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Explicit Solutions to a Class of Linear Partial Difference Equations

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Abstract: In this paper, we present explicit solutions of a class of linear partial difference equations with constant coefficients, and two kinds of linear partial difference equations with constant coefficients are discussed and their explicit solutions are obtained. As an application we give examples to show the efficiency of the solutions.

Keywords: *Partial difference equation; combinatorial enumeration; induction.*

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1 Introduction

Difference equations often appear in the study of numerical methods, combinatorial enumeration and system analysis [5, 6]. There are many methods for solving the linear difference equations of one argument with constant coefficients such as method of generating functions, method of Z transformation and that similar to the methods for solving the linear differential equations [1, 2]. But there are few papers on the difference equations of two variables or partial difference equations. In this paper two kinds of linear partial difference equations with constant coefficients are discussed and their explicit solutions are obtained.

2 Definitions and a Lemma

Definition 1 The following *difference equation* is called *first order linear partial difference equation with constant coefficients*

$$\alpha u(t,s) + \beta u(t,s-1) + \gamma u(t-1,s) + \delta u(t-1,s-1) + \lambda = 0, \tag{1}$$

where α, β, γ and δ are all constants, t and s are positive integers, and $\lambda = \lambda(t, s)$ is a given function.

For convenience, we will classify equations (1) as follows:

If $\alpha \neq 0,\ \gamma \neq 0$, then (1) can be written as

$$u(t, s) = -\frac{\alpha}{\beta}\, u(t, s-1) - \frac{\gamma}{\alpha}\, u(t-1, s) - \frac{\delta}{\alpha}\, u(t-1, s-1) - \frac{\lambda}{\alpha}. \tag{2}$$

If $\alpha \neq 0,\ \gamma = 0$, then (1) can be written as

$$u(t, s) = -\frac{\beta}{\alpha}\, u(t, s-1) - \frac{\delta}{\alpha}\, u(t-1, s-1) - \frac{\lambda}{\alpha}. \tag{3}$$

If $\alpha = 0,\ \beta \neq 0,\ \gamma \neq 0$, then (1) can be written as

$$u(t, s-1) = -\frac{\gamma}{\beta} u(t-1, s) - \frac{\delta}{\beta} u(t-1, s-1) - \frac{\lambda}{\beta}. \tag{4}$$

Equation (1) also can be expressed by

$$u(t, s) = -\frac{\gamma}{\beta}\, u(t-1, s+1) - \frac{\delta}{\beta}\, u(t-1, s) - \frac{\lambda}{\beta}. \tag{5}$$

If the term $u(t-1, s)$ of (4) is placed on the left-hand side, the other terms are moved to the right-hand side, and $t-1$ is replaced by t, then (4) can be written as

$$u(t, s) = -\frac{\beta}{\gamma} u(t+1, s-1) - \frac{\delta}{\gamma} u(t, s-1) - \frac{\lambda}{\gamma}. \tag{6}$$

If $\alpha = 0,\ \beta \neq 0,\ \gamma = 0$ or if $\alpha = 0,\ \beta = 0,\ \gamma \neq 0$, the equations (1) are both actually difference equations of one variable, which is not discussed here. If $\alpha = \beta = \gamma = 0$, $\delta \neq 0$, then the equation (1) is trivial one, which is also not considered here.

Without loss of generality the equations (2)−(6) can be classified in following four types:

$$\text{(A)} \qquad \begin{aligned} & u(s, t) = au(t, s-1) + bu(t-1, s) + cu(t-1, s-1) + d(t, s), \\ & u(t, 0) = F(t), \quad u(0, s) = E(s). \end{aligned} \tag{7}$$

$$\text{(B)} \qquad \begin{aligned} & u(s, t) = au(t, s-1) + bu(t-1, s-1) + d(t, s), \\ & u(t, 0) = F(t). \end{aligned} \tag{8}$$

$$\text{(C)} \qquad \begin{aligned} & u(s, t) = au(t-1, s+1) + bu(t-1, s) + d(t, s), \\ & u(0, s) = E(s). \end{aligned} \tag{9}$$

$$\text{(D)} \qquad \begin{aligned} & u(s, t) = au(t+1, s-1) + bu(t, s-1) + d(t, s), \\ & u(t, 0) = F(t). \end{aligned} \tag{10}$$

where a, b, c are constants and $d(t, s)$ is a given function of t and s.

Definition 2 The following *difference equation* is called *second order linear partial difference equation with constant coefficients*

$$au(t, s)+bu(t, s-1)+cu(t, s-2)+d(t-1, s-1)+eu(t-1, s)+fu(t-2, s)+g(t, s) = 0. \tag{11}$$

where a, b, c, d, e, f are all constants, $t \geq 2$, $s \geq 2$ and $g = g(t, s)$ is a given function.

In this paper we only discuss equation of type (E)

$$\text{(E)} \qquad \begin{aligned} & u(t, s) = au(t-2, s) + bu(t, s-2) + g(t, s), \\ & u(t, 0) = F_0(t), \quad u(t, 1) = F_1(t), \quad u(0, s) = E_0(s), \quad u(1, s) = E_1(s). \end{aligned} \tag{12}$$

Lemma *For the given constant a, b, c and nonnegative integers k and n, we set*

$$R(n,k) = \sum_{\substack{i_1+i_2+\cdots+i_n=k \\ i_r\geq 0\ (r=1,2\dots,n)}} \prod_{r=1}^{n} (H * b^{i_r-1})$$

where

$$R(n,k) = \begin{cases} 0, & k>0; \\ 1, & k=0. \end{cases}, \qquad H = ab+c, \qquad H * b^{m-1} = \begin{cases} a, & m=0; \\ Hb^{m-1}, & m\geq 1. \end{cases}$$

Then for the given function $f(t)$, $R(n,k)$ satisfies the following equation

$$\sum_{k=0}^{t} R(m,k) \sum_{j=0}^{t-k} R(1,j) f(t-k-j) = \sum_{k=0}^{t} R(m+1,\,k) f(t-k).$$

The lemma is easy to verify by induction, so the proof is omitted.

3 Solution of Explicit Expressions of the Partial Difference Equations

In the sequel, we shall give main results of this paper.

Theorem 1 *For $t\geq 1$ and $s\geq 1$,*

(1) *the solution of type* (A) *is*

$$u(t,s) = \sum_{k=0}^{t-1} R(s,k)F(t-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) M(t-k,\,s-j),$$

where

$$\begin{aligned} M(t,s) &= b^{t-1}[cu(0,\,s-1) + bu(0,s)] + \sum_{i=0}^{t-1} b^i d(t-i,\,s) \\ &= b^{t-1}[cE(s-1) + bE(s)] + \sum_{i=0}^{t-1} b^i d(t-i,\,s); \end{aligned}$$

(2) *the solution of type* (B) *is*

$$u(t,s) = \sum_{k=0}^{s} C_s^k a^{s-k} b^k F(t-k) + \sum_{k=0}^{s-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-k,\,s-j);$$

(3) *the solution of type* (C) *is*

$$u(t,s) = \sum_{k=0}^{t} C_t^k a^{t-k} b^k E(t+s-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-k,\,s+k-j);$$

(4) *the solution of type* (D) *is*

$$u(t,s) = \sum_{k=0}^{s} C_s^k a^{s-k} b^k F(t+s-k) + \sum_{k=0}^{s-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t+k-j,\,s-k).$$

Proof We only prove (1) and (3), the other proofs are similar to each other, hence omitted.

(1) For the problem (A), $b \neq 0$. Let us set $N(t,s) = au(t,\, s-1) + cu(t-1,\, s-1) + d(t,s)$, then

$$\begin{aligned} u(t,s) &= N(t,s) + bu(t-1,s) = N(t,s) + b[N(t-1,\, s) + bu(t-2,\, s)] \\ &= N(t,s) + bN(t-1,\, s) + b^2 u(t-2,\, s) = \dots \end{aligned} \tag{13}$$

By induction we have

$$\begin{aligned} u(t,s) &= \sum_{k=0}^{t-1} b^k N(t-k,s) + b^t u(0,s) = \sum_{k=0}^{t-1} H * b^{k-1} u(t-k,s-1) + M(t,s) \\ &= \sum_{k=0}^{t-1} R(1,k) u(t-k,s-1) + M(t,s). \end{aligned} \tag{14}$$

For any positive integer i $(1 \leq i \leq s)$, it is easy to be tested by induction also that

$$u(t,s) = \sum_{k=0}^{t-1} R(i,k) u(t-k,\, s-i) + \sum_{j=0}^{i-1} \sum_{k=0}^{t-1} R(j,k) M(t-k,\, s-j), \tag{15}$$

where $i = s$ (15) is the solution to (A).

(3) The situation "$\gamma = 0$ in (1)" is equivalent to the situation "$b = 0$ in (A)". But the proof (1) is under the condition $b \neq 0$, so the solution of problem (B) can be not the solution of (A) by letting $b = 0$. Now we will prove that for any positive integer i $(1 \leq i \leq s)$ the following holds:

$$u(t,s) = \sum_{k=0}^{i} C_i^k a^{i-k} b^k u(t-k,\, s-i) + \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k). \tag{16}$$

In fact, while $i = 1$, (16) becomes (8). Suppose (16) holds for i, then for $i+1$, substituting $u(t-k,\, s-i)$ in (16) by (8) we have

$$\begin{aligned} u(t,s) = &\sum_{k=0}^{i} C_i^k a^{i-k} b^k [au(t-k,\, s-i-1) + bu(t-k-1,\, s-i-1) + d(t-k,s-i)] \\ &+ \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k) \\ = &\sum_{k=0}^{i} C_i^k a^{i+1-k} b^k u(t-k,\, s-i-1) + \sum_{k=0}^{i} C_i^k a^{i-k} b^{k+1} u(t-k-1,\, s-i-1) \\ &+ \sum_{k=0}^{i} C_i^k a^{i-k} b^k d(t-k,s-i) + \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,s-k). \end{aligned} \tag{17}$$

The first two terms of (17) can be merged into one term by using formula $C_i^k + C_i^{k-1} = C_{i+1}^k$, the third term can be merged into the last term by substituting k by j and i by k respectively, hence

$$u(t,s) = \sum_{k=0}^{i+1} C_{i+1}^k a^{i+1-k} b^k u(t-k,\, s-i-1) + \sum_{k=0}^{(i+1)-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k),$$

which shows that (16) is proved and by letting $i = s$ (16) becomes the solution of (2).

Theorem 2 *The solution to* (E) *is*

$$\begin{aligned} u(t,s) = a^{t_1} \sum_{k=0}^{s_1-1} C_{k+t_1-1}^k b^k u(\delta(t),\, s-2k) + b^{s_1} \sum_{j=0}^{t_1-1} C_{j+s_1-1}^j a^j u(t-2j,\, \delta(s)) \\ + \sum_{k=0}^{s_1-1} \sum_{j=0}^{t_1-1} C_{k+j}^k a^j b^k g(t-2j,\, s-2k), \end{aligned} \tag{18}$$

where $t_1 = \left[\dfrac{t-1}{2}\right]$, $s_1 = \left[\dfrac{s-1}{2}\right]$, $[x]$ *expresses the minimum integer which is greater than or equals to* x*,*

$$\delta(x) = \begin{cases} 1, & \text{if } x \text{ is odd;} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

Proof If $b = 0$ in (12), then the equation becomes the second order difference equation of one argument t. The solution can be easily calculated as follows:

$$u(t,s) = a^{t_1} u(\delta(t), s) + \sum_{k=0}^{t_1-1} a^k g(t-2k,\, s). \tag{19}$$

If $b \neq 0$ in (12), taking $u(t, s-2)$ in (12) as an iterative term calculated successively one obtains

$$\begin{aligned} u(t,s) &= au(t-2,\, s) + bu(t,\, s-2) + g(t,s) \\ &= au(t-2,\, s) + b[au(t-2,\, s-2) + bu(t,\, s-4) + g(t,\, s-2)] + g(t,s) \\ &= au(t-2,\, s) + abu(t-2,\, s-2) + b^2[au(t-2,\, s-4) \\ &\quad + bu(t,\, s-6) + g(t,\, s-4)] + bg(t,\, s-2) + g(t,s) \\ &= au(t-2,\, s) + abu(t-2,\, s-2) + ab^2u(t-2,\, s-4) + b^3u(t,\, s-6) \\ &\quad + b^2g(t, s-4) + bg(t, s-2) + g(t,s) \end{aligned}$$

Hence one can get the following by induction

$$u(t,s) = a \sum_{k=0}^{s_1-1} b^k u(t-2,\, s-2k) + b^{s_1} u(t, \delta(s)) + \sum_{k=0}^{s_1-1} b^k g(t,\, s-2k). \tag{20}$$

Now we will prove by induction again that for any positive integer i $(1 \le i \le t_1)$

$$\begin{aligned} u(t,s) = a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k u(t-2i,\, s-2k) + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s)) \\ + \sum_{k=0}^{s_1-1} \sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k). \end{aligned} \tag{21}$$

In fact, when $i = 1$ (21) becomes (20). Suppose (21) holds for i, then for $i+1$, substituting the term $u(t-2i,\, s-2k)$ in (21) by (20) one can get:

$$\begin{aligned} u(t,s) = a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \Bigg[a \sum_{j=0}^{\left[\frac{s-2k-1}{2}\right]-1} b^j u(t-2i-2,\, s-2k-2j) \\ + b^{\left[\frac{s-2k-1}{2}\right]-1} u(t-2i,\, \delta(s-2k)) + \sum_{j=0}^{\left[\frac{s-2k-1}{2}\right]-1} b^j g(t-2i,\, s-2k-2j) \Bigg] \\ + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s)) + \sum_{k=0}^{s_1-1} \sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k). \end{aligned} \tag{22}$$

Because

$$\left[\frac{s-2k-1}{2}\right] = \left[\frac{s-1}{2} - k\right] = \left[\frac{s-1}{2}\right] - k = s_1 - k, \quad \delta(s-2k) = \delta(s),$$

$$\sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j = \sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k, \qquad \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} = C_{s_1+i-1}^{i-1},$$

hence (22) can be written as

$$\begin{aligned} u(t,s) &= a^{i+1} \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j u(t-2i-2,\, s-2k-2j) \\ &\quad + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^{s_1} u(t-2i,\, \delta s) + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j g(t-2i, s-2k-2j) \\ &\quad + \sum_{k=0}^{s_1-1} \sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j, s-2k) + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j, \delta s) \\ &= a^{i+1} \sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k u(t-2(i+1),\, s-2k) + b^{s_1} \sum_{j=0}^{(i+1)-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta s) \\ &\quad + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{(i+1)-1} b^k g(t-2i,\, s-2k) + \sum_{k=0}^{s_1-1} \sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k) \end{aligned}$$

$$= a^{i+1}\sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k u(t-2(i+1),\, s-2k) + b^{s_1}\sum_{j=0}^{(i+1)-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s))$$
$$+ \sum_{k=0}^{s_1-1}\sum_{j=0}^{(i+1)-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k).$$

One obtains (21). With $i = t_1 = \left[\dfrac{t-1}{2}\right]$, (21) becomes (18). It is easy to know that (19) is the exception of (18), thus the proof of Theorem 2 is completed.

4 Examples

Example 1 Find the numbers of the shortest lattice paths with diagonal steps [3].

On the coordinate plane, the number of the shortest lattice paths with diagonal steps is called Delannoy number [3], which satisfies the difference equation:

$$\begin{gathered} D(t,s) = D(t,\, s-1)D(t-1,\, s) + D(t-1,\, s-1), \\ D(t,0) = D(0,s) = 1. \end{gathered} \tag{23}$$

Problem (23) is the type of (A), where $a = b = c = 1$, $d = 0$ and its solution is

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k)F(t-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k)M(t-k,\, s-j),$$

where $F(t) = D(t,0) = 1$, $M(t,s) = D(0,\, s-1) + D(0,s) = 2$, from which one gets

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k) + 2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k). \tag{24}$$

Because

$$H = ab + c = 2, \quad H * b^{m-1} = \begin{cases} 1, & m = 0; \\ 2, & m \ge 1; \end{cases}$$

$$R(n,k) = \sum_{\substack{i_1+i_2+\dots+i_n=k,\\ i_r\ge 0,\ (r=1,2\dots,n)}} \prod_{r=1}^{n} (H * b^{i_r-1}) = \sum_{m=0}^{n} C_n^m \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} \prod_{r=1}^{n-m} (H * b^{i_r-1}) \right)$$
$$= \sum_{m=0}^{n} C_n^m \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} 2^{n-m} \right) = \sum_{m=0}^{n} C_n^m 2^{n-m} \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} 1 \right).$$

From the literature [4] we know that the number of the natural number solution to the equation $x_1 + x_2 + \dots + x_{n-m} = k$ is C_{k-1}^{n-m-1}, hence

$$R(n,k) = \sum_{m=0}^{n} 2^{n-m} C_n^m C_{k-1}^{n-m-1}.$$

It follows that

$$\sum_{k=0}^{t-1} R(s,k) = \sum_{k=0}^{t-1}\sum_{m=0}^{s} 2^{s-m} C_s^m C_{k-1}^{s-m-1} = \sum_{m=0}^{s} 2^{s-m} C_s^m \left(\sum_{k=0}^{t-1} C_{k-1}^{s-m-1} \right)$$
$$= \sum_{m=0}^{s} 2^{s-m} C_s^m \left(\sum_{k=s-m}^{t-1} C_{k-1}^{s-m-1} \right) = \sum_{m=0}^{s} 2^{s-m} C_s^m C_{t-1}^{s-m} = \sum_{m=0}^{s} 2^m C_s^m C_{t-1}^m,$$

where

$$\sum_{k=s-m}^{t-1} C_{k-1}^{s-m-1} = C_{t-1}^{s-m}.$$

Calculating shows that

$$2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) = 2\sum_{j=0}^{s-1}\left(\sum_{k=0}^{t-1} R(j,k)\right) = 2\sum_{j=0}^{s-1}\sum_{m=0}^{j} 2^m C_j^m C_{t-1}^m$$
$$= 2\sum_{m=0}^{s-1}\sum_{j=m}^{s-1} 2^m C_j^m C_{t-1}^m = 2\sum_{m=0}^{s-1} 2^m C_{t-1}^m \left(\sum_{j=m}^{s-1} C_j^m\right)$$
$$= 2\sum_{m=0}^{s-1} 2^m C_{t-1}^m C_s^{m+1} = \sum_{m=0}^{s-1} 2^{m+1} C_{t-1}^m C_s^{m+1},$$

where

$$\sum_{j=m}^{s-1} C_j^m = C_s^{m+1}.$$

With the above result one gets

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k) + 2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) = \sum_{m=0}^{s} 2^m C_s^m C_{t-1}^m + \sum_{m=0}^{s-1} 2^{m+1} C_s^{m+1} C_{t-1}^m$$
$$= \sum_{m=0}^{s} 2^m C_s^m (C_{t-1}^m + C_{t-1}^{m-1}) = \sum_{m=0}^{s} 2^m C_s^m C_t^m.$$

This is the very result given out of [3].

Example 2 Find all eigenvalues of the following matrix B

$$B = \begin{pmatrix} \alpha & & & \gamma & & \\ & \ddots & & & & \\ & & \alpha & & & \beta \\ & & & \ddots & & \\ & \gamma & & & \ddots & \\ & & \beta & & & \alpha \end{pmatrix},$$

where the matrix B has the following characteristics:

1. All the main diagonal elements of B are α.

2. All the skew diagonal elements of the downright square submatrix of B are β. If the number of the submatrix's order is odd, then the cross-element of the main diagonal line of the submatrix and its skew diagonal line is still α (the line connecting the upright corner of a square matrix with its downleft corner is called skew diagonal line).

3. The remained elements of B are all γ.

To find the eigenvalues of the matrix B is to find the roots of equation $\det(\lambda I - B) = 0$, where I is a unit matrix of order equivalent to B. Let us first calculate $\det B$. In fact, the matrix B is similar to matrix $A(t,s)$ where

$$A(t,s) = \begin{pmatrix} \alpha & \beta & & & & & & & & \\ \beta & \alpha & & & & & & & & \\ & & \alpha & \beta & & & & & & \\ & & \beta & \alpha & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \alpha & \beta & & & \\ & & & & & \beta & \alpha & & & \\ & & & & & & & \alpha & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \alpha \end{pmatrix}_{(t+s)\times(t+s)} \tag{25}$$

$(t \ge s \ge 0)$ is a matrix of order $(t+s)\times(t+s)$, where there are s submatrixes of order 2: $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, and a submatrix of order $(t-s)$: $\begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix}$, the unlisted elements are all γ, which is shown above.

It is easy to know that

$$\begin{gathered} \det B = \det A(t,s) = 2(\alpha-\beta)\det A(t,s-1) - (\alpha-\beta)^2 \det A(t-1,\, s-1), \\ t \ge s \ge 1. \end{gathered} \tag{26}$$

The initial condition is that

$$\det A(t,0) = (\alpha-\gamma)^{t-1}[\alpha + (t-1)\gamma], \quad \det A(0,0) = 1.$$

This is the very type of problem (B), where

$$a = 2(\alpha-\beta), \quad b = -(\alpha-\beta)^2, \quad c = 0, \quad F(t) = (\alpha-\gamma)^{t-1}[\alpha + (t-1)\gamma].$$

The solution of it can be calculated as follows

$$\begin{aligned} \det A(t,s) = (\alpha-\beta)^s(\alpha+\beta-2\gamma)^{s-1}(\alpha-\gamma)^{t-s-1} \\ \times \{(\alpha+\beta-2\gamma)[\alpha+(t-1)\gamma] + s\gamma(\alpha-\beta)\}. \end{aligned} \tag{27}$$

Substituting elements α, β and γ of matrix B by $\lambda - \alpha$, $-\beta$ and $-\gamma$ respectively, then we can get immediately eigenvalues of B:

(i) If $t > s > 1$, then

$\lambda_1 = (\alpha - \beta)$ is a root of s-multiplicity;

$\lambda_2 = (\alpha - \gamma)$ is a root of $(t - s - 1)$-multiplicity;

$\lambda_3 = (\alpha + \beta - 2\gamma)$ is a root of $(s - 1)$-multiplicity;

$$\lambda_4 = \frac{1}{2}\Big\{[(\alpha+\beta-2\gamma)+(\alpha+(t-1)\gamma)+s\gamma]$$
$$+\sqrt{[(\alpha+\beta-2\gamma)+(\alpha+(t+s-1)\gamma)]^2-4[(\alpha+\beta-2\gamma)(\alpha+(t-1)\gamma+s\gamma(\alpha-\beta))]}\Big\},$$
$$\lambda_5 = \frac{1}{2}\Big\{[(\alpha+\beta-2\gamma)+(\alpha+(t-1)\gamma)+s\gamma]$$
$$-\sqrt{[(\alpha+\beta-2\gamma)+(\alpha+(t+s-1)\gamma)]^2-4[(\alpha+\beta-2\gamma)(\alpha+(t-1)\gamma+s\gamma(\alpha-\beta))]}\Big\}$$

are two single roots.

(ii) If $t = s \geq 1$, then

$$\det A(t,s) = (\alpha-\beta)^s(\alpha+\beta-2\gamma)^{s-1}(\alpha+\beta+(s-2)\gamma),$$

$\lambda_1 = (\alpha - \beta)$ is a root of s-multiplicity;
$\lambda_2 = (\alpha + \beta - 2\gamma)$ is a root of $(s - 1)$-multiplicity;
$\lambda_3 = (\alpha + \beta - 2\gamma)$ is a single root.

(iii) If $t > s = 0$, then

$$\det A(t,s) = (\alpha-\gamma)^{t-1}(\alpha+(t-1)\gamma);$$

$\lambda_1 = \alpha - \gamma$ is a root of $(t - 1)$-multiplicity;
$\lambda_2 = \alpha + (t - 1)\gamma$ is a single root.

5 Conclusions

This paper has been focused on the study of the solution's explicit expressions of some kind of partial difference equations. The method is very simple, but the results can be used in the study of some kind of combinatorial enumerations and some other related fields.

References

[1] Saber, N. *An Introduction to Difference Equations.* New York: Springer, 1999.

[2] Lakshmikantham V. and Trigiante, D. *Theory of Difference Equations: Numerical Methods and Applications.* New York: Academic Press, 1998.

[3] Comtete, L. *Advanced Combinatorics.* D. Reidel Publishing Company, 1974.

[4] Xu, L. and Wang, X. *Selected Examples and Methods of Mathematic Analysis.* Beijing: Higher Education Press, 1985.

[5] Giannakopoulos, F. and Zapp, A. Stability and Hopf bifurcation in difference equations with one delay. *Nonlinear Dynamics and Systems Theory* **1**(2) (2001) 145–158.

[6] Zemtsova, N.I. Stability of the stationary solutions of the difference equations of restricted Newtonian problem with in complete symmetry. *Nonlinear Dynamics and Systems Theory* **3**(1) (2003) 105-118.