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Asymptotic Behavior in Some Classes of Functional Differential Equations

M. Mahdavi

Department of Mathematics, Bowie State University, 14000 Jericho Park Road, Bowie, Maryland 20715, U.S.A.

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Abstract: In this paper we shall investigate the asymptotic behavior (at $+\infty$) of certain classes of functional differential equations, involving causal (abstract Volterra) operators. Vast literature exists on this subject, mainly in the case of ordinary differential equations, delay equations and integro-differential equations. We mention here the book, non-linear differential equations, by G. Sansone and R. Conti for classical results. More recent contributions can be found in the books by C. Corduneanu, with pertinent references. See also the papers by A.R. Aftabizadeh, and C. Corduneanu and M. Mahdavi.

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1 A Result of Global Existence

Let us consider the first order functional differential equation

$$\dot{x}(t) + (Lx)(t) = (Mx)(t), \tag{1}$$

on the positive half-axis $R_+ = [0, \infty)$. In equation (1), L and M stand for causal operators acting on convenient function spaces (to be specified below), with L assumed to be linear and continuous, while M is in general nonlinear. As we know (see [3], Chapter 3), a local solution does exist for (1), under suitable conditions, satisfying the initial condition

$$x(0) = x_0, \tag{2}$$

and being defined on some interval $[0, T)$, $T \leq \infty$. For instance, if we also assume the linearity and continuity of M on the underlying space, then necessarily $T = \infty$ (see [3]).

We shall now obtain an upper estimate for the solution of (1), (2). This estimate will allow us to conclude that all solutions of (1), (2) are defined on the whole half-axis R_+.

In view of obtaining the estimate, we shall formulate and utilize certain assumptions on the data. Also, we need to choose the space on which the causal operators L and M are acting.

The assumptions are:

(H_1) The operators L and M in (1) are causal operators on the space $C(R_+, R^n)$, with L linear and continuous, while M is continuous and nonlinear.

(H_2) The operator L satisfies the condition

$$\int_0^t \langle (Lx)(s), x(s)\rangle \, ds \ \geq \lambda(t) \int_0^t |x(s)|^2 \, ds, \tag{3}$$

for each $t \in R_+$, with $\lambda(t)$ a non-increasing function on R_+.

(H_3) There exists a function $m \in L^2(R_+, R)$, such that

$$|(Mx)(t)| \leq m(t), \quad a.e. \quad \text{on} \quad R_+, \tag{4}$$

for every $x \in C(R_+, R^n)$.

Remark 1 Condition (H_3) is certainly very restrictive, and we shall use it in obtaining an estimate for the solution of (1), (2). It is particularly useful with regard to the boundedness of solutions. As we know, there are nonlinear maps/operators, such as

$$(M_1 x)(t) = \exp\{-|x(t)|\}, \quad x \in C(R_+, R),$$

or

$$(M_2 x)(t) = \tan^{-1} x(t), \quad x \in C(R_+, R),$$

which can be easily used to get operators satisfying (4). For instance, in the case $n = 1$, one can take $(Mx)(t) = m(t)(M_1 x)(t)$, with $m \in L^2(R_+, R)$, i.e.,

$$\int_0^\infty m^2(t) \, dt < \infty. \tag{5}$$

Let us now consider equation (1), under hypotheses $(H_1)-(H_3)$ and let $x(t)$ be a local solution of (1), (2). The existence of such a solution is guaranteed by our hypotheses (see [3], Chapter 3). On the existence interval, we multiply scalarly (in R^n) both sides of (1) by $x(t)$:

$$\langle x(t), \dot{x}(t)\rangle \ + \ \langle (Lx)(t), x(t)\rangle \ = \ \langle (Mx)(t), x(t)\rangle \, . \tag{6}$$

Integrating (6) from 0 to t, we obtain

$$\int_0^t \langle x(s), \dot{x}(s)\rangle \, ds + \int_0^t \langle (Lx)(s), x(s)\rangle \, ds = \int_0^t \langle (Mx)(s), x(s)\rangle \, ds. \tag{7}$$

Taking into account (3) and (4), and noticing that for each $\epsilon > 0$,

$$\left| \int_0^t \langle (Mx)(s), x(s) \rangle \, ds \right| \le \frac{1}{2\epsilon} \int_0^t m^2(s) \, ds + \frac{\epsilon}{2} \int_0^t |x(s)|^2 \, ds. \tag{8}$$

We derive from (7) the inequality:

$$\frac{1}{2}(|x(t)|^2 - |x_0|^2) + \lambda(t) \int_0^t |x(s)|^2 \, ds \le \frac{1}{2\epsilon} \int_0^t m^2(s) \, ds + \frac{\epsilon}{2} \int_0^t |x(s)|^2 \, ds. \tag{9}$$

From (9) we easily derive the Gronwall's type inequality

$$|x(t)|^2 + [2\lambda(t) - \epsilon] \int_0^t |x(s)|^2 \, ds \le |x_0|^2 + \frac{1}{\epsilon} \int_0^t m^2(s) \, ds. \tag{10}$$

The inequality (10) provides, in the usual way, an estimate for $x(t)$ on the interval of existence. In particular, under our assumptions, we have the fact that each solution of (1), (2) remains bounded on the interval of (local) existence, which implies that all solutions of (1), (2) are defined on the whole positive half-axis R_+.

The conclusion of the above carried discussion can be formulated as follows:

Theorem 1 *Consider the initial value problem (1), (2), under hypotheses* (H_1) *–* (H_3)*, and for arbitrary* $x_0 \in R^n$*. Then, there exists a solution* $x(t)$ *of this problem, defined on the whole positive half-axis* R_+*.*

Remark 2 If one looks at the inequality (10), it is easily seen that the conclusion of Theorem 1 remains valid if (H_3) is substituted by the weaker condition

$$m \in L^2_{loc}(R_+, R). \tag{11}$$

Indeed, in this case $\int_0^t m^2(s) \, ds$ is bounded for any $t \in R_+$, which means the right hand side in (10) remains bounded on any finite interval of R_+.

2 Dissipativity Conditions for System (1)

We shall now try to exploit further the inequality (10), under our hypotheses. Let us assume that a strengthened version of the inequality (3), in the hypothesis (H_2), is valid. Namely,

(H_4) There exists $\lambda_0 > 0$, such that

$$\int_0^t \langle (Lx)(s), x(s) \rangle \, ds \ge \lambda_0 \int_0^t |x(s)|^2 ds, \tag{12}$$

for any $t \in R_+$, and $x \in C(R_+, R^n)$.

Returning now to the inequality (10), we notice it becomes

$$|x(t)|^2 + (2\lambda_0 - \epsilon) \int_0^t |x(s)|^2 \, ds \le |x_0|^2 + \frac{1}{\epsilon} \int_0^\infty m^2(s) \, ds, \tag{13}$$

taking into account (H_3). This is true as far as $x(t)$ is defined.

Let us now consider the problem (1), (2), under hypotheses (H_1), (H_3), and (H_4). Since $\epsilon > 0$ is arbitrary in (13), we will choose it small enough, such that

$$2\lambda_0 - \epsilon > 0. \tag{14}$$

The right hand side in (13) is a constant, which means that the left hand side of (13) must be bounded on R_+. This means

$$|x(t)| \le C_1, \quad t \in R_+, \tag{15}$$

and

$$\int_0^t |x(s)|^2 \, ds \le C_2, \quad t \in R_+,$$

which actually means $x \in L^\infty(R_+, R^n) \bigcap L^2(R_+, R^n)$, or

$$\int_0^\infty |x(s)|^2 ds \le C_2. \tag{16}$$

From (15) and (16) we shall derive

$$\lim |x(t)| = 0, \quad \text{as} \quad t \to \infty, \tag{17}$$

which proves the dissipativity of the system (1), under our hypotheses.

We need, yet, one more condition in order to derive (17). We shall formulate this condition as hypothesis:

(H_5) The space $L^\infty(R_+, R^n)$ is invariant for the operator L in (1).

Under hypotheses (H_1), (H_3), (H_4), and (H_5), the equation (1) shows that $\dot{x}(t)$ can be represented as a sum of two terms, one in $L^\infty(R_+, R^n)$ and the second in $L^2(R_+, R^n)$. Say, $\dot{x}(t) = u(t) + v(t)$, with $u \in L^\infty$ and $v \in L^2$. Since for any t, $s \in R_+$ we have

$$|x(t) - x(s)| = \left| \int_s^t \dot{x}(\tau) \, d\tau \right| \le \left| \int_s^t u(\tau) \, d\tau \right| + \left| \int_s^t v(\tau) \, d\tau \right|, \tag{18}$$

we can write

$$\left| \int_s^t u(\tau) \, d\tau \right| \le k \, |t - s|, \tag{19}$$

for $k = ess - \sup|u(t)|,\ t \in R_+$, and

$$\left| \int_s^t v(\tau)d\tau \right| \le |t-s|^{\frac{1}{2}} \left(\int_0^\infty |v(\tau)|^2 d\tau \right)^{\frac{1}{2}}. \tag{20}$$

The inequalities (18) – (20) lead to

$$|x(t) - x(s)| \le k\,|t-s| + \left(\int_0^\infty |v(\tau)|^2 d\tau \right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}, \tag{21}$$

which proves the uniform continuity of $x(t)$ on R_+. Moreover, since $x(t) \in L^\infty(R_+, R^n)$, we see that $|x(t)|^2$ is also uniformly continuous on R_+.

Now using a fact which is known as Barbalat's lemma (see, for instance, [4]), the property (17) is proven.

Summing up the above discussion, we can formulate

Theorem 2 *If hypotheses (H_1), (H_3), (H_4), and (H_5) are satisfied, then any solution of the problem (1), (2) tends to zero at infinity.*

Remark 3 We cannot term the result in Theorem 2 as a stability result of the zero solution, because (1) may not admit the zero solution. It is rather a result of dissipativity of the system (1).

With further assumptions, we can estimate $|x(t) - y(t)|$ in terms of $|x_0 - y_0|$, and other data of the system (1). We shall not pursue this problem here.

3 A Second Order Functional Equation

Results similar to those given in Theorems 1 and 2, but for second order functional differential equations, can be found in the references [1] and [6]. We shall now consider a second order equation, namely

$$\ddot{x}(t) + (L\dot{x})(t) = (Vx)(t), \tag{22}$$

where L stands for a linear causal operator, continuous on the space $C(R_+, R^n)$, and V is also causal on $C(R_+, R^n)$, generally nonlinear.

The initial conditions for (22) will be the classical Cauchy conditions, namely

$$x(0) = x^0, \quad \dot{x}(0) = v^0, \tag{23}$$

with $x^0,\ v^0 \in R^n$.

Unlike the procedure in [4], at this time, we shall transform the problem (22), (23) into an integral functional equation.

First, we notice that (22), under the second initial condition (23), is equivalent to the first order equation

$$\dot{x}(t) = X(t,0)\,v^0 + \int_0^t X(t,s)\,(Vx)(s)\,ds, \tag{24}$$

where $X(t,s)$ is the Cauchy matrix associated with the linear operator L. This is obtained by using the variation of parameters formula for equations of the form $\dot{y}(t) = (Ly)(t) + f(t)$, see [2], or [3].

Integrating both sides of (24) from 0 to t, $t \in R_+$, and considering the first condition (23), we obtain the functional integral equation

$$x(t) = x^0 + \int_0^t X(s,0)\, v^0\, ds + \int_0^t \int_0^s X(s,u)\, (Vx)(u)\, du\, ds. \tag{25}$$

The equation (25) can be processed in a similar manner to that used in [5]. In order to obtain local or global existence, it suffices to impose a growth condition on the operator V. In [5], the following condition has been used:

$$\int_0^t \left(\int_0^s |(Vx)(u)|\, du \right)^2 ds \leq \lambda(t) \int_0^t |x(s)|^2\, ds + \mu(t), \tag{26}$$

with λ and μ non-decreasing on R_+. Condition (26) is also sufficient for assuring the local existence of solutions to (22), under conditions (23). We shall not pursue this direction here. Instead, we will consider the problem of obtaining an upper estimate for the norms of the solution to (22), (23).

In order to obtain this upper estimate, we need the following assumptions on the data:

(i) There exists $M > 0$, such that

$$\int_0^t |X(s,0)| ds \leq M, \qquad \int_0^t |X(t,s)| ds \leq M, \tag{27}$$

for $0 \leq s \leq t < \infty$.

(ii)
$$|(Vx)(t)| \leq \lambda(t) \sup_{0 \leq s \leq t} |x(s)|, \quad t \in R_+, \tag{28}$$

where $\lambda(t)$ is a non-negative non-decreasing function on R_+.

Now from (25) we derive

$$|x(t)| \leq |x^0| + \int_0^t |X(s,0)|\, |v^0| ds + \int_0^t \int_0^s |X(s,u)|\, |(Vx)(u)|\, du\, ds, \tag{29}$$

on the interval of existence for $x(t)$.

The inequality (29) leads to

$$|x(t)| \leq (|x^0| + M|v^0|) + M \int_0^t \sup_{0 \leq u \leq s} |(Vx)(u)|\, ds, \tag{30}$$

and taking into account (28), we obtain

$$|x(t)| \le (|x^0| + M|v^0|) + M \int_0^t \lambda(s) \sup_{0\le u\le s} |x(u)|\, ds. \tag{31}$$

Let us denote,

$$X(t) = \sup |x(s)|, \quad 0 \le s \le t. \tag{32}$$

Since the right hand side of (31) is non-decreasing in t, (31) and (32) imply

$$X(t) \le (|x^0| + M|v^0|) + M \int_0^t \lambda(s)\, X(s)\, ds, \tag{33}$$

for all $t > 0$ for which $x(t)$ is defined.

The inequality (33) is a Gronwall type integral inequality, and implies

$$X(t) \le (|x^0| + M|v^0|) \exp\left(M \int_0^t \lambda(s)\, ds \right), \tag{34}$$

which means, on behalf of (32),

$$|x(t)| \le (|x^0| + M|v^0|) \exp\left(M \int_0^t \lambda(s) ds \right).$$

The estimate (35) is actually valid on R_+, because it also implies the possibility of continuing a local solution to (25) on the semi-axis R_+.

Let us point out the fact that the inequality (29) for $|x(t)|$ could be also exploited in obtaining other estimates for $|x(t)|$.

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