dx\\ &-\int\limits_{\partial\Omega}\mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\mathbf{n}(x)\phi(x,t)da.\end{aligned} \tag{3.12}$$

Following [4], we put $\phi(x,t)=\phi_\delta(x,t)=\phi_\delta(y)\in C_0^\infty(R)$, a monotonic decreasing function of $y=|x-x_0|-\rho-\chi(T-t)$, with $\rho>0$, $(x_0,T)\in\Omega\times\mathbf{R}^+$, χ given by

$(3.7)_2$, such that $\phi_\delta(y) = 1$ for all $y \le -\delta$, $\phi_\delta(y) = 0$ for all $y > \delta$, $\phi'_\delta(y) \le 0$, $\dot{\phi}_\delta(x,t) = \chi\phi'_\delta(y)$, $\nabla\phi_\delta(x,t) = \phi'_\delta(y)\nabla|x-x_0|$, for any $(x,t) \in \Omega \times (0,T)$. Thus, from (3.12), using (3.7) and the properties of ϕ_δ, it follows an inequality, which, integrated over $(0,T)$, yields

$$\begin{aligned} E_{\phi_\delta}(\Omega,T) - E_{\phi_\delta}(\Omega,0) \le \int_0^T \int_\Omega &[\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma + \mathbf{g}(x,t)\cdot\mathbf{H}(x,t) \\ &- \mathbf{f}(x,t)\cdot\mathbf{E}(x,t)]\phi_\delta(x,t)\,dx\,dt, \end{aligned} \tag{3.13}$$

whose limit as δ tends to zero gives (3.10), since ϕ_δ tends to the characteristic function of the subset $B(x_0, \rho + \chi(T-t))$.

From this theorem a useful estimate of the energy can be derived as follows.

Corollary 3.1 *For any solution* $(\mathbf{E},\mathbf{H},\mathbf{J})$ *of the problem* P *we have this inequality*

$$E(\Omega,t) \le e^T\left\{E(\Omega,0) + M\int_0^T\int_\Omega [\mathbf{f}^2(x,t) + \mathbf{g}^2(x,t) + \mathbf{l}^2(x,t)]\,dx\,dt\right\}, \tag{3.14}$$

where $M = \max\{2/\varepsilon, 2/\mu, 2/(\alpha\sigma)\}$ *and* $t \in (0,T)$.

Proof If ρ is large enough, (3.10) yields

$$E(\Omega,t) - E(\Omega,0) \le \int_0^t\int_\Omega [\mathbf{l}(x,\tau)\cdot\mathbf{J}(x,\tau)/\sigma + \mathbf{g}(x,\tau)\cdot\mathbf{H}(x,\tau) - \mathbf{f}(x,\tau)\cdot\mathbf{E}(x,\tau)]\,dx\,d\tau, \tag{3.15}$$

where $t \in (0,T)$.

Applications of Schwarz's inequality allow us to increase the integral as follows

$$\begin{aligned} &\int_0^t\int_\Omega (\mathbf{l}\cdot\mathbf{J}/\sigma + \mathbf{g}\cdot\mathbf{H} - \mathbf{f}\cdot\mathbf{E})\,dx\,d\tau \le \int_0^t \left(\int_\Omega \frac{1}{\sigma}\,\mathbf{l}^2\,dx\right)^{1/2}\left(\int_\Omega \frac{1}{\sigma}\,\mathbf{J}^2\,dx\right)^{1/2} d\tau \\ &+ \int_0^t \left(\int_\Omega \mathbf{g}^2\,dx\right)^{1/2}\left(\int_\Omega \mathbf{H}^2\,dx\right)^{1/2} d\tau + \int_0^t \left(\int_\Omega \mathbf{f}^2\,dx\right)^{1/2}\left(\int_\Omega \mathbf{E}^2\,dx\right)^{1/2} d\tau \\ &\le \left(\int_0^t \frac{2}{\alpha\sigma}\int_\Omega \mathbf{l}^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t \frac{1}{2}\int_\Omega \frac{\alpha}{\sigma}\,\mathbf{J}^2\,dx\,d\tau\right)^{1/2} + \left(\int_0^t \frac{2}{\mu}\int_\Omega \mathbf{g}^2\,dx\,d\tau\right)^{1/2} \\ &\times \left(\int_0^t \frac{1}{2}\int_\Omega \mu\mathbf{H}^2\,dx\,d\tau\right)^{1/2} + \left(\int_0^t \frac{2}{\varepsilon}\int_\Omega \mathbf{f}^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t \frac{1}{2}\int_\Omega \varepsilon\mathbf{E}^2\,dx\,d\tau\right)^{1/2} \\ &\quad \le \left[\int_0^t \frac{1}{2}\int_\Omega \left(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2 + \frac{\alpha}{\sigma}\,\mathbf{J}^2\right)dx\,d\tau\right]^{1/2}\left[\left(\int_0^t \frac{2}{\varepsilon}\int_\Omega \mathbf{f}^2\,dx\,d\tau\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_0^t \frac{2}{\mu}\int_\Omega \mathbf{g}^2\,dx\,d\tau\right)^{1/2} + \left(\int_0^t \frac{2}{\alpha\sigma}\int_\Omega \mathbf{l}^2\,dx\,d\tau\right)^{1/2}\right]. \end{aligned} \tag{3.16}$$

Using the elementary inequality $2ab \le a^2 + b^2$, we have $2(ab)^{1/2} \le a + b$; then, $(a^{1/2} + b^{1/2} + c^{1/2})^2 = a + b + c + 2(ab)^{1/2} + 2(ac)^{1/2} + 2(bc)^{1/2} \le 4(a + b + c)$ and hence $a^{1/2} + b^{1/2} + c^{1/2} \le 2(a + b + c)^{1/2}$.

Therefore, (3.16) becomes

$$\int_0^t \int_\Omega (\mathbf{l} \cdot \mathbf{J}/\sigma + \mathbf{g} \cdot \mathbf{H} - \mathbf{f} \cdot \mathbf{E}) dx\, d\tau \tag{3.17}$$

$$\le \Big[\int_0^t E(\Omega, \tau)\, d\tau \Big]^{1/2} 2 \Big(\int_0^t \frac{2}{\varepsilon} \int_\Omega \mathbf{f}^2\, dx\, d\tau + \int_0^t \frac{2}{\mu} \int_\Omega \mathbf{g}^2\, dx\, d\tau + \int_0^t \frac{2}{\alpha\sigma} \int_\Omega \mathbf{l}^2\, dx\, d\tau \Big)^{1/2}$$

$$\le \int_0^t E(\Omega, \tau)\, d\tau + \int_0^t \int_\Omega \Big(\frac{2}{\varepsilon} \mathbf{f}^2 + \frac{2}{\mu} \mathbf{g}^2 + \frac{2}{\alpha\sigma} \mathbf{l}^2 \Big) dx\, d\tau$$

and (3.15), with

$$\xi(t) = \int_0^t E(\Omega, \tau)\, d\tau, \quad \xi'(t) = E(\Omega, t), \quad \xi'(0) = E(\Omega, 0), \tag{3.18}$$

can be written as follows

$$\xi'(t) - \xi'(0) \le \xi(t) + M \int_0^T \int_\Omega (\mathbf{f}^2 + \mathbf{g}^2 + \mathbf{l}^2)\, dx\, dt. \tag{3.19}$$

Putting

$$a = \xi'(0) + M \int_0^T \int_\Omega (\mathbf{f}^2 + \mathbf{g}^2 + \mathbf{l}^2)\, dx\, dt, \tag{3.20}$$

(3.19) reduces to

$$\xi'(t) \le \xi(t) + a \quad \forall\, t \in (0, T). \tag{3.21}$$

From the last inequality, integrating with $\xi(0) = 0$, we have

$$\xi(t) \le a(e^t - 1), \tag{3.22}$$

which allows us to derive from (3.21)

$$\xi'(t) \le a e^t \tag{3.23}$$

and hence to obtain (3.14).

4 Existence and Uniqueness Theorem

To study the existence and uniqueness of the solution to the problem P, we consider the following function spaces

$$\begin{aligned}
I_1(\Omega) &= \{\mathbf{E} \in I(\Omega) \colon \nabla \times \mathbf{E} \in L^2(\Omega)\},\\
H^1_\alpha(\Omega,(0,T)) &= L^2((0,T);I_1(\Omega)) \cap H^1((0,T);L^2(\Omega)),\\
H^1_\beta(\Omega,(0,T)) &= H^1((0,T);L^2(\Omega)),\\
\mathcal{H}(\Omega,(0,T)) &= \{(\mathbf{E},\mathbf{H},\mathbf{J}) \in H^1_\alpha(\Omega,(0,T)) \times H^1_\alpha(\Omega,(0,T)) \times H^1_\beta(\Omega,(0,T)) \colon\\
&\qquad (\mathbf{E},\mathbf{H}) \ \textit{satisfies (3.6) on} \ \ \partial\Omega \times (0,T)\},\\
\mathcal{W}(\Omega,(0,T)) &= \{(\mathbf{E},\mathbf{H},\mathbf{J}) \in L^2((0,T);I(\Omega)) \times L^2((0,T);I(\Omega))\\
&\qquad \times L^2((0,T);L^2(\Omega)) \colon (\mathbf{E},\mathbf{H}) \ \textit{satisfies (3.6) on} \ \ \partial\Omega \times (0,T)\},
\end{aligned}$$

together with

$$\begin{aligned}
\mathcal{W}_0(\Omega,(0,T)) &= L^2((0,T);L^2(\Omega)) \times L^2((0,T);I(\Omega)) \times L^2((0,T);L^2(\Omega)),\\
\mathcal{W}_1(\Omega,(0,T)) &= L^2((0,T);I_1(\Omega)) \times L^2((0,T);I_1(\Omega)) \times L^2((0,T);L^2(\Omega)),
\end{aligned}$$

where $(0,T) \subset R^+$.

Definition 4.1 We call *strong solution of* P with sources $(\mathbf{f},\mathbf{g},\mathbf{l}) \in \mathcal{W}_0(\Omega,(0,T))$ and *initial data* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I(\Omega) \times I(\Omega) \times L^2(\Omega)$ any triplet $(\mathbf{E},\mathbf{H},\mathbf{J}) \in \mathcal{H}(\Omega,(0,T))$ which satisfies almost everywhere (3.1)–(3.3) in $\Omega \times (0,T)$ and (3.5) in Ω.

Definition 4.2 We call *weak solution of* P with sources $(\mathbf{f},\mathbf{g},\mathbf{l}) \in \mathcal{W}_0(\Omega,(0,T))$ and *initial data* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I(\Omega) \times I(\Omega) \times L^2(\Omega)$ any triplet $(\mathbf{E},\mathbf{H},\mathbf{J}) \in \mathcal{W}(\Omega,(0,T))$ such that the following identity

$$\begin{aligned}
&\int_0^T \int_\Omega \Big\{[\varepsilon\dot{\mathbf{e}}(x,t) - \nabla \times \mathbf{h}(x,t) + \mathbf{p}(x,t)] \cdot \mathbf{E}(x,t) + [\mu\dot{\mathbf{h}}(x,t) + \nabla \times \mathbf{e}(x,t)]\\
&\quad \cdot \mathbf{H}(x,t) + \left[\frac{\alpha}{\sigma}\,\dot{\mathbf{p}}(x,t) - \frac{1}{\sigma}\,\mathbf{p}(x,t) - \mathbf{e}(x,t)\right] \cdot \mathbf{J}(x,t) - \mathbf{f}(x,t) \cdot \mathbf{e}(x,t)\\
&\quad + \mathbf{g}(x,t) \cdot \mathbf{h}(x,t) + \mathbf{l}(x,t) \cdot \frac{1}{\sigma}\,\mathbf{p}(x,t)\Big\}\,dx\,dt - \int_0^T \int_{\partial\Omega} [\mathbf{e}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x)\\
&\quad + \mathbf{E}(x,t) \times \mathbf{h}(x,t) \cdot \mathbf{n}(x)]\,dx\,dt + \int_\Omega [\varepsilon\mathbf{E}_0(x) \cdot \mathbf{e}(x,0) + \mu\mathbf{H}_0(x) \cdot \mathbf{h}(x,0)\\
&\quad + \frac{\alpha}{\sigma}\,\mathbf{J}_0(x) \cdot \mathbf{p}(x,0)]\,dx = 0,
\end{aligned} \tag{4.1}$$

holds for any $(\mathbf{e},\mathbf{h},\mathbf{p}) \in \mathcal{H}(\Omega,(0,T))$ such that

$$\mathbf{e}(x,T) = \mathbf{0}, \quad \mathbf{h}(x,T) = \mathbf{0}, \quad \mathbf{p}(x,T) = \mathbf{0}. \tag{4.2}$$

We now prove the uniqueness and the existence of the weak solution.

Theorem 4.1 (Uniqueness) *The problem* P *has at most one solution in the sense of Definition 4.2.*

Proof The identity (4.1) must hold for any $(\mathbf{e}, \mathbf{h}, \mathbf{p}) \in \mathcal{H}(\Omega, (0,T))$; therefore, following [6], we can choose

$$\mathbf{e}(x,t) = \begin{cases} (\tau - t)\mathbf{a}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases},$$
$$\mathbf{h}(x,t) = \begin{cases} (\tau - t)\mathbf{b}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases}, \tag{4.3}$$
$$\mathbf{p}(x,t) = \begin{cases} (\tau - t)\mathbf{c}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases},$$

where τ is a fixed value of $(0,T)$ and $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an arbitrary triplet of $I_1(\Omega) \times I_1(\Omega) \times L^2(\Omega)$.

Substituting (4.3) into (4.1), we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big\{ (\tau - t)[\nabla \times \mathbf{a}(x) \cdot \mathbf{H}(x,t) - \nabla \times \mathbf{b}(x) \cdot \mathbf{E}(x,t) - \mathbf{J}(x,t) \cdot \mathbf{c}(x)/\sigma \\
&- (\mathbf{J}(x,t) + \mathbf{f}(x,t)) \cdot \mathbf{a}(x) + \mathbf{g}(x,t) \cdot \mathbf{b}(x) + (\sigma \mathbf{E}(x,t) + \mathbf{l}(x,t)) \cdot \mathbf{c}(x)/\sigma] \\
&\quad - \Big[\varepsilon \mathbf{a}(x) \cdot \mathbf{E}(x,t) + \mu \mathbf{b}(x) \cdot \mathbf{H}(x,t) + \frac{\alpha}{\sigma} \mathbf{c}(x) \cdot \mathbf{J}(x,t) \Big] \Big\} dx\, dt \\
&\quad - \int_0^\tau \int_{\partial\Omega} (\tau - t)[\mathbf{a}(x) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x) + \mathbf{E}(x,t) \times \mathbf{b}(x) \cdot \mathbf{n}(x)] da\, dt \\
&\quad + \tau \int_\Omega \Big[\varepsilon \mathbf{E}_0(x) \cdot \mathbf{a}(x) + \mu \mathbf{H}_0(x) \cdot \mathbf{b}(x) + \frac{\alpha}{\sigma} \mathbf{J}_0(x) \cdot \mathbf{c}(x) \Big] dx = 0.
\end{aligned} \tag{4.4}$$

Hence, differentiating with respect to τ, we have an identity, which, on introducing the following notation for the fields in (4.4)

$$\Phi_1(x,\tau) = \int_0^\tau \Phi(x,t)\, dt, \tag{4.5}$$

becomes

$$\begin{aligned}
&\int_\Omega \{ \nabla \times \mathbf{a}(x) \cdot \mathbf{H}_1(x,\tau) - \nabla \times \mathbf{b}(x) \cdot \mathbf{E}_1(x,\tau) - [\mathbf{J}_1(x,\tau) + \mathbf{f}_1(x,\tau)] \cdot \mathbf{a}(x) \\
&\quad + \mathbf{g}_1(x,\tau) \cdot \mathbf{b}(x) + [\sigma \mathbf{E}_1(x,\tau) - \mathbf{J}_1(x,\tau) + \mathbf{l}_1(x,\tau)] \cdot \mathbf{c}(x)/\sigma \} dx \\
&\quad\quad - \int_{\partial\Omega} [\mathbf{a}(x) \times \mathbf{H}_1(x,\tau) \cdot \mathbf{n}(x) + \mathbf{E}_1(x,\tau) \times \mathbf{b}(x) \cdot \mathbf{n}(x)] da
\end{aligned} \tag{4.6}$$

$$+\int_\Omega [\varepsilon\mathbf{E}_0(x)\cdot\mathbf{a}(x)+\mu\mathbf{H}_0(x)\cdot\mathbf{b}(x)+\frac{\alpha}{\sigma}\,\mathbf{J}_0(x)\cdot\mathbf{c}(x)]dx$$

$$-\int_\Omega \left[\varepsilon\mathbf{a}(x)\cdot\frac{d}{d\tau}\,\mathbf{E}_1(x,\tau)+\mu\mathbf{b}(x)\cdot\frac{d}{d\tau}\,\mathbf{H}_1(x,\tau)+\frac{\alpha}{\sigma}\,\mathbf{c}(x)\cdot\frac{d}{d\tau}\,\mathbf{J}_1(x,\tau)\right]dx=0.$$

This identity must be applied to the case with $(\mathbf{f},\mathbf{g},\mathbf{l})=(\mathbf{0},\mathbf{0},\mathbf{0})$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)=(\mathbf{0},\mathbf{0},\mathbf{0})$, which corresponds to the homogeneous system with zero initial data.

We observe that the relation derived in such a case must hold for every $(\mathbf{a},\mathbf{b},\mathbf{c})\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$; therefore, in particular, it follows that both $\mathbf{E}_1$ and $\mathbf{H}_1$ belong to $I_1(\Omega)$.

Thus, we can put

$$\mathbf{a}(x)=\mathbf{E}_1(x,\tau),\quad \mathbf{b}(x)=\mathbf{H}_1(x,\tau),\quad \mathbf{c}(x)=\mathbf{J}_1(x,\tau), \tag{4.7}$$

in the modified relation (4.6), which reduces to

$$\begin{aligned}&\frac{d}{d\tau}\frac{1}{2}\int_\Omega \left[\varepsilon\mathbf{E}_1^2(x,\tau)+\mu\mathbf{H}_1^2(x,\tau)+\frac{\alpha}{\sigma}\,\mathbf{J}_1^2(x,\tau)\right]dx\\ &=-\int_\Omega \frac{1}{\sigma}\,\mathbf{J}_1^2(x,\tau)\,dx-\int_{\partial\Omega}\mathbf{E}_1(x,\tau)\times\mathbf{H}_1(x,\tau)\cdot\mathbf{n}(x)\,da\le 0,\end{aligned} \tag{4.8}$$

on account of (3.6) too.

Since $\mathbf{E}_1(x,0)=\mathbf{0}$, $\mathbf{H}_1(x,0)=\mathbf{0}$, $\mathbf{J}_1(x,0)=\mathbf{0}$, by integrating (4.8) over $(0,\tau)$ we get

$$\int_\Omega \left[\varepsilon\mathbf{E}_1^2(x,\tau)+\mu\mathbf{H}_1^2(x,\tau)+\frac{\alpha}{\sigma}\,\mathbf{J}_1^2(x,\tau)\right]dx\le 0, \tag{4.9}$$

from which we have

$$\mathbf{E}_1(x,\tau)=\mathbf{0},\quad \mathbf{H}_1(x,\tau)=\mathbf{0},\quad \mathbf{J}_1(x,\tau)=\mathbf{0} \tag{4.10}$$

for almost all $\tau\in(0,T)$; therefore, it follows that

$$\mathbf{E}(x,t)=\mathbf{0},\quad \mathbf{H}(x,t)=\mathbf{0},\quad \mathbf{J}(x,t)=\mathbf{0} \tag{4.11}$$

in $\Omega\times(0,T)$, i.e. the uniqueness of the weak solution.

For the existence of the weak solution we first give this theorem.

Theorem 4.2 *Let us consider the sets*

$$\begin{aligned}\mathcal{R}=\big\{(\mathbf{f},\mathbf{g},\mathbf{l})\in\mathcal{W}_0(\Omega,(0,T))\colon \mathbf{f}=\nabla\times\mathbf{H}-\varepsilon\dot{\mathbf{E}}-\mathbf{J},\ \mathbf{g}=\nabla\times\mathbf{E}\\ +\mu\dot{\mathbf{H}},\ \mathbf{l}=\alpha\dot{\mathbf{J}}+\mathbf{J}-\sigma\mathbf{E}\quad\forall\,(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))\big\},\end{aligned} \tag{4.12}$$

$$\mathcal{S}=\{(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)\}, \tag{4.13}$$

$$\mathcal{T}=\{(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in I(\Omega)\times I(\Omega)\times L^2(\Omega)\}, \tag{4.14}$$

$\mathcal{R}\times\mathcal{S}$ is dense in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$.

Proof To prove the density of $\mathcal{R}\times\mathcal{S}$ in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$, we consider its closure $\overline{\mathcal{R}\times\mathcal{S}}$, which is a closed linear subspace of $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$, and we prove that its orthogonal complement $\mathcal{C}$ contains the null element only.

If we suppose that a non-zero element $((\mathbf{f}^*,\mathbf{g}^*,\mathbf{l}^*),(\mathbf{E}_0^*,\mathbf{H}_0^*,\mathbf{J}_0^*))\in\mathcal{C}$ exists, it follows that for any $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}$ the following equality

$$\begin{aligned}\int_0^T\int_\Omega\Big[(\nabla\times\mathbf{H}-\varepsilon\dot{\mathbf{E}}-\mathbf{J})\cdot\mathbf{f}^*+(\nabla\times\mathbf{E}+\mu\dot{\mathbf{H}})\cdot\mathbf{g}^*+\frac{1}{\sigma}(\alpha\dot{\mathbf{J}}+\mathbf{J}-\sigma\mathbf{E})\cdot\mathbf{l}^*\Big]dx\,dt\\ +\int_\Omega\Big(\varepsilon\mathbf{E}_0^*\cdot\mathbf{E}_0+\mu\mathbf{H}_0^*\cdot\mathbf{H}_0+\frac{\alpha}{\sigma}\mathbf{J}_0^*\cdot\mathbf{J}_0\Big)dx=0.\end{aligned}\tag{4.15}$$

must hold.

In this identity the arbitrariness of $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}$ allows us to take first $\mathbf{H}\equiv\mathbf{0}$, $\mathbf{J}\equiv\mathbf{0}$ and $\mathbf{E}_0=\mathbf{0}$, then $\mathbf{E}\equiv\mathbf{0}$, $\mathbf{J}\equiv\mathbf{0}$ and $\mathbf{H}_0=\mathbf{0}$ and finally $\mathbf{E}\equiv\mathbf{0}$, $\mathbf{H}\equiv\mathbf{0}$ and $\mathbf{J}_0=\mathbf{0}$ and to obtain

$$\int_0^T\int_\Omega(\varepsilon\dot{\mathbf{E}}\cdot\mathbf{f}^*-\nabla\times\mathbf{E}\cdot\mathbf{g}^*+\mathbf{E}\cdot\mathbf{l}^*)dx\,dt=0,\quad \mathbf{E}_0=0,\tag{4.16}$$

$$\int_0^T\int_\Omega(\mu\dot{\mathbf{H}}\cdot\mathbf{g}^*+\nabla\times\mathbf{H}\cdot\mathbf{f}^*)dx\,dt=0,\qquad \mathbf{H}_0=0,\tag{4.17}$$

$$\int_0^T\int_\Omega\Big[\frac{1}{\sigma}(\alpha\dot{\mathbf{J}}+\mathbf{J})\cdot\mathbf{l}^*-\mathbf{J}\cdot\mathbf{f}^*\Big]dx\,dt=0,\qquad \mathbf{J}_0=0,\tag{4.18}$$

respectively.

The initial conditions, which must be considered in these three identities, suggest to proceed as we have done for the uniqueness theorem, assuming now

$$\begin{aligned}\mathbf{E}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{A}(x) & \tau\le t\le T\end{cases},\\ \mathbf{H}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{B}(x) & \tau\le t\le T\end{cases},\\ \mathbf{J}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{C}(x) & \tau\le t\le T\end{cases},\end{aligned}\tag{4.19}$$

where τ is a fixed value in $(0,T)$, for every $(\mathbf{A},\mathbf{B},\mathbf{C})\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$, such that (3.6) holds; therefore, with this choice $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$.

Substituting (4.19) into (4.16)–(4.18), we get three relations where the range of integration $(0,T)$ reduces to (τ,T). Then, differentiating with respect to τ the relations so derived and putting

$$\mathbf{f}_i^*(x,\tau) = \int_\tau^T \mathbf{f}^*(x,t)dt \quad \text{with} \quad \frac{d}{d\tau}\,\mathbf{f}_i^*(x,\tau) = -\mathbf{f}^*(x,\tau) \tag{4.20}$$

and analogous expressions for $\mathbf{g}^*$ and $\mathbf{l}^*$, we obtain the following system

$$\begin{aligned}\varepsilon\int_\Omega \mathbf{A}(x)\cdot\frac{d}{d\tau}\,\mathbf{f}_i^*(x,\tau)\,dx + \int_\Omega [\nabla\times\mathbf{A}(x)\cdot\mathbf{g}_i^*(x,\tau)\\ - \mathbf{A}(x)\cdot\mathbf{l}_i^*(x,\tau)]\,dx = 0,\end{aligned} \tag{4.21}$$

$$\mu\int_\Omega \mathbf{B}(x)\cdot\frac{d}{d\tau}\,\mathbf{g}_i^*(x,\tau)\,dx - \int_\Omega \nabla\times\mathbf{B}(x)\cdot\mathbf{f}_i^*(x,\tau)\,dx = 0, \tag{4.22}$$

$$\frac{\alpha}{\sigma}\int_\Omega \mathbf{C}(x)\cdot\frac{d}{d\tau}\,\mathbf{l}_i^*(x,\tau)\,dx + \int_\Omega \mathbf{C}(x)\cdot\left[\mathbf{f}_i^*(x,\tau) - \frac{1}{\sigma}\,\mathbf{l}_i^*(x,\tau)\right]dx = 0. \tag{4.23}$$

We observe that this system must hold for every $(\mathbf{A},\mathbf{B},\mathbf{C}) \in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$; hence, in particular, it follows that $\mathbf{f}_i^*$ and $\mathbf{g}_i^*$ belong to $I_1(\Omega)$ and are equal to zero on $\partial\Omega$ for the absence of any surface integral in the system.

Thus, we can put

$$\mathbf{A}(x) = \mathbf{f}_i^*(x,\tau), \quad \mathbf{B}(x) = \mathbf{g}_i^*(x,\tau), \quad \mathbf{C}(x) = \mathbf{l}_i^*(x,\tau) \tag{4.24}$$

and, adding (4.21)–(4.23), we get

$$\begin{aligned}&\frac{d}{d\tau}\frac{1}{2}\int_\Omega \left\{\varepsilon[\mathbf{f}_i^*(x,\tau)]^2 + \mu[\mathbf{g}_i^*(x,\tau)]^2 + \frac{\alpha}{\sigma}[\mathbf{l}_i^*(x,\tau)]^2\right\}dx\\ &= \frac{1}{\sigma}\int_\Omega [\mathbf{l}_i^*(x,\tau)]^2\,dx - \int_\Omega [\nabla\times\mathbf{f}_i^*(x,\tau)\cdot\mathbf{g}_i^*(x,\tau) - \nabla\times\mathbf{g}_i^*(x,\tau)\cdot\mathbf{f}_i^*(x,\tau)]\,dx.\end{aligned} \tag{4.25}$$

Hence, taking account of the previous observation, since from $(4.20)_1$ we have $\mathbf{f}_i^*(x,T) = \mathbf{0}$, $\mathbf{g}_i^*(x,T) = \mathbf{0}$, $\mathbf{l}_i^*(x,T) = \mathbf{0}$, the integral over (τ,T) yields

$$\frac{1}{2}\int_\Omega \left\{\varepsilon[\mathbf{f}_i^*(x,\tau)]^2 + \mu[\mathbf{g}_i^*(x,\tau)]^2 + \frac{\alpha}{\sigma}[\mathbf{l}_i^*(x,\tau)]^2\right\}dx + \int_\tau^T\int_\Omega \frac{1}{\sigma}[\mathbf{l}_i^*(x,\xi)]^2\,dx\,d\xi = 0. \tag{4.26}$$

Therefore, we get

$$\mathbf{f}^*(x,t) = \mathbf{0}, \quad \mathbf{g}^*(x,t) = \mathbf{0}, \quad \mathbf{l}^*(x,t) = \mathbf{0}. \tag{4.27}$$

Thus, (4.15) reduces to

$$\int_\Omega \left(\varepsilon\mathbf{E}_0^*\cdot\mathbf{E}_0 + \mu\mathbf{H}_0^*\cdot\mathbf{H}_0 + \frac{\alpha}{\sigma}\,\mathbf{J}_0^*\cdot\mathbf{J}_0\right)dx = 0 \quad \forall\,(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}, \tag{4.28}$$

from which, choosing $\mathbf{E}(t)\equiv\mathbf{E}_0$, $\mathbf{H}(t)\equiv\mathbf{0}$, $\mathbf{J}(t)\equiv\mathbf{0}$, then $\mathbf{H}(t)\equiv\mathbf{H}_0$, $\mathbf{E}(t)\equiv\mathbf{0}$, $\mathbf{J}(t)\equiv\mathbf{0}$ and finally $\mathbf{J}(t)\equiv\mathbf{J}_0$, $\mathbf{E}(t)\equiv\mathbf{0}$, $\mathbf{H}(t)\equiv\mathbf{0}$, we get

$$\mathbf{E}_0^* = \mathbf{0}, \quad \mathbf{H}_0^* = \mathbf{0}, \quad \mathbf{J}_0^* = \mathbf{0}. \tag{4.29}$$

Equations (4.27) and (4.29) are contrary to the assumed hypothesis and hence $\mathcal{R}\times\mathcal{S}$ is dense in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$.

Theorem 4.3 (Existence) *There exists the solution of the problem P in the sense of Definition 4.2 for all data* $(\mathbf{f}, \mathbf{g}, \mathbf{l}) \in \mathcal{W}_0(\Omega, (0,T))$ *and* $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J}_0) \in \mathcal{T}$.

Proof To show the theorem we must prove that $\mathcal{R} \times \mathcal{S}$ is closed in $\mathcal{W}_0(\Omega, (0,T)) \times \mathcal{T}$. Let $(\mathbf{f}^{(n)}, \mathbf{g}^{(n)}, \mathbf{l}^{(n)}) \in \mathcal{R}$ and $(\mathbf{E}_0^{(n)}, \mathbf{H}_0^{(n)}, \mathbf{J}_0^{(n)}) \in \mathcal{S}$, $n = 1, 2, ...,$ be two sequences convergent to $(\mathbf{f}, \mathbf{g}, \mathbf{l}) \in \mathcal{W}_0(\Omega, (0,T))$ and $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J}_0) \in \mathcal{S}$, respectively; we denote by $(\mathbf{E}^{(n)}, \mathbf{H}^{(n)}, \mathbf{J}^{(n)}) \in \mathcal{H}(\Omega, (0,T))$, $n = 1, 2...,$ the corresponding solutions.

Applying Corollary 3.1 to the differences $\mathbf{E}^{(n)} - \mathbf{E}^{(m)}$, $\mathbf{H}^{(n)} - \mathbf{H}^{(m)}$, $\mathbf{J}^{(n)} - \mathbf{J}^{(m)}$ yields

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \left[\varepsilon |\mathbf{E}^{(n)} - \mathbf{E}^{(m)}|^2 + \mu |\mathbf{H}^{(n)} - \mathbf{H}^{(m)}|^2 + \frac{\alpha}{\sigma} |\mathbf{J}^{(n)} - \mathbf{J}^{(m)}|^2 \right] dx \\
&\leq e^T \Bigg\{ \left[\frac{1}{2} \int_\Omega \left[\varepsilon |\mathbf{E}_0^{(n)} - \mathbf{E}_0^{(m)}|^2 + \mu |\mathbf{H}_0^{(n)} - \mathbf{H}_0^{(m)}|^2 + \frac{\alpha}{\sigma} |\mathbf{J}_0^{(n)} - \mathbf{J}_0^{(m)}|^2 \right] dx \right. \\
&\quad + M \int_0^T \int_\Omega [|\mathbf{f}^{(n)} - \mathbf{f}^{(m)}|^2 + |\mathbf{g}^{(n)} - \mathbf{g}^{(m)}|^2 + |\mathbf{l}^{(n)} - \mathbf{l}^{(m)}|^2] dx\, dt \Bigg\}
\end{aligned} \tag{4.30}$$

and hence it follows that $(\mathbf{E}^{(n)}, \mathbf{H}^{(n)}, \mathbf{J}^{(n)})$, $n = 1, 2...,$ is a Cauchy sequence; thus, there exists the limit

$$\lim_{n \to \infty} (\mathbf{E}^{(n)}, \mathbf{H}^{(n)}, \mathbf{J}^{(n)}) = (\mathbf{E}, \mathbf{H}, \mathbf{J}) \in \mathcal{H}(\Omega, (0,T)). \tag{4.31}$$

Substituting the solutions and the corresponding sources into equations (3.1) – (3.3) gives a sequence of identities; the limit as $n \to +\infty$ is an analogous identity expressed in terms of $(\mathbf{f}, \mathbf{g}, \mathbf{l})$ and $(\mathbf{E}, \mathbf{H}, \mathbf{J})$, which is the solution of our problem.

We can now prove the uniqueness and the existence of the strong solution.

Theorem 4.4 *There exists a unique strong solution of the problem P in the sense of Definition 4.1 for all data* $(\mathbf{f}, \mathbf{g}, \mathbf{l}) \in \mathcal{W}_1(\Omega, (0,T))$ *and* $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J}_0) \in I_1(\Omega) \times I_1(\Omega) \times L^2(\Omega)$.

Proof We observe that a strong solution, when it exists, coincides with the weak solution of the problem P. In fact, let $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{J}}) \in \mathcal{H}(\Omega, (0,T))$ be such a strong solution, corresponding to given initial conditions $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{J}_0) \in I_1(\Omega) \times I_1(\Omega) \times L^2(\Omega)$ and sources $(\mathbf{f}, \mathbf{g}, \mathbf{l}) \in \mathcal{W}_1(\Omega, (0,T))$, then it satisfies the system (3.1) – (3.3) almost everywhere. It is enough to take the integrals over Ω and $(0,T)$ of the inner product of each equation of the system with any $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{p}/\sigma$, respectively, such that $(\mathbf{e}, \mathbf{h}, \mathbf{p}) \in \mathcal{H}(\Omega, (0,T))$ and $\mathbf{e}(\mathbf{x}, T) = \mathbf{0}$, $\mathbf{h}(\mathbf{x}, T) = \mathbf{0}$, $\mathbf{p}(\mathbf{x}, T) = \mathbf{0}$, and subtract the second and the third relations from the first one, to arrive at (4.1), which characterizes weak solutions.

Therefore, applying Theorem 4.1, the uniqueness of the strong solution follows at once.

For the existence of the strong solution, let $(\mathbf{E}', \mathbf{H}', \mathbf{J}') \in \mathcal{W}(\Omega, (0,T))$ be the weak solution to the problem P, whose existence and uniqueness have been already proved, corresponding to suitable data $(\mathbf{f}', \mathbf{g}', \mathbf{l}') \in \mathcal{W}_0(\Omega, (0,T))$ and $(\mathbf{E}'_0, \mathbf{H}'_0, \mathbf{J}'_0) \in I(\Omega) \times I(\Omega) \times L^2(\Omega)$. This solution satisfies (4.1) for any $(\mathbf{e}, \mathbf{h}, \mathbf{p}) \in \mathcal{H}(\Omega, (0,T))$ such that $\mathbf{e}(\mathbf{x}, T) = \mathbf{0}$, $\mathbf{h}(\mathbf{x}, T) = \mathbf{0}$, $\mathbf{p}(\mathbf{x}, T) = \mathbf{0}$; therefore, as we have done to prove the uniqueness theorem, we can choose the form (4.3) for $(\mathbf{e}, \mathbf{h}, \mathbf{p})$ and derive a relation analogous to (4.6), which allows us to conclude that both $\mathbf{E}_1'$ and $\mathbf{H}_1'$, defined by (4.5), belong to $I_1(\Omega)$ and that $(\mathbf{E}', \mathbf{H}', \mathbf{J}')$ satisfies the following system

$$\varepsilon\dot{\mathbf{E}}_1' = \nabla \times \mathbf{H}_1' - \mathbf{J}_1' - \mathbf{f}_1' + \varepsilon\mathbf{E}_0', \tag{4.32}$$

$$\mu\dot{\mathbf{H}}_1' = -\nabla \times \mathbf{E}_1' + \mathbf{g}_1' + \mu\mathbf{H}_0', \tag{4.33}$$

$$\alpha\dot{\mathbf{J}}_1' = -\mathbf{J}_1' + \sigma\mathbf{E}_1' + \mathbf{l}_1' + \alpha\mathbf{J}_0'. \tag{4.34}$$

We can now fix the suitable data as follows

$$\mathbf{f}' = -\frac{1}{\mu}\,\nabla \times \mathbf{g} + \frac{1}{\alpha}\,\mathbf{l}, \qquad \mathbf{g}' = \frac{1}{\varepsilon}\,\nabla \times \mathbf{f}, \qquad \mathbf{l}' = -\frac{\sigma}{\varepsilon}\,\mathbf{f} - \frac{1}{\alpha}\,\mathbf{l}, \tag{4.35}$$

$$\mathbf{E}_0' = \frac{1}{\varepsilon}(\nabla \times \mathbf{H}_0 - \mathbf{J}_0), \qquad \mathbf{H}_0' = -\frac{1}{\mu}\,\nabla \times \mathbf{E}_0, \qquad \mathbf{J}_0' = \frac{1}{\alpha}(\sigma\mathbf{E}_0 - \mathbf{J}_0). \tag{4.36}$$

Then, we put

$$\tilde{\mathbf{E}} = \mathbf{E}_1' - \frac{1}{\varepsilon}\,\mathbf{f}_1 + \mathbf{E}_0, \quad \tilde{\mathbf{H}} = \mathbf{H}_1' + \frac{1}{\mu}\,\mathbf{g}_1 + \mathbf{H}_0, \quad \tilde{\mathbf{J}} = \mathbf{J}_1' + \frac{1}{\alpha}\,\mathbf{l}_1 + \mathbf{J}_0, \tag{4.37}$$

which yield

$$\dot{\mathbf{E}}_1' = \dot{\tilde{\mathbf{E}}} + \frac{1}{\varepsilon}\,\mathbf{f}, \qquad \dot{\mathbf{H}}_1' = \dot{\tilde{\mathbf{H}}} - \frac{1}{\mu}\,\mathbf{g}, \qquad \dot{\mathbf{J}}_1' = \dot{\tilde{\mathbf{J}}} - \frac{1}{\alpha}\,\mathbf{l} \tag{4.38}$$

and

$$\tilde{\mathbf{E}}(x,0) = \mathbf{E}_0(x), \quad \tilde{\mathbf{H}}(x,0) = \mathbf{H}_0(x), \quad \tilde{\mathbf{J}}(x,0) = \mathbf{J}_0(x). \tag{4.39}$$

Substituting (4.38), (4.36) and the expressions of $(\mathbf{E}_1', \mathbf{H}_1', \mathbf{J}_1')$, derived from (4.37), together with the expressions of $(\mathbf{f}_1', \mathbf{g}_1', \mathbf{l}_1')$, which follow from (4.35), we have

$$\varepsilon\dot{\tilde{\mathbf{E}}} = \nabla \times \tilde{\mathbf{H}} - \tilde{\mathbf{J}} - \mathbf{f}, \tag{4.40}$$

$$\mu\dot{\tilde{\mathbf{H}}} = -\nabla \times \tilde{\mathbf{E}} + \mathbf{g}, \tag{4.41}$$

$$\alpha\dot{\tilde{\mathbf{J}}} = -\tilde{\mathbf{J}} + \sigma\tilde{\mathbf{E}} + \mathbf{l} \tag{4.42}$$

and hence see that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{J}})$ is the strong solution of the problem P.

5 Asymptotic Stability

The problem P can be transformed into an equivalent one characterized by zero initial data, by putting

$$\breve{\mathbf{E}}(x,t) = \mathbf{E}(x,t) - \mathbf{u}(x,t), \qquad \breve{\mathbf{H}}(x,t) = \mathbf{H}(x,t) - \mathbf{v}(x,t),$$
$$\breve{\mathbf{J}}(x,t) = \mathbf{J}(x,t) - \mathbf{w}(x,t),$$

where $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ are regular fields with support compact in $\Omega \times \mathbf{R}^+$ such that

$$\nabla \cdot \mathbf{u}(x,t) = 0, \qquad \nabla \cdot \mathbf{v}(x,t) = 0$$

and

$$\mathbf{u}(x,0)=\mathbf{E}_0(x),\qquad \mathbf{v}(x,0)=\mathbf{H}_0(x),\qquad \mathbf{w}(x,0)=\mathbf{J}_0(x).$$

On substituting the expressions of $(\mathbf{E},\mathbf{H},\mathbf{J})$ in terms of $(\breve{\mathbf{E}},\breve{\mathbf{H}},\breve{\mathbf{J}})$ and $(\mathbf{u},\mathbf{v},\mathbf{w})$, equations (3.1) – (3.3) assume a similar form but with the terms

$$\begin{aligned}\mathbf{F}(x,t)&=-\mathbf{f}(x,t)-\mathbf{w}(x,t)+\nabla\times\mathbf{v}(x,t)-\varepsilon\dot{\mathbf{u}}(x,t),\\ \mathbf{G}(x,t)&=\mathbf{g}(x,t)-\nabla\times\mathbf{u}(x,t)-\mu\dot{\mathbf{v}}(x,t),\\ \mathbf{I}(x,t)&=\mathbf{l}(x,t)+\sigma\mathbf{u}(x,t)-\mathbf{w}(x,t)-\alpha\dot{\mathbf{w}}(x,t)\end{aligned}$$

to be considered as three new sources in the corresponding equations.

Therefore, without changing the notation of the fields $(\mathbf{E},\mathbf{H},\mathbf{J})$, the new problem is given by

$$\varepsilon\dot{\mathbf{E}}(x,t)-\nabla\times\mathbf{H}(x,t)+\mathbf{J}(x,t)=\mathbf{F}(x,t), \tag{5.1}$$

$$\mu\dot{\mathbf{H}}(x,t)+\nabla\times\mathbf{E}(x,t)=\mathbf{G}(x,t), \tag{5.2}$$

$$\alpha\dot{\mathbf{J}}(x,t)+\mathbf{J}(x,t)-\sigma\mathbf{E}(x,t)=\mathbf{I}(x,t) \tag{5.3}$$

with (3.4), the new initial conditions

$$\mathbf{E}_0(x)=\mathbf{0},\quad \mathbf{H}_0(x)=\mathbf{0},\quad \mathbf{J}_0(x)=\mathbf{0} \tag{5.4}$$

and the boundary condition (3.6), which holds because of the hypotheses on $\mathbf{u}$ and $\mathbf{v}$, equal to zero on $\partial\Omega$.

We introduce the Fourier transform of any $f\colon \mathbf{R}^+\to\mathbf{R}^n$, identified with the causal extension on $(-\infty,0)$, where f is put equal to zero, i.e.

$$\hat{f}(\omega)=\int_{-\infty}^{+\infty} f(t)\exp[-i\omega t]dt, \tag{5.5}$$

and recall that if $f,\ f'\in L^2(\mathbf{R}^+)$ then $\hat{f},\ \hat{f}'\in L^2(\mathbf{R})$ and we have

$$\hat{f}'(\omega)=i\omega\hat{f}(\omega)-f(0),\qquad f(0)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\,d\omega. \tag{5.6}$$

We denote by P' the new problem (5.1) – (5.4) with the boundary condition (3.6), for which, since it holds for any $t\in\mathbf{R}^+$, Plancherel's theorem justifies the assumption that

$$\int_{\partial\Omega}\hat{\mathbf{E}}(x,\omega)\times\hat{\mathbf{H}}^*(x,\omega)\cdot\mathbf{n}(x)\,da\geq 0\quad \forall\,(\hat{\mathbf{E}},\hat{\mathbf{H}})\in\hat{\Sigma}',\quad \forall\,(x,\omega)\in\partial\Omega\times\mathbf{R}, \tag{5.7}$$

where * denotes the complex conjugate and $\hat{\Sigma}'$ is the set of Fourier's transforms of the electromagnetic fields $(\mathbf{E},\mathbf{H})\in\Sigma'$.

We consider the function spaces of Section 4, where $(0,T)$ is changed into $\mathbf{R}^+$, and, in particular, we introduce

$$\mathcal{W}_2(\Omega,\mathbf{R}^+) = \Big\{(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_0(\Omega,\mathbf{R}^+)\colon \frac{\partial^{n+1}}{\partial t^{n+1}}(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_0(\Omega,\mathbf{R}^+), \\ \left[\frac{\partial^n}{\partial t^n}(\mathbf{F},\mathbf{G},\mathbf{I})\right]_{t=0} = 0 \quad (n=0,1,2,3)\Big\},$$

where the last conditions, on the initial values of the new sources and of their derivatives with respect to time, are satisfied by choosing the derivatives of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ at $t=0$ opportunely.

When the Fourier transforms with respect to time are considered, the function spaces can be distinguished with a superposed $\hat{}$; in particular $\mathcal{W}(\Omega,\mathbf{R}^+)$ becomes

$$\hat{\mathcal{W}}(\Omega,\mathbf{R}) = \big\{(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}}) \in L^2(\mathbf{R};I(\Omega)) \times L^2(\mathbf{R};I(\Omega)) \times L^2(\mathbf{R};L^2(\Omega))\colon i\omega\hat{\mathbf{E}}, \\ i\omega\hat{\mathbf{H}} \in L^2(\mathbf{R};I(\Omega)) \text{ and } (\hat{\mathbf{E}},\hat{\mathbf{H}}) \text{ satisfies (5.7) on } \partial\Omega\times\mathbf{R}\big\}$$

and analogously for $\hat{\mathcal{W}}_2(\Omega,\mathbf{R})$.

Theorem 5.1 *If*

$$\mathcal{I}(\omega) = \int_\Omega \big(|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{H}}|^2 + |\hat{\mathbf{J}}|^2 + |\nabla\times\hat{\mathbf{E}}|^2 + |\nabla\times\hat{\mathbf{H}}|^2\big)\,dx, \tag{5.8}$$

under suitable conditions on the material constants the following inequality

$$(\min\{\varepsilon,\mu\})^2\mathcal{I}(\omega) \le \delta^2(\omega)\int_\Omega \big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx \tag{5.9}$$

holds with $\delta(\omega)$ a positive function of the material constants for any $\omega\in\mathbf{R}$.

Proof Application of Fourier's transform to the system (5.1)–(5.3), taking account of $(5.6)_1$ and (5.4), yields

$$i\omega\varepsilon\hat{\mathbf{E}}(x,\omega) - \nabla\times\hat{\mathbf{H}}(x,\omega) + \hat{\mathbf{J}}(x,\omega) = \hat{\mathbf{F}}(x,\omega), \tag{5.10}$$

$$i\omega\mu\hat{\mathbf{H}}(x,\omega) + \nabla\times\hat{\mathbf{E}}(x,\omega) = \hat{\mathbf{G}}(x,\omega), \tag{5.11}$$

$$(1+i\omega\alpha)\hat{\mathbf{J}}(x,\omega) - \sigma\hat{\mathbf{E}}(x,\omega) = \hat{\mathbf{I}}(x,\omega). \tag{5.12}$$

From this system, the integrals over Ω of the inner products of the first equation with $\hat{\mathbf{E}}^*$, $\hat{\mathbf{J}}^*$ and $\nabla\times\hat{\mathbf{H}}^*$ yield

$$i\omega\varepsilon\int_\Omega |\hat{\mathbf{E}}|^2\,dx - \int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{E}}^*\,dx + \int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^*\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx, \tag{5.13}$$

$$i\omega\varepsilon\int_\Omega \hat{\mathbf{E}}\cdot\hat{\mathbf{J}}^*\,dx - \int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{J}}^*\,dx + \int_\Omega |\hat{\mathbf{J}}|^2\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx, \tag{5.14}$$

$$i\omega\varepsilon\int_\Omega \hat{\mathbf{E}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx - \int_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx + \int_\Omega \hat{\mathbf{J}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx; \tag{5.15}$$

analogously, the inner products of the conjugate of the second equation with $\hat{\mathbf{H}}$ and $\nabla \times \hat{\mathbf{E}}$ give

$$-i\omega\mu \int_\Omega |\hat{\mathbf{H}}|^2\, dx + \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx = \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx, \tag{5.16}$$

$$-i\omega\mu \int_\Omega \hat{\mathbf{H}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx + \int_\Omega |\nabla \times \hat{\mathbf{E}}|^2\, dx = \int_\Omega \hat{\mathbf{G}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx, \tag{5.17}$$

and finally from the inner products of the conjugate of the third equation with $\hat{\mathbf{J}}$ and $\hat{\mathbf{E}}$ it follows that

$$(1 - i\omega\alpha) \int_\Omega |\hat{\mathbf{J}}|^2\, dx - \sigma \int_\Omega \hat{\mathbf{E}}^* \cdot \hat{\mathbf{J}}\, dx = \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}}\, dx, \tag{5.18}$$

$$(1 - i\omega\alpha) \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx - \sigma \int_\Omega |\hat{\mathbf{E}}|^2\, dx = \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{E}}\, dx. \tag{5.19}$$

Let $\omega \neq 0$. The real parts of (5.19), (5.18), (5.17) and (5.15) yield

$$\sigma \int_\Omega |\hat{\mathbf{E}}|^2\, dx = -\operatorname{Re} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{E}}\, dx + \operatorname{Re} \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx + \omega\alpha \operatorname{Im} \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx, \tag{5.20}$$

$$\int_\Omega |\hat{\mathbf{J}}|^2\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}}\, dx + \sigma \operatorname{Re} \int_\Omega \hat{\mathbf{E}}^* \cdot \hat{\mathbf{J}}\, dx, \tag{5.21}$$

$$\int_\Omega |\nabla \times \hat{\mathbf{E}}|^2\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx - \omega\mu \operatorname{Im} \int_\Omega \hat{\mathbf{H}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx, \tag{5.22}$$

$$\begin{aligned} \int_\Omega |\nabla \times \hat{\mathbf{H}}|^2\, dx &= -\operatorname{Re} \int_\Omega \hat{\mathbf{F}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx \\ &\quad - \omega\varepsilon \operatorname{Im} \int_\Omega \hat{\mathbf{E}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx + \operatorname{Re} \int_\Omega \hat{\mathbf{J}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx; \end{aligned} \tag{5.23}$$

while the imaginary part of (5.16) gives

$$\mu \int_\Omega |\hat{\mathbf{H}}|^2\, dx = \frac{1}{\omega}\Big(-\operatorname{Im} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx + \operatorname{Im} \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx \Big). \tag{5.24}$$

In (5.20) – (5.24) we have some quantities to be derived.

First, from the real part of (5.16) we have at once

$$\operatorname{Re} \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx, \tag{5.25}$$

which, taking account of (5.7), allows us to derive from the real part of (5.13)

$$\operatorname{Re}\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^*\,dx = \operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da; \tag{5.26}$$

therefore, (5.21) becomes

$$\begin{aligned}\int_\Omega |\hat{\mathbf{J}}|^2\,dx = \operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx + \sigma\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big).\end{aligned} \tag{5.27}$$

Then, we consider (5.19), whose imaginary part, on account of (5.26), yields

$$\begin{aligned}\operatorname{Im}\int_\Omega \hat{\mathbf{J}}^*\cdot\hat{\mathbf{E}}\,dx = \operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\alpha\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),\end{aligned} \tag{5.28}$$

whence, taking account of (5.26) too, (5.20) assumes the following form

$$\begin{aligned}\sigma\int_\Omega |\hat{\mathbf{E}}|^2\,dx = -\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\alpha\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ + (1+\omega^2\alpha^2)\Big[\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big].\end{aligned} \tag{5.29}$$

Substituting this relation into the imaginary part of (5.13) we get

$$\begin{aligned}\operatorname{Im}\int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{E}}^*\,dx = -\operatorname{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \Big(\omega^2\,\frac{\alpha\varepsilon}{\sigma} - 1\Big)\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ - \omega\,\frac{\varepsilon}{\sigma}\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2) - \alpha\Big]\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)\end{aligned} \tag{5.30}$$

useful to rewrite (5.24) and (5.22) as follows

$$\begin{aligned}\mu\int_\Omega |\hat{\mathbf{H}}|^2\,dx = -\frac{1}{\omega}\Big(\operatorname{Im}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx + \operatorname{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\Big) - \frac{\varepsilon}{\sigma}\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ + \frac{1}{\omega}\Big(\omega^2\frac{\alpha\varepsilon}{\sigma} - 1\Big)\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2) - \alpha\Big]\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),\end{aligned} \tag{5.31}$$

$$\begin{aligned}\int_\Omega |\nabla\times\hat{\mathbf{E}}|^2\,dx &= \mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx+\omega\mu\Big\{-\mathrm{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\\ &-\omega\,\frac{\varepsilon}{\sigma}\;\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\Big(\omega^2\,\frac{\alpha\varepsilon}{\sigma}-1\Big)\,\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\omega\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2)-\alpha\Big]\\ &\times\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)\Big\}.\end{aligned}\tag{5.32}$$

Finally, for (5.23) we must derive its last term, which follows from the real part of (5.14), using (5.27) and (5.28), i.e.

$$\begin{aligned}\mathrm{Re}\int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{J}}^*\,dx &= -\,\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx-\omega\varepsilon\,\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx\\ &+(\sigma-\omega^2\varepsilon\alpha)\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),\end{aligned}\tag{5.33}$$

hence, we have

$$\begin{aligned}\int_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx &= -\,\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx+\omega\varepsilon\Big[-\mathrm{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\\ &+\omega\,\frac{\varepsilon}{\sigma}\Big(-\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\omega\alpha\,\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx\Big)\Big]-\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx\\ &-2\omega\varepsilon\,\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx+\Big\{\sigma+\omega^2\varepsilon\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2)-2\alpha\Big]\Big\}\\ &\times\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big).\end{aligned}\tag{5.34}$$

Thus, from (5.29) multiplied by ε/σ, (5.31), (5.27), (5.32) and (5.34) we have

$$\begin{aligned}\int_\Omega \big(\varepsilon|\hat{\mathbf{E}}|^2+\mu|\hat{\mathbf{H}}|^2+|\hat{\mathbf{J}}|^2+|\nabla\times\hat{\mathbf{E}}|^2+|\nabla\times\hat{\mathbf{H}}|^2\big)dx &= \Big\{\frac{\varepsilon}{\sigma}(1+\alpha^2\omega^2)[2+(\varepsilon+\mu)\omega^2]\\ &+2\sigma-\alpha[1+(2\varepsilon+\mu)\omega^2]\Big\}\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx\\ &-\int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)-\frac{1}{\omega}[1+(\varepsilon+\mu)\omega^2]\,\mathrm{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx-\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx\\ &-\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx-\frac{1}{\omega}\;\mathrm{Im}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx\end{aligned}\tag{5.35}$$

$$
-\frac{\varepsilon}{\sigma}[2+(\varepsilon+\mu)\omega^2]\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\frac{1}{\omega}\Big\{\frac{\varepsilon\alpha}{\sigma}\,\omega^2[2+(\varepsilon+\mu)\omega^2]
$$

$$
-[1+(2\varepsilon+\mu)\omega^2]\Big\}\,\mathrm{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+2\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx.
$$

In this equality we have the presence of a surface integral, which must satisfy the boundary condition (5.7). This term can be neglected if its coefficient is not positive, that is when

$$
\varphi(\xi)\equiv\frac{\varepsilon}{\sigma}\,\alpha^2(\varepsilon+\mu)\xi^2+\left[\frac{\varepsilon}{\sigma}(\varepsilon+\mu+2\alpha^2)-\alpha(2\varepsilon+\mu)\right]\xi+2\frac{\varepsilon}{\sigma}+2\sigma-\alpha\geq 0 \tag{5.36}
$$

for all $\omega\in\mathbf{R}$, with $\xi=\omega^2$.

We give some sufficient conditions to can neglect this boundary term in (5.35).

We first examine the case when all the coefficients in (5.36) are positive or null; thus, we impose that the following system

$$
\begin{cases} 2\varepsilon\alpha^2-(2\varepsilon+\mu)\sigma\alpha+\varepsilon(\varepsilon+\mu)\geq 0,\\ 2\sigma^2-\alpha\sigma+2\varepsilon\geq 0 \end{cases} \tag{5.37}
$$

must be satisfied for all positive values of the material constants.

If we consider the first inequality in function of α and the second of σ, the system is always satisfied if the discriminants are not greater than zero, i.e. when σ and α satisfy these inequalities

$$
\sigma\leq\frac{2\varepsilon}{2\varepsilon+\mu}\,\sqrt{2(\varepsilon+\mu)},\qquad \alpha\leq 4\sqrt{\varepsilon}. \tag{5.38}
$$

Moreover, some other particular cases can be considered by imposing that be positive or null the sum of the first two terms or the sum of the second and the third term of each inequality in (5.37). Thus, we see that if one of the following conditions, relative to $(5.37)_1$,

$$
\frac{\alpha}{\sigma}\geq\frac{2\varepsilon+\mu}{2\varepsilon}\qquad\text{or}\qquad \alpha\sigma\leq\frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu} \tag{5.39}
$$

is satisfied together with one of the other two conditions, corresponding now to the second inequality of (5.37),

$$
\alpha\leq 2\sigma\quad\text{or}\quad \alpha\sigma\leq 2\varepsilon, \tag{5.40}
$$

then the system (5.37) holds and the boundary term is negligible in (5.35).

Finally, another interesting condition on the parameters can be easily derived by neglecting the boundary terms in (5.27) and (5.29), since their coefficients are negative for all $\omega\in\mathbf{R}$, and by assuming $\varepsilon/\sigma\geq\alpha$, which allows us to neglect the boundary terms also in (5.31) and (5.32), while in (5.34) we can consider $\varepsilon/\sigma\geq 2\alpha$. Therefore, the other sufficient condition is the following one

$$
\alpha\sigma\leq\frac{\varepsilon}{2}. \tag{5.41}
$$

This inequality is a simpler restriction on the product of α and σ for it is expressed in term of ε only. However, if we consider that the previous case, when $(5.39)_2$ holds together with $(5.40)_2$, since $2\varepsilon > \frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}$, is expressed by the unique condition

$$\alpha\sigma \le \frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}, \tag{5.42}$$

we see that (5.41) is more restrictive than (5.42), where also μ is interested, being $\dfrac{\varepsilon}{2} < \dfrac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}$.

Thus, the equality (5.35) becomes an inequality whenever the boundary term can be neglected. In these cases, let us consider the sum of the moduli of the coefficients of the real and imaginary parts of the same integral; then we denote by $\gamma(\omega)$ the maximum of these quantities and we get

$$\begin{aligned}
\min\{\varepsilon,\mu\}\mathcal{I}(\omega) \le \gamma(\omega)\Bigg[&\Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{E}}|^2\,dx\Big)^{1/2}\\
&+\Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{J}}|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx\Big)^{1/2}\\
&+\Big(\int_\Omega |\hat{\mathbf{G}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{H}}|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\hat{\mathbf{G}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla\times\hat{\mathbf{E}}|^2\,dx\Big)^{1/2}\\
&+\Big(\int_\Omega |\hat{\mathbf{I}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{E}}|^2\,dx\Big)^{1/2}+\Big(\int_\Omega |\hat{\mathbf{I}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{J}}|^2\,dx\Big)^{1/2}\Bigg]\\
&\le 7\gamma(\omega)\Big[\int_\Omega \big(|\hat{\mathbf{F}}|^2+|\hat{\mathbf{G}}|^2+|\hat{\mathbf{I}}|^2\big)\,dx\Big]^{1/2}[\mathcal{I}(\omega)]^{1/2}.
\end{aligned} \tag{5.43}$$

Hence we have (5.9) immediately.

Let $\omega = 0$. In this case we are interested in finding static solutions, consequently (5.10)$-$(5.12) must be considered with $\omega = 0$, as well as (5.13)$-$(5.19). Proceeding as we have done previously, we see that (5.35) reduces to

$$\begin{aligned}
\mathcal{I}_0(0) \equiv \int_\Omega &\big(|\hat{\mathbf{E}}|^2+|\hat{\mathbf{J}}|^2+|\nabla\times\hat{\mathbf{E}}|^2+|\nabla\times\hat{\mathbf{H}}|^2\big)\,dx\\
=\Big(\frac{1}{\sigma}+2\sigma\Big)&\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx\Big)-\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx\\
-\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx&+\mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx-\frac{1}{\sigma}\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx\\
+2\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx&-\Big(2\sigma+\frac{1}{\sigma}\Big)\int_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da
\end{aligned} \tag{5.44}$$

$$\leq c\Big[\int_\Omega \big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx\Big]^{1/2} [\mathcal{I}_0(0)]^{1/2},$$

where all the fields are functions of $(x,0)$ and c is a constant [9]. Hence it follows that

$$\mathcal{I}_0(0) \leq C \int_\Omega \big[|\hat{\mathbf{F}}(x,0)|^2 + |\hat{\mathbf{G}}(x,0)|^2 + |\hat{\mathbf{I}}(x,0)|^2\big]\,dx, \tag{5.45}$$

i.e. a relation similar to (5.9) with a constant C.

Theorem 5.2 *Let the sources be* $(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_2(\Omega,\mathbf{R}^+)$*, then the inverse Fourier transforms of* $(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}}) \in \hat{H}(\Omega,\mathbf{R})$ *exist and are* L^2*-functions with zero initial data.*

Proof In (5.9) $\delta(\omega)$ is a positive function of $\omega \in \mathbf{R}$ and approaches infinity as ω^4; such a condition, together with the hypotheses on the sources, states that the integral on $\mathbf{R}$ exists for the right-hand side of (5.9), that is

$$\int_{-\infty}^{+\infty}\int_\Omega \delta^2(\omega)\big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx\,d\omega < +\infty. \tag{5.46}$$

Therefore, (5.9) gives

$$\int_{-\infty}^{+\infty} \mathcal{I}(\omega)\,d\omega \leq \int_{-\infty}^{+\infty}\int_\Omega \left(\frac{\delta(\omega)}{\min\{\varepsilon,\mu\}}\right)^2 \big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx\,d\omega, \tag{5.47}$$

i.e. there exists finite the integral over $\mathbf{R}$ of $\mathcal{I}(\omega)$.

Application of Plancherel's theorem yields the existence of the inverse Fourier transforms of $(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}})$; moreover, these solutions have the asymptotic behaviour which follows by belonging to the space $\hat{H}(\Omega,\mathbf{R})$.

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