Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 105–118

Stability of the Stationary Solutions of the Differential Equations of Restricted Newtonian Problem with Incomplete Symmetry

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Received: July 10, 2002; Revised: December 23, 2002

Abstract: We investigate the Lyapunov stability of the stationary solutions of the differential equations of restricted six-body problem with the gravitational centre. The gravitational field is created by bodies P_0, P_1, P_2, P_3 and P_4 with masses m_0, m_1, m_2, m_3 and m_4, respectively. In this gravitational field the movement of a body P with zero mass $(m = 0)$ is investigated. The bodies P_1, P_2, P_3 and P_4 form a rhombus, rotating uniformly around the centre of gravity P_0. In the article we have formulated necessary and sufficient conditions of Lyapunov stability and instability of equilibrium point of this model. All necessary analytical calculations are executed in the system of symbolical calculations (SSC) "Mathematica".

Keywords: *Hamiltonian systems; stability.*

Mathematics Subject Classification (2000): 37J25, 37J40.

1 Introduction

It is known, that the restricted Newtonian many-body problem is very important for a wide class of applications, from theoretical physics to celestial mechanics and astrodynamics [1, 6]. It is well known [4, 5], that the differential equations of this problem are in general not integrable, therefore Poincaré considered the first problem should be the search for the exact particular solutions and the research of their stability [1]. The latter problem is the most difficult in the qualitative theory of the differential equations and can be solved within the framework of the Kolmogorov-Arnold-Mozer (·) theory [12, 13].
With occurrence of the systems of symbolic calculations, for example, Mathematica [10], possibilities of performance of symbolic calculations have essentially increased. Such calculations are necessary for correct application of the well known Arnold-Mozer theorem [13, 15]. Let's consider the following restricted 6-body problem in Grebenikov-Elmabsout

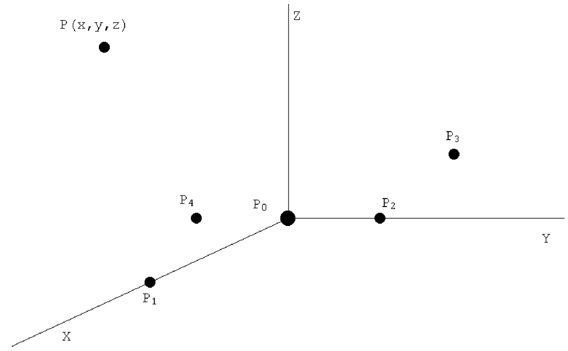

Figure 1.1

model [3, 7]. In the non-inertial Euclidean space P_0xyz there are six bodies P_0, P_1, P_2, P_3, P_4 and P with masses m_0, m_1, m_2, m_3, m_4 and m. It is shown, that in this model the bodies P_1, P_2, P_3 and P_4 move in one plane and form a rhombus, rotating uniformly around a body P_0 [2]. In this gravitational newtonian field, produced by mutual gravitation of five bodies, we investigate the motion of a body P with zero mass $m = 0$ (Figure 1.1).

The purpose of our work is the definition of the stationary solutions (states of equilibrium) of differential equations, describing this model, and the research of their Lyapunov stability by methods of computer algebra. It has been proved, that exact rhombus-like solutions do exist in this physical model, if the following conditions are executed [8]: a) the masses, located in the opposite vertices of a rhombus, are equal among themselves:

$$m_1 = m_3; \qquad m_2 = m_4; \tag{1}$$

b) relations of diagonals ρ_1, ρ_2, and masses m_1, m_2 of a rhombus are correlated as:

$$\lambda = \frac{\rho^3\left[8 - (1+\rho^2)^{3/2}\right]}{8\rho^3 - (1+\rho^2)^{3/2}}, \tag{2}$$

where $\dfrac{\rho_1}{\rho_2} = \rho$, $\dfrac{m_1}{m_2} = \lambda$.

2 Definition of Equilibrium State

Without loss of generality, it is possible to assume, that the gravitational rhombus rotates always in a plane P_0XY around an axis Z with a constant angular velocity ω.

It is obvious, that the sizes of a rhombus can be arbitrary, therefore we shall define coordinates of a rhombus as follows: $P_1(\alpha, 0)$, $P_2(0, 1)$, $P_3(-\alpha, 0)$, $P_4(0, -1)$.

In [9] it is shown that

$$m_2 = \frac{4m_0(1+\alpha^2)^{3/2}(\alpha^3 - 1) + m_1(8\alpha^3 - (1+\alpha^2)^{3/2})}{\alpha^3(8 - (1+\alpha^2)^{3/2})}, \tag{3}$$

therefore from conditions $\alpha > 0,\ m_0 > 0,\ m_1 > 0,\ m_2 > 0$ we receive admissible values of the parameter α: $1/\sqrt{3} < \alpha < \sqrt{3}$. For $\alpha \geq 1$ the masses can take any values and in the range $1/\sqrt{3} < \alpha < 1$ the relation

$$m_1 > \frac{4(1+\alpha^2)^{3/2}(1-\alpha^3)}{8\alpha^3 - (1+\alpha^2)^{3/2}}\, m_0 \tag{4}$$

should be satisfied. The angular velocity of rotation of the rhombus $P_1P_2P_3P_4$ is defined by the formula [2]

$$\omega = \sqrt{\frac{4fm_0(1+\alpha^2)^{3/2}(8\alpha^3 - (1+\alpha^2)^{3/2}) + fm_1(64\alpha^3 - (1+\alpha^2)^{3/2})}{4\alpha^3(1+\alpha^2)^{3/2}(8-(1+\alpha^2)^{3/2})}}, \tag{5}$$

where f is a gravitation constant.

Further for convenience we shall consider, that $f = 1$ and $m_0 = 0$.

The differential equations of motion of passive gravitating point P ($m = 0$) in uniformly rotating Cartesian frame P_0XYZ are [4]:

$$\begin{aligned}
\frac{d^2X}{dt^2} &= \omega^2 X + 2\omega\,\frac{dY}{dt} - \frac{X}{r^3} + \frac{\partial R}{\partial X},\\
\frac{d^2Y}{dt^2} &= \omega^2 Y 2\omega\,\frac{dX}{r^3} - \frac{Y}{r^3} + \frac{\partial R}{\partial Y},\\
\frac{d^2Z}{dt^2} &= -\frac{Z}{r^3} + \frac{\partial R}{\partial Z},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
R &= \sum_{j=1}^{4} m_j\left(\frac{1}{\Delta_j} - \frac{XX_j + YY_j + ZZ_j}{r_j^3}\right),\\
\Delta_j^2 &= (X-X_j)^2 + (Y-Y_j)^2 + (Z-Z_j)^2,\\
r^2 &= X^2 + Y^2 + Z^2, \qquad r_j^2 = X_j^2 + Y_j^2 + Z_j^2,
\end{aligned} \tag{7}$$

$X,\ Y,\ Z$ are the coordinates of the zero mass (point P), $X_j,\ Y_j,\ Z_j$ are the given coordinates of points P_j, ω is the angular velocity of rotation of the rhombus $P_1P_2P_3P_4$ around P_0. System (6) is not integrable in a general form, therefore we shall search for partial solutions, such as "equilibrium state". For this purpose we shall introduce a 6-dimensional phase space $x = X,\ y = Y,\ z = Z,\ u = \dfrac{dX}{dt},\ v = \dfrac{dY}{dt},\ w = \dfrac{dZ}{dt}$. Then the system (6) becomes

$$\begin{aligned}
\frac{dx}{dt} &= u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w,\\
\frac{du}{dt} &= \omega^2 x + 2\omega v - \frac{x}{r^3} + \frac{\partial R}{\partial x},\\
\frac{dv}{dt} &= \omega^2 y - 2\omega u - \frac{y}{r^3} + \frac{\partial R}{\partial y},\\
\frac{dw}{dt} &= -\frac{z}{r^3} + \frac{\partial R}{\partial z}.
\end{aligned} \tag{8}$$

Finding of equilibrium state of system (8) is reduced to the solution of the system of equations

$$
\begin{aligned}
&u = 0, \quad v = 0, \quad , w = 0,\\
&\omega^2 x - \frac{x}{(x^2+y^2+z^2)^{3/2}} - \Bigg(\frac{m_1(x-\alpha)}{((x-\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_2 x}{(x^2+(y-1)^2+z^2)^{3/2}}\\
&\qquad + \frac{m_3(x+\alpha)}{((x+\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_4 x}{(x^2+(y+1)^2+z^2)^{3/2}} \Bigg) = 0,\\
&\omega^2 y - \frac{y}{(x^2+y^2+z^2)^{3/2}} - \Bigg(\frac{m_1 y}{((x-\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_2(y-1)}{(x^2+(y-1)^2+z^2)^{3/2}}\\
&\qquad + \frac{m_3 y}{((x+\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_4(y+1)}{(x^2+(y+1)^2+z^2)^{3/2}} \Bigg) = 0,\\
&\frac{z}{(x^2+y^2+z^2)^{3/2}} - \Bigg(\frac{m_1 z}{((x-\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_2 z}{(x^2+(y-1)^2+z^2)^{3/2}}\\
&\qquad + \frac{m_3 z}{((x+\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_4 z}{(x^2+(y+1)^2+z^2)^{3/2}} \Bigg) = 0.
\end{aligned}
\tag{9}
$$

From the last equation it follows that $z = 0$, that is all stationary solutions lay in the plane P_0xy, and the solution of the system (9) is reduced to the solution of the following system

$$
\begin{aligned}
&\omega^2 x - \frac{x}{(x^2+y^2+z^2)^{3/2}} - \Bigg(\frac{m_1(x-\alpha)}{((x-\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_2 x}{(x^2+(y-1)^2+z^2)^{3/2}}\\
&\qquad + \frac{m_3(x+\alpha)}{((x+\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_4 x}{(x^2+(y+1)^2+z^2)^{3/2}} \Bigg) = 0,\\
&\omega^2 y - \frac{y}{(x^2+y^2+z^2)^{3/2}} - \Bigg(\frac{m_1 y}{((x-\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_2(y-1)}{(x^2+(y-1)^2+z^2)^{3/2}}\\
&\qquad + \frac{m_3 y}{((x+\alpha)^2+y^2+z^2)^{3/2}} + \frac{m_4(y+1)}{(x^2+(y+1)^2+z^2)^{3/2}} \Bigg) = 0.
\end{aligned}
\tag{10}
$$

The following theorem takes place.

Theorem 2.1 *Necessary and sufficient condition of existence of the stationary solutions of the restricted six-body problem is decidability of system (10) with respect to the unknown x and y.*

The equations (10) are nonlinear, therefore the question on their decidability can be studied by graphic and iteration techniques. In terms of the "Mathematica" system a graphic solution of system (10) is constructed. For example, for $m_1 = 0.5$ and $\alpha = 0.95$ the two curves are shown on Figure 2.1.

On this figure the bold points denote points P_0, P_1, P_2, P_3, P_4. Cross-points of the curves, laying on axes of coordinates, are denoted by N_i, other cross-points – by S_i. The points N_i and S_i are the equilibrium solutions of system (10). The calculations show, that the quantity of equilibrium states essentially depends both on the gravitational

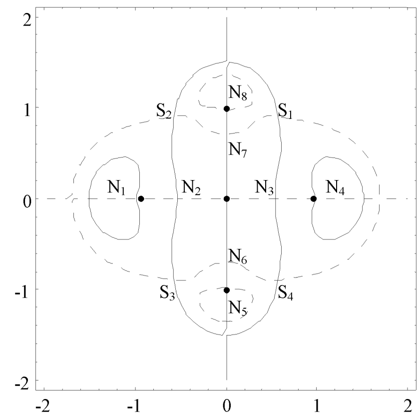

Figure 2.1

parameter m_1 and on the size of the diagonal α. Using Newton iteration method, we determine the coordinates of the equilibrium state for various values of the parameters m_1 and α.

3 Research of the Linear Stability of Equilibrium State

To investigate the linear stability of the equilibrium solutions of system of the differential equations (8) it is necessary to construct a linearized system of the differential equations in the neighborhood of points N_i and S_i, (Figure 2.1) with coordinates x_i^*, y_i^*, $z_i^* = 0$, and to study properties of eigenvalues of a matrix of this system. Denoting by x the phase vector $x = (u - u_i^*, v - v_i^*, w - w_i^*, x - x_i^*, y - y_i^*, z - z_i^*)$ and executing the procedure of linearization of the right parts of system (8) in the neighborhood of a phase point $x = 0$ in SSC "Mathematica", we shall get the system of linear differential equations

$$\frac{dx}{dt} = Ax. \tag{11}$$

Six-dimensional matrix A is

$$A = \begin{bmatrix} [0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & 0 & 0 & 2\omega & 0 \\ b & c & 0 & -2\omega & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

The elements a, b, c, d of matrix A depend on the values x_i^*, y_i^*, m_1, α, whose expressions

m	α	N_1		S_1	
		λ_1, λ_2	λ_3, λ_4	λ_1, λ_2	λ_3, λ_4
0.001	0.99985	±2.35155	±1.97710i	±0.994867i	±0.099199i
	0.9999	±2.35146	±1.97703i	±0.990894i	±0.130508i
	1.0031	±2.34591	±1.97251i	±0.744772i	±0.651662i
	1.0032	±2.34574	±1.97237i	±0.34841 + 0.70044i	±0.34841 - 0.70044i
0.003	0.99954	±2.28897	±1.94015i	±0.98424i	±0.173624i
	0.9999	±2.28842	±1.93971i	±0.95927i	±0.273309i
	1.0022	±2.28494	±1.9369i	±0.736147i	±0.661479i
	1.0023	±2.28479	±1.93678i	±0.04689 + 0.70116i	±0.04689 - 0.70116i
0.009	0.99863	±2.20639	±1.89254i	±0.948869i	±0.310446i
	0.999	±2.20593	±1.89218i	±0.906597i	±0.411469i
	1.0001	±2.20458	±1.89111i	±0.73989i	±0.658941i
	1.0002	±2.20446	±1.89101i	±0.051454 + 0.7022i	±0.051454 - 0.7022i
0.01	0.9985	±2.19734	±1.88743i	±0.937409i	±0.341927i
	0.9998	±2.19578	±1.88619i	±0.738788i	±0.660504i
	0.9999	±2.19566	±1.8861i	±0.0537 + 0.7025i	±0.0537 - 0.7025i
0.03	0.99546	±2.09331	±1.831071i	±0.14454 + 0.71804i	±0.14454 - 0.71804i
	0.9955	±2.09328	±1.83104i	±0.16925 + 0.7227i	±0.16925 - 0.7227i
0.09	0.986537	±1.98254	±1.78116i	±0.47268 + 0.84128i	±0.47268 - 0.84128i
	0.987	±1.98234	±1.78102i	±0.50501 + 0.85608i	±0.50501 - 0.85608i
0.1	0.98507	±1.97277	±1.77794i	±0.49967 + 0.85574i	±0.49967 - 0.85574i
	0.99	±1.97088	±1.77667i	±0.62882 + 0.91929i	±0.62882 - 0.91929i
0.9	0.8874	±2.02623	±2.00395i	±1.1724 + 1.3235i	±1.1724 - 1.3235i
	0.89	±2.02549	±2.00323i	±1.1821 + 1.3243i	±1.1821 - 1.3243i

Table 3.1. Eigenvalues of matrix A.

are quite cumbersome, therefore we shall present the expressions for a and d:

$$
\begin{aligned}
a = \omega^2 &+ \frac{3x_i^{*2}}{(x_i^{*2}+y_i^{*2})^{5/2}} - \frac{1}{(x_i^{*2}+y_i^{*2})^{3/2}} + \frac{3x_i^{*2}m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} \\
&- \frac{6x_i^*\alpha m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} + \frac{3\alpha m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} - \frac{m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{3/2}} \\
&+ \frac{3x_i^{*2}m_2}{(x_i^{*2}+(y_i^*-1)^2)^{5/2}} - \frac{m_2}{(x_i^{*2}+(y_i^*-1)^2)^{5/2}} + \frac{3x_i^{*2}m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} \qquad (13) \\
&- \frac{m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{3/2}} + \frac{6x_i^{*2}\alpha m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} + \frac{3\alpha^2 m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} \\
&+ \frac{3x_i^{*2}m_4}{(x_i^{*2}+(y_i^*+1)^2)^{5/2}} - \frac{m_4}{(x_i^{*2}+(y_i^*+1)^2)^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
d = &- \frac{1}{(x_i^{*2}+y_i^{*2})^{3/2}} - \frac{m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{3/2}} - \frac{m_2}{(x_i^{*2}+(y_i^*-1)^2)^{3/2}} \qquad (14) \\
&- \frac{m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{3/2}} - \frac{m_4}{(x_i^{*2}+(y_i^*+1)^2)^{3/2}}.
\end{aligned}
$$

From the formula (14) it is clear, that $d < 0$. The eigenvalues of a matrix A are defined from the characteristic equation

$$\det(A - \lambda E) = (\lambda^2 - d)(\lambda^4 + (4\omega^2 - a - c)\lambda^2 + ac - b^2) = 0. \qquad (15)$$

First multiplier of the equation (15) gives two pure imaginary eigenvalues, for example, λ_5 and λ_6. Using the instruction "Eigenvalues" of SSC "Mathematica" for calculation of eigenvalues, we have received other eigenvalues λ_1, λ_2, λ_3, λ_4 of matrix A at points N_1 and S_1 for various values of m_1 and α. Some of them are given in Table 3.1.

From Table 3.1 it is clear, that at point N_1 for any values of the parameters m_1 and α the eigenvalues of matrix A are not pure imaginary. The similar result is received for other points N_i.

At point S_1 for small enough values of m_1 and α, close to unit, the eigenvalues of matrix A are pure imaginary, that is the equilibrium solutions S_1 are stable in the first approximation. By the iterative method we calculate an interval of stability for m_1

$$0 < m_1 < m_1^{**} = 0.0250344906\dots. \tag{16}$$

The interval of stability for α depends on m_1, for each values of m_1 there is an interval of variation of $\alpha\colon (\alpha^*, \alpha^{**})$. The calculated values of α^* and α^{**} for different m_1 are given in the following table:

m_1	α^*	α^{**}		m_1	α^*	α^{**}
0.001	0.9998476686	1.0031639276		0.002	0.9996953824	1.0026710278
0.003	0.9995431685	1.0022385906		0.004	0.9993910147	1.0018413555
0.009	0.9986311638	1.0001378981		0.01	0.9984793776	0.9998343232
0.02	0.9969648966	0.9972354376		0.025	0.9962099616	0.9962207882

Table 3.2.

The calculations executed for other points S_i, give the similar result. The carried out analysis allows to formulate the statements, following from the classical Lyapunov theorem on stability in the first approximation.

Theorem 3.1 *The stationary solutions of the differential equations of the restricted six-body problem, located on rotating axes of coordinates, are unstable for any values of mass m_1 and for any values of the relations of rhombus diagonals α.*

Theorem 3.2 *The stationary solutions of the differential equations of the restricted six-body problem, not located on the axes of coordinates, are stable in the first approximation for any value of parameter m_1 from the interval (15) and any value of parameter α from the interval $\alpha^* < \alpha < \alpha^{**}$.*

4 Research of Lyapunov Stability

The restricted 6-body problem is typically Hamiltonian, and, hence, differential equations, describing dynamics of our model, can be written a canonical form. Hence it follows, in particular, that the problem of stability of the stationary solutions S_1, S_2, S_3, S_4 in the sense of Lyapunov [5] can be solved only in the framework of KAM-theory [6, 15] on the basis of the well known Arnold-Mozer theorem [12, 13]. Now we formulate this theorem [6].

Theorem 4.1 *Let a Hamiltonian system*

$$\begin{aligned} \frac{dq_1}{dt} &= \frac{\partial K}{\partial p_1}, & \frac{dp_1}{dt} &= -\frac{\partial K}{\partial q_1}, \\ \frac{dq_2}{dt} &= \frac{\partial K}{\partial p_2}, & \frac{dp_2}{dt} &= -\frac{\partial K}{\partial q_2} \end{aligned} \tag{17}$$

by given with the Hamiltonian

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + \dots,$$

and let the origin be a singular point, such as the equilibrium state of system (17). Besides, let a canonical transformation

$$(q_1, q_2, p_1, p_2) \to (\psi_1, \psi_2, T_1, T_2)$$

exist, which yields

$$K(q_1, q_2, p_1, p_2) \equiv W(\psi_1, \psi_2, T_1, T_2),$$

where

$$\begin{gathered} W(\psi_1, \psi_2, T_1, T_2) = W_2(T_1, T_2) + W_4(T_1, T_2) + W_5(\psi_1, \psi_2, T_1, T_2) + \dots, \\ W_2 = \sigma_1 T_1 + \sigma_2 T_2, \qquad W_4 = c_{20} T_1^2 + c_{11} T_1 T_2 + c_{02} T_2^2. \end{gathered} \tag{18}$$

If:

(1) *eigenvalues of a matrix of linearized system (17) are the imaginary numbers* $\pm i\sigma_1$, $\pm i\sigma_2$*;*
(2) $n_1\sigma_1 + n_2\sigma_2 \neq 0$ *for* $|n_1| + |n_2| \leq 4$*;*
(3) $c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2 \neq 0$*,*

then the equilibrium

$$T_1 = T_2 = \psi_1 = \psi_2 = 0$$

of the Hamiltonian system

$$\begin{aligned} \frac{d\psi_1}{dt} &= \frac{\partial W}{\partial T_1}, & \frac{dT_1}{dt} &= -\frac{\partial W}{\partial \psi_1}, \\ \frac{d\psi_2}{dt} &= \frac{\partial W}{\partial T_2}, & \frac{dT_2}{dt} &= -\frac{\partial W}{\partial \psi_2} \end{aligned}$$

with the Hamiltonian (18) is Lyapunov stable.

Now we turn to a four-dimensional-phase space of Lagrangian coordinates and impulses (x, y, p_x, p_y). We shall get the Hamiltonian system of the 4-th order, equivalent to system (8):

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial h}{\partial p_x}, & \frac{dy}{dt} &= \frac{\partial h}{\partial p_y}, \\ \frac{dp_x}{dt} &= -\frac{\partial h}{\partial x}, & \frac{dp_y}{dt} &= -\frac{\partial h}{\partial y}, \end{aligned} \tag{19}$$

where the Hamiltonian h is expressed by the formula (see [9]):

$$\begin{aligned} h &= \omega(yp_x - xp_y) + \frac{1}{2}(p_x^2 + p_y^2) - (x^2 + y^2)^{-1/2} \\ &- m_1((x-\alpha)^2 + y^2)^{-1/2} - m_2((x^2 + (y-1)^2)^{-1/2} \\ &- m_3((x+\alpha)^2 + y^2)^{-1/2} - m_4((x^2 + (y+1)^2)^{-1/2}. \end{aligned} \tag{20}$$

The elementary transformation makes any point S_i with coordinates x^*, y^* the beginning of coordinates: $X = x - x^*$, $Y = y - y^*$, $P_x = p_x - p_{x^*}$, $P_y = p_y - p_{y^*}$. For the Hamiltonian we get the expression:

$$\begin{aligned} H &= \omega((Y + y_*)(P_X + p_{x^*}) - (X + x^*)(P_Y + p_{y^*})) \\ &+ \frac{1}{2}((P_X + p_*)^2 + (P_Y + p_{y^*})^2) - ((X + x^*)^2 + (Y + y^*)^2)^{-1/2} \\ &- m_1((X + x^* - \alpha)^2 + (Y + y^*)^2)^{-1/2} - m_2((X + x^*)^2 \\ &+ (Y + y^* - 1)^2)^{-1/2} - m_3((X + x^* + \alpha)^2 + (Y + y^*)^2)^{-1/2} \\ &- m_4((X + x^*)^2 + (Y + y^* + 1)^2)^{-1/2}. \end{aligned} \tag{21}$$

In the new variables the Hamiltonian differential equations of motion have the form

$$\begin{aligned} \frac{dX}{dt} &= \frac{\partial H}{\partial P_X}, & \frac{dY}{dt} &= \frac{\partial H}{\partial P_Y}, \\ \frac{dP_X}{dt} &= -\frac{\partial H}{\partial X}, & \frac{dP_Y}{dt} &= -\frac{\partial H}{\partial Y}. \end{aligned} \tag{22}$$

The formulated Arnold-Mozer theorem is in applicable to system (22), as the Hamiltonian (22) is not a positively definite function of the variable (X, Y, P_X, P_Y) [5]. It is necessary to execute its further transformations. For this purpose it is necessary to construct Birkhoff normalization. This normalization will be executed for a certain equilibrium position. For example, we shall consider the point S_1, stable in the first approximation, with coordinates

$$x^* = 0.37355, \quad y^* = 0.971439,$$

calculated for $m_1 = 0.001$ and $\alpha = 0.99985$.

We build a sequence of Hamiltonian transformations, necessary for fulfilment of conditions of the Arnold-Mozer theorem.

4.1 Transformation 1

In a sufficiently small neighborhood of the point S_1 the analytical Hamiltonian (21) is presented in the form of a convergent power series:

$$H = H_2(X, Y, P_X, P_Y) + H_3(X, Y) + H_4(X, Y) + \ldots,$$

where $H_k (k = 2, 3, \dots)$ is a homogeneous form of k-th degree, in our case

$$\begin{aligned} H_2 &= 0.414231X^2 - 0.915163Y^2 + 0.5(P_X^2 + P_Y^2) - 0.690873XY \\ &\quad + \omega(YP_X - XP_Y)), \\ H_3 &= -0.317928X^3 + 0.835341Y^3 - 1.049161X^2Y + 1.316579XY^2, \\ H_4 &= -0.19571X^4 - 0.73835Y^4 + 1.52426X^3Y + 1.65464X^2Y^2 \\ &\quad - 2.09451XY^3. \end{aligned} \tag{23}$$

Expression (23) indicates, that the quadratic form $H_2(X, Y, P_X, P_Y)$ contains the term $\omega(YP_X - XP_Y)$, which is the first obstacle on the way of investigation of the Lyapunov stability.

4.2 Transformation 2

Let's execute the linear transformation

$$[X, Y, P_X, P_Y] = B[q_1, q_2, p_1, p_2], \tag{24}$$

where symplectic matrix [10] B is defined so, that in the new transformed Hamiltonian

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + \dots$$

the quadratic form has a normal Birkhoff form [6, 14]

$$K_2 = \frac{1}{2}\,\sigma_1(q_1^2 + p_1^2) - \frac{1}{2}\,\sigma_2(q_2^2 + p_2^2),$$

where frequencies σ_1, σ_2: $\sigma_1 = |\lambda_1| = |\lambda_2|$, $\sigma_2 = |\lambda_3| = |\lambda_4|$, λ_1, λ_2, λ_3, λ_4 are the eigenvalues of linearized system for system (8) at point S_1.

Finding of elements of a matrix B is reduced to the solution of system of linear algebraic equations of the 16-th order. For the examined point S_1 $\sigma_1 = 0.994537$, $\sigma_2 = 0.102242$, and the matrix B has the form

$$B = \begin{bmatrix} 0 & 0 & 1.98114 & 5.328999 \\ -1.03191 & 0.38363 & -0.35829 & -1.29568 \\ -0.93804 & 0.16108 & 0.35842 & 1.29614 \\ 0.356334 & -0.13247 & 0.95557 & 5.29166 \end{bmatrix}. \tag{25}$$

The application of canonical transformation (24) with matrix (25) to Hamiltonian H gives the following expressions for the forms K_2, K_3 and K_4:

$$\begin{aligned} K_2 &= 0.49727(p_1^2 + q_1^2) - 0.05112(p_2^2 + q_2^2), \\ K_3 &= 0.700344p_1^3 - 3.771015p_1^2p_2 - 4.896871p_1p_2^2 + 0.452007p_2^3 \\ &\quad + 5.846024p_1^2q_1 + 32.621649p_1p_2q_1 + 45.164737p_2^2q_1 + 1.821347p_1q_1^2 \\ &\quad + 4.013408p_2q_1^2 - 0.917884q_1^3 - 2.173359p_1^2q_2 - 12.127657p_1p_2q_2 \end{aligned}$$

$$
\begin{aligned}
&- 16.790077p_2^2q_2 - 1.354234p_1q_1q_2 - 2.984107p_2q_1q_2 + 1.023717q_1^2q_2\\
&+ 0.251729p_1q_2^2 + 0.554696p_2q_2^2 - 0.380585q_1q_2^2 + 0.047163q_2^3,\\
K_4 = &- 6.249106p_1^4 - 69.140864p_1^3p_2 - 285.506731p_1^2p_2^2 - 521.541002p_1p_2^3\\
&- 355.623129p_2^4 - 5.919479p_1^3q_1 - 40.652926p_1^2p_2q_1 - 89.158571p_1p_1^2q_1\\
&- 61.002839p_2^3q_1 + 11.059209p_1^2q_1^2 + 62.773252p_1p_2q_1^2 + 88.314296p_2^2q_1^2\\
&+ 3.396809p_1q_1^3 + 8.059740p_2q_1^3 - 0.837196q_1^4 + 2.200668p_1^3q_2\\
&+ 15.113422p_1^2p_2q_2 + 33.146227p_1p_2^2q_2 + 22.678852p_2^3q_2 - 8.222901p_1^2q_1q_2\\
&- 46.674065p_1p_2q_1q_2 - 65.664708p_2^2q_1q_2 - 3.788466p_1q_1^2q_2 - 8.98904p_2q_1^2q_2\\
&+ 1.244968q_1^3q_2 + 1.528502p_1^2q_2^2 + 8.675942p_1p_2q_2^2 + 12.205991p_2^2q_2^2\\
&+ 1.408427p_1q_1q_2^2 + 3.341829p_2q_1q_2^2 - 0.694257q_1^2q_2^2 - 0.174536p_1q_2^3\\
&- 0.414127p_2q_2^3 + 0.172068q_1q_2^3 - 0.015992q_2^4.
\end{aligned}
$$

4.3 Transformation 3

Let's pass from the canonical variables (q_1, q_2, p_1, p_2) to the new canonical variables according to the Birkhoff formulas [14]

$$
\begin{aligned}
q_1 &= \sqrt{2\tau_1}\sin\theta_1, \qquad q_2 = \sqrt{2\tau_2}\sin\theta_2,\\
p_1 &= \sqrt{2\tau_1}\cos\theta_1, \qquad p_2 = \sqrt{2\tau_2}\cos\theta_2.
\end{aligned} \tag{26}
$$

Transformation (26) eliminates expressions with the coordinates θ_1, θ_2 from the quadratic part of the new Hamiltonian F and leaves expressions, dependent only on the new variables τ_1, τ_2. If we present new Hamiltonian F in the form

$$
F(\theta_1,\theta_2,\tau_1,\tau_2) = F_2(\tau_1,\tau_2) + F_3(\theta_1,\theta_2,\tau_1,\tau_2) + F_4(\theta_1,\theta_2,\tau_1,\tau_2) + \dots,
$$

then after necessary symbolical transformations we shall receive

$$
\begin{aligned}
F_2 &= \sigma_1\tau_1 - \sigma_2\tau_2 = 0.994537\tau_1 - 0.102242\tau_2,\\
F_3 &= (0.197768\cos\theta_1 - 1.7831\cos 3\theta_1 + 2.18664\sin\theta_1 + 4.78281\sin 3\theta_1)\tau^{3/2}\\
&+ (6.46201\cos(2\theta_1-\theta_2) + 0.342795\cos\theta_2 - 4.54683\cos(2\theta_1+\theta_2)\\
&+ 25.3277\sin(2\theta_1-\theta_2) - 1.62584\sin\theta_2 + 20.806\sin(2\theta_1+\theta_2))\tau_1\sqrt{\tau_2}\\
&- (6.5692\cos\theta_1 + 5.7507\cos(\theta_1-2\theta_2) + 1.53053\cos(\theta_1+2\theta_2)\\
&- 63.3344\sin\theta_1 - 40.781\sin(\theta_1-2\theta_2) - 23.6299\sin(\theta_1+2\theta_2))\sqrt{\tau_1}\tau_2\\
&+ (1.35108\cos\theta_2 - 0.072612\cos 2\theta_2 - 11.7728\sin\theta_2 - 11.9062\sin 3\theta_2)\tau_1^{3/2},\\
F_4 &= (-5.09985 - 10.8238\cos 2\theta_1 - 9.07276\cos 4\theta_1 - 2.52267\sin 2\theta_1\\
&- 4.65814\sin 4\theta_1)\tau_1^2 - (74.5687\cos(\theta_1-\theta_2) + 70.691\cos(3\theta_1-\theta_2)
\end{aligned}
$$

$$+ 70.081\cos(\theta_1+\theta_2) + 61.223\cos(3\theta_1+\theta_2) + 9.64362\sin(\theta_1-\theta_2)$$
$$+ 27.3509\sin(3\theta_1-\theta_2) + 6.8308\sin(\theta_1+\theta_2) + 21.362\sin(3\theta_1+\theta_2))\tau^{3/2}\sqrt{\tau_2}$$
$$-(196.358\tau_1\tau_2 + 371.598\cos 2\theta_1 + 211.36\cos(2\theta_1-2\theta_2) + 198.027\cos 2\theta_2$$
$$+ 164.69\cos(2\theta_1+2\theta_2) + 87.7501\sin 2\theta_1 + 57.3347\sin(2\theta_1-2\theta_2)$$
$$- 6.1244\sin 2\theta_2 + 33.2323(2\theta_1+2\theta_2))\tau_1\tau_2 - (298.027\cos(\theta_1-3\theta_2)$$
$$+ 810.55\cos(\theta_1-\theta_2) + 745.399\cos(\theta_1+\theta_2) + 232.19\cos(\theta_1+3\theta_2)$$
$$+ 48.8327\sin(\theta_1-3\theta_2) + 106.145\sin(\theta_1-\theta_2) + 73.522\sin(\theta_1+\theta_2)$$
$$+ 15.512\sin(\theta_1+3\theta_2))\sqrt{\tau_1}\tau_2^{3/2} - (527.356 + 711.214\cos 2\theta_2$$
$$+ 183.923\cos 4\theta_2 - 22.2647\sin 2\theta_2 - 11.5465\sin 4\theta_2)\tau_2^2.$$

4.4 Transformation 4

Let's construct the final canonical transformation

$$(\theta_1,\theta_2,\tau_1,\tau_2) \to (\psi_1,\psi_2,T_1,T_2) \tag{27}$$

which sets to zero the form of order of $3/2\,F_3(\theta_1,\theta_2,\tau_1,\tau_2)$, and excludes phase angles from the second-order form $F_4(\theta_1,\theta_2,\tau_1,\tau_2)$. Besides, the quadratic form $F_2(\tau_1,\tau_2)$ does not change. So, the transformed Hamiltonian should be

$$W(\psi_1,\psi_2,T_1,T_2) = W_2(T_1,T_2) + W_4(T_1,T_2) + W_5(\psi_1,\psi_2,T_1,T_2) + \ldots. \tag{28}$$

We shall search the given transformation as

$$\begin{aligned}
\theta_1 &= \psi_1 + V_{13}(\psi_1,\psi_2,T_1,T_2) + V_{14}(\psi_1,\psi_2,T_1,T_2),\\
\theta_2 &= \psi_2 + V_{23}(\psi_1,\psi_2,T_1,T_2) + V_{24}(\psi_1,\psi_2,T_1,T_2),\\
\tau_1 &= T_1 + U_{13}(\psi_1,\psi_2,T_1,T_2) + U_{14}(\psi_1,\psi_2,T_1,T_2),\\
\tau_2 &= T_2 + U_{23}(\psi_1,\psi_2,T_1,T_2) + U_{24}(\psi_1,\psi_2,T_1,T_2),
\end{aligned} \tag{29}$$

where V_{13}, V_{14}, V_{23}, V_{24}, U_{13}, U_{14}, U_{23}, U_{24} are determined from some linear partial differential equations. For example, the equation for the unknown function U_{13} has the form

$$\frac{\partial U_{13}}{\partial\psi_1}\,\sigma_1 + \frac{\partial U_{13}}{\partial\psi_2}\,\sigma_2 = A_{13}(\psi_1,\psi_2,T_1,T_2),$$

where A_{13} is expressed by partial derivative of forms $F_3(\theta_1,\theta_2,\tau_1,\tau_2)$ and $F_4(\theta_1,\theta_2,\tau_1,\tau_2)$, in which the replacement (27) is executed. The solution, which guarantees the form (28) for the new Hamiltonian $W(\psi_1,\psi_2,T_1,T_2)$, is to be found by the method of characteristics [10] and has the form

$$\begin{aligned}
U_{13} = {} & (0.198854\cos\psi_1 + 1.7929\cos 3\psi_1 - 2.19865\sin\psi_1 - 4.809\sin 3\psi_1)T_1^{3/2}\\
& + (6.179\cos(2\psi_1-\psi_2) + 4.819\cos(2\psi_1+\psi_2) - 24.222\sin(2\psi_1-\psi_2)\\
& - 22.054\sin(2\psi_1+\psi_2))T_1\sqrt{T}_2 + (6.605\cos\psi_1 + 4.79616\cos(\psi_1-2\psi_2)\\
& - 63.6823\sin\psi_1 + 1.937\cos(\psi_1+2\psi_2) - 34.012\sin(\psi_1-2\psi_2)\\
& - 29.909\sin(\psi_1+2\psi_2))\sqrt{T}_1T_2.
\end{aligned}$$

Carrying out transformation (29) with the found functions V_{13}, V_{14}, V_{23}, V_{24}, U_{13}, U_{14}, U_{23}, U_{24}, we receive for transformed Hamiltonian the final form (28), where

$$W_2 = \sigma_1 T_1 - \sigma_2 T_2 = 0.99453 T_1 - 0.102242 T_2,$$
$$W_4 = -197.657 T_1^2 - 5539.05 T_1 T_2 + 2591.95 T_2^2.$$

As a result of the executed transformations it is possible to assert the following.

1. The intervals for m_1: $(0, m_1^{**})$ and α: (α^*, α^{**}) are found. At each point of these intervals the linear system is stable.
2. The resonant curves are determined

$$\begin{cases} f_{1,-2}(m_1,\alpha) = \sigma_1(m_1,\alpha) - 2\sigma_2(m_1,\alpha), \\ f_{1,-2}(m_1,\alpha) = 0, \end{cases}$$
$$\begin{cases} f_{1,-3}(m_1,\alpha) = \sigma_1(m_1,\alpha) - 3\sigma_2(m_1,\alpha), \\ f_{1,-3}(m_1,\alpha) = 0, \end{cases} \tag{30}$$

which should be excluded from the set of intervals of stability.

3. Third condition of the theorem is also executed, thus for the point S_1 the value of function $W_4(\sigma_1, \sigma_2)$ is equal to 2018.72.

The executed calculations for points S_2, S_3, S_4 give similar results.

Thus, the following statement is valid.

Theorem 4.1 *The equilibrium points, not lying on coordinate axes, are Lyapunov stable for any values of parameters m_1 from interval of stability $0 < m_1 < m_1^{**} = 0.0250344906\ldots$ and any values of α from interval $\alpha^* < \alpha < \alpha^{**}$, except for the points, belonging to two resonant curves (30).*

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