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Hierarchical Lyapunov Functions for Stability Analysis of Discrete-Time Systems with Applications to the Neural Networks

T.A. Lukyanova and A.A. Martynyuk

S.P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Nesterov str. 3, 03057, Kiev-57, Ukraine

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Abstract: In the paper the application of hierarchical Lyapunov functions is proposed for qualitative analysis of solutions of discrete-time system. General results of analysis of quasi-linear discrete system are applied to the analysis of robust stability of large-scale neural system in the case of unperturbed and perturbed equilibrium state. The obtained results are compared with those obtained via the application of vector Lyapunov function in this problem. It is shown that the application of hierarchical Lyapunov function allows us to extend the boundaries of the parameter values of the neural network for which the exponential stability of its solutions takes place. The examples illustrating the efficiency of the proposed approach are given.

Keywords: *Discrete-time system; large-scale system; neural system; exponential stability; hierarchical Lyapunov function.*

Mathematics Subject Classification (2000): 39A11, 93C55, 93D30, 92B20.

1 Introduction

Discrete-time uncertain systems are satisfactory models for investigation of real phenomena in populational dynamics, macroeconomics, for simulation of chemical reactions, and also for analysis of discrete Markov processes, finite and probabilistic automata and others.

One of the most actively developed areas in recent years is the dynamics of neural systems [1 − 3] which are described by discrete-time equations (see [4, 5] and the references therein). Along with the investigation of such systems under different assumptions there has been a considerable interest in the development of general approaches in stability analysis of discrete-time uncertain systems, which will be admissible in the stability analysis of neural networks.

The aim of this paper is to develop a method of analysis of exponential stability of neural systems with nonperturbed and perturbed equilibrium states based on hierarchical Lyapunov function.

The paper is arranged as follows.

In Section 2 the uncertain quasilinear system is considered. To decrease the order of subsystems this system is decomposed. For each component and subsystem auxiliary norm-like functions are constructed and robust bounds are given.

In Section 3 the uncertain neural system with nonperturbed equilibrium state is linearized and the results of Section 2 are applied.

In Section 4 similar problem is solved for the uncertain neural system with perturbed equilibrium state.

In final Section 5 two numerical examples are given.

2 Uncertain Quasilinear System

We consider the discrete-time system with uncertainties and perturbations of the form

$$S\colon \qquad x(\tau+1) = (A+\Delta A)x(\tau) + g(x(\tau)), \tag{2.1}$$

where $\tau \in \mathcal{T}_\tau = \{t_0 + k, k = 0,1,2,\dots\}$, $t_0 \in R$, $x \in R^n$, $x_e \equiv 0$ is an equilibrium of (2.1), $g\colon U \to R^n$ is a continuous vector function, $U \subseteq R^n$ is an open subset containing x_e. $A \in R^{n\times n}$ is a constant matrix, $\Delta A \in R^{n\times n}$ is an uncertain matrix. The only knowledge we have regarding the matrix ΔA is that it lies in the known compact set $S \subset R^{n\times n}$. In paper [5] robust stability results were established for the system (2.1) via scalar quadratic Lyapunov function. Unlike this paper we shall apply vector and hierarchical Lyapunov functions composed of norm-like components.

2.1 Vector approach

Assume that the system (2.1) is decomposed into two interconnected subsystems

$$\begin{gathered}\hat{S}_i\colon \quad x_i(\tau+1) = (A_i + \Delta A_i)\,x_i(\tau) + (B_i + \Delta B_i)\,x_j(\tau) + g_i(x(\tau)),\\ i,\, j = 1,2, \quad i \neq j.\end{gathered} \tag{2.2}$$

Here $x_i \in R^{n_i}$, A_i, B_i and ΔA_i, ΔB_i are submatrices of the known and uncertain matrices

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \quad \text{and} \quad \Delta A = \begin{pmatrix} \Delta A_1 & \Delta B_1 \\ \Delta B_2 & \Delta A_2 \end{pmatrix},$$

respectively, with $A_i, \Delta A_i \in R^{n_i\times n_i}$, $B_i, \Delta B_i \in R^{n_i \times n_j}$, $i,j = 1,2$, $i \neq j$, $g = (g_1^{\mathrm{T}}, g_2^{\mathrm{T}})^{\mathrm{T}}$, $g_i\colon U \to R^{n_i}$ are continuous vector functions.

From (2.2) we extract the independent subsystems

$$S_i\colon \quad x_i(\tau+1) = (A_i + \Delta A_i)\,x_i(\tau), \quad i = 1,2, \tag{2.3}$$

with the same designations of variables as in system (2.2).

Assumption 2.1 *We assume that:*

(1) *there exist unique symmetric and positive definite matrices* $P_i \in R^{n_i \times n_i}$*, which satisfy Lyapunov matrix equations*

$$A_i^{\mathrm{T}} P_i A_i - P_i = -G_i, \qquad i = 1, 2, \tag{2.4}$$

where $G_i \in R^{n_i \times n_i}$ *are arbitrary symmetric and positive definite matrices;*

(2) *there exists a constant* $\gamma \in (0; 1)$ *such that*

$$\|B_1\|\,\|B_2\| < \gamma^2 \mu_1 \mu_2,$$

where $\mu_i = \big(\sigma_M^{\frac{1}{2}}(P_i - I_{n_i})\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i)\big)^{-1}$*,* P_i *are solutions of the Lyapunov matrix equations (2.4) for the matrices* $G_i = I_{n_i}$*,* I_{n_i} *are* $n_i \times n_i$ *identity matrices,* $i = 1, 2$*;*

(3) $\lim_{\|x\| \to 0} \|g(x)\|/\|x\| = 0$.

Here $\|B_i\| = \sup_{\|x_i\| \le 1} \|B_i x_i\|$, $\|x_i\| = (x_i^{\mathrm{T}} x_i)^{\frac{1}{2}}$ are the Euclidean norms of vectors x_i, and $\sigma_M(P_i)$ are the maximum eigenvalues of P_i.

Let P_i be determined as solutions of the Lyapunov matrix equations (2.4) for $G_i = I_{n_i}$. We define the constants

$$\begin{aligned}
&\alpha_i = \sigma_M^{\frac{1}{2}}(P_i)\mu_i = \big(\sigma_M^{\frac{1}{2}}(P_i - I_{n_i}) + \sigma_M^{\frac{1}{2}}(P_i)\big)^{-1}, \quad i = 1, 2,\\
&a = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2), \quad b = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)(\|B_1\| + \|B_2\|),\\
&c = \gamma^2 \alpha_1 \alpha_2 - \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)\|B_1\|\,\|B_2\|, \ \ \epsilon = ((b^2 + 4ac)^{\frac{1}{2}} - b)/2a.
\end{aligned} \tag{2.5}$$

Theorem 2.1 *We assume that for the uncertain system (2.1) the decomposition (2.2), (2.3) takes place and all conditions of Assumption 2.1 are satisfied. If the inequalities*

$$\|\Delta A_i\| \le (1 - \gamma)\mu_i \quad \textit{and} \quad \|B_i\| < \epsilon, \quad i = 1, 2,$$

are true, then the equilibrium $x_e = 0$ *of (2.1) is global exponentially stable.*

Proof For nominal subsystems

$$x_i(\tau + 1) = A_i x_i(\tau), \quad i = 1, 2,$$

we construct the norm-like functions

$$v_i(x_i) = (x_i^{\mathrm{T}} P_i x_i)^{\frac{1}{2}}, \qquad i = 1, 2, \tag{2.6}$$

and the function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2),$$

where d_1, d_2 are some positive constants.

Similarly to the proof of Theorem 3.1 from paper [6] for the first differences $\Delta v_i(x_i)$ of the functions (2.6) along the solutions of the system (2.1) we obtain the estimates

$$\Delta v_i(x_i)\big|_{\hat{S}_i} \le -(\alpha_i - \sigma_M^{\frac{1}{2}}(P_i)\|\Delta A_i\|)\|x_i\| + \sigma_M^{\frac{1}{2}}(P_i)(\|B_i\| + \|\Delta B_i\|)\|x_j\| + \sigma_M^{\frac{1}{2}}(P_i)\|g_i(x)\|,$$

$i, j = 1, 2,\ i \neq j$, and the estimate

$$\Delta v(x)\big|_S = \tilde{d}^{\mathrm{T}} W z + \tilde{g}(x),$$

where $\tilde{d} = (d_1, d_2)^{\mathrm{T}},\ z = (\|x_1\|, \|x_2\|)^{\mathrm{T}},\ W \in R^{2\times 2}$ is a matrix with the elements

$$w_{ij} = \begin{cases} \alpha_i - \sigma_M^{\frac{1}{2}}(P_i)\|\Delta A_i\|, & \text{if} \quad i = j, \\ -\sigma_M^{\frac{1}{2}}(P_i)(\|B_i\| + \|\Delta B_i\|), & \text{if} \quad i \neq j, \end{cases}$$

the function $\tilde{g}\colon R^n \to R_+$ is such that $\lim_{\|x\|\to 0} \|\tilde{g}(x)\|/\|x\| = 0$.

It was shown [6] that condition (2) of Assumption 2.1 implies that the matrix W is an M-matrix [7]. Then there exist positive constants d_1 and d_2 such that the vector $\tilde{d}^{\mathrm{T}} W$ has positive elements [8].

Using the trivial inequalities $\|x_i\| \geq v_i(x_i)/\sigma_M^{\frac{1}{2}}(P_i),\ i = 1, 2$, for the first difference of the function $v(x)$ along the solutions of the system (2.1) we get

$$\begin{aligned} \Delta v(x)\big|_S &\leq -\Big[\mu_1 - \|\Delta A_1\| - \frac{d_2 \sigma_M^{\frac{1}{2}}(P_2)}{d_1 \sigma_M^{\frac{1}{2}}(P_1)}(\|B_2\| + \|\Delta B_2\|)\Big] d_1 v_1(x_1) \\ &- \Big[\mu_2 - \|\Delta A_2\| - \frac{d_1 \sigma_M^{\frac{1}{2}}(P_1)}{d_2 \sigma_M^{\frac{1}{2}}(P_2)}(\|B_1\| + \|\Delta B_1\|)\Big] d_2 v_2(x_2) + \tilde{g}(x) \\ &\leq -\omega(d_1 v_1(x_1) + d_2 v_2(x_2)) + \tilde{g}(x) = -\omega v(x) + \tilde{g}(x), \end{aligned} \tag{2.7}$$

where

$$\omega = \min_{i,j=1,2,\, i\neq j} \Big\{ \mu_i - \|\Delta A_i\| - \frac{d_j \sigma_M^{\frac{1}{2}}(P_j)}{d_i \sigma_M^{\frac{1}{2}}(P_i)}(\|B_j\| + \|\Delta B_j\|) \Big\}.$$

The choice of the constants d_1 and d_2 implies $\omega > 0$. Let us assume that $\omega \geq 1$, then

$$\mu_i - \|\Delta A_i\| - \frac{d_j \sigma_M^{\frac{1}{2}}(P_j)}{d_i \sigma_M^{\frac{1}{2}}(P_i)}(\|B_j\| + \|\Delta B_j\|) \geq 1, \quad i, j = 1, 2, i \neq j. \tag{2.8}$$

If $\|\Delta A_1\| = \|\Delta A_2\| = \|\Delta B_1\| = \|\Delta B_2\| = \|B_1\| = \|B_2\| = 0$, the system (2.1) is written in the form

$$x_i(\tau + 1) = A_i x_i(\tau) + g_i(x(\tau)), \quad i = 1, 2.$$

It is known [9] that the equilibrium $x = 0$ of this system is exponentially stable.

Let at least one of the numbers $\|\Delta A_i\|, \|\Delta B_i\|$, or $\|B_i\|$ be not equal to zero, for example, $\|\Delta A_1\|$. Then the inequalities (2.8) give

$$\mu_1 \geq 1 + \|\Delta A_1\| + \frac{d_2 \sigma_M^{\frac{1}{2}}(P_2)}{d_1 \sigma_M^{\frac{1}{2}}(P_1)}(\|B_2\| + \|\Delta B_2\|) > 1,$$

but

$$\mu_1 = \frac{1}{\sigma_M^{\frac{1}{2}}(P_1 - I_{n_1})\sigma_M^{\frac{1}{2}}(P_1) + \sigma_M(P_1)} \leq 1,$$

since $\sigma_M(P_1) \geq 1$. We get the contradiction, from which it follows that $0 < \omega < 1$.

Using the condition (3) of Assumption 2.1 for the function $\tilde{g}(x)$ we get the estimate

$$
\begin{aligned}
\tilde{g}(x) &= d_1\sigma_M^{\frac{1}{2}}(P_1)\|g_1(x_1)\| + d_2\sigma_M^{\frac{1}{2}}(P_2)\|g_2(x_2)\| \leq (d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2))\|g(x)\| \\
&\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\|x\| \leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)(\|x_1\| + \|x_2\|) \\
&\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\left(\frac{v_1(x_1)}{\sigma_m^{\frac{1}{2}}(P_1))} + \frac{v_2(x_2)}{\sigma_m^{\frac{1}{2}}(P_2))}\right) \\
&\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\max\left\{\frac{1}{d_1\sigma_m^{\frac{1}{2}}(P_1))}, \frac{1}{d_2\sigma_m^{\frac{1}{2}}(P_2)}\right\}v(x),
\end{aligned}
$$

where $\sigma_m^{\frac{1}{2}}(P_i)$ are minimum eigenvalues of the matrices P_i, α is a small positive number such that for the constant

$$
\tilde{\omega} = \omega - \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\max\left\{\frac{1}{d_1\sigma_m^{\frac{1}{2}}(P_1))}, \frac{1}{d_2\sigma_m^{\frac{1}{2}}(P_2)}\right\}
$$

the inequality $0 < \tilde{\omega} < 1$ holds.

Using (2.7) we get the estimate

$$
\Delta v(x)\big|_S \leq -\tilde{\omega}v(x)
$$

for all x belonging to sufficiently small neighborhood of the origin $\widetilde{U} \subseteq U$, which implies global exponential stability of the equilibrium $x_e = 0$ of (2.1) (see [10]).

The proof of Theorem 2.1 is complete.

2.2 Hierarchical approach

Now in the framework of hierarchical approach we decompose each subsystem (2.3) into two interconnected components

$$
\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\,x_{ij}(\tau) + (B_{ij} + \Delta B_{ij})\,x_{ik}(\tau), \tag{2.9}
$$

where $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $A_{ij}, \Delta A_{ij} \in R^{n_{ij}\times n_{ij}}$, $B_{i1}, \Delta B_{i1} \in R^{n_{i1}\times n_{i2}}$, $B_{i2}, \Delta B_{i2} \in R^{n_{i2}\times n_{i1}}$, $i,j,k = 1,2$, $j \neq k$,

$$
A_i = \begin{pmatrix} A_{i1} & B_{i1} \\ B_{i2} & A_{i2} \end{pmatrix}, \qquad \Delta A_i = \begin{pmatrix} \Delta A_{i1} & \Delta B_{i1} \\ \Delta B_{i2} & \Delta A_{i2} \end{pmatrix}.
$$

We assume that the matrices B_i and ΔB_i have a block structure:

$$
B_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)} \end{pmatrix}, \qquad \Delta B_i = \begin{pmatrix} \Delta M_{11}^{(i)} & \Delta M_{12}^{(i)} \\ \Delta M_{12}^{(i)} & \Delta M_{22}^{(i)} \end{pmatrix},
$$

where $M_{jk}^{(i)}, \Delta M_{jk}^{(i)} \in R^{n_{ij}\times n_{lk}}$, $i,j,k,l = 1,2$, $i \neq l$.

We take from (2.9) the independent components

$$
C_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\,x_{ij}(\tau), \quad i,j = 1,2. \tag{2.10}
$$

with the same designations of variables as in system (2.9).

We denote $g_i = (g_{i1}^{\mathrm{T}}, g_{i2}^{\mathrm{T}}, \dots, g_{in_i}^{\mathrm{T}})^{\mathrm{T}}$ and introduce the following assumptions.

Assumption 2.2 *We assume that:*

(1) *there exist unique symmetric and positive definite matrices P_{ij}, which satisfy the Lyapunov matrix equations*

$$A_{ij}^{\mathrm{T}} P_{ij} A_{ij} - P_{ij} = -G_{ij}, \qquad i = 1, 2, \tag{2.11}$$

where G_{ij} are arbitrary symmetric and positive definite matrices;

(2) *there exist constants $\gamma_i \in (0,1)$ such that*

$$\|B_{i1}\| \, \|B_{i2}\| < \gamma_i^2 \mu_{i1} \mu_{i2}, \quad i = 1, 2,$$

where $\mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{n_{ij}}) \sigma_M^{\frac{1}{2}}(P_{ij}) + \sigma_M(P_{ij})\big)^{-1}$. Here and over P_{ij} are solutions of the Lyapunov matrix equations (2.11) for the matrices $G_{ij} = I_{n_{ij}}$, $I_{n_{ij}}$ are $n_{ij} \times n_{ij}$ identity matrices.

We construct the auxiliary functions v_i on the base of the functions $v_{ij}(x_{ij}) = (x_{ij}^{\mathrm{T}} P_{ij} x_{ij})^{\frac{1}{2}}$ by formulae $v_i(x_i) = d_{i1} v_{i1}(x_{i1}) + d_{i2} v_{i2}(x_{i2})$, $i = 1, 2$.

We consider 2×2 matrices $W_i = (w_{jk}^{(i)})$ with the elements

$$w_{jk}^{(i)} = \begin{cases} \gamma_i \alpha_{ij}, & \text{for} \quad j = k, \\ -\sigma_M^{\frac{1}{2}}(P_{ij})(\|B_{ij}\| + \overline{\epsilon}_i), & \text{for} \quad j \neq k. \end{cases}$$

Here $0 < \overline{\epsilon}_i < \epsilon_i$,

$$\begin{aligned} &\alpha_{ij} = \sigma_M^{\frac{1}{2}}(P_{ij}) \mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{ij}) + \sigma_M^{\frac{1}{2}}(P_{ij})\big)^{-1}, \\ &\mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{ij}) \sigma_M^{\frac{1}{2}}(P_{ij}) + \sigma_M(P_{ij})\big)^{-1}, \\ &\epsilon_i = ((b_i^2 + 4 a_i c_i)^{\frac{1}{2}} - b_i)/2a_i, \quad a_i = \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2}), \\ &c_i = \gamma_i^2 \alpha_{i1} \alpha_{i2} - \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2}) \|B_{i1}\| \, \|B_{i2}\|, \quad i, j = 1, 2, \\ &b_i = \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2})(\|B_{i1}\| + \|B_{i2}\|). \end{aligned} \tag{2.12}$$

Let us denote

$$\begin{gathered} \pi_i = \min\{d_{i1} w_{11}^{(i)} + d_{i2} w_{21}^{(i)}; d_{i1} w_{12}^{(i)} + d_{i2} w_{22}^{(i)}\}, \quad i = 1, 2, \\ m = \frac{1}{2} \left(\frac{\pi_1 \pi_2}{\big(d_{11} \sigma_M^{\frac{1}{2}}(P_{11}) + d_{12} \sigma_M^{\frac{1}{2}}(P_{12})\big) \big(d_{21} \sigma_M^{\frac{1}{2}}(P_{21}) + d_{22} \sigma_M^{\frac{1}{2}}(P_{22})\big)} \right)^{\frac{1}{2}} \end{gathered} \tag{2.13}$$

A method of optimal choice of the constant d_{ij}, $i, j = 1, 2$, is given in [6].

Assumption 2.3 *Let $\lim\limits_{\|x\| \to 0} \|g(x)\| / \|x\| = 0$ and for the submatrices $M_{jk}^{(i)}$ of the matrices B_i the inequalities $\overline{m} = \max \|M_{jk}^{(i)}\| < m$ be realized for all i, j, $k = 1, 2$.*

Theorem 2.2 *We assume that for the uncertain system (2.1) the two-level decomposition (2.2), (2.3), (2.9), (2.10) is realized and all conditions of Assumptions 2.2 and 2.3 are satisfied. If the inequalities*

$$\|\Delta A_{ij}\| \le (1-\gamma_i)\mu_{ij}, \quad \|\Delta B_{ij}\| \le \overline{\epsilon}_i, \quad \|\Delta M_{jk}^{(i)}\| < m - \overline{m}$$

are fulfilled for all $i,j,k = 1,2$, then the equilibrium $x_e = 0$ of the system (2.1) is global exponentially stable.

Proof Under the hypotheses of Theorem 2.2 analogous to the proof of Theorem 4.1 from [6] for the function $v(x) = d_1 v_1(x_1) + d_2 v_2(x_2)$ we get the estimates:

$$\Delta v(x)\big|_S = d_1 \Delta v_1(x_1)\big|_{\widehat{S}_1} + d_2 \Delta v_2(x_2)\big|_{\widehat{S}_2} \le -\hat{d}^{\mathrm{T}} W\, z + \widetilde{g}(x), \tag{2.14}$$

where $\hat{d} = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$, $g_i = (g_{i1}^{\mathrm{T}}, g_{i2}^{\mathrm{T}})^{\mathrm{T}}$ and W is 2×2 matrix with the elements

$$w_{jk} = \begin{cases} \pi_j, & \text{for} \quad j = k, \\ -d_{j1}\sigma_M^{\frac12}(P_{j1})(2\overline{m} + \|\Delta M_{11}^{(j)}\|) + \|\Delta M_{12}^{(j)}\|) & \\ \qquad -d_{j2}\sigma_M^{\frac12}(P_{j2})(2\overline{m} + \|\Delta M_{21}^{(j)}\|) + \|\Delta M_{22}^{(j)}\|), & \text{for} \quad j \ne k, \end{cases}$$

$$\begin{aligned} \widetilde{g}(x) = {} & d_1\Big(d_{11}\sigma_M^{\frac12}(P_{11})\|g_{11}(x)\| + d_{12}\sigma_M^{\frac12}(P_{12})\|g_{12}(x)\|\Big) \\ & + d_2\Big(d_{21}\sigma_M^{\frac12}(P_{21})\|g_{11}(x)\| + d_{22}\sigma_M^{\frac12}(P_{22})\|g_{22}(x)\|\Big). \end{aligned}$$

Under the hypotheses of Theorem 2.2 the matrix W is the M-matrix and, according to [7] there exist positive constants d_1 and d_2 such that the vector $\hat{d}^{\mathrm{T}} W$ has positive components. That is

$$\begin{aligned} \hat{d}^{\mathrm{T}} W\, z &= (\pi_1 d_1 - \omega_{21} d_2)\|x_1\| + (\pi_2 d_2 - \omega_{12} d_1)\|x_2\| \\ &\ge \sum_{i,j=1,2,\, i\ne j} (\pi_i d_i - \omega_{ji} d_j)(\|x_{i1}\| + \|x_{i2}\|)/\sqrt{2} \\ &\ge \sum_{i,j=1,2,\, i\ne j} \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}} \left(\frac{v_{i1}(x_{i1})}{\sigma_M^{\frac12}(P_{i1})} + \frac{v_{i2}(x_{i2})}{\sigma_M^{\frac12}(P_{i2})}\right) \\ &\ge \sum_{i,j=1,2,\, i\ne j} \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i} \left(\frac{1}{d_{i1}\sigma_M^{\frac12}(P_{i1})} d_i d_{i1} v_{i1} + \frac{1}{d_{i2}\sigma_M^{\frac12}(P_{i2})} d_i d_{i2} v_{i2}\right) \ge \omega v(x), \end{aligned}$$

where

$$\omega = \min_{i,j=1,2,\, i\ne j} \left\{ \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i d_{i1} \sigma_M^{\frac12}(P_{i1})}, \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i d_{i2} \sigma_M^{\frac12}(P_{i2})} \right\}.$$

As the matrix W is the M-matrix, $\gamma_1 \in (0,1)$ and $\mu_{11} \le 1$,

$$\begin{aligned} & \frac{\pi_1 d_1 - \omega_{21} d_2}{\sqrt{2}\, d_1 d_{11} \sigma_M^{\frac12}(P_{11})} \le \frac{\pi_1}{\sqrt{2}\, d_{11} \sigma_M^{\frac12}(P_{11})} \\ & \le \frac{d_{11}\omega_{11}^{(1)} + d_{12}\omega_{21}^{(1)}}{\sqrt{2}\, d_{11}\sigma_M^{\frac12}(P_{11})} \le \frac{\omega_{11}^{(1)}}{\sqrt{2}\, \sigma_M^{\frac12}(P_{11})} = \frac{\gamma_1 \alpha_{11}}{\sqrt{2}\, \sigma_M^{\frac12}(P_{11})} = \frac{\gamma_1 \mu_{11}}{\sqrt{2}} < 1 \end{aligned}$$

and $0 < \omega < 1$.

It follows from (2.14) that

$$\Delta v(x)\big|_S \le -\omega v(x) + \widetilde{g}(x).$$

As for sufficiently small $\alpha > 0$ the estimate

$$\begin{aligned}\widetilde{g}(x) &\le \alpha \sum_{i=1}^{2} d_i\Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)\|x\| \\ &\le \alpha\beta^{-1} \sum_{i=1}^{2} d_i\Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)v(x)\end{aligned}$$

is realized in some neighborhood of zero $\widetilde{U}$,

$$\Delta v(x)\big|_S \le -\widetilde{\omega} v(x),$$

where $\widetilde{\omega} = \omega - \alpha\beta^{-1} \sum\limits_{i=1}^{2} d_i\Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)$, $0 < \widetilde{\omega} < 1$.

These conditions are sufficient [10] for the global exponential stability of the equilibrium $x = 0$ of (2.1). The proof of Theorem 2.2 is complete.

3 Neural System with Nonperturbed Equilibrium

We consider discrete-time neural networks described by

$$x(\tau + 1) = Gx(\tau) + Cs(Tx(\tau) + I), \tag{3.1}$$

where $\tau \in \mathcal{T}_\tau = \{t_0 + k, k = 0, 1, 2, \dots\}$, $t_0 \in R$, $x \in R^n$, $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, x_i is the state of ith neuron, $x_i \in R$, $s\colon R^n \to R^n$, $s(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, $s_i\colon R \to (-1, 1)$, $T \in R^{n\times n}$, $G = \operatorname{diag}\{g_1, g_2, \dots g_n\}$, $g_i \in [-1, 1]$, $C = \operatorname{diag}\{c_1, c_2, \dots c_n\}$, $c_i \ne 0$ for all $i = 1, 2, \dots, n$. The functions s_i are twice continuously differentiable functions, they are monotonically increasing and odd.

Together with the system (3.1) we consider an uncertain system

$$x(\tau + 1) = (G + \Delta G)x(\tau) + (C + \Delta C)s\big((T + \Delta T)x(\tau) + (I + \Delta I)\big), \tag{3.2}$$

where $\Delta G, \Delta C, \Delta T \in R^{n\times n}$, $\Delta I \in R^n$ are uncertain matrices and a vector.

3.1 Vector approach

In the framework of vector approach we decompose the neural system (3.1) into two interconnected subsystems

$$x_i(\tau+1) = G_i x_i(\tau) + C_i s_i(T_{i1}x_1(\tau) + T_{i2}x_2(\tau) + I_i), \;\; i = 1, 2, \tag{3.3}$$

where $x_i \in R^{n_i}$, $x_i = (x_{i1}, x_{i2}, \dots x_{in_i})^{\mathrm{T}}$, x_{ij} represents the state of the ijth neuron, $x_{ij} \in R$, $s_i \colon R^{n_i} \to R^{n_i}$, $s_i(x_i) = (s_{i1}(x_{i1}), s_{i2}(x_{i2}), \dots, s_{in_i}(x_{in_i}))^{\mathrm{T}}$, $s_{ij} \colon R \to (-1;1)$, $T_{ij} \in R^{n_i \times n_j}$, $G_i = \operatorname{diag}\{g_{i1},\, g_{i2}, \dots g_{in_i}\}$, $g_{ij} \in [-1;1]$, $C_i = \operatorname{diag}\{c_{i1}, c_{i2}, \dots\; c_{in_i}\}$, $c_{ij} \neq 0$ for all $i = 1, 2$, $j = 1, 2, \dots, n_i$. The functions s_{ij} are twice continuously differentiable functions, they are monotonically increasing and odd.

Together with the system (3.1) we decompose the uncertain system (3.2)

$$\begin{aligned} x_i(\tau+1) = (G_i + \Delta G_i)x_i(\tau) + (C_i + \Delta C_i)s_i\big((T_{i1} + \Delta T_{i1})x_1(\tau) \\ + (T_{i2} + \Delta T_{i2})x_2(\tau) + (I_i + \Delta I_i)\big). \end{aligned} \tag{3.4}$$

Here $\Delta G_i, \Delta C_i \in R^{n_i \times n_i}$, $\Delta T_{ij} \in R^{n_i \times n_j}$, $\Delta I_i \in R^{n_i}$ are uncertain matrices and vector.

Let $x_e = (x_{1e}^{\mathrm{T}}, x_{2e}^{\mathrm{T}})^{\mathrm{T}}$ denote the equilibrium state of (3.1), $s_i'(x_i) = \operatorname{diag}\{s_{i1}'(x_{i1}), s_{i2}'(x_{i2}), \dots, s_{in_i}'(x_{in_i})\}$, $s_i''(x_i) = \operatorname{diag}\{s_{i1}''(x_{i1}), s_{i2}''(x_{i2}), \dots, s_{in_i}''(x_{in_i})\}$, $L_{i1} = \sup_{x_i \in R^{n_i}} \|s_i'(x_i)\|$, $L_{i2} = \sup_{x_i \in R^{n_i}} \|s_i''(x_i)\|$.

All above assumptions concerning the matrices G_i, C_i, T_{ij}, the vectors I_i and the functions s_i are similar to the assumptions under which a scalar Lyapunov function is applied to the neural systems of (3.1) type in paper [5]. Further we need assumptions connected just with the decomposition of neural system.

Let us introduce the matrices

$$\begin{aligned} A_i &= G_i + C_i s_i'(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i)T_{ii}, \\ B_i &= C_i s_i'(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i)T_{ij}, \quad i,j = 1,2,\, i \neq j, \end{aligned} \tag{3.5}$$

and the following assumptions.

Assumption 3.1 *Assume that:*

(1) *for the matrices (3.5) the conditions (1) and (2) of Assumption 2.1 are satisfied;*
(2) x_e *is an equilibrium state of both (3.3) and (3.4).*

We set

$$\begin{aligned} \beta_i &= 1 + \big(\|C_i\| + \|T_{ii}\|\big)L_{i1} + \big(1 + \|x_{1e}\| + \|x_{2e}\|\big)\|C_i\|\|T_{ii}\|L_{i2}, \\ \delta_i &= \big(\|C_i\| + \|T_{ij}\|\big)L_{i1} + \big(1 + \|x_{1e}\| + \|x_{2e}\|\big)\|C_i\|\|T_{ij}\|L_{i2}, \\ K_i &= \min\left\{\frac{1}{2L_{i1}}\big((\beta^2 + 4(1-\gamma)\mu_i L_{i1})^{\frac12} - \beta_i\big),\; \frac{1}{2L_{i1}}\big((\delta^2 + 4\epsilon L_{i1})^{\frac12} - \delta_i\big)\right\} \end{aligned} \tag{3.6}$$

where $i,j = 1,2$, $i \neq j$, the constants μ_i, ϵ are computed by (2.5) for the matrices (3.5).

Theorem 3.1 *Let for the system (3.2) the decomposition (3.4) take place and all conditions of Assumption 3.1 be satisfied. If the inequalities*

$$\max\{\,\|\Delta G_i\|,\,\|\Delta C_i\|,\,\|\Delta T_{i1}\|,\,\|\Delta T_{i2}\|,\,\|\Delta I_i\|\,\} < K_i, \quad i=1,2, \tag{3.7}$$

are true, then the equilibrium x_e of (3.2) is global exponentially stable.

Proof We denote

$$\begin{aligned} f_i(x) &= G_i x_i + C_i s_i\big(T_{i1}x_1 + T_{i2}x_2 + I_i\big),\\ h_i(x) &= \Delta G_i x_i + (C_i+\Delta C_i)s_i\big((T_{i1}+\Delta T_{i1})x_1 + (T_{i2}+\Delta T_{i2})x_2 + (I_i+\Delta I_i)\big)\\ &\qquad - C_i s_i\big(T_{i1}x_1 + T_{i2}x_2 + I_i\big). \end{aligned}$$

As the functions f_i and h_i are twice continuously differentiable functions in the neighborhood of the equilibrium x_e, the equations (3.4) can be written in the equivalent form

$$\begin{aligned} x_i(\tau+1) - x_e &= f_i(x_(\tau)) + h_i(x(\tau)) - f_i(x_e) - h_i(x_e)\\ &= \frac{\partial f_i(x_e)}{\partial x_i}(x_i(\tau)-x_{ie}) + \frac{\partial f_i(x_e)}{\partial x_j}(x_j(\tau)-x_{je})\\ &\quad + \frac{\partial h_i(x_e)}{\partial x_i}(x_i(\tau)-x_{ie}) + \frac{\partial h_i(x_e)}{\partial x_j}(x_j(\tau)-x_{je}) + g_i(x(\tau)-x_e), \end{aligned} \tag{3.8}$$

where $g_i(x(\tau)-x_e)$ are the higher-order terms with respect to $(x(\tau)-x_e)$,

$$\frac{\partial f_i}{\partial x_i} = \begin{pmatrix} \frac{\partial f_{i1}}{\partial x_{i1}} & \frac{\partial f_{i1}}{\partial x_{i2}} & \cdots & \frac{\partial f_{i1}}{\partial x_{in_i}}\\ \frac{\partial f_{i2}}{\partial x_{i1}} & \frac{\partial f_{i2}}{\partial x_{i2}} & \cdots & \frac{\partial f_{i2}}{\partial x_{in_i}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_{in_i}}{\partial x_{i1}} & \frac{\partial f_{in_i}}{\partial x_{i2}} & \cdots & \frac{\partial f_{in_i}}{\partial x_{in_i}} \end{pmatrix}, \qquad \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial f_{i1}}{\partial x_{j1}} & \frac{\partial f_{i1}}{\partial x_{j2}} & \cdots & \frac{\partial f_{i1}}{\partial x_{jn_j}}\\ \frac{\partial f_{i2}}{\partial x_{i1}} & \frac{\partial f_{i2}}{\partial x_{j2}} & \cdots & \frac{\partial f_{i2}}{\partial x_{jn_j}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_{in_i}}{\partial x_{j1}} & \frac{\partial f_{in_i}}{\partial x_{j2}} & \cdots & \frac{\partial f_{in_i}}{\partial x_{jn_j}} \end{pmatrix},$$

and $\frac{\partial h_i}{\partial x_i}$, $\frac{\partial h_i}{\partial x_j}$ have an analogous form. We get

$$\begin{aligned} \frac{\partial f_i(x_e)}{\partial x_i} &= G_i + C_i s_i'\big(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i\big)T_{ii} = A_i,\\ \frac{\partial f_i(x_e)}{\partial x_j} &= C_i s_i'\big(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i\big)T_{ij} = B_i, \qquad i,j=1,2,\ i\neq j, \end{aligned}$$

If we denote

$$\Delta A_i = \frac{\partial h_i(x_e)}{\partial x_i}, \qquad \Delta B_i = \frac{\partial h_i(x_e)}{\partial x_j}, \qquad y(\tau) = x(\tau) - x_e,$$

then the equations (3.8) are written in the form

$$y_i(\tau+1) = (A_i+\Delta A_i)\,y_i(\tau) + (B_i+\Delta B_i)\,y_j(\tau) + g_i(y(\tau)) \tag{3.9}$$

and the state $y=0$ will be the equilibrium of the system (3.9). Letting

$$\begin{aligned} z_i &= (T_{i1}+\Delta T_{i1})x_{1e} + (T_{i2}+\Delta T_{i2})x_{2e} + (I_i+\Delta I_i),\\ t_i &= T_{i1}x_{1e} + T_{i2}x_{2e} + I_i, \quad i=1,2. \end{aligned}$$

we find

$$\begin{aligned}\Delta A_i &= \Delta G_i + (C_i + \Delta C_i)s_i'(z_i)(T_{ii} + \Delta T_{ii}) - C_i s_i'(t_i)T_{ii}\\ &= \Delta G_i + C_i s_i'(z_i)\Delta T_{ii} + \Delta C_i s_i'(z_i)(T_{ii} + \Delta T_{ii}) + C_i(s_i'(z_i) - s_i'(t_i))T_{ii}\\ &= \Delta G_i + C_i s_i'(z_i)\Delta T_{ii} + \Delta C_i s_i'(z_i)(T_{ii} + \Delta T_{ii}) + C_i Q_i(z_i, t_i)\Lambda_i(z_i - t_i)T_{ii}.\end{aligned}$$

Similarly to [5] here we have used the formula

$$f(a) - f(b) = (a-b)\int_0^1 f'(a + \xi(b-a))d\xi$$

for the functions $f = s_{ij}$,

$$\begin{aligned}Q_i(z_i, t_i) = \operatorname{diag}\Bigg\{ &\int_0^1 s_{i1}''(z_i + \xi(t_i - z_i))\,d\xi, \int_0^1 s_{i2}''(z_i + \xi(t_i - z_i))\,d\xi, \dots,\\ &\int_0^1 s_{in_i}''(z_i + \xi(t_i - z_i))\,d\xi\Bigg\},\\ \Lambda_i(z_i - t_i) = \operatorname{diag}&\Big\{z_{i1} - t_{i1}, z_{i2} - t_{i2}, \dots, z_{in_i} - t_{in_i}\Big\}, \qquad i = 1, 2.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\|Q_i(z_i, t_i)\| &= \sup_{j=1,2,\dots,n_i}\left|\int_0^1 s_{ij}''(z_i + \xi(t_i - z_i))\,d\xi\right| \le L_{i2},\\ \|\Lambda_i(z_i - t_i)\| &\le \|z_i - t_i\| \le (1 + \|x_{1e}\| + \|x_{2e}\|)K_i.\end{aligned} \tag{3.10}$$

Using (3.6), (3.7) and (3.10), we get

$$\begin{aligned}\|\Delta A_i\| &< K_i + \|C_i\|L_{i1}K_i + L_{i1}(\|T_{ii}\| + K_i)K_i + \|C_i\|L_{i2}\|z_i - t_i\|\,\|T_{ii}\|\\ &\le L_{i1}K_i^2 + (1 + (\|C_i\| + \|T_{ii}\|)L_{i1} + \|C_i\|\|T_{ii}\|L_{i2}(1 + \|x_{1e}\| + \|x_{2e}\|))K_i\\ &= L_{i1}K_i^2 + \beta_i K_i \le (1-\gamma)\mu_i.\end{aligned} \tag{3.11}$$

Similarly for $i \ne j$

$$\begin{aligned}\Delta B_i &= (C_i + \Delta C_i)s_i'(z_i)(T_{ij} + \Delta T_{ij}) - C_i s_i'(t_i)T_{ij}\\ &= C_i s_i'(z_i)\Delta T_{ij} + \Delta C_i s_i'(z_i)(T_{ij} + \Delta T_{ij}) + C_i(s_i'(z_i)s_i'(t_i))T_{ij}\\ &= C_i s_i'(z_i)\Delta T_{ij} + \Delta C_i s_i'(z_i)(T_{ij} + \Delta T_{ij}) + C_i Q_i(z_i, t_i)\Lambda_i(z_i - t_i)T_{ij}\end{aligned}$$

and

$$\begin{aligned}\|\Delta B_i\| &< L_{i1}K_i^2 + ((\|C_i\| + \|T_{ij}\|)L_{i1} + \|C_i\|T_{ij}L_{i2}(1 + \|x_{1e}\| + \|x_{2e}\|))K_i\\ &= L_{i1}K_i^2 + \delta_i K_i \le \epsilon.\end{aligned} \tag{3.12}$$

It follows from (3.11), (3.12) and Assumption 3.1 that, for the system (3.9) all conditions of Theorem 2.1 are satisfied. Hence the equilibrium $y = 0$ of the system (3.9) is global exponentially stable, and it is equivalent to global exponential stability of equilibrium x_e of the system (3.2). Theorem 3.1 is proved.

3.2 Hierarchical approach

In the framework of hierarchical approach we decompose each subsystem (3.3) into two interconnected components

$$\begin{aligned} x_{ij}(\tau+1) = G_{ij}x_{ij}(\tau) + C_{ij}s_{ij}\Big(&T^{i1}_{j1}x_{11}(\tau) + T^{i1}_{j2}x_{12}(\tau) \\ &+ T^{i2}_{j1}x_{21}(\tau) + T^{i2}_{j2}x_{22}(\tau) + I_{ij}\Big), \quad i,j = 1,2. \end{aligned} \tag{3.13}$$

Here $x_i = (x^{\mathrm T}_{i1}, x^{\mathrm T}_{i2})^{\mathrm T}$, $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $x_{ij} = (x_{ij1}, x_{ij2}, \dots, x_{ijn_{ij}})^{\mathrm T}$, x_{ijl} represents the state of the ijlth neuron, $x_{ijl} \in R$,

$$T_{ij} = \begin{pmatrix} T^{ij}_{11} & T^{ij}_{12} \\ T^{ij}_{21} & T^{ij}_{22} \end{pmatrix}, \; G_i = \begin{pmatrix} G_{i1} & 0 \\ 0 & G_{i2} \end{pmatrix}, C_i = \begin{pmatrix} C_{i1} & 0 \\ 0 & C_{i2} \end{pmatrix}, \; I_i = \left(I^{\mathrm T}_{i1}, I^{\mathrm T}_{i2}\right)^{\mathrm T},$$

$T^{ip}_{jk} \in R^{n_{ij} \times n_{pk}}$, $I_{ij} \in R^{n_{ij}}$, $G_{ij} = \operatorname{diag}\{g_{ij1}, g_{ij2}, \dots g_{ijn_{ij}}\}$, $g_{ijl} \in [-1,1]$, $C_{ij} = \operatorname{diag}\{c_{ij1},\ c_{ij2}, \dots c_{ijn_{ij}}\}$, $c_{ijl} \neq 0$, $s_i(x) = (s_{i1}(x_{i1})^{\mathrm T},\ s_{i2}(x_{i2})^{\mathrm T})^{\mathrm T}$, $s_{ij} \colon R^{n_{ij}} \to R^{n_{ij}}$, $s_{ij}(x_{ij}) = (s_{ij1}(x_{ij1}),\ s_{ij2}(x_{ij2}), \dots, s_{ijn_{ij}}(x_{ijn_{ij}}))^{\mathrm T}$, the functions $s_{ijl} \colon\ R \to (-1,1)$, s_{ijl} are twice continuously differentiable, increasing and odd, $i,j,k,p = 1,2$, $l = 1,2,\dots,n_{ij}$.

Together with the system (3.3) we decompose the system (3.4) into interconnected components

$$\begin{aligned} x_{ij}(\tau+1) = (G_{ij} + \Delta G_{ij})x_{ij}(\tau) + (C_{ij} + \Delta C_{ij})s_{ij}\Big(&(T^{i1}_{j1} + \Delta T^{i1}_{j1})x_{11}(\tau) \\ &+ (T^{i1}_{j2} + \Delta T^{i1}_{j2})x_{12}(\tau) + (T^{i2}_{j1} + \Delta T^{i2}_{j1})x_{21}(\tau) \\ &+ (T^{i2}_{j2} + \Delta T^{i2}_{j2})x_{22}(\tau) + (I_{ij} + \Delta I_{ij})\Big), \quad i,j = 1,2. \end{aligned} \tag{3.14}$$

Here $\Delta G_{ij}, \Delta C_{ij}$, ΔT^{ik}_{jp}, ΔI_{ij} are unknown matrices and a vector of corresponding dimensions. The only knowledge about it is that it lies in some known compact sets.

Let us denote by $x_e = (x^{e\,\mathrm T}_{11}, x^{e\,\mathrm T}_{12}, x^{e\,\mathrm T}_{21}, x^{e\,\mathrm T}_{22})^{\mathrm T}$ the equilibrium of system (3.13), and

$$\begin{aligned} s'_{ij}(x_{ij}) &= \operatorname{diag}\{s'_{ij1}(x_{ij1}), s'_{ij2}(x_{ij2}), \dots, s'_{ijn_{ij}}(x_{ijn_{ij}})\}, \\ s''_{ij}(x_{ij}) &= \operatorname{diag}\{s''_{ij1}(x_{ij1}), s''_{ij2}(x_{ij2}), \dots, s''_{ijn_{ij}}(x_{ijn_{ij}})\}, \\ L^1_{ij} &= \sup_{x_{ij} \in R^{n_{ij}}} \|s''_{ij}(x_{ij})\|, \quad L^2_{ij} = \sup_{x_{ij} \in R^{n_{ij}}} \|s''_{ij}(x_{ij})\|, \\ t_{ij} &= T^{i1}_{j1}x_{11} + T^{i1}_{j2}x_{12} + T^{i2}_{j1}x_{21} + T^{i2}_{j2}x_{22} + I_{ij}, \\ z_{ij} &= (T^{i1}_{j1} + \Delta T^{i1}_{j1})x_{11} + (T^{i1}_{j2} + \Delta T^{i1}_{j2})x_{12} \\ &\quad + (T^{i2}_{j1} + \Delta T^{i2}_{j1})x_{21} + (T^{i2}_{j2} + \Delta T^{i2}_{j2})x_{22} + (I_{ij} + \Delta I_{ij}) \\ t^e_{ij} &= T^{i1}_{j1}x^e_{11} + T^{i1}_{j2}x^e_{12} + T^{i2}_{j1}x^e_{21} + T^{i2}_{j2}x^e_{22} + I_{ij}, \\ z^e_{ij} &= (T^{i1}_{j1} + \Delta T^{i1}_{j1})x^e_{11} + (T^{i1}_{j2} + \Delta T^{i1}_{j2})x^e_{12} \\ &\quad + (T^{i2}_{j1} + \Delta T^{i2}_{j1})x^e_{21} + (T^{i2}_{j2} + \Delta T^{i2}_{j2})x^e_{22} + (I_{ij} + \Delta I_{ij}). \end{aligned}$$

and the matrices

$$
\begin{aligned}
A_{ij} &= G_{ij} + C_{ij} s'_{ij}(t^e_{ij}) T^{ii}_{jj},\\
B_{ij} &= C_{ij} s'_{ij}(t^e_{ij}) T^{ii}_{jk}, \quad j \neq k,\\
M^{(i)}_{jl} &= C_{ij} s'_{ij}(t^e_{ij}) T^{ip}_{jl}, \quad i \neq p.
\end{aligned}
\tag{3.15}
$$

We need the following assumptions.

Assumption 3.2 *We assume that:*

(1) *all conditions of Assumption 2.2 are satisfied for the matrices (3.15);*
(2) *the state x_e is an equilibrium of both (3.13) and (3.14);*
(3) $\overline{m} = \max \|M^{(i)}_{jl}\| < m$, *where the constant m is computed by formula (2.13).*

Let us denote

$$
\begin{gathered}
\beta_{ij} = 1 + \big(\|C_{ij}\| + \|T^{ii}_{jj}\|\big) L^1_{ij} + \|C_{ij}\| \|T^{ii}_{jj}\| L^2_{ij} R_e,\\
\delta^{pk}_{ij} = \big(\|C_{ij}\| + \|T^{ip}_{jk}\|\big) L^1_{ij} + \|C_{ij}\| \|T^{ip}_{jk}\| L^2_{ij} R_e, \ i \neq p \ \text{ﬀ} \quad j \neq k,\\
R_e = 1 + \|x^e_{11}\| + \|x^e_{12}\| + \|x^e_{21}\| + \|x^e_{22}\|,\\
\alpha^1_{ij} = \big((\beta^2_{ij} + 4(1-\gamma_i)\mu_{ij} L^1_{ij})^{\frac12} - \beta_{ij}\big)/2L^1_{ij},\\
\alpha^2_{ij} = \big(((\delta^{ik}_{ij})^2 + 4\overline{\epsilon}_i L^1_{ij})^{\frac12} - \delta^{ik}_{ij}\big)/2L^1_{ij}, \quad j \neq k,\\
\alpha^3_{ijl} = \big(((\delta^{pl}_{ij})^2 + 4\widetilde{m} L^1_{ij})^{\frac12} - \delta^{pl}_{ij}\big)/2L^1_{ij}, \quad i \neq p,\\
K_{ij} = \min\big\{\alpha^1_{ij}, \alpha^2_{ij}, \alpha^3_{ij1}, \alpha^3_{ij2}\big\} \quad i,j,k,p,l = 1,2.
\end{gathered}
$$

Here $0 < \widetilde{m} < m - \overline{m}$ and $0 < \overline{\epsilon}_i < \epsilon_i$ the constants μ_{ij}, ϵ_i are computed for the matrices (3.15) by formula (2.12).

Theorem 3.2 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 3.2 be satisfied. If the inequalities*

$$
\max_{p,k=1,2} \Big\{ \|\Delta G_{ij}\|, \|\Delta C_{ij}\|, \|\Delta T^{ip}_{jk}\|, \|\Delta I_{ij}\| \Big\} \le K_{ij}, \quad i,j = 1,2,
$$

are true, then the equilibrium x_e of (3.2) is global exponentially stable.

Proof We denote

$$
\begin{aligned}
f_{ij}(x) &= G_{ij} x_{ij} + C_{ij} s_{ij}\big(t_{ij}\big),\\
h_{ij}(x) &= \Delta G_{ij} x_{ij} + (C_{ij} + \Delta C_{ij}) s_{ij}\big(z_{ij}\big) - C_{ij} s_{ij}\big(t_{ij}\big).
\end{aligned}
$$

For the functions f_{ij} we get

$$
\frac{\partial f_{ij}(x_e)}{\partial x_{pk}} = \begin{cases}
G_{ij} + C_{ij} s'_{ij}\big(t^e_{ij}\big) T^{ii}_{jj} = A_{ij}, & i = p, j = k,\\
C_{ij} s'_{ij}\big(t^e_{ij}\big) T^{ii}_{jk} = B_{ij}, & i = p, j \neq k,\\
C_{ij} s'_{ij}\big(t^e_{ij}\big) T^{ip}_{jk} = M^{(i)}_{jk}, & i \neq p.
\end{cases}
$$

Since the functions f_{ij} and h_{ij} are twice continuously differentiable in the neighborhood of the equilibrium x_e, the equations (3.14) can be written in the equivalent form

$$
\begin{aligned}
x_{ij}(\tau+1) - x_{ij}^e &= f_{ij}(x(\tau)) + h_{ij}(x(\tau)) - f_{ij}(x_e) - h_{ij}(x_e) = \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}(x_{ij}(\tau) - x_{ij}^e) \\
&+ \frac{\partial f_{ij}(x_e)}{\partial x_{ik}}(x_{ik}(\tau) - x_{ik}^e) + \frac{\partial f_{ij}(x_e)}{\partial x_{p1}}(x_{p1}(\tau) - x_{p1}^e) + \frac{\partial f_{ij}(x_e)}{\partial x_{p2}}(x_{p2}(\tau) - x_{p2}^e) \\
&+ \frac{\partial h_{ij}(x_e)}{\partial x_{ij}}(x_{ij}(\tau) - x_{ij}^e) + \frac{\partial h_{ij}(x_e)}{\partial x_{ik}}(x_{ik}(\tau) - x_{ik}^e) + \frac{\partial h_{ij}(x_e)}{\partial x_{p1}}(x_{p1}(\tau) - x_{p1}^e) \\
&+ \frac{\partial h_{ij}(x_e)}{\partial x_{p2}}(x_{p2}(\tau) - x_{p2}^e) + g_{ij}(x(\tau) - x_e), \qquad i,j,k,p = 1,2, \quad i \neq p, j \neq k,
\end{aligned} \tag{3.16}
$$

where $g_{ij}(x(\tau) - x_e)$ are the higher-order terms with respect to $x(\tau) - x_e$. If we denote

$$
\Delta A_{ij} = \frac{\partial h_{ij}(x_e)}{\partial x_{ij}}, \quad \Delta B_{ij} = \frac{\partial h_{ij}(x_e)}{\partial x_{ik}}, \; y(\tau) = x(\tau) - x_e,
$$

$$
\Delta M_{jl}^{(i)} = \frac{\partial h_{ij}(x_e)}{\partial x_{pl}}, \quad i,j,k,p,l = 1,2, \quad i \neq p, \quad j \neq k,
$$

the equations (3.16) are written in the form

$$
\begin{gathered}
y_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\, y_{ij}(\tau) + (B_{ij} + \Delta B_{ij})\, y_{ik}(\tau) \\
+ (M_{j1}^{(i)} + \Delta M_{j1}^{(i)})\, y_{p1}(\tau) + (M_{j2}^{(i)} + \Delta M_{j2}^{(i)})\, y_{p2}(\tau) + g_{ij}(y(\tau)),
\end{gathered} \tag{3.17}
$$

$i \neq p$, $j \neq k$, and the state $y = 0$ is an equilibrium of (3.17).

Then, as in proof of Theorem 3.1, we have

$$
\begin{gathered}
\|\Delta A_{ij}\| \leq L_{ij}^1 K_{ij}^2 + \Big(1 + (\|C_{ij}\| + \|T_{jj}^{ii}\|) L_{ij}^1 \\
+ \|C_{ij}\| \|T_{jj}^{ii}\| L_{ij}^2 R_e\Big) K_{ij} = L_{ij}^1 K_{ij}^2 + \beta_{ij} K_{ij} \leq (1 - \gamma_i)\mu_{ij},
\end{gathered} \tag{3.18}
$$

$$
\begin{gathered}
\|\Delta B_{ij}\| < L_{ij}^1 K_{ij}^2 + \Big((\|C_{ij}\| + \|T_{jk}^{ii}\|) L_{ij}^1 + \|C_{ij}\| T_{jk}^{ii} L_{ij}^2 R_e)\Big) K_{ij} \\
= L_{ij}^1 K_{ij}^2 + \delta_{ij}^{ik} K_{ij} \leq \overline{\epsilon}_i, \quad j \neq k,
\end{gathered} \tag{3.19}
$$

$$
\begin{gathered}
\|\Delta M_{jl}^{(i)}\| < L_{ij}^1 K_{ij}^2 + \Big((\|C_{ij}\| + \|T_{jl}^{ip}\|) L_{ij}^1 + \|C_{ij}\| \|T_{jl}^{ip}\| L_{ij}^2 R_e)\Big) K_{ij} \\
= L_{ij}^1 K_{ij}^2 + \delta_{ij}^{pl} K_{ij} \leq \widetilde{m} < m - \overline{m}, \quad i \neq p.
\end{gathered} \tag{3.20}
$$

It follows from (3.18)–(3.20) and conditions of Assumption 3.2, that for the system (3.17) all conditions of Theorem 2.2 are fulfilled. Hence, the equilibrium $y = 0$ of (3.17) is global exponentially stable, and it is equivalent to global exponential stability of the equilibrium x_e of (3.14). Theorem 3.2 is proved.

4 Neural System with Perturbed Equilibrium

The approach considered below may be used for the investigation of uncertain neural systems with perturbed equilibrium.

Assumption 4.1 *We assume that:*

(1) *for the matrices (3.5) the conditions (1) and (2) of Assumption 2.1 are satisfied;*
(2) x_e *is an equilibrium of (3.1),* $\overline{x}_e$ *is an equilibrium of (3.2),* $x_e \neq \overline{x}_e$.

We denote

$$\begin{aligned}
&\overline{K}_i = \min\left\{\frac{1}{2L_{i1}}\big((\beta_i^2 + 2(1-\gamma)\mu_i L_{i1})^{\frac12} - \beta_i\big),\ \frac{1}{2L_{i1}}\big((\delta_i^2 + 2\epsilon L_{i1})^{\frac12} - \delta_i\big)\right\},\\
&r_{ij} = L_{i2}\big(\|C_i\| + K_i\big)\big(\|T_{ij}\| + K_i\big)\big(\|T_{i1}\| + \|T_{i2}\| + 2K_i\big), \qquad i,j = 1,2, \\
&\Delta < \min\left\{\frac{(1-\gamma)\mu_1}{2r_{11}},\ \frac{(1-\gamma)\mu_2}{2r_{22}},\ \frac{\epsilon}{2r_{12}},\ \frac{\epsilon}{2r_{21}}\right\},
\end{aligned} \tag{4.1}$$

where the constants μ_1, μ_2 and ϵ are computed by (2.5) for the matrices (3.5).

Theorem 4.1 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 4.1 be satisfied. If the inequalities*

$$\max\{\,\|\Delta G_i\|,\ \|\Delta C_i\|,\ \|\Delta T_{i1}\|,\ \|\Delta T_{i2}\|,\ \|\Delta I_i\|\,\} < \overline{K}_i,\ \ \|x_{ie} - \overline{x}_{ie}\| < \Delta,\ \ i = 1,2,$$

are true, then the equilibrium $\overline{x}_e$ *of (3.2) is global exponentially stable.*

Proof In the neighborhood $\overline{x}_e$ the equations (3.4) can be written in the equivalent form

$$\begin{aligned}
x_i(\tau+1) - \overline{x}_e &= f_i(x(\tau)) + h_i(x(\tau)) - f_i(\overline{x}_e) - h_i(\overline{x}_e)\\
&\quad + \left(\frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j}\right)(x_j(\tau) - \overline{x}_{je}) + g_i(x(\tau) - \overline{x}_e)\\
&= \left(\frac{\partial f_i(x_e)}{\partial x_i} + \frac{\partial f_i(\overline{x}_e)}{\partial x_i} + \frac{\partial h_i(\overline{x}_e)}{\partial x_i} - \frac{\partial f_i(x_e)}{\partial x_i}\right)(x_i(\tau) - \overline{x}_{ie})\\
&\quad + \left(\frac{\partial f_i(x_e)}{\partial x_j} + \frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j} - \frac{\partial f_i(x_e)}{\partial x_j}\right)(x_j(\tau) - \overline{x}_{je})\\
&\quad + g_i(x(\tau) - \overline{x}_e), \qquad i,j = 1,2,\ i \neq j,
\end{aligned}$$

where $g_{ij}(x(\tau) - \overline{x}_e)$ are higher-order terms. If we denote

$$\begin{gathered}
\Delta A_i = \frac{\partial f_i(\overline{x}_e)}{\partial x_i} + \frac{\partial h_i(\overline{x}_e)}{\partial x_i} - \frac{\partial f_i(x_e)}{\partial x_i}, \qquad \Delta B_i = \frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j} - \frac{\partial f_i(x_e)}{\partial x_j},\\
y(\tau) = x(\tau) - \overline{x}_e, \qquad i,j = 1,2,\ i \neq j,
\end{gathered}$$

then the equations (3.4) are written in the form (3.9) and further the proof of Theorem 4.1 is analogous to the proof of Theorem 3.1.

Assumption 4.2 *We assume that:*

(1) *for matrices (3.15) the conditions (1) and (2) of Assumption 2.2 are satisfied;*
(2) x_e *is an equilibrium state of (3.1),* $\overline{x}_e$ *is an equilibrium state of (3.2),* $x_e \neq \overline{x}_e$;
(3) $\overline{m} = \max\|M_{jk}^{(i)}\| < m$, *the constant* m *is computed by formula (2.13).*

Let us denote

$$
\begin{aligned}
\alpha_{ij}^1 &= \big((\beta_{ij}^2 + 2(1-\gamma_i)\mu_{ij}L_{ij}^1)^{\frac{1}{2}} - \beta_{ij}\big)/2L_{ij}^1,\\
\alpha_{ij}^2 &= \big(((\delta_{ij}^{ik})^2 + 2\overline{\epsilon}_i L_{ij}^1)^{\frac{1}{2}} - \delta_{ij}^{ik}\big)/2L_{ij}^1, \quad j \neq k,\\
\alpha_{ijl}^3 &= \big(((\delta_{ij}^{pl})^2 + 2\widetilde{m}L_{ij}^1)^{\frac{1}{2}} - \delta_{ij}^{pl}\big)/2L_{ij}^1, \quad i \neq p,\\
\overline{K}_{ij} &= \min\big\{\alpha_{ij}^1, \alpha_{ij}^2, \alpha_{ij1}^3, \alpha_{ij2}^3\big\}, \quad i,j,k,p,l = 1,2. \qquad (4.2)\\
r_{jl}^{ip} &= L_{i2}\big(\|C_{ij}\| + \overline{K}_{ij}\big)\big(\|T_{jl}^{ip}\| + \overline{K}_{ij}\big)\left(\sum_{\nu,\xi=1}^{2}\|T_{j\xi}^{i\nu}\| + 4\overline{K}_i\right),\\
\Phi &= \min\left\{\frac{(1-\gamma_i)\mu_{ij}}{2r_{jj}^{ii}}, \frac{\overline{\epsilon}_i}{2r_{jk}^{ii}}, \frac{\widetilde{m}}{2r_{jl}^{ip}}\right\}, \quad i \neq p, \quad j \neq k,
\end{aligned}
$$

where $0 < \widetilde{m} < m - \overline{m}$, $0 < \overline{\epsilon}_i < \epsilon_i$, the constants μ_{ij} and ϵ_i are computed by (2.12) for the matrices (3.15).

Theorem 4.2 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 4.2 be satisfied. If the inequalities*

$$
\max_{p,k=1,2}\Big\{\|\Delta G_{ij}\|, \|\Delta C_{ij}\|, \|\Delta T_{jk}^{ip}\|, \|\Delta I_{ij}\|\Big\} \leq \overline{K}_{ij}, \quad \|x_{ij}^e - \overline{x}_{ij}^e\| \leq \Phi, \quad i,j = 1,2,
$$

are true, then the equilibrium $\overline{x}_e$ of (3.2) is global exponentially stable.

Proof As in the proof of Theorem 4.5, the equations (3.14) are written in the equivalent form

$$
\begin{aligned}
x_{ij}(\tau+1) - \overline{x}_{ij}^e &= f_{ij}(x(\tau)) + h_{ij}(x(\tau)) - f_{ij}(\overline{x}_e) - h_{ij}(\overline{x}_e)\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{ij}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ij}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}\right)(x_{ij}(\tau) - \overline{x}_{ij}^e)\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{ik}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ik}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ik}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ik}}\right)(x_{ik}(\tau) - \overline{x}_{ik}^e)\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{p1}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{p1}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{p1}}\right)(x_{p1}(\tau) - \overline{x}_{p1}^e)\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{p2}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{p2}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{p2}} - \frac{\partial f_{ij}(x_e)}{\partial x_{p2}}\right)(x_{p2}(\tau) - \overline{x}_{p2}^e)\\
&\quad + g_{ij}(x(\tau) - \overline{x}_e), \qquad i,j,k,p = 1,2, \quad i \neq p, \quad j \neq k.
\end{aligned}
$$

If we denote

$$
\Delta A_{ij} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ij}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}, \quad \Delta B_{ij} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ik}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ik}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ik}},
$$

$$
\Delta M_{jl}^{(i)} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{pl}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{pl}} - \frac{\partial f_{ij}(x_e)}{\partial x_{pl}}, \quad y(\tau) = x(\tau) - \overline{x}_e,
$$

$$
i,j,k,p,l = 1,2, \quad i \neq p, \quad j \neq k.
$$

then the equations (3.14) take the form (3.17) and further the proof is carried out analogously to the proof of Theorem 3.2.

5 Numerical Results

Example 5.1 Let us consider the system

$$\begin{cases} x_1(\tau+1) = \dfrac{4}{\pi}\arctan\left(x_1(\tau) + 0.005\,x_2(\tau) - 0.005\right), \\ x_2(\tau+1) = -0.75\,x_2(\tau) + \dfrac{7}{\pi}\arctan\left(\dfrac{1}{350}\,x_1(\tau) + x_2(\tau) - \dfrac{1}{350}\right), \end{cases}$$

with the equilibrium $x_e = (1;1)^{\mathrm{T}}$ and the perturbed system

$$\begin{cases} x_1(\tau+1) = \dfrac{4.04}{\pi}\arctan\left(1.02\,x_1(\tau) - 0.0052\,x_2(\tau) - 0.025\right), \\ x_2(\tau+1) = -0.75\,x_2(\tau) + \dfrac{7}{\pi}\arctan\left(\dfrac{1}{350}\,x_1(\tau) + 1.02\,x_2(\tau) - \dfrac{801}{35000}\right) \end{cases} \tag{5.1}$$

with the equilibrium $\overline{x}_e = (1.01;1)^{\mathrm{T}}$.

In the framework of scalar approach (see [5]), we have

$$K_1 = 0.0167, \quad \overline{\epsilon}_1 = 0.0942, \quad \Delta G = \mathrm{diag}\,\{0;0\}, \quad \Delta C = \mathrm{diag}\,\{0.02;0\},$$

$$\Delta I = \left(-0.02; -\frac{701}{35000}\right)^{\mathrm{T}}, \qquad \Delta T = \begin{pmatrix} 0.02 & -0.0102 \\ 0 & 0.02 \end{pmatrix}.$$

As $|\Delta T|_\infty = 0.0302 > K_1$, we can not make the conclusion about exponential stability of equilibrium state $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1).

In the framework of vector approach the constants computed by (4.1) are

$$K_1 = 0.0215, \quad K_2 = 0.0205, \quad \Phi = 0.1054.$$

Then

$$\max\{\,\|\Delta G_1\|,\, \|\Delta C_1\|,\, \|\Delta T_{11}\|,\, \|\Delta T_{12}\|,\, \|\Delta I_1\|\,\} = 0.02 < K_1,$$
$$\max\{\,\|\Delta G_2\|,\, \|\Delta C_2\|,\, \|\Delta T_{21}\|,\, \|\Delta T_{22}\|,\, \|\Delta I_2\|\,\} = 0.02002 < K_2,$$
$$\|\overline{x}_{1e} - x_{1e}\| = 0.01 < \Phi, \qquad \|\overline{x}_{2e} - x_{2e}\| = 0 < \Phi,$$

and, by Theorem 4.1, we can conclude that the equilibrium $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1) is global exponentially stable.

In the framework of vector approach the constants computed by (4.1) are

$$K_1 = 0.0215, \quad K_2 = 0.0205, \quad \Phi = 0.1054.$$

Then

$$\max\{\|\Delta G_1\|, \|\Delta C_1\|, \|\Delta T_{11}\|, \|\Delta T_{12}\|, \|\Delta I_1\|\} = 0.02 < K_1,$$
$$\max\{\|\Delta G_2\|, \|\Delta C_2\|, \|\Delta T_{21}\|, \|\Delta T_{22}\|, \|\Delta I_2\|\} = 0.02002 < K_2,$$
$$\|\overline{x}_{1e} - x_{1e}\| = 0.01 < \Phi, \qquad \|\overline{x}_{2e} - x_{2e}\| = 0 < \Phi,$$

and, by Theorem 4.1, we can conclude that the equilibrium $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1) is global exponentially stable.

Example 5.2 Let the system have the form

$$\begin{cases} x_{11}(\tau+1) = \dfrac{4}{\pi}\arctan\left(x_{11}(\tau) - 0.01\,x_{21}(\tau) - 0.01\right), \\ x_{12}(\tau+1) = -0.2\,x_{12}(\tau) + \dfrac{1.2}{\pi}\arctan(x_{12}(\tau) + 0.01x_{21}(\tau) + 0.01), \\ x_{21}(\tau+1) = 0.6 - \dfrac{1.6}{\pi}\arctan\left(0.01x_{11}(\tau) + x_{21}(\tau) - 1.99\right), \\ x_{22}(\tau+1) = x_{22}(\tau) - \dfrac{3}{\pi}\arctan(0.01x_{12}(\tau) + x_{22}(\tau)), \end{cases}$$

$x = (x_{11}, x_{12}, x_{21}, x_{22})^{\mathrm{T}} \in R^4$. $x_e = (1; 0; -1; 0)^{\mathrm{T}}$,

$$s(x) = \frac{2}{\pi}(\arctan x_{11},\ \arctan x_{12},\ \arctan x_{21},\ \arctan x_{22})^{\mathrm{T}},$$

$$G = \mathrm{diag}\,\{0;\ -0.2;\ 0.6;\ 1\},\ C = \mathrm{diag}\,\{2;\ 0.6;\ -0.8;\ -1.5\}, I = (-0.01;\ 0.01;\ 1.99;\ 0)^{\mathrm{T}},$$

$$T = \begin{pmatrix} 1 & 0 & -0.01 & 0 \\ 0 & 1 & 0.01 & 0 \\ 0.01 & 0 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \dfrac{2}{\pi} & 0 & -\dfrac{0.02}{\pi} & 0 \\ 0 & \dfrac{6-\pi}{5\pi} & \dfrac{0.012}{\pi} & 0 \\ -\dfrac{0.008}{\pi} & 0 & \dfrac{3\pi-4}{5\pi} & 0 \\ 0 & -\dfrac{0.03}{\pi} & 0 & \dfrac{\pi-3}{\pi} \end{pmatrix}.$$

As a result of hierarchical decomposition we get

$$A_{11} = \frac{2}{\pi}, \quad A_{12} = \frac{6-\pi}{5\pi}, \quad A_{21} = \frac{3\pi-4}{5\pi}, \quad A_{22} = \frac{\pi-3}{\pi},$$
$$B_{11} = B_{12} = B_{21} = B_{22} = 0,$$
$$M_{11}^{(1)} = -\frac{0.02}{\pi}, \quad M_{21}^{(1)} = \frac{0.012}{\pi}, \quad M_{11}^{(2)} = -\frac{0.008}{\pi}, \quad M_{22}^{(2)} = -\frac{0.03}{\pi},$$
$$M_{12}^{(1)} = M_{22}^{(1)} = M_{12}^{(2)} = M_{21}^{(2)} = 0,$$
$$R_e = 3, \quad L_{jk}^i = \frac{2}{\pi}, \quad \overline{m} = 0.0095.$$

We get $\gamma_1 = \gamma_2 = 0.5$, $\overline{\epsilon}_1 = \overline{\epsilon}_2 = 0.1$ and do relevant computations

$$\begin{array}{llll} P_{11} = 1.6814, & P_{12} = 1.0342, & P_{21} = 1.1354, & P_{22} = 1.0020, \\ \mu_{11} = 0.3633, & \mu_{12} = 0.8180, & \mu_{21} = 0.6546, & \mu_{22} = 0.9549, \\ a_1 = 1.3187, & c_1 = 0.0980, & a_2 = 1.0666, & c_2 = 0.1667, \\ \epsilon_1 = 0.2726, & \epsilon_2 = 0.3953, & m = 0.0504. & \end{array}$$

If $\overline{\epsilon}_1 = \overline{\epsilon}_2 = 0.1$, $\widetilde{m} = m - \overline{m} = 0.0409$ then we get the robust bounds

$$\overline{K}_{11} = 0.0269, \quad \overline{K}_{12} = 0.0407, \quad \overline{K}_{21} = 0.0304, \quad \overline{K}_{22} = 0.0393.$$

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