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Synchronization of Time-Delay Chua's Oscillator with Application to Secure Communication

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Abstract: In this paper, we use a Generalized Hamiltonian systems approach to synchronize the time-delay-feedback Chua's oscillator (hyperchaotic circuit with multiple positive Lyapunov exponents). Synchronization is thus between the transmitter and the receiver dynamics with the receiver being given by an observer. We apply this approach to transmit private analog and binary information signals in which the quality of the recovered signal is higher than in traditional observer techniques while the encoding remains potentially secure.

Keywords: *Synchronization; time-delay-feedback Chua's oscillator; hyperchaos; passivity based observers; secure communication.*

Mathematics Subject Classification (2000): 37N35, 65P20, 68P25, 70K99, 93D20, 94A99.

1 Introduction

Recently, chaotic synchronization has received much attention. Many synchronization methods for chaotic oscillators have been proposed in the literature (see e.g. (Pecora and Carroll 1990; Wu and Chua 1993; Feldmann, *et al.* 1996; Nijmeijer and Mareels 1997; Special Issue, 1997; 2000; Fradkov, *et al.* 1998; Chen and Dong 1998; Cruz and Nijmeijer 1999; 2000; Sira-Ramírez and Cruz 2001; Pikovsky, *et al.* 2001; Aguilar and Cruz 2002) and references therein). Data encryption using chaotic dynamics was reported in the early 1990s as a new approach for signal encoding which differs from the conventional methods using numerical algorithms as the encryption key. As a result,

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chaotic synchronization plays an important role in chaotic communications. Different methods have been developed in order to hide the contents of a message using chaotic synchronization, such as *chaotic additive masking* (Cuomo, *et al.* 1993), *chaotic switching* (Parlitz, *et al.* 1992; Cuomo, *et al.* 1993; Dedieu, *et al.* 1993), and *chaotic parameter modulation* (Yang and Chua 1996). However, it has been shown see e.g., (Short 1994; Pérez and Cerdeira 1995) that encrypted signals by means of comparatively simple chaos with only one positive Lyapunov exponent does not ensure a sufficient level of security. For higher security the hyperchaotic oscillators characterized by more than one positive exponents are advantageous over simple chaotic oscillators. Two factors that are of primordial importance in security considerations related to chaotic communication systems. These are; the dimensionality of the chaotic attractor, and the effort required to obtain the necessary parameters for the matching of a receiver dynamics.

One way to enhance the level of security of the communication system can consist in applying proper cryptographic techniques to the information signal see e.g., (Yang, *et al.* 1997). Another way to solve this security problem is to encode the message by using high dimensional chaotic attractors, or hyperchaotic attractors, which take advantage of the increased randomness and unpredictably of the higher dimensional dynamics. In such option one generally encounters *multiple positive Lyapunov exponents*. However, the synchronization of hyperchaotic oscillators is a much more difficult problem (see e.g. (Brucoli, *et al.* 1999; Peng, *et al.* 1996; Cruz, *et al.* 2002) and Aguilar and Cruz 2002 for the discrete-time context). Most of the previous work done on hyperchaotic synchronization has been concentrated on finite-dimensional oscillators described by ordinary differential equations. Thus, the number of positive Lyapunov exponents is limited by dimension of the state space.

As alternative way of constructing synchronized hyperchaotic oscillators can be based on delay differential equations, such oscillators have an infinite-dimensional state space and can produce hyperchaotic dynamics with an arbitrarily large number of positive Lyapunov exponents. It has been known that even a very simple first-order oscillator with a time-delay-feedback can produce extremely complex hyperchaotic behaviors (Mackey and Glass 1977; Farmer 1982; Lu and He 1996). This property has already stimulated the work on design of systems for secure communication which claimed to have low detectability (Mensour and Longtin 1998; Pyragas 1998).

The objective of this paper is to extend the approach developed in (Sira-Ramírez and Cruz 2001) to the synchronization of time-delay-feedback Chua's oscillator (hyperchaotic circuit with multiple positive Lyapunov exponents) through a Generalized Hamiltonian systems and observer approach. Moreover, we apply this method to transmit and retrieve private/secure analog and binary information signals using hyperchaotic additive masking and hyperchaotic switching, respectively. We can enumerate several advantages over the existing synchronization methods:

- It enables synchronization be achieved in a systematic way;
- It can be successfully applied to several well-known chaotic or hyperchaotic oscillators;
- It does not require the computation of any Lyapunov exponent;
- It does not require initial conditions belonging to the same basin of attraction.

The organization of the paper is as follows. In Section 2, we obtain the synchronization of the time-delay-feedback Chua's oscillator through a Generalized Hamiltonian systems and observer approach. In Section 3, we present the stability analysis related to the synchronization error. In Section 4, we give an application to secure communication of private analog and binary information signals. Finally, Section 5 is devoted to some concluding remarks and suggestions for further work.

2 Synchronization of Time-Delay Chaotic Oscillator

Consider the following dynamical system described by

$$\dot{x} = f\left(x,\, x\left(t-\tau\right)\right), \tag{1}$$

where $x\left(t\right) = \left(x_1, \dots, x_n\right)^T \in \mathbb{R}^n$ is the state vector, f is a nonlinear function, and τ is the time-delay. The system (1) provides an example of infinite-dimensional oscillator with *multiple positive Lyapunov exponents* (generating extremely complex hyperchaotic signals). Following the approach developed in (Sira-Ramírez and Cruz 2001), the time-delay oscillator described by equation (1) can be written in the following *Generalized Hamiltonian canonical form*,

$$\dot{x} = \mathcal{J}(x)\,\frac{\partial H}{\partial x} + \mathcal{S}(x)\,\frac{\partial H}{\partial x} + \mathcal{F}(x, x\left(t-\tau\right)), \qquad x \in \mathbb{R}^n \tag{2}$$

where $H(x)$ denotes a smooth *energy function* which is globally positive definite in $\mathbb{R}^n$. The column *gradient vector* of H, denoted by $\partial H/\partial x$, is assumed to exist everywhere. We use *quadratic* energy function $H(x) = \frac{1}{2}\,x^T\mathcal{M}x$ with $\mathcal{M}$ being a, constant, symmetric positive definite matrix. In such case, $\partial H/\partial x = \mathcal{M}x$. The square matrices, $\mathcal{J}(x)$ and $\mathcal{S}(x)$ satisfy, for all $x \in \mathbb{R}^n$, the following properties, which clearly depict the *energy managing* structure of the system, $\mathcal{J}(x) + \mathcal{J}^T(x) = 0$, and $\mathcal{S}(x) = \mathcal{S}^T(x)$. The vector field $\mathcal{J}(x)\,\partial H/\partial x$ exhibits the *conservative* part of the system and it is also referred to as the *workless* part, or *work-less* forces of the system; and $\mathcal{S}(x)$ depicting the *working* or *nonconservative* part of the system. For certain systems, $\mathcal{S}(x)$ is *negative definite* or *negative semidefinite*. In such cases, the vector field is addressed to as the *dissipative* part of the system. If, on the other hand, $\mathcal{S}(x)$ is positive definite, positive semidefinite, or indefinite, it clearly represents, respectively, the global, semi-global, and local *destabilizing* part of the system. In the last case, we can always (although nonuniquely) discompose such an indefinite symmetric matrix into the sum of a symmetric negative semidefinite matrix $\mathcal{R}(x)$ and a symmetric positive semidefinite matrix $\mathcal{N}(x)$. And where $\mathcal{F}(x, x(t-\tau))$ represents a *locally destabilizing* vector field.

In the context of observer design, we consider a special class of Generalized Hamiltonian systems with destabilizing vector field and linear output map, $y(t)$, given by

$$\begin{aligned} \dot{x} &= \mathcal{J}(y)\,\frac{\partial H}{\partial x} + (\mathcal{I} + \mathcal{S})\,\frac{\partial H}{\partial x} + \mathcal{F}(y, y(t-\tau)), \qquad x \in \mathbb{R}^n, \\ y &= \mathcal{C}\,\frac{\partial H}{\partial x}, \qquad y \in \mathbb{R}^m, \end{aligned} \tag{3}$$

where $\mathcal{S}$ is a constant symmetric matrix, not necessarily of definite sign. The matrix $\mathcal{I}$ is a constant skew symmetric matrix. The vector variable $y(t)$ is referred to as the system *output*. The matrix $\mathcal{C}$ is a constant matrix.

We denote the *estimate* of the state vector $x(t)$ by $\xi(t)$, and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of H in terms of $\xi(t)$. Similarly, we denote by $\eta(t)$ the estimated output, computed in terms of the estimated state $\xi(t)$. The gradient vector $\partial H(\xi)/\partial\xi$ is, naturally, of the form $\mathcal{M}\xi$ with $\mathcal{M}$ being a, constant, symmetric positive definite matrix.

A dynamic *nonlinear state observer* for the Generalized Hamiltonian system (3) is readily obtained as

$$\begin{aligned} \dot{\xi} &= \mathcal{J}(y)\,\frac{\partial H}{\partial \xi} + (\mathcal{I}+\mathcal{S})\,\frac{\partial H}{\partial \xi} + \mathcal{F}(y, y(t-\tau)) + K(y-\eta), \qquad \xi \in \mathbb{R}^n, \\ \eta &= \mathcal{C}\,\frac{\partial H}{\partial \xi}, \end{aligned} \tag{4}$$

K is a constant matrix, known as the *observer gain*.

The *state estimation error*, defined as $e(t) = x(t) - \xi(t)$ and the output estimation error, defined as $e_y(t) = y(t) - \eta(t)$, are governed by

$$\begin{aligned} \dot{e} &= \mathcal{J}(y)\,\frac{\partial H}{\partial e} + (\mathcal{I}+\mathcal{S}-K\mathcal{C})\,\frac{\partial H}{\partial e}, \qquad e \in \mathbb{R}^n, \\ e_y &= \mathcal{C}\,\frac{\partial H}{\partial e}, \qquad e_y \in \mathbb{R}^m, \end{aligned} \tag{5}$$

where the vector, $\partial H/\partial e$ actually stands, with some abuse of notation, for the gradient vector of the *modified* energy function, $\partial H(e)/\partial e = \partial H/\partial x - \partial H/\partial \xi = \mathcal{M}(x-\xi) = \mathcal{M}e$. We set, when needed, $\mathcal{I}+\mathcal{S} = \mathcal{W}$.

Remark 1 Note that the error state dynamics described by equation (5) is independent of time-delay τ, i.e. equation (5) is a simple linear ordinary differential equation.

Definition 1 Synchronization problem: We say that the receiver dynamics (4) *synchronizes* with the transmitter dynamics (3), if

$$\lim_{t\to\infty} \|x(t) - \xi(t)\| = 0, \tag{6}$$

no matter which initial conditions $x(0)$ and $\xi(0)$ have. Where the state estimation error $e(t) = x(t) - \xi(t)$ represents the *synchronization error*.

Example 1 Time-delay-feedback Chua's oscillator The state equations of "standard" or "classic" Chua's oscillator are given by

$$\begin{aligned} C_1\dot{x}_1 &= G(x_2 - x_1) - F(x_1), \\ C_2\dot{x}_2 &= G(x_1 - x_2) + x_3, \\ L\dot{x}_3 &= -x_2 - R_0 x_3, \end{aligned} \tag{7}$$

with nonlinear function

$$F(x_1) = bx_1 + \frac{1}{2}\,(a-b)(|x_1+1| - |x_1-1|), \quad a,\, b < 0. \tag{8}$$

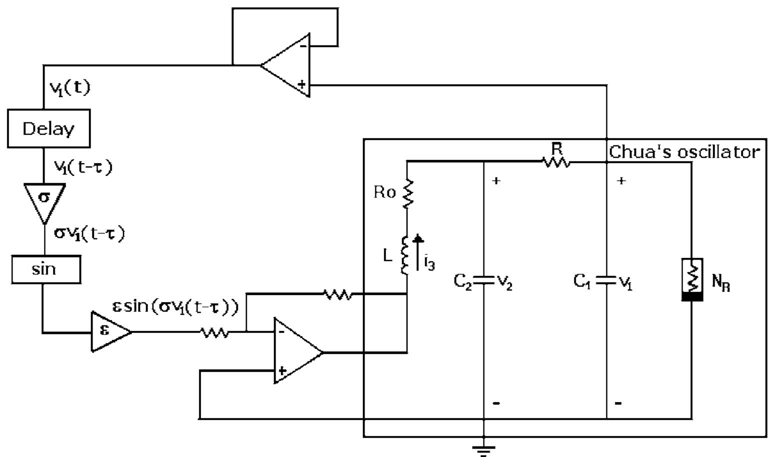

Figure 2.1. Time-delay-feedback Chua's oscillator.

The classic Chua's oscillator (7) – (8) can only produce low-dimensional chaos with one positive Lyapunov exponent. The time-delay-feedback Chua's oscillator considered in this paper is shown in Figure 2.1, and can be described by (Wang, *et al.* 2001):

$$\begin{aligned} C_1\dot{x}_1 &= G(x_2 - x_1) - F(x_1), \\ C_2\dot{x}_2 &= G(x_1 - x_2) + x_3, \\ L\dot{x}_3 &= -x_2 - R_0 x_3 - w(x_1(t-\tau)), \end{aligned} \tag{9}$$

with $F(x_1)$ given by (8) and where the time-delay term is taken as

$$w(x_1(t-\tau)) = \varepsilon\, \sin(\sigma x_1(t-\tau)), \tag{10}$$

with ε and σ are two positive constants, and τ represents the time-delay. Clearly, the maximum amplitude of the time-delay term is ε, i.e.

$$|w(x_1(t-\tau))| \le \varepsilon. \tag{11}$$

For arbitrarily given $\varepsilon > 0$, the time-delay-feedback Chua's oscillator (9) can be chaotic for sufficiently large σ and τ, even if the corresponding standard Chua's oscillator (7) has stable period orbits.

The time-delay-feedback Chua's oscillator is characterized by the following parameters: $R = 1910\,\Omega,\ R_0 = 16\,\Omega,\ C_1 = 10\,nF,\ C_2 = 100\,nF,\ L = 18.68\,mH,\ G_a = -0.75\,mS,$ $G_b = -0.41\,mS,\ B_p = 1\,V$, and $\tau = 0.001$. These values assure the existence of very complex hyperchaotic behavior. Figure 2.2 shows several different types of attractors from the time-delay-feedback Chua's oscillator for $\tau = 0.001$:

(a) $\varepsilon = 0.07$ and $\sigma = 0.4$;
(b) $\varepsilon = 0.2$ and $\sigma = 0.5$;
(c) $\varepsilon = 0.5$ and $\sigma = 3$;
(d) $\varepsilon = 1$ and $\sigma = 1$.

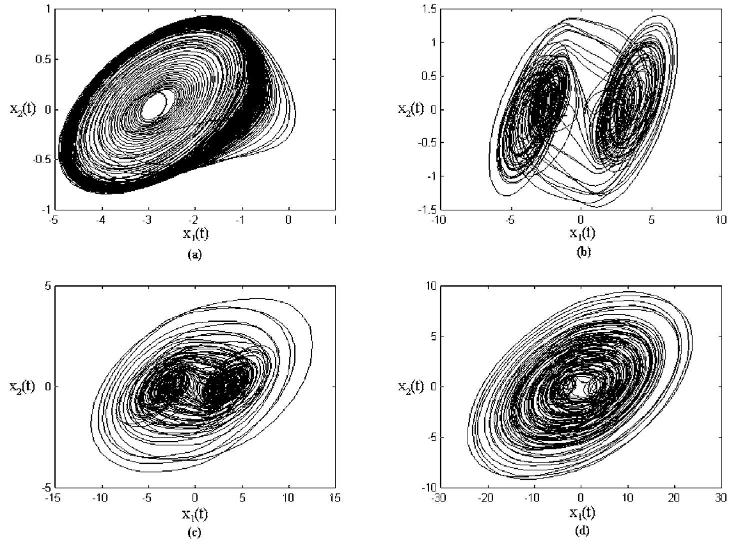

Figure 2.2. Several different types of chaotic attractors from the time-delay-feedback Chua's oscillator for $\tau = 0.001$: (a) $\varepsilon = 0.07$ and $\sigma = 0.4$. (b) $\varepsilon = 0.2$ and $\sigma = 0.5$. (c) $\varepsilon = 0.5$ and $\sigma = 3$. (d) $\varepsilon = 1$ and $\sigma = 1$.

The state equations describing the time-delay-feedback Chua's oscillator (9) in Hamiltonian canonical form with a destabilizing vector field (*transmitter circuit*) is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1 C_2} & 0 \\ \frac{G}{C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & -\frac{R_0}{L^2} \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} -\frac{1}{C_1} F(x_1) \\ 0 \\ -\frac{1}{L} w(x_1(t-\tau)) \end{bmatrix} \tag{12}$$

taking as the Hamiltonian energy function

$$H(x) = \frac{1}{2}\left[C_1 x_1^2 + C_2 x_2^2 + L x_3^2\right] \tag{13}$$

and gradient vector as

$$\frac{\partial H}{\partial x} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_1 x_1 \\ C_2 x_2 \\ L x_3 \end{bmatrix}.$$

The destabilizing vector field evidently calls for $x_1(t)$ to be used as the output, $y_1(t)$, of the transmitter circuit (12). The matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$, are given by

$$\mathcal{C} = \left[\tfrac{1}{C_1} \quad 0 \quad 0\right], \qquad \mathcal{S} = \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1 C_2} & 0 \\ \frac{G}{C_1 C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & -\frac{R_0}{L^2} \end{bmatrix}, \qquad \mathcal{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is neither observable nor detectable. However, the pair $(\mathcal{C}, \mathcal{W})$ is observable. The system lacks damping in the $x_3(t)$ variable, and either in the $x_1(t)$ or

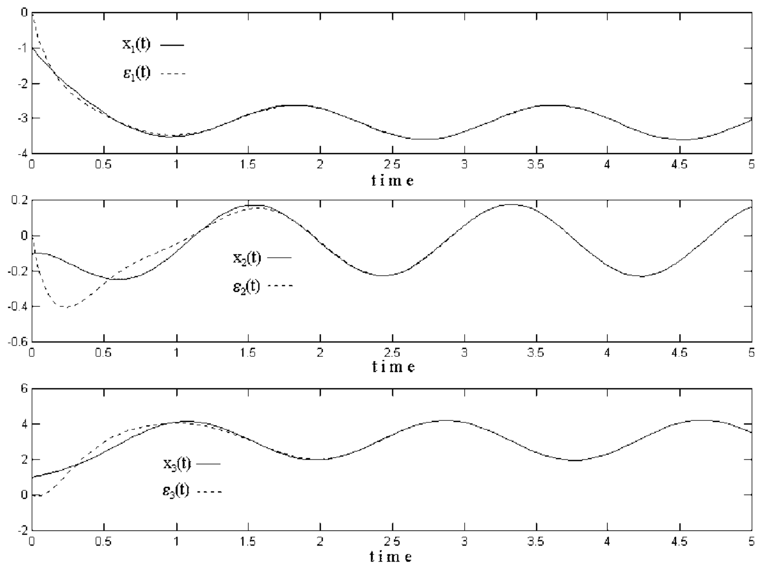

Figure 2.3. Time-delay-feedback Chua's oscillator state trajectories and synchronized receiver trajectories.

the $x_2(t)$ variable as inferred from the negative semi-definite nature of the dissipation structure matrix, $\mathcal{S}$. If $x_1(t)$ is used as output, then the output error injection term can enhance the dissipation in the error state dynamics. The *receiver circuit* is designed as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{LC_2} \\ 0 & -\frac{1}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\frac{G}{C_1^2} & \frac{G}{C_1C_2} & 0 \\ \frac{G}{C_1C_2} & -\frac{G}{C_2^2} & 0 \\ 0 & 0 & -\frac{R_0}{L^2} \end{bmatrix} \frac{\partial H}{\partial \xi} + \begin{bmatrix} -\frac{1}{C_1}F(y) \\ 0 \\ -\frac{1}{L}w(y(t-\tau)) \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} e_y, \tag{14}$$

where the gain vector $K = (k_1, k_2, k_3)^T$ is chosen in order to guarantee the asymptotic exponential stability to zero of the state reconstruction error trajectories (synchronization error $e(t)$). From (12) and (14) the synchronization error dynamics is governed by

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_2}{2C_1} & \frac{k_3}{2C_1} \\ -\frac{k_2}{2C_1} & 0 & \frac{2}{LC_2} \\ -\frac{k_3}{2C_1} & -\frac{2}{LC_2} & 0 \end{bmatrix} \frac{\partial H}{\partial e} + \begin{bmatrix} -\frac{G+C_1k_1}{C_1^2} & \frac{2G-C_2k_2}{2C_1C_2} & -\frac{k_3}{2C_1} \\ \frac{2G-C_2k_3}{2C_1C_2} & -\frac{G}{C_2^2} & 0 \\ -\frac{k_3}{2C_1} & 0 & -\frac{R_0}{L} \end{bmatrix} \frac{\partial H}{\partial e}. \tag{15}$$

With $x(0) = (-1, -0.1, 1)$ and $\xi(0) = (0, 0, 0)$ we obtain the following numerical results. Figure 2.3 shows the time-delay-feedback Chua's oscillator state trajectories and synchronized receiver trajectories. Figure 2.4 illustrates the time behaviors of the synchronization error trajectories $e_i(t) = x_i(t) - \xi_i(t)$, $i = 1, 2, 3$ for $k_1 = k_2 = k_3 = 5$. To ease the numerical simulations we resorted the following normalized version of the time-delay-feedback Chua's oscillator:

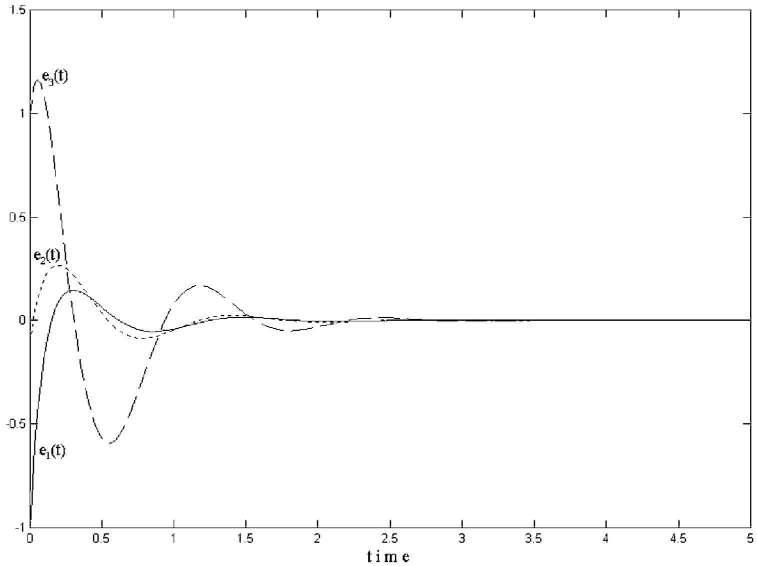

Figure 2.4. Synchronization error trajectories $e_i(t) = x_i(t) - \xi_i(t)$, $i = 1, 2, 3$.

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1 - f(x_1)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3 - \beta\,\varepsilon \sin(\sigma x_1(t-\tau)), \end{aligned} \tag{16}$$

where the nonlinear function is given by

$$f(x_1) = bx_1 + \frac{1}{2}\,(a-b)(|x_1+1| - |x_1-1|)$$

with $\alpha = 10$, $\beta = 19.53$, $\gamma = 0.1636$, $a = -1.4325$, $b = -0.7831$, $\sigma = 0.5$, $\varepsilon = 0.2$, and $\tau = 0.001$.

3 Synchronization Stability Analysis

In this section, we examine the stability of the synchronization error (15) between time-delay-feedback Chua's oscillator in Hamiltonian canonical form (12) and nonlinear state observer (14).

Theorem 1 (Sira-Ramírez and Cruz 2001) *The state $x(t)$ of the nonlinear system (12) can be globally, exponentially, asymptotically estimated by the state $\xi(t)$ of an observer of the form (14), if the pair of matrices $(\mathcal{C}, \mathcal{W})$, or the pair $(\mathcal{C}, \mathcal{S})$, is either observable or, at least, detectable.*

An observability condition on either of the pairs $(\mathcal{C}, \mathcal{W})$, or $(\mathcal{C}, \mathcal{S})$, is clearly a sufficient but not necessary condition for asymptotic state reconstruction. A necessary and sufficient condition for global asymptotic stability to zero of the estimation error is given by the following theorem.

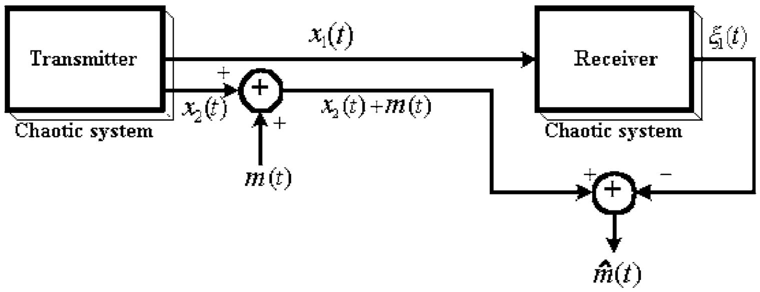

Figure 4.1. Chaotic secure communication system with two transmission channels: $m(t)$ is the private message to be hidden and transmitted. $x_1(t)$ is the synchronizing signal. $x_2(t) + m(t)$ is a hyperchaotic signal, and $\hat{m}(t)$ is the retrieved message at the receiver end.

Theorem 2 (Sira-Ramírez and Cruz 2001) *The state $x(t)$ of the nonlinear system (12) can be globally, exponentially, asymptotically estimated, by the state $\xi(t)$ of the observer (14) if and only if there exists a constant matrix K such that the symmetric matrix*

$$[\mathcal{W} - K\mathcal{C}] + [\mathcal{W} - K\mathcal{C}]^T = [\mathcal{S} - K\mathcal{C}] + [\mathcal{S} - K\mathcal{C}]^T = 2\left[\mathcal{S} - \frac{1}{2}(K\mathcal{C} + \mathcal{C}^T K^T)\right]$$

is negative definite.

4 Application to Chaotic Communications

In this section, we apply the Hamiltonian synchronization of time-delay-feedback Chua's oscillator to send secret messages. In particular, we propose two hyperchaotic communication schemes to transmit analog and binary information signals using two transmission channels and using a single transmission channel, respectively.

4.1 Secret communication using two transmission channels

The secret analog communication system is achieved by using the hyperchaotic additive masking technique. With this scheme, we obtain faster synchronization and higher privacy; one channel is used to send the hyperchaotic synchronizing signal $x_1(t)$ from the transmitter (12), with no connection with the secret message $m(t)$. While the other channel is used to transmit hidden message $m(t)$ which is recovered at the receiver end by means of the comparison between the signals $x_2(t) + m(t)$ and $\xi_2(t)$. Figure 4.1 shows the hyperchaotic secure communication system with two transmission channels. Figure 4.2 shows the secret message communication of an audio message: the private signal information to be hidden and transmitted $m(t)$, audio message (top of figure), the transmitted hyperchaotic signal $x_2(t) + m(t)$ (middle of figure), and the recovered audio message $\hat{m}(t)$ at the receiver end (bottom of figure) which is obtained after a short transient behavior.

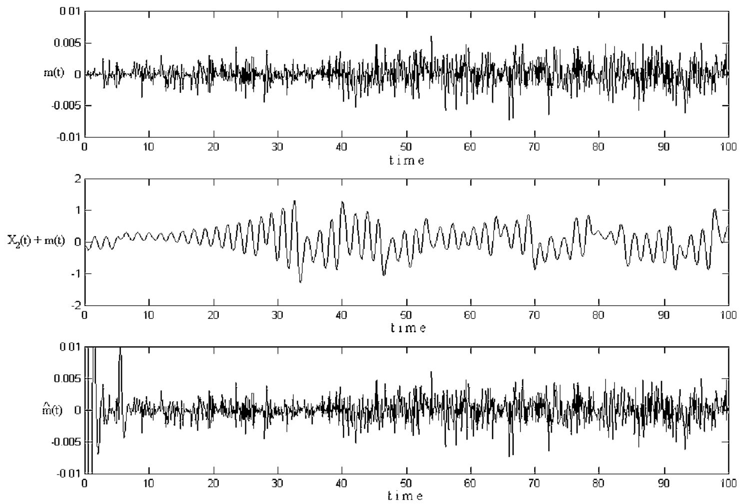

Figure 4.2. Transmission and recovering of an audio message: Private message to be hidden and transmitted (top of figure). Transmitted hyperchaotic signal $x_2(t) + m(t)$ (middle of figure). Recovered audio message at the receiver end $\hat{m}(t)$ (bottom of figure).

4.2 Secret communication using a single transmission channel

As second communication scheme, we propose the secret binary communication system using a single transmission channel, this objective is achieved by hyperchaotic switching technique (see e.g. Parlitz, *et al.* 1992; Cuomo, *et al.* 1993; Dedieu, *et al.* 1993 for chaotic systems). In this technique, the binary message $m(t)$ is used to modulate one or more parameter of the (switching) transmitter, i.e. $m(t)$ controls a switch whose action changes the parameter values of the transmitter. Thus, according to the value of $m(t)$ at any given time t, the transmitter has either the parameter set value p or the parameter set value p'. At the receiver $m(t)$ is decoded by using the synchronization error to decide whether the received signal corresponds to one parameter value, or the other (it can be interpreted as an 'one' or 'zero'). To transmit $m(t)$, let β be the parameter to be modulated in the hyperchaotic Chua transmitter (16), the parameter α and γ were fixed. We use a 'modulation rule' to modulate $m(t)$ as follows

$$\beta(t) = \beta + r \cdot m(t),$$

where $r = 0.41$ while the private message is defined as

$$m(t) = 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0 \cdots .$$

The following results illustrate the transmission of $m(t)$ for $t = 10$ sec, when β is switched between $\beta(1) = \beta' = 19.53$ and $\beta(0) = \beta = 19.12$. Figure 4.3 shows the chaotic secure communication system by chaotic switching. While Figure 4.4 shows the transmission and recovering of secret binary message: the private message $m(t)$ (top of figure), the transmitted hyperchaotic signal $x_1(t)$ (middle of figure), and the

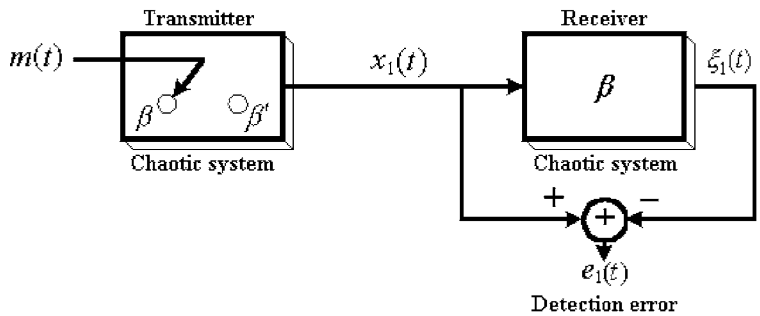

Figure 4.3. Chaotic secure communication system using chaotic switching.

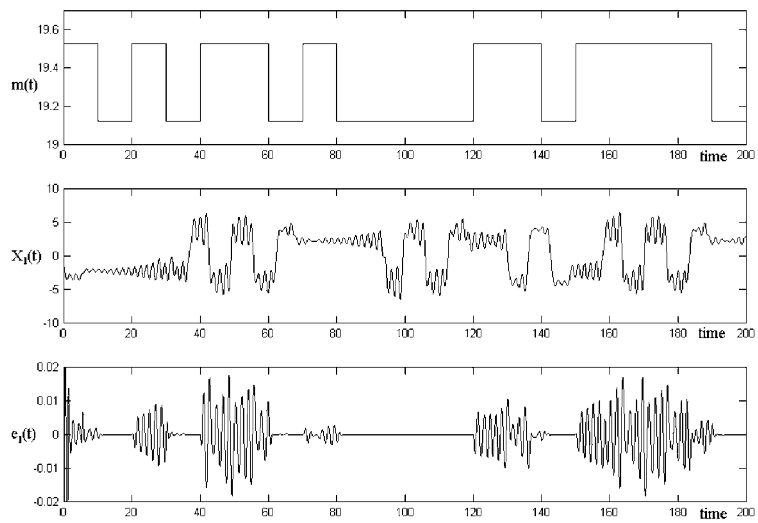

Figure 4.4. Transmission and recovery of a secret binary message: Binary private signal to be hidden and transmitted $m(t)$ (top of figure). Transmitted hyperchaotic signal $x_1(t)$ (middle of figure). Recovered binary message at the receiver end by synchronization error detection $e_1(t) = x_1(t) - \xi_1(t)$ (bottom of figure).

recovered binary message at the receiver end (bottom of figure) by synchronization error detection $e_1(t) = x_1(t) - \xi_1(t)$.

5 Conclusions

In this paper, we have approached the problem of synchronization of time-delay-feedback Chua's circuit from the perspective of Generalized Hamiltonian systems developed in (Sira-Ramírez and Cruz 2001). The approach allows one to give a simple design procedure for the receiver circuit given by a nonlinear observer, and clarifies the issue of deciding on the nature of the output signal to be transmitted. The suggested approach has been successfully applied to a secure communication schemes to transmit analog and binary secret messages. Simulation results have been reported to illustrate the capability of the proposed approach, and shows great potential for actual private/secure communication systems in which the encoding is required to be secure. Because of the increased complexity of the transmitted signal as well as the adoption of infinite-dimensional Chua's oscillator with multiple positive Lyapunov exponents.

In a forthcoming article we will be concerned with a physical implementation of the method with a specific quantization of the degree of safety of the proposal in actual communication systems.

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Global Exponential Stabilization for Several Classes of Uncertain Nonlinear Systems with Time-Varying Delay

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Abstract: In this paper, exponential stabilization for three classes of uncertain nonlinear systems with time-varying delay is investigated. A continuous feedback control is constructed for each class of systems, under which global exponential stability of the feedback-controlled system can be guaranteed. Our results are shown to be generalizations of several results reported in recent literature. A numerical example is provided to illustrate the use of our main results.

Keywords: *Global exponential stabilization; uncertain systems; time-varying delay; Lyapunov function.*

Mathematics Subject Classification (2000): 15A60, 93B52, 93C30, 93D05, 93D15.

1 Introduction

Any physical dynamic system inherently contains, more or less, some delay phenomena because energy in the system propagates with a finite speed. Typical systems with time delays include turbojet engine, microwave oscillator, control of epidemics, inferred grinding model, and population dynamics model [6, 7, 17]. It is noted that for many stable systems the introduction of arbitrarily small time delay into the loop of systems can cause instability [4, 10]. Furthermore, the system uncertainties could be present due to mathematical model errors, temperature varying, and element life. Thus, feedback control of uncertain time-delay systems is crucial for practical design of control systems; see, e.g., [2, 7 – 9, 13 –17, 19, 20] and the references therein. We wish to point out that many systems whose dynamics contain a term that is affine-linear in control have been investigated in the past; see, e.g., [1, 3, 11, 14, 18 – 20]. The generalization allowing some systems whose dynamics contain a term which depends on the square of the control in addition to an affine term has been considered in [12].

Uncertain input nonlinearity considered in this paper is a general expression on uncertainties and control, that is a generalization of uncertain input nonlinearity for many time-delay systems. Moreover, we allow the dynamics of the system containing the term which depends on the control order up to $(r-1)$, where r is a positive integer greater than three.

The paper is organized as follows. In Section 2, some preliminaries are provided. Exponential stabilization for three classes of uncertain nonlinear systems are considered in Section 3 to Section 5, respectively. An illustrative example is provided in Section 6 to demonstrate the use of our main results. Finally, summary follows in Section 7.

2 Preliminaries

For convenience, we define some notation that will be used throughout this paper as follows:

R_+	– Set of all nonnegative reals.
R	– Set of all real numbers.
R^n	– n-dimensional real space.
$R^{n\times m}$	– Set of all real n by m matrices.
I	– Unit matrix.
A^{T}	– Transpose of matrix A.
$\|A\|$	– Spectral norm of matrix A.
$\|x\|$	– Euclidean norm of $x \in R^n$.
$\lambda_{\min}(P)$	– Minimal eigenvalue of symmetric matrix P.
$\lambda_{\max}(P)$	– Maximal eigenvalue of symmetric matrix P.
C	– Set of all continuous functions from $[-H,0]$ to R^n.
$\nabla_x V(t,x)$	– Gradient of smooth scalar function $V(t,x)$.
$\|a\|$	– Absolute value of real number a.
$\forall$	– Means "for every."

Consider the following nonlinear time-delay dynamic system:

$$\dot{x}(t) = f(t, x_t), \quad \forall\, t \ge t_0 \ge 0, \tag{1}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H,\, 0], \tag{2}$$

where $x \in R^n$, $x_t(s) = x(t+s)$, $\forall\, s \in [-H,\, 0]$, $H \ge 0$, with $\|x_t\|_s = \sup\limits_{-H\le\tau\le 0} \|x(t+\tau)\|$, and $\theta \in C$ is a given initial function. The function $f\colon R_+ \times C \to R^n$ is supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t_0,\theta)(t)$ through (t_0,θ) is continuous in (t_0,θ,t), $t \ge t_0 \ge 0$, in the domain of definition of the function [8].

Definition 2.1 *System* (1) is said to be *globally exponentially stable with convergence rate* $\alpha > 0$ if, for each $\theta \in C$ and $t_0 \in R_+$, we have

$$\|x(t_0,\theta)(t)\| \le c(t_0, \|\theta\|_s)\, \exp(-\alpha(t-t_0)) \quad \text{for all} \quad t \ge t_0 \ge 0,$$

where $c(\cdot)$ is a bounded function depending on t_0 and $\|\theta\|_s$.

Lemma 2.1 *Assume there exist a sufficiently smooth function $V(t,x)$ and positive constants λ_1, λ_2, λ_3, p, ε, and β, with $\beta > \lambda_3/\lambda_2$, such that for all $x \in R^n$, $t \ge t_0 \ge 0$,*

$$\lambda_1 \|x\|^p \le V(t,x) \le \lambda_2 \|x\|^p, \tag{3}$$

and the derivative of V along solutions of (1) satisfies

$$\frac{dV(t,\, x(t))}{dt} = \nabla_t V(t,\, x(t)) + \nabla_x^{\mathrm{T}} V(t,\, x(t)) \cdot f(t, x_t) \le -\lambda_3 \|x(t)\|^p + \varepsilon e^{-\beta t}, \tag{4}$$

then system (1) is globally exponentially stable with guaranteed convergence rate

$$\eta = \lambda_3/(\lambda_2 p).$$

Proof See Appendix A.

Lemma 2.2 *For any nonnegative integers i and j, define $0! = 1$, $i! = 1\times 2\times 3\times \ldots \times i$, and $C_j^i = i!/[j!(i-j)!]$, $i \ge j$. Then, for any nonnegative integers i, j, and r, the following inequalities are true:*

(a) $C_{j-3}^{r-3} \ge C_{j-i}^{r-i}, \quad \forall\, 3 \le i \le j \le r,$

(b) $3\,C_{j-2}^{r-2} - 2\,C_{j-1}^{r-2} + C_j^{r-2} \ge 3\,C_{j-3}^{r-3}, \quad \forall\, 3 \le j \le r-2.$

Proof See Appendix B.

3 First Class of Systems

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) \\ &\quad + G(t, x(t), x(t-h(t)))\Delta\Psi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \end{aligned} \tag{5}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H,\, 0], \tag{6}$$

where $x \in R^n$, $h(t)$ is a delay argument with $0 \le h(t) \le H$, $u \in R^m$ is the input vector, and $\theta \in C$ is a given initial function. The functions F, G, and $\Delta\Psi$ (uncertain input nonlinearity) are assumed to be continuous with $F(t,0) = 0$, $t \ge t_0 \ge 0$.

Before presenting our main results, we make some assumptions as follows.

Assumption (A1) [18] There exist a sufficiently smooth function $W(t, x(t))$ and positive constants λ_1, λ_2, λ_3, and p such that, for all $x \in R^n$, $t \ge t_0 \ge 0$,

$$\lambda_1 \|x\|^p \le W(t,x) \le \lambda_2 \|x\|^p, \tag{7}$$

and the derivative of W along solutions of $\dot{x}(t) = F(t, x(t))$ satisfies

$$\frac{dW(t, x(t))}{dt} = \nabla_t W(t, x(t)) + \nabla_x^{\mathrm{T}} W(t, x(t)) \cdot F(t, x(t)) \le -\lambda_3 \|x(t)\|^p. \tag{8}$$

Assumption (A2) [12] There exist positive continuous functions $f_1(t,x,y)$, $f_2(t,x,y)$, and nonnegative continuous functions $f_3(t,x,y)$, $\ldots$, $f_r(t,x,y)$ such that for all $t \geq t_0 \geq 0$, $x, y \in R^n$, and $u \in R^m$,

$$u^{\mathrm{T}} \cdot \Delta\Psi(t,x,y,u) \geq -f_1(t,x,y)\|u\| + f_2(t,x,y)\|u\|^2 - \sum_{j=3}^{r} f_j(t,x,y)\|u\|^j, \tag{9}$$

with

$$f_2^{r-1}(t,x,y) \geq \sum_{j=3}^{r} 2^{j-1} f_1^{j-2}(t,x,y) f_2^{r-j}(t,x,y) f_j(t,x,y). \tag{10}$$

Remark 3.1 For $r=3$, $r=4$, and $r=5$, inequality (10) becomes, respectively,

$$f_2^2 \geq 4f_1f_3, \tag{11}$$

$$f_2^3 \geq 4f_1f_2f_3 + 8f_1^2f_4, \tag{12}$$

$$f_2^4(t,x,y) \geq 4f_1f_2^2f_3 + 8f_1^2f_2f_4 + 16f_1^3f_5. \tag{13}$$

It is interesting to note that (13) reduces to (12) by setting $f_5 = 0$ and (12) reduces to (11) by setting $f_4 = 0$. Similar statement can be made for higher r.

Theorem 3.1 *System (5) satisfying Assumptions (A1)∼(A2) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_3/(\lambda_2 p)$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(t,\, x(t),\, x(t-h(t))), \tag{14}$$

where

$$\gamma(t,\, x(t),\, x(t-h(t))) = \frac{2f_1^2(t,x(t),\, x(t-h(t)))}{f_2(t,x(t),x(t-h(t)))[f_1(t,x(t),x(t-h(t)))\|K(t,x(t),x(t-h(t)))\| + \varepsilon^*(t)]}, \tag{15}$$

$$\varepsilon^*(t) = 3\exp(-\beta t), \tag{16}$$

$$K(t,x(t),x(t-h(t))) = G^{\mathrm{T}}(t,x(t),x(t-h(t)))\nabla_x W(t,x(t)), \tag{17}$$

with $\beta > \lambda_3/\lambda_2$.

Proof Let $W(t,x(t))$, satisfying (7)–(8), be a Lyapunov function candidate of the system (5) with (14)–(17). The time derivative of $W(t,x(t))$ along trajectories of the closed-loop system is given by

$$\dot{W} = \nabla_t W + \nabla_x^{\mathrm{T}} W(F + G\cdot\Delta\Psi) \leq -\lambda_3\|x\|^p + \nabla_x^{\mathrm{T}} W G \cdot \Delta\Psi. \tag{18}$$

From (9) and (14), we have

$$-\gamma K^{\mathrm{T}} \cdot \Delta\Psi \geqslant -f_1\gamma\|K\| + f_2\gamma^2\|K\|^2 - \sum_{j=3}^{r} f_j\gamma^j\|K\|^j,$$

which implies

$$K^{\mathrm{T}} \cdot \Delta\Psi \le f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j. \tag{19}$$

Applying (19) to (18) with (15) – (17), we have

$$\dot W \le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j.$$

In the following, the proof is made by setting $r = 3$, $r = 4$, and $r \ge 5$, respectively.

For $r = 3$, we have

$$\begin{aligned}
\dot W &\le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 \\
&= -\lambda_3\|x\|^p + f_1\|K\| - \frac{2f_2f_1^2\|K\|^2}{f_2(f_1\|K\|+\varepsilon^*)} + \frac{4f_3f_1^4\|K\|^3}{f_2^2(f_1\|K\|+\varepsilon^*)^2} + (-e^{-\beta t} + e^{-\beta t}) \\
&= \frac{f_2^2(f_1\|K\| - e^{-\beta t})(f_1\|K\|+\varepsilon^*)^2 - 2f_2^2f_1^2\|K\|^2(f_1\|K\|+\varepsilon^*)}{f_2^2(f_1\|K\|+\varepsilon^*)^2} \\
&\quad + \frac{4f_3f_1^4\|K\|^3}{f_2^2(f_1\|K\|+\varepsilon^*)^2} - \lambda_3\|x\|^p + e^{-\beta t} \\
&= \frac{-\|K\|^3f_1^3(f_2^2 - 4f_1f_3) - \|K\|^2f_1^2f_2^2e^{-\beta t} + 3\|K\|f_1f_2^2e^{-2\beta t} - 9f_2^2e^{-3\beta t}}{f_2^2(f_1\|K\|+\varepsilon^*)^2} \\
&\quad - \lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned} \tag{20}$$

By using the fact that $2ab \le a^2 + b^2$ for any a, $b \ge 0$, one has

$$3\|K\|f_1f_2^2e^{-2\beta t} = f_2^2e^{-\beta t}\left[2\,f_1\|K\|\,\frac{3e^{-\beta t}}{2}\right] \le f_2^2e^{-\beta t}\left[(f_1\|K\|)^2 + \frac{9e^{-2\beta t}}{4}\right]. \tag{21}$$

From (11), (20), and (21), we have

$$\dot W \le \frac{-\|K\|^3f_1^3(f_2^2 - 4f_1f_3) - (27/4)f_2^2e^{-3\beta t}}{f_2^2(f_1\|K\|+\varepsilon^*)^2} - \lambda_3\|x\|^p + e^{-\beta t} \le -\lambda_3\|x\|^p + e^{-\beta t}.$$

Next for $r = 4$, we have

$$\begin{aligned}
\dot W &\le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 + f_4\gamma^3\|K\|^4 \\
&= -\frac{\|K\|^4f_1^4(f_2^3 - 4f_1f_2f_3 - 8f_1^2f_4) + \|K\|^3f_1^3e^{-\beta t}(4f_2^3 - 4f_1f_2f_3(3)) + 27f_2^3e^{-4\beta t}}{f_2^3(f_1\|K\|+\varepsilon^*)^3} \\
&\quad - \lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned} \tag{22}$$

From (12) and (22), we have

$$\dot{W} \le -\frac{\|K\|^4 f_1^4(f_2^3-4f_1f_2f_3-8f_1^2f_4)+4\|K\|^3 f_1^3 e^{-\beta t}(f_2^3-4f_1f_2f_3)+27f_2^3e^{-4\beta t}}{f_2^3(f_1\|K\|+\varepsilon^*)^3} -\lambda_3\|x\|^p+e^{-\beta t} \le -\lambda_3\|x\|^p+e^{-\beta t}.$$

Finally for $r \ge 5$, we have

$$\begin{aligned}
\dot{W} &\le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j \\
&= -\lambda_3\|x\|^p + f_1\|K\| - \frac{2f_2f_1^2\|K\|^2}{f_2(f_1\|K\|+\varepsilon^*)} + \sum_{j=3}^{r}\frac{(2)^{j-1}f_jf_1^{2(j-1)}\|K\|^j}{f_2^{j-1}(f_1\|K\|+\varepsilon^*)^{j-1}} + (-e^{-\beta t}+e^{-\beta t}) \\
&= \frac{f_2^{r-1}(f_1\|K\|-e^{-\beta t})(f_1\|K\|+3e^{-\beta t})^{r-1} - 2f_2^{r-1}f_1^2\|K\|^2(f_1\|K\|+3e^{-\beta t})^{r-2}}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&+ \frac{\sum_{j=3}^{r}(2)^{j-1}f_jf_1^{2(j-1)}\|K\|^j f_2^{r-j}(f_1\|K\|+3e^{-\beta t})^{r-j}}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t} \\
&= -\frac{\|K\|^r f_1^r\left(f_2^{r-1} - \sum_{j=3}^{r}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j\right)}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{\|K\|^{r-1}e^{-\beta t}f_1^{r-1}\left[f_2^{r-1}(3C_{r-3}^{r-2}-2C_{r-2}^{r-2}) - \sum_{j=3}^{r-1}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j(3C_{r-1-j}^{r-j})\right]}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{\sum_{j=3}^{r-2}\|K\|^j e^{-(r-j)\beta t}3^{r-j-1}f_1^j\left[f_2^{r-1}(3C_{j-2}^{r-2}-2C_{j-1}^{r-2}+C_j^{r-2}) - \sum_{i=3}^{j}2^{(i-1)}f_1^{i-2}f_2^{r-i}f_i(3C_{j-i}^{r-i})\right]}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{\|K\|^2 f_2^{r-1}f_1^2 3^{r-3}e^{-(r-2)\beta t}(3-2C_1^{r-2}+C_2^{r-2}) + \|K\|f_2^{r-1}f_1(3)^{r-2}e^{-(r-1)\beta t}(-2+C_1^{r-2})}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{f_2^{r-2}(3)^{r-1}e^{-r\beta t}}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t} \\
&= -\frac{\|K\|^r f_1^r\left(f_2^{r-1} - \sum_{j=3}^{r}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j\right)}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{\|K\|^{r-1}e^{-\beta t}f_1^{r-1}\left[f_2^{r-1}(3r-8) - \sum_{j=3}^{r-1}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j(3(r-j))\right]}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}} \\
&- \frac{\sum_{j=3}^{r-2}\|K\|^j e^{-(r-j)\beta t}3^{r-j-1}f_1^j\left[f_2^{r-1}(3C_{j-2}^{r-2}-2C_{j-1}^{r-2}+C_j^{r-2}) - \sum_{i=3}^{j}2^{(i-1)}f_1^{i-2}f_2^{r-i}f_i(3C_{j-i}^{r-i})\right]}{f_2^{r-1}(f_1\|K\|+3e^{-\beta t})^{r-1}}
\end{aligned}$$

$$- \frac{\|K\|^2 f_2^{r-1} f_1^2 (3)^{r-3} e^{-(r-2)\beta t}[(r-4)(r-5)] + \|K\| f_2^{r-1} f_1 (3)^{r-2} e^{-(r-1)\beta t}(r-4)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$- \frac{f_2^{r-2}(3)^{r-1} e^{-r\beta t}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3 \|x\|^p + e^{-\beta t}. \tag{23}$$

By (23), Lemma 2.2, and the fact that $3r-8 > 3(r-j)$, $\forall\, 3 \le j \le r-1$, one has

$$\dot{W} \le -\frac{\|K\|^r f_1^r \left(f_2^{r-1} - \sum\limits_{j=3}^{r} 2^{(j-1)} f_1^{j-2} f_2^{r-j} f_j \right)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$- (3r-8) \cdot \frac{\|K\|^{r-1} e^{-\beta t} f_1^{r-1} \left[f_2^{r-1} - \sum\limits_{j=3}^{r-1} 2^{(j-1)} f_1^{j-2} f_2^{r-j} f_j \right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$- \frac{\sum\limits_{j=3}^{r-2} \|K\|^j e^{-(r-j)\beta t} 3^{r-j-1} f_1^j (3C_{j-2}^{r-2} - 2C_{j-1}^{r-2} + C_j^{r-2}) \left[f_2^{r-1} - \sum\limits_{i=3}^{j} 2^{(i-1)} f_1^{i-2} f_2^{r-i} f_i \right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$- \frac{\|K\|^2 f_2^{r-1} f_1^2 (3)^{r-3} e^{-(r-2)\beta t}[(r-4)(r-5)] + \|K\| f_2^{r-1} f_1 (3)^{r-2} e^{-(r-1)\beta t}(r-4)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$- \frac{f_2^{r-2}(3)^{r-1} e^{-r\beta t}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3 \|x\|^p + e^{-\beta t} \le -\lambda_3 \|x\|^p + e^{-\beta t}.$$

This completes our proof in view of Lemma 2.1.

Remark 3.2 In [18], exponential stability can be guaranteed via nonlinear state feedback control for system (4) with $G(t, x(t), x(t-h(t))) = \widetilde{G}(t, x(t))$, $\Delta\Psi(t, x(t), x(t-h(t)),\ u(t) = u(t) + \xi(t, x(t)),\ \|\xi(t, x(t))\| \le \rho(t, x(t)),\ p = 2$, where $\xi(\cdot,\cdot)\colon R \times R^n \to R^m$ and $\rho(\cdot,\cdot)$ is a nonnegative continuous function. In this case, there exists a positive continuous function $\widetilde{\rho}(t,x) \ge \rho(t,x)$, $\forall\, t \ge t_0 \ge 0$, $x \in R^n$, such that

$$u^{\mathrm{T}} \Delta\Psi(t, x, x(t-h), u) = \|u\|^2 + u^{\mathrm{T}} \xi(t,x) \ge -\widetilde{\rho}(t,x)\|u\| + \|u|^2.$$

In view of (9), we have

$$f_1 = \widetilde{\rho}(t,x), \quad f_2 = 1, \quad f_3 = 0, \quad \dots, \quad f_r = 0,$$

and (10) is satisfied. Hence, global exponential stability can be guaranteed by memoryless controller (14) with $f_1 = \widetilde{\rho}(t,x)$ and $f_2 = 1$ by Theorem 3.1. However, our global exponential stability result holds for more general systems.

4 Second Class of Systems

Consider the following uncertain system with a time-varying delay:

$$\dot{x}(t) = Ax(t) + B\Delta\Phi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \tag{24}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H, 0], \tag{25}$$

where $x \in R^n$, $h(t)$ is a delay argument with $0 \le h(t) \le H$, (A, B) is stabilizable, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $\Delta\Phi$, representing uncertain input nonlinearity, is assumed to be a continuous function satisfying the following assumption.

Assumption (A3) There exist positive continuous functions $\tilde{f}_1(t,x,y)$ and $\tilde{f}_2(t,x,y)$ such that for all $t \geq t_0 \geq 0$, $x, y \in R^n$, and $u \in R^m$,

$$u^{\mathrm{T}}\Delta\Phi(t,x,y,u) \geq -\tilde{f}_1(t,x,y)\|u\| + \tilde{f}_2(t,x,y)\|u\|^2.$$

Since the pair (A,B) is assumed to be stabilizable, we can find a matrix $M \in R^{m\times n}$ such that $\tilde{A} = A - BM$ is a Hurwitz matrix and the following Lyapunov equation

$$\tilde{A}^{\mathrm{T}}P + P\tilde{A} = -2I, \tag{26}$$

has a unique positive definite symmetric solution P.

Theorem 4.1 *System (24) satisfying Assumption (A3) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_{\max}(P)^{-1}$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(x(t)), \tag{27}$$

where

$$\gamma(t, x(t),\, x(t-h(t))) = \frac{2[\tilde{f}_1(t,x(t),x(t-h(t))) + \|Mx(t)\|]^2}{\tilde{f}_2(t,x(t),x(t-h(t)))[(\tilde{f}_1(t,x(t),x(t-h(t))) + \|Mx(t)\|)\|K(x(t))\| + \varepsilon^*(t)]},$$

$$\varepsilon^*(t) = 3\exp(-\beta t),$$

$$K(x(t)) = 2B^{\mathrm{T}}Px(t),$$

with $\beta > 2/\lambda_{\max}(P)$, *and* P *is the solution of (26).*

Proof System (24) can be rewritten as

$$\dot{x}(t) = \tilde{A}x(t) + B[\Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u(t)) + Mx(t)], \quad t \geq t_0 \geq 0.$$

In the following, we use Theorem 3.1 to prove this theorem. Choose

$$F(t,x) = \tilde{A}x, \quad G(t,\, x(t),\, x(t-h(t))) = B,$$
$$\Delta\Psi(t,\, x(t),\, x(t-h(t)),\, u) = Mx + \Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u).$$

Let $W(t,x(t)) = x^{\mathrm{T}}(t)Px(t)$, then we have

$$\lambda_{\min}(P)\|x\|^2 \leq W(t,x) \leq \lambda_{\max}(P)\|x\|^2, \quad \forall\, t \geq t_0 \geq 0, \quad x \in R^n, \tag{28}$$

and the derivative of W along solutions of $\dot{x}(t) = \tilde{A}x(t)$ is given by

$$\frac{dW(t,x(t))}{dt} = x(t)^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}P + P\tilde{A})x(t) = -2\|x(t)\|^2. \tag{29}$$

In view of Assumption (A3), we obtain

$$\begin{aligned} u^{\mathrm{T}}\Delta\Psi(t,\, x,\, x(t-h(t)),\, u) &= u^{\mathrm{T}}Mx + u^{\mathrm{T}}\Delta\Phi(t,\, x,\, x(t-h(t)),\, u) \\ &\geq -[\|Mx\| + \tilde{f}_1(t,\, x(t),\, x(t-h(t)))]\cdot\|u\| + \tilde{f}_2(t,x(t),x(t-h(t)))\cdot\|u\|^2. \end{aligned} \tag{30}$$

Comparing (28) – (30) with (7) – (9), one obtains

$$\lambda_1 = \lambda_{\min}(P), \quad \lambda_2 = \lambda_{\max}(P), \quad \lambda_3 = 2, \quad p = 2,$$
$$f_1 = \|Mx\| + \tilde{f}_1, \quad f_2 = \tilde{f}_2, \quad f_3 = 0, \quad \dots, \quad f_r = 0,$$

and (10) is satisfied. The rest follows immediately from Theorem 3.1.

In the following remarks, we show that the preceding result is a generalization of several results reported in recent literature.

Remark 4.1 In [1], practical stability can be guaranteed for system (20) with $\Delta\Phi(t, x, x(t-h(t)), u) = D(q)x + u + E(q)u + v(q)$, $\|E(q)\| < 1$, where A is a Hurwitz matrix, $D(\cdot)$, $E(\cdot)$, and $v(\cdot)$ depend continuously on their arguments, and the uncertainty q belongs to a compact set Q. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(D(q)x + u + E(q)u + v(q)) \ge -(\rho_D\|x\| + \rho_v)\cdot\|u\| + (1-\rho_E)\cdot\|u\|^2 \\ &\ge -(\rho_D\|x\| + \widetilde{\rho}_v)\cdot\|u\| + (1-\rho_E)\cdot\|u\|^2, \end{aligned}$$

where $\widetilde{\rho}_v > 0$, $\widetilde{\rho}_v \ge \rho_v = \max\limits_{q\in Q}\|v(q)\|$, $\rho_E = \max\limits_{q\in Q}\|E(q)\| < 1$, and $\rho_D = \max\limits_{q\in Q}\|D(q)\|$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = \rho_D\|x\| + \widetilde{\rho}_v$ and $\tilde{f}_2 = 1 - \rho_E$.

Remark 4.2 In [3], global practical stability can be guaranteed for system (24) with $\Delta\Phi(t, x, x(t-h(t)), u) = u + e(t,x,u)$ and $\|e(t,x,u)\| \le k_0 + k_1\|x\| + k_2\|u\|$, where $k_0, k_1, k_2 \in R_+$ and $k_2 < 1$. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(u + e(t,x,u)) \ge -(k_0 + k_1\|x\|)\cdot\|u\| + (1-k_2)\cdot\|u\|^2 \\ &\ge -(k_3 + k_1\|x\|)\cdot\|u\| + (1-k_2)\cdot\|u\|^2, \end{aligned}$$

where $k_3 > 0$ and $k_3 \ge k_0$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = k_3 + k_1\|x\|$ and $\tilde{f}_2 = 1 - k_2$.

Remark 4.3 In [11], global exponential stability can be guaranteed by a linear control for system (24) with $\Delta\Phi = D(q)x + u + E(q)u$, $\|D(q)\| \le \delta$, $\delta \in R_+$, and $\delta_E = \lambda_{\min}(E^{\mathrm{T}}(q) + E(q)) > -1$, $\forall\, q \in Q$, where $D(\cdot)$ and $E(\cdot)$ depend continuously on their arguments, and the uncertainty q belongs to a compact set Q. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(D(q)x + u + E(q)u) \\ &\ge -(\delta\|x\|)\cdot\|u\| + (1+\delta_E/2)\cdot\|u\|^2 \\ &\ge -(\delta_1 + \delta\|x\|)\cdot\|u\| + (1+\delta_E/2)\cdot\|u\|^2, \end{aligned}$$

where δ_1 is a any positive constant. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = \delta_1 + \delta\|x\|$ and $\tilde{f}_2 = 1 + \delta_E/2$.

Remark 4.4 In [14], global practical stability can be guaranteed via a linear control for system (24) with $\Delta\Phi(t, x, x(t-h(t)), u) = \phi(u) + a(t,x,u)$, $\gamma_1\|u\|^2 \le u^{\mathrm{T}}\phi(u)$, and

$\|a(t,x,u)\| \le k_0 + k_1\|x\| + k_2\|u\|$, where $k_0, k_1, k_2 \in R_+$ and $k_2 < \gamma_1$. In this case, we have

$$\begin{aligned} u^{\mathrm T}\Delta\Phi &= u^{\mathrm T}(\phi(u) + a(t,x,u)) \\ &\ge -(k_0 + k_1\|x\|)\cdot\|u\| + (\gamma_1 - k_2)\cdot\|u\|^2 \\ &\ge -(k_3 + k_1\|x\|)\cdot\|u\| + (\gamma_1 - k_2)\cdot\|u\|^2, \end{aligned}$$

where $k_3 > 0$ and $k_3 \ge k_0$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde f_1 = k_3 + k_1\|x\|$ and $\tilde f_2 = \gamma_1 - k_2$.

Remark 4.5 In [19], global exponential stability can be guaranteed by a composite control for system (24) with $\Delta\Phi = u + \xi(t,x) + \xi^h(t, x(t-h(t)))$, $\|\xi(t,x)\| \le \rho(t,x)$, and $\|\xi^h(t, x(t-h(t)))\| \le \delta\|x(t-h(t))\|$, where $\delta > 0$ and $\rho(\cdot,\cdot)\colon R_+ \times R^n \to R_+$ is a bounded continuous function. In this case, we have

$$\begin{aligned} u^{\mathrm T}\Delta\Phi &= u^{\mathrm T}(u + \xi(t,x) + \xi^h(t, x(t-h(t)))) \\ &\ge -(\rho(t,x) + \delta\|x(t-h(t))\|)\cdot\|u\| + \|u\|^2 \\ &\ge -(\rho_1(t,x) + \delta\|x(t-h(t))\|)\cdot\|u\| + \|u\|^2, \end{aligned}$$

where $\rho_1(t,x) \ge \rho(t,x)$ and $\rho_1(t,x) > 0$, $\forall\, t \ge t_0 \ge 0$, $x \in R^n$. By the preceding theorem, global exponential stability can also be guaranteed by controller (27) with $\tilde f_1 = \rho_1(t,x) + \delta\|x(t-h(t))\|$ and $\tilde f_2 = 1$.

5 Third Class of Systems

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned} \dot x(t) = Ax(t) &+ A_1 x(t-h(t)) + \Delta f(t, x(t), x(t-h(t))) \\ &+ B\Delta\Phi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \end{aligned} \tag{31}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H, 0], \tag{32}$$

where $x \in R^n$, $h(t)$ is a delay argument with $0 \le h(t) \le H$, $u \in R^m$ is the input vector, $\theta \in C$ is a given initial function, $A, A_1 \in R^{n\times n}$, and $B \in R^{n\times m}$ are constant matrices, (A,B) is stabilizable, $\mathrm{rank}(B) = n$, $\Delta\Phi$ is assumed to be continuous and satisfies Assumption (A3), and the mismatch uncertainty Δf is assumed to be continuous and satisfies the following assumption.

Assumption (A4) There exists a nonnegative continuous function $q(t,x,y)$ such that for all $t \ge t_0 \ge 0$ and $x, y \in R^n$,

$$\|\Delta f(t,x,y)\| \le q(t,x,y).$$

Theorem 5.1 *System (31) satisfying Assumptions (A3) and (A4) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_{\max}(P)^{-1}$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(x(t)), \tag{33}$$

where

$$\gamma(t,\, x(t),\, x(t-h(t))) = \frac{2\hat{f}_1^2}{\tilde{f}_2(t,x(t),x(t-h(t)))[\hat{f}_1\|K(x(t))\| + \varepsilon^*(t)]},$$
$$\hat{f}_1 = \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}A_1x(t-h(t))\| + \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}\|q + \|Mx\| + \tilde{f}_1,$$
$$\varepsilon^*(t) = 3\exp(-\beta t),$$
$$K(x(t)) = 2B^{\mathrm{T}}Px(t),$$

with $\beta > 2/\lambda_{\max}(P)$*,* M *is a matrix such that* $\tilde{A} = A - BM$ *is Hurwitz, and* P *is the solution of (26).*

Proof Since $\mathrm{rank}(B) = n$, the matrix BB^{T} is nonsingular. System (31) can be rewritten as

$$\dot{x}(t) = \tilde{A}x(t) + BB^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}[A_1x(t-h(t)) + \Delta f(t,x(t),x(t-h(t)))]$$
$$+ B[\Delta\Phi(t,x(t),x(t-h(t)),u(t)) + Mx(t)], \quad t \ge t_0 \ge 0.$$

Define

$$\Delta\Psi(t,x(t),x(t-h(t)),u) = B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}[A_1x(t-h(t)) + \Delta f(t,\, x(t),\, x(t-h(t)))]$$
$$+ Mx + \Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u),$$

then we have

$$u^{\mathrm{T}}\Delta\Psi(t,\, x,\, x(t-h(t)),\, u) \ge -\big[\|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}A_1x(t-h(t))\|$$
$$+ \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}\|\, q + \|Mx\| + \tilde{f}_1\big]\|u\| + \tilde{f}_2\|u\|^2.$$

Hence the result follows in view of Theorem 4.1.

Remark 5.1 In [13], global exponential stabilization has been considered for a class of uncertain systems with multiple time-varying delays and input deadzone nonlinearities. If they consider only single time-varying delay, their system can be put in the form of (26) with $q(t,x,y) = a_0\|x\| + a_1\|y\|$, $\Delta\Phi(t,x,y,u) = \Delta\Phi_3(t,x,y) + \phi(u)$, $\|\Delta\Phi_3(t,x,y)\| \le f(t,x,y)$, where $y = x(t-h(t))$, $a_0, a_1 \in R_+$, $\Delta\Phi_3(\cdot)$ and $f(\cdot)$ depend continuously on their arguments, $\phi(u) = [\phi_1(u_1),\dots,\phi_m(u_m)]^{\mathrm{T}}$ with each $\phi_i(u_i) \in D(u_i,d_1,d_2)$ representing the input deadzone nonlinearity, and $D(u_i,d_1,d_2)$ is defined in [13] with $d_1 \ge 0$, $d_2 > 0$. In this case, we have

$$u^{\mathrm{T}}\Delta\Phi = u^{\mathrm{T}}[\Delta\Phi_3 + \phi(u) - d_2u + d_2u] \ge -\|u\|\cdot[\|\Delta\Phi_3\| + \|\phi(u) - d_2u\|] + d_2\|u\|^2$$
$$\ge -\|u\|\cdot[f + md_1d_2] + d_2\|u\|^2 \ge -\|u\|\cdot[f + m\tilde{d}_1d_2] + d_2\|u\|^2,$$

where $\tilde{d}_1 > 0$, $\tilde{d}_1 \ge d_1$. By the preceding theorem, global exponential stability can also be guaranteed by controller (33) with $\tilde{f}_1 = f + m\tilde{d}_1d_2$ and $\tilde{f}_2 = d_2$.

6 Example

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned}\dot{x}(t) = & \begin{bmatrix} -2x_1 + 2x_1x_2^2 + 2x_1x_2\sqrt{|x_1x_2|} \\ -2x_2 - x_1^2x_2 - x_1^2\sqrt{|x_1x_2|} \end{bmatrix} \\ & + \begin{bmatrix} x_2(t-h(t)) \\ -x_1^2 \end{bmatrix} [a(t) + (b(t) + c(t)|x_1(t-h(t))|)u + d(t)u^3],\end{aligned} \tag{34}$$

where $u \in R,\ x = [x_1, x_2]^{\mathrm{T}} \in R^2,\ h(t) = 2 + \cos(2t),\ -1 \le a(t) \le 1,\ 4 \le b(t) \le 4.5,$ $1 \le c(t) \le 2,\ -2 \le d(t) \le 2$ for all $t \ge t_0 \ge 0$. Comparing (34) with (5), one has

$$F(t, x(t)) = \begin{bmatrix} -2x_1 + 2x_1x_2^2 + 2x_1x_2\sqrt{|x_1x_2|} \\ -2x_2 - x_1^2x_2 - x_1^2\sqrt{|x_1x_2|} \end{bmatrix},$$

$$G(t,\, x(t),\, x(t-h(t))) = \begin{bmatrix} x_2(t-h(t)) \\ -x_1^2 \end{bmatrix},$$

$$\Delta\Psi(t, x(t), x(t-h(t)), u) = a(t) + (b(t) + c(t)|x_1(t-h(t))|)u + d(t)u^3.$$

Choose a simple quadratic functional

$$W(t, x) = x^{\mathrm{T}} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} x.$$

Then (7) and (8) are evidently satisfied with $\lambda_1 = 2,\ \lambda_2 = 4,\ p = 2,$ and $\lambda_3 = 8$. In view of (9), we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Psi(t, x, x(t-h(t)), u) &= a(t)u + (b(t) + c(t)|x_1(t-h(t))|)u^2 + d(t)u^4 \\ &\ge -|u| + (4 + |x_1(t-h(t))|)|u|^2 - 2|u|^4. \end{aligned}$$

This suggests that in (9) we choose $f_1 = 1,\ f_2 = 4 + |x_1(t-h(t))|,\ f_3 = 0,$ and $f_4 = 2$.

It is easy to show that (10) is satisfied with $r = 4$. According to (16) with $\beta = 2.1 > \lambda_3/\lambda_2 = 2$, we have

$$\varepsilon^*(t) = 3\exp(-2.1t).$$

By (17) and (15), we obtain

$$K(t,\, x(t),\, x(t-h(t))) = 4x_1(t)x_2(t-h(t)) - 8x_1^2(t)x_2(t),$$

$$\gamma(t, x, x(t-h(t))) = \frac{2}{(4 + |x_1(t-h(t))|)(|K(t, x(t), x(t-h(t)))| + \varepsilon^*(t))}.$$

Finally, owing to (14), it can be readily obtained that

$$u = -\gamma(t, x,\, x(t-h(t)))(4x_1(t)x_2(t-h(t)) - 8x_1^2x_2). \tag{35}$$

By Theorem 3.1, we conclude that system (34) with control (35) is globally exponentially stable with guaranteed convergence rate $\eta = 1$. With, e.g., $a(t) = 1,\ b(t) = 4,$ $c(t) = 1,\ d(t) = 2,$ and $x_1(t) = 4,\ x_2(t) = -2,\ \forall\, t \in [-3, 0]$, state trajectories of the

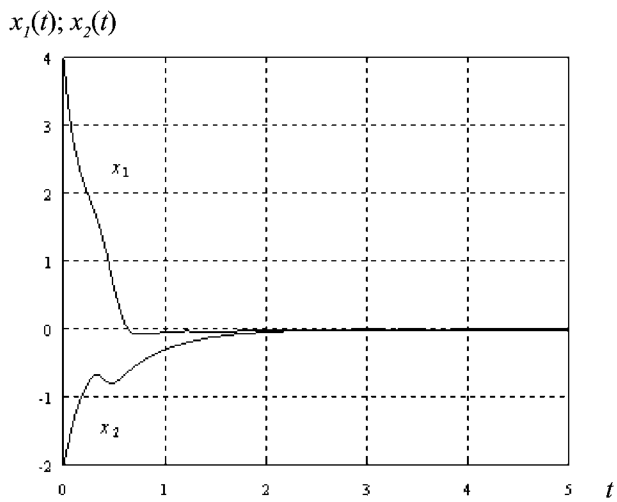

Figure 6.1. State trajectories of feedback-controlled system of (28).

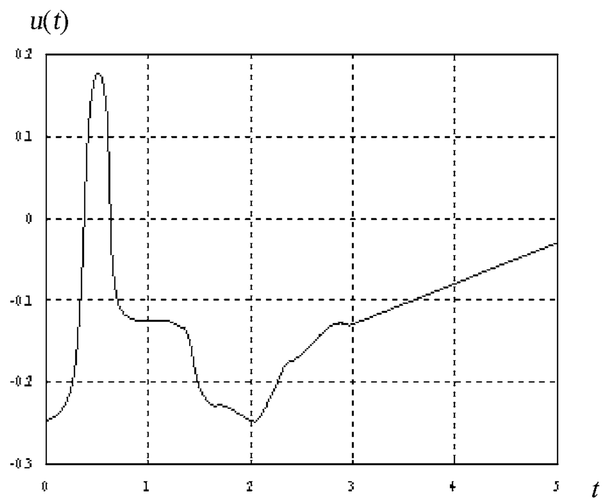

Figure 6.2. Typical control signal for system (28).

feedback-controlled system and control signal are depicted in Figure 6.1 and Figure 6.2, respectively.

7 Summary

In this paper, exponential stabilization for three classes of uncertain nonlinear systems with time-varying delay has been considered. A continuous state feedback control has been proposed in each case for exponential stability of feedback-controlled systems. Guaranteed convergence rate has also been provided. Our results have also been shown to be generalizations of several results reported in recent literature. Finally, a numerical example has been provided to illustrate the use of our main results. It is interesting to consider the problem of exponential stabilization for more general uncertain systems with time-varying delay.

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Appendix A

Proof of Lemma 2.1 Let

$$Q(t,x) = V(t,x)\exp(\lambda_3 t/\lambda_2). \tag{B1}$$

From (3), (4) and (B1), we have

$$\begin{aligned}\frac{d\,Q(t,x)}{dt} &= \frac{d\,V(t,x)}{dt}\exp(\lambda_3 t/\lambda_2) + \lambda_3 Q/\lambda_2 \\ &\le [(-\lambda_3/\lambda_2)V + \varepsilon\exp(-\beta t)]\exp(\lambda_3 t/\lambda_2) + \lambda_3 Q/\lambda_2 \\ &= \varepsilon\exp[-(\beta - \lambda_3/\lambda_2)t].\end{aligned} \tag{B2}$$

Set $\delta = \beta - \lambda_3/\lambda_2 > 0$. Integrating both sides of (B2), we have, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}Q(t,x(t)) - Q(t_0,x(t_0)) &\le -\varepsilon\delta^{-1}[\exp(-\delta t) - \exp(-\delta t_0)] \\ &= \varepsilon\delta^{-1}[\exp(-\delta t_0) - \exp(-\delta t)] \le \varepsilon\delta^{-1}\exp(-\delta t_0) \le \varepsilon\delta^{-1} = \varepsilon(\beta - \lambda_3/\lambda_2)^{-1}.\end{aligned}$$

This implies that, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}Q(t,x(t)) &\le Q(t_0,x(t_0)) + \varepsilon(\beta - \lambda_3/\lambda_2)^{-1} \\ &= V(t_0,x(t_0))\,\exp(\lambda_3 t_0/\lambda_2) + \varepsilon(\beta - \lambda_3/\lambda_2)^{-1} \\ &\le \lambda_2\|\theta\|_s^p\exp(\lambda_3 t_0/\lambda_2) + \varepsilon(\beta - \lambda_3/\lambda_2)^{-1} = a(t_0,\|\theta\|_s).\end{aligned} \tag{B3}$$

From (3), (B1), and (B3), we have, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}\|x(t_0,\theta)(t)\| &\le [(1/\lambda_1)\,V(t,x(t))]^{1/p} = [(1/\lambda_1)\exp(-\lambda_3 t/\lambda_2)\cdot Q(t,x(t))]^{1/p} \\ &\le [(a/\lambda_1)\exp(-\lambda_3 t/\lambda_2)]^{1/p} = c(t_0,\|\theta\|_s)\,\exp(-\eta t) \\ &\le c(t_0,\|\theta\|_s)\,\exp[-\eta(t-t_0)],\end{aligned}$$

where $c(t_0,\|\theta\|_s) = [a(t_0,\|\theta\|_s)/\lambda_1]^{1/p}$ and $\eta = \lambda_3/(\lambda_2 p) > 0$.

This completes our proof.

Appendix B

Proof of Lemma 2.2 For any integers i, j, and r such that $3 \le i \le j < r$, one has

$$[(r-3)\times(r-4)\times\ldots\times(j-3+1)] \ge [(r-i)\times(r-i-1)\times\ldots\times(j-i+1)].$$

This implies

$$C_{j-3}^{r-3} = \frac{(r-3)!}{(j-3)!(r-j)!} \geq \frac{(r-i)!}{(j-i)!(r-j)!} = C_{j-i}^{r-i}, \quad \forall\, 3 \leq i \leq j < r,$$

and

$$C_{j-3}^{r-3} = 1 = C_{j-i}^{r-i}, \quad \forall\, 3 \leq i \leq j = r.$$

Hence statement (a) is true. Now for any integers j, r such that $r \geq 5$ and $3 \leq j \leq r-2$, one has

$$\begin{aligned}
&3\, C_{j-2}^{r-2} - 2\, C_{j-1}^{r-2} + C_{j}^{r-2} - 3\, C_{j-3}^{r-3} \\
&\quad = \frac{(r-2)!}{j!(r-j)!}\left[3j(j-1) - 2j(r-j) + (r-j)(r-j-1) - 3\frac{j(j-1)(j-2)}{(r-2)}\right] \\
&\quad = \frac{(r-2)!}{j!(r-j)!(r-2)}[r^3 + r^2(-4j-3) + r(6j^2+6j+2) - (3j^3+3j^2+2j)] \\
&\quad = \frac{(r-2)!}{j!(r-j)!(r-2)}[-3j^3 + (6r-3)j^2 - (4r^2-6r+2)j + (r^3-3r^2+2r)].
\end{aligned}$$

For any given $r \geq 5$, consider the following continuous function

$$g(y) = -3y^3 + (6r-3)y^2 - (4r^2-6r+2)y + (r^3-3r^2+2r), \quad y \in [3,\, r-2].$$

The derivative of $g(\cdot)$ is given by

$$\frac{d}{dy}g(y) = -9y^2 + (12r-6)y - (4r^2-6r+2).$$

Furthermore, the roots of the equation $\dot{g}(y) = 0$ is given by

$$a = \frac{2r-1-\sqrt{2r-1}}{3}, \quad b = \frac{2r-1+\sqrt{2r-1}}{3}.$$

With given $r \geq 5$, define

$$\begin{aligned}
g_1(r) &= g(a) = \frac{1}{9}\left(r^3 - 3r^2 + 4 - 2(2r-1)\sqrt{2r+1}\right), \\
g_2(r) &= g(b) = \frac{1}{9}\left(r^3 - 3r^2 + 4 + 2(2r-1)\sqrt{2r+1}\right), \\
g_3(r) &= g(3) = r^3 - 15r^2 + 74r - 114, \\
g_4(r) &= g(r-2) = 2r^2 - 12r + 16 = 2(r-2)(r-4).
\end{aligned}$$

Clearly we have

$$g_1(5) = 0,$$

$$\frac{d}{dr}g_1(r) = \frac{1}{9}\left[3r^2 - 6r - 6\sqrt{2r-1}\right] > \frac{1}{9}\left[3r(r-5) + (9-6\sqrt{2})r\right] > 0, \quad \forall\, r \geq 5,$$

$$g_2(r) \geq g_1(r) \geq g_1(5) = 0, \quad \forall\, r \geq 5,$$

$$g_4(r) > 0, \quad \forall\, r \geq 5.$$

Moreover, by Sturm's theorem [5], it is easy to show that $g_3(r) > 0$, $\forall\, r \geq 5$. Consequently, $g(y) \geq 0$ for all $y \in [3,\, r-2]$ and for each $r \geq 5$. This completes the proof of statement (b).

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Hierarchical Lyapunov Functions for Stability Analysis of Discrete-Time Systems with Applications to the Neural Networks

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Abstract: In the paper the application of hierarchical Lyapunov functions is proposed for qualitative analysis of solutions of discrete-time system. General results of analysis of quasi-linear discrete system are applied to the analysis of robust stability of large-scale neural system in the case of unperturbed and perturbed equilibrium state. The obtained results are compared with those obtained via the application of vector Lyapunov function in this problem. It is shown that the application of hierarchical Lyapunov function allows us to extend the boundaries of the parameter values of the neural network for which the exponential stability of its solutions takes place. The examples illustrating the efficiency of the proposed approach are given.

Keywords: *Discrete-time system; large-scale system; neural system; exponential stability; hierarchical Lyapunov function.*

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1 Introduction

Discrete-time uncertain systems are satisfactory models for investigation of real phenomena in populational dynamics, macroeconomics, for simulation of chemical reactions, and also for analysis of discrete Markov processes, finite and probabilistic automata and others.

One of the most actively developed areas in recent years is the dynamics of neural systems [1 – 3] which are described by discrete-time equations (see [4, 5] and the references therein). Along with the investigation of such systems under different assumptions there has been a considerable interest in the development of general approaches in stability analysis of discrete-time uncertain systems, which will be admissible in the stability analysis of neural networks.

The aim of this paper is to develop a method of analysis of exponential stability of neural systems with nonperturbed and perturbed equilibrium states based on hierarchical Lyapunov function.

The paper is arranged as follows.

In Section 2 the uncertain quasilinear system is considered. To decrease the order of subsystems this system is decomposed. For each component and subsystem auxiliary norm-like functions are constructed and robust bounds are given.

In Section 3 the uncertain neural system with nonperturbed equilibrium state is linearized and the results of Section 2 are applied.

In Section 4 similar problem is solved for the uncertain neural system with perturbed equilibrium state.

In final Section 5 two numerical examples are given.

2 Uncertain Quasilinear System

We consider the discrete-time system with uncertainties and perturbations of the form

$$S\colon \qquad x(\tau+1) = (A+\Delta A)x(\tau) + g(x(\tau)), \tag{2.1}$$

where $\tau \in \mathcal{T}_\tau = \{t_0 + k, k = 0, 1, 2, \dots\}$, $t_0 \in R$, $x \in R^n$, $x_e \equiv 0$ is an equilibrium of (2.1), $g\colon U \to R^n$ is a continuous vector function, $U \subseteq R^n$ is an open subset containing x_e. $A \in R^{n\times n}$ is a constant matrix, $\Delta A \in R^{n\times n}$ is an uncertain matrix. The only knowledge we have regarding the matrix ΔA is that it lies in the known compact set $S \subset R^{n\times n}$. In paper [5] robust stability results were established for the system (2.1) via scalar quadratic Lyapunov function. Unlike this paper we shall apply vector and hierarchical Lyapunov functions composed of norm-like components.

2.1 Vector approach

Assume that the system (2.1) is decomposed into two interconnected subsystems

$$\hat{S}_i\colon \quad x_i(\tau+1) = (A_i + \Delta A_i)\, x_i(\tau) + (B_i + \Delta B_i)\, x_j(\tau) + g_i(x(\tau)), \quad i, j = 1, 2, \quad i \neq j. \tag{2.2}$$

Here $x_i \in R^{n_i}$, A_i, B_i and ΔA_i, ΔB_i are submatrices of the known and uncertain matrices

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \quad \text{and} \quad \Delta A = \begin{pmatrix} \Delta A_1 & \Delta B_1 \\ \Delta B_2 & \Delta A_2 \end{pmatrix},$$

respectively, with $A_i, \Delta A_i \in R^{n_i\times n_i}$, $B_i, \Delta B_i \in R^{n_i\times n_j}$, $i, j = 1, 2$, $i \neq j$, $g = (g_1^{\mathrm{T}}, g_2^{\mathrm{T}})^{\mathrm{T}}$, $g_i\colon U \to R^{n_i}$ are continuous vector functions.

From (2.2) we extract the independent subsystems

$$S_i\colon \quad x_i(\tau+1) = (A_i + \Delta A_i)\, x_i(\tau), \quad i = 1, 2, \tag{2.3}$$

with the same designations of variables as in system (2.2).

Assumption 2.1 *We assume that:*

(1) *there exist unique symmetric and positive definite matrices* $P_i \in R^{n_i \times n_i}$, *which satisfy Lyapunov matrix equations*

$$A_i^{\mathrm{T}} P_i A_i - P_i = -G_i, \qquad i = 1, 2, \tag{2.4}$$

where $G_i \in R^{n_i \times n_i}$ *are arbitrary symmetric and positive definite matrices;*

(2) *there exists a constant* $\gamma \in (0;1)$ *such that*

$$\|B_1\|\,\|B_2\| < \gamma^2 \mu_1 \mu_2,$$

where $\mu_i = \big(\sigma_M^{\frac{1}{2}}(P_i - I_{n_i})\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i)\big)^{-1}$, P_i *are solutions of the Lyapunov matrix equations (2.4) for the matrices* $G_i = I_{n_i}$, I_{n_i} *are* $n_i \times n_i$ *identity matrices,* $i = 1, 2$;

(3) $\lim_{\|x\| \to 0} \|g(x)\| / \|x\| = 0$.

Here $\|B_i\| = \sup_{\|x_i\| \le 1} \|B_i x_i\|$, $\|x_i\| = (x_i^{\mathrm{T}} x_i)^{\frac{1}{2}}$ are the Euclidean norms of vectors x_i, and $\sigma_M(P_i)$ are the maximum eigenvalues of P_i.

Let P_i be determined as solutions of the Lyapunov matrix equations (2.4) for $G_i = I_{n_i}$. We define the constants

$$\begin{aligned}
&\alpha_i = \sigma_M^{\frac{1}{2}}(P_i)\mu_i = \big(\sigma_M^{\frac{1}{2}}(P_i - I_{n_i}) + \sigma_M^{\frac{1}{2}}(P_i)\big)^{-1}, \quad i = 1, 2, \\
&a = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2), \quad b = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)(\|B_1\| + \|B_2\|), \\
&c = \gamma^2 \alpha_1 \alpha_2 - \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)\|B_1\|\,\|B_2\|, \quad \epsilon = ((b^2 + 4ac)^{\frac{1}{2}} - b)/2a.
\end{aligned} \tag{2.5}$$

Theorem 2.1 *We assume that for the uncertain system (2.1) the decomposition (2.2), (2.3) takes place and all conditions of Assumption 2.1 are satisfied. If the inequalities*

$$\|\Delta A_i\| \le (1 - \gamma)\mu_i \quad \text{and} \quad \|B_i\| < \epsilon, \quad i = 1, 2,$$

are true, then the equilibrium $x_e = 0$ *of (2.1) is global exponentially stable.*

Proof For nominal subsystems

$$x_i(\tau + 1) = A_i x_i(\tau), \quad i = 1, 2,$$

we construct the norm-like functions

$$v_i(x_i) = (x_i^{\mathrm{T}} P_i x_i)^{\frac{1}{2}}, \qquad i = 1, 2, \tag{2.6}$$

and the function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2),$$

where d_1, d_2 are some positive constants.

Similarly to the proof of Theorem 3.1 from paper [6] for the first differences $\Delta v_i(x_i)$ of the functions (2.6) along the solutions of the system (2.1) we obtain the estimates

$$\Delta v_i(x_i)\big|_{\hat{S}_i} \le -(\alpha_i - \sigma_M^{\frac{1}{2}}(P_i)\|\Delta A_i\|)\|x_i\| + \sigma_M^{\frac{1}{2}}(P_i)(\|B_i\| + \|\Delta B_i\|)\|x_j\| + \sigma_M^{\frac{1}{2}}(P_i)\|g_i(x)\|,$$

$i,j=1,2,\ i\neq j$, and the estimate

$$\Delta v(x)\big|_S = \tilde{d}^{\mathrm T} W z + \tilde{g}(x),$$

where $\tilde{d}=(d_1,d_2)^{\mathrm T},\ z=(\|x_1\|,\|x_2\|)^{\mathrm T},\ W\in R^{2\times 2}$ is a matrix with the elements

$$w_{ij}=\begin{cases}\alpha_i-\sigma_M^{\frac12}(P_i)\|\Delta A_i\|, & \text{if} \quad i=j,\\ -\sigma_M^{\frac12}(P_i)(\|B_i\|+\|\Delta B_i\|), & \text{if} \quad i\neq j,\end{cases}$$

the function $\tilde{g}\colon R^n\to R_+$ is such that $\lim_{\|x\|\to 0}\|\tilde{g}(x)\|/\|x\|=0$.

It was shown [6] that condition (2) of Assumption 2.1 implies that the matrix W is an M-matrix [7]. Then there exist positive constants d_1 and d_2 such that the vector $\tilde{d}^{\mathrm T}W$ has positive elements [8].

Using the trivial inequalities $\|x_i\|\geq v_i(x_i)/\sigma_M^{\frac12}(P_i),\ i=1,2$, for the first difference of the function $v(x)$ along the solutions of the system (2.1) we get

$$\begin{aligned}\Delta v(x)\big|_S &\leq -\Big[\mu_1-\|\Delta A_1\|-\frac{d_2\sigma_M^{\frac12}(P_2)}{d_1\sigma_M^{\frac12}(P_1)}(\|B_2\|+\|\Delta B_2\|)\Big]d_1v_1(x_1)\\ &-\Big[\mu_2-\|\Delta A_2\|-\frac{d_1\sigma_M^{\frac12}(P_1)}{d_2\sigma_M^{\frac12}(P_2)}(\|B_1\|+\|\Delta B_1\|)\Big]d_2v_2(x_2)+\tilde{g}(x)\\ &\leq -\omega(d_1v_1(x_1)+d_2v_2(x_2))+\tilde{g}(x)=-\omega v(x)+\tilde{g}(x),\end{aligned}\tag{2.7}$$

where

$$\omega=\min_{i,j=1,2,\,i\neq j}\Big\{\mu_i-\|\Delta A_i\|-\frac{d_j\sigma_M^{\frac12}(P_j)}{d_i\sigma_M^{\frac12}(P_i)}(\|B_j\|+\|\Delta B_j\|)\Big\}.$$

The choice of the constants d_1 and d_2 implies $\omega>0$. Let us assume that $\omega\geq 1$, then

$$\mu_i-\|\Delta A_i\|-\frac{d_j\sigma_M^{\frac12}(P_j)}{d_i\sigma_M^{\frac12}(P_i)}(\|B_j\|+\|\Delta B_j\|)\geq 1,\quad i,j=1,2,i\neq j.\tag{2.8}$$

If $\|\Delta A_1\|=\|\Delta A_2\|=\|\Delta B_1\|=\|\Delta B_2\|=\|B_1\|=\|B_2\|=0$, the system (2.1) is written in the form

$$x_i(\tau+1)=A_ix_i(\tau)+g_i(x(\tau)),\quad i=1,2.$$

It is known [9] that the equilibrium $x=0$ of this system is exponentially stable.

Let at least one of the numbers $\|\Delta A_i\|,\|\Delta B_i\|$, or $\|B_i\|$ be not equal to zero, for example, $\|\Delta A_1\|$. Then the inequalities (2.8) give

$$\mu_1\geq 1+\|\Delta A_1\|+\frac{d_2\sigma_M^{\frac12}(P_2)}{d_1\sigma_M^{\frac12}(P_1)}(\|B_2\|+\|\Delta B_2\|)>1,$$

but

$$\mu_1=\frac{1}{\sigma_M^{\frac12}(P_1-I_{n_1})\sigma_M^{\frac12}(P_1)+\sigma_M(P_1)}\leq 1,$$

since $\sigma_M(P_1) \geq 1$. We get the contradiction, from which it follows that $0 < \omega < 1$.

Using the condition (3) of Assumption 2.1 for the function $\tilde{g}(x)$ we get the estimate

$$\begin{aligned}\tilde{g}(x) &= d_1\sigma_M^{\frac{1}{2}}(P_1)\|g_1(x_1)\| + d_2\sigma_M^{\frac{1}{2}}(P_2)\|g_2(x_2)\| \leq (d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2))\|g(x)\| \\ &\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\|x\| \leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)(\|x_1\| + \|x_2\|) \\ &\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\left(\frac{v_1(x_1)}{\sigma_m^{\frac{1}{2}}(P_1))} + \frac{v_2(x_2)}{\sigma_m^{\frac{1}{2}}(P_2))}\right) \\ &\leq \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\max\left\{\frac{1}{d_1\sigma_m^{\frac{1}{2}}(P_1))}, \frac{1}{d_2\sigma_m^{\frac{1}{2}}(P_2)}\right\}v(x),\end{aligned}$$

where $\sigma_m^{\frac{1}{2}}(P_i)$ are minimum eigenvalues of the matrices P_i, α is a small positive number such that for the constant

$$\tilde{\omega} = \omega - \alpha\big(d_1\sigma_M^{\frac{1}{2}}(P_1) + d_2\sigma_M^{\frac{1}{2}}(P_2)\big)\max\left\{\frac{1}{d_1\sigma_m^{\frac{1}{2}}(P_1))}, \frac{1}{d_2\sigma_m^{\frac{1}{2}}(P_2)}\right\}$$

the inequality $0 < \tilde{\omega} < 1$ holds.

Using (2.7) we get the estimate

$$\Delta v(x)\big|_S \leq -\tilde{\omega}v(x)$$

for all x belonging to sufficiently small neighborhood of the origin $\widetilde{U} \subseteq U$, which implies global exponential stability of the equilibrium $x_e = 0$ of (2.1) (see [10]).

The proof of Theorem 2.1 is complete.

2.2 Hierarchical approach

Now in the framework of hierarchical approach we decompose each subsystem (2.3) into two interconnected components

$$\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\,x_{ij}(\tau) + (B_{ij} + \Delta B_{ij})\,x_{ik}(\tau), \tag{2.9}$$

where $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $A_{ij}, \Delta A_{ij} \in R^{n_{ij}\times n_{ij}}$, $B_{i1}, \Delta B_{i1} \in R^{n_{i1}\times n_{i2}}$, $B_{i2}, \Delta B_{i2} \in R^{n_{i2}\times n_{i1}}$, $i,j,k = 1,2$, $j \neq k$,

$$A_i = \begin{pmatrix} A_{i1} & B_{i1} \\ B_{i2} & A_{i2}\end{pmatrix}, \qquad \Delta A_i = \begin{pmatrix} \Delta A_{i1} & \Delta B_{i1} \\ \Delta B_{i2} & \Delta A_{i2}\end{pmatrix}.$$

We assume that the matrices B_i and ΔB_i have a block structure:

$$B_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)}\end{pmatrix}, \qquad \Delta B_i = \begin{pmatrix} \Delta M_{11}^{(i)} & \Delta M_{12}^{(i)} \\ \Delta M_{12}^{(i)} & \Delta M_{22}^{(i)}\end{pmatrix},$$

where $M_{jk}^{(i)}, \Delta M_{jk}^{(i)} \in R^{n_{ij}\times n_{lk}}$, $i,j,k,l = 1,2$, $i \neq l$.

We take from (2.9) the independent components

$$C_{ij}: \quad x_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\,x_{ij}(\tau), \quad i,j = 1,2. \tag{2.10}$$

with the same designations of variables as in system (2.9).

We denote $g_i = (g_{i1}^{\mathrm{T}}, g_{i2}^{\mathrm{T}}, \ldots, g_{in_i}^{\mathrm{T}})^{\mathrm{T}}$ and introduce the following assumptions.

Assumption 2.2 *We assume that:*

(1) *there exist unique symmetric and positive definite matrices P_{ij}, which satisfy the Lyapunov matrix equations*

$$A_{ij}^{\mathrm{T}} P_{ij} A_{ij} - P_{ij} = -G_{ij}, \qquad i = 1, 2, \tag{2.11}$$

where G_{ij} are arbitrary symmetric and positive definite matrices;

(2) *there exist constants $\gamma_i \in (0,1)$ such that*

$$\|B_{i1}\| \, \|B_{i2}\| < \gamma_i^2 \mu_{i1} \mu_{i2}, \quad i = 1, 2,$$

where $\mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{n_{ij}}) \sigma_M^{\frac{1}{2}}(P_{ij}) + \sigma_M(P_{ij})\big)^{-1}$. Here and over P_{ij} are solutions of the Lyapunov matrix equations (2.11) for the matrices $G_{ij} = I_{n_{ij}}$, $I_{n_{ij}}$ are $n_{ij} \times n_{ij}$ identity matrices.

We construct the auxiliary functions v_i on the base of the functions $v_{ij}(x_{ij}) = (x_{ij}^{\mathrm{T}} P_{ij} x_{ij})^{\frac{1}{2}}$ by formulae $v_i(x_i) = d_{i1} v_{i1}(x_{i1}) + d_{i2} v_{i2}(x_{i2})$, $i = 1, 2$.

We consider 2×2 matrices $W_i = (w_{jk}^{(i)})$ with the elements

$$w_{jk}^{(i)} = \begin{cases} \gamma_i \alpha_{ij}, & \text{for} \quad j = k, \\ -\sigma_M^{\frac{1}{2}}(P_{ij})(\|B_{ij}\| + \overline{\epsilon}_i), & \text{for} \quad j \neq k. \end{cases}$$

Here $0 < \overline{\epsilon}_i < \epsilon_i$,

$$\begin{aligned} &\alpha_{ij} = \sigma_M^{\frac{1}{2}}(P_{ij}) \mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{ij}) + \sigma_M^{\frac{1}{2}}(P_{ij})\big)^{-1}, \\ &\mu_{ij} = \big(\sigma_M^{\frac{1}{2}}(P_{ij} - E_{ij}) \sigma_M^{\frac{1}{2}}(P_{ij}) + \sigma_M(P_{ij})\big)^{-1}, \\ &\epsilon_i = ((b_i^2 + 4a_i c_i)^{\frac{1}{2}} - b_i)/2a_i, \quad a_i = \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2}), \\ &c_i = \gamma_i^2 \alpha_{i1} \alpha_{i2} - \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2}) \|B_{i1}\| \, \|B_{i2}\|, \quad i, j = 1, 2, \\ &b_i = \sigma_M^{\frac{1}{2}}(P_{i1}) \sigma_M^{\frac{1}{2}}(P_{i2})(\|B_{i1}\| + \|B_{i2}\|). \end{aligned} \tag{2.12}$$

Let us denote

$$\begin{aligned} &\pi_i = \min\{d_{i1} w_{11}^{(i)} + d_{i2} w_{21}^{(i)}; d_{i1} w_{12}^{(i)} + d_{i2} w_{22}^{(i)}\}, \quad i = 1, 2, \\ &m = \frac{1}{2} \left(\frac{\pi_1 \pi_2}{\big(d_{11} \sigma_M^{\frac{1}{2}}(P_{11}) + d_{12} \sigma_M^{\frac{1}{2}}(P_{12})\big) \big(d_{21} \sigma_M^{\frac{1}{2}}(P_{21}) + d_{22} \sigma_M^{\frac{1}{2}}(P_{22})\big)} \right)^{\frac{1}{2}} \end{aligned} \tag{2.13}$$

A method of optimal choice of the constant d_{ij}, $i, j = 1, 2$, is given in [6].

Assumption 2.3 *Let $\lim\limits_{\|x\| \to 0} \|g(x)\| / \|x\| = 0$ and for the submatrices $M_{jk}^{(i)}$ of the matrices B_i the inequalities $\overline{m} = \max \|M_{jk}^{(i)}\| < m$ be realized for all i, j, $k = 1, 2$.*

Theorem 2.2 *We assume that for the uncertain system (2.1) the two-level decomposition (2.2), (2.3), (2.9), (2.10) is realized and all conditions of Assumptions 2.2 and 2.3 are satisfied. If the inequalities*

$$\|\Delta A_{ij}\| \le (1-\gamma_i)\mu_{ij}, \quad \|\Delta B_{ij}\| \le \bar{\epsilon}_i, \quad \|\Delta M_{jk}^{(i)}\| < m - \overline{m}$$

are fulfilled for all $i,j,k = 1,2$, then the equilibrium $x_e = 0$ of the system (2.1) is global exponentially stable.

Proof Under the hypotheses of Theorem 2.2 analogous to the proof of Theorem 4.1 from [6] for the function $v(x) = d_1 v_1(x_1) + d_2 v_2(x_2)$ we get the estimates:

$$\Delta v(x)\big|_S = d_1 \Delta v_1(x_1)\big|_{\widehat{S}_1} + d_2 \Delta v_2(x_2)\big|_{\widehat{S}_2} \le -\hat{d}^{\mathrm{T}} W z + \widetilde{g}(x), \tag{2.14}$$

where $\hat{d} = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$, $g_i = (g_{i1}^{\mathrm{T}}, g_{i2}^{\mathrm{T}})^{\mathrm{T}}$ and W is 2×2 matrix with the elements

$$w_{jk} = \begin{cases} \pi_j, & \text{for} \quad j = k, \\ -d_{j1}\sigma_M^{\frac{1}{2}}(P_{j1})(2\overline{m} + \|\Delta M_{11}^{(j)}\|) + \|\Delta M_{12}^{(j)}\|) & \\ \qquad -d_{j2}\sigma_M^{\frac{1}{2}}(P_{j2})(2\overline{m} + \|\Delta M_{21}^{(j)}\|) + \|\Delta M_{22}^{(j)}\|), & \text{for} \quad j \ne k, \end{cases}$$

$$\widetilde{g}(x) = d_1\Big(d_{11}\sigma_M^{\frac{1}{2}}(P_{11})\|g_{11}(x)\| + d_{12}\sigma_M^{\frac{1}{2}}(P_{12})\|g_{12}(x)\|\Big)$$
$$+ d_2\Big(d_{21}\sigma_M^{\frac{1}{2}}(P_{21})\|g_{11}(x)\| + d_{22}\sigma_M^{\frac{1}{2}}(P_{22})\|g_{22}(x)\|\Big).$$

Under the hypotheses of Theorem 2.2 the matrix W is the M-matrix and, according to [7] there exist positive constants d_1 and d_2 such that the vector $\hat{d}^{\mathrm{T}} W$ has positive components. That is

$$\hat{d}^{\mathrm{T}} W z = (\pi_1 d_1 - \omega_{21} d_2)\|x_1\| + (\pi_2 d_2 - \omega_{12} d_1)\|x_2\|$$
$$\ge \sum_{i,j=1,2,\, i\ne j} (\pi_i d_i - \omega_{ji} d_j)(\|x_{i1}\| + \|x_{i2}\|)/\sqrt{2}$$
$$\ge \sum_{i,j=1,2,\, i\ne j} \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}} \left(\frac{v_{i1}(x_{i1})}{\sigma_M^{\frac{1}{2}}(P_{i1})} + \frac{v_{i2}(x_{i2})}{\sigma_M^{\frac{1}{2}}(P_{i2})} \right)$$
$$\ge \sum_{i,j=1,2,\, i\ne j} \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i} \left(\frac{1}{d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1})} d_i d_{i1} v_{i1} + \frac{1}{d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})} d_i d_{i2} v_{i2} \right) \ge \omega v(x),$$

where

$$\omega = \min_{i,j=1,2,\, i\ne j} \left\{ \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i d_{i1} \sigma_M^{\frac{1}{2}}(P_{i1})},\ \frac{\pi_i d_i - \omega_{ji} d_j}{\sqrt{2}\, d_i d_{i2} \sigma_M^{\frac{1}{2}}(P_{i2})} \right\}.$$

As the matrix W is the M-matrix, $\gamma_1 \in (0,1)$ and $\mu_{11} \le 1$,

$$\frac{\pi_1 d_1 - \omega_{21} d_2}{\sqrt{2}\, d_1 d_{11} \sigma_M^{\frac{1}{2}}(P_{11})} \le \frac{\pi_1}{\sqrt{2}\, d_{11} \sigma_M^{\frac{1}{2}}(P_{11})}$$
$$\le \frac{d_{11}\omega_{11}^{(1)} + d_{12}\omega_{21}^{(1)}}{\sqrt{2}\, d_{11} \sigma_M^{\frac{1}{2}}(P_{11})} \le \frac{\omega_{11}^{(1)}}{\sqrt{2}\, \sigma_M^{\frac{1}{2}}(P_{11})} = \frac{\gamma_1 \alpha_{11}}{\sqrt{2}\, \sigma_M^{\frac{1}{2}}(P_{11})} = \frac{\gamma_1 \mu_{11}}{\sqrt{2}} < 1$$

and $0 < \omega < 1$.

It follows from (2.14) that

$$\Delta v(x)\big|_S \leq -\omega v(x) + \widetilde{g}(x).$$

As for sufficiently small $\alpha > 0$ the estimate

$$\begin{aligned}\widetilde{g}(x) &\leq \alpha \sum_{i=1}^{2} d_i \Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)\|x\| \\ &\leq \alpha\beta^{-1} \sum_{i=1}^{2} d_i \Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)v(x)\end{aligned}$$

is realized in some neighborhood of zero $\widetilde{U}$,

$$\Delta v(x)\big|_S \leq -\widetilde{\omega} v(x),$$

where $\widetilde{\omega} = \omega - \alpha\beta^{-1} \sum\limits_{i=1}^{2} d_i \Big(d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1}) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\Big)$, $0 < \widetilde{\omega} < 1$.

These conditions are sufficient [10] for the global exponential stability of the equilibrium $x = 0$ of (2.1). The proof of Theorem 2.2 is complete.

3 Neural System with Nonperturbed Equilibrium

We consider discrete-time neural networks described by

$$x(\tau + 1) = Gx(\tau) + Cs(Tx(\tau) + I), \tag{3.1}$$

where $\tau \in \mathcal{T}_\tau = \{t_0 + k, k = 0, 1, 2, \dots\}$, $t_0 \in R$, $x \in R^n$, $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, x_i is the state of ith neuron, $x_i \in R$, $s\colon R^n \to R^n$, $s(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, $s_i\colon R \to (-1, 1)$, $T \in R^{n\times n}$, $G = \operatorname{diag}\{g_1, g_2, \dots g_n\}$, $g_i \in [-1, 1]$, $C = \operatorname{diag}\{c_1, c_2, \dots c_n\}$, $c_i \neq 0$ for all $i = 1, 2, \dots, n$. The functions s_i are twice continuously differentiable functions, they are monotonically increasing and odd.

Together with the system (3.1) we consider an uncertain system

$$x(\tau + 1) = (G + \Delta G)x(\tau) + (C + \Delta C)s\big((T + \Delta T)x(\tau) + (I + \Delta I)\big), \tag{3.2}$$

where $\Delta G, \Delta C, \Delta T \in R^{n\times n}$, $\Delta I \in R^n$ are uncertain matrices and a vector.

3.1 Vector approach

In the framework of vector approach we decompose the neural system (3.1) into two interconnected subsystems

$$x_i(\tau+1) = G_i x_i(\tau) + C_i s_i(T_{i1}x_1(\tau) + T_{i2}x_2(\tau) + I_i),\ i = 1,2, \tag{3.3}$$

where $x_i \in R^{n_i}$, $x_i = (x_{i1}, x_{i2}, \dots x_{in_i})^{\mathrm{T}}$, x_{ij} represents the state of the ijth neuron, $x_{ij} \in R$, $s_i\colon R^{n_i} \to R^{n_i}$, $s_i(x_i) = (s_{i1}(x_{i1}), s_{i2}(x_{i2}), \dots, s_{in_i}(x_{in_i}))^{\mathrm{T}}$, $s_{ij}\colon R \to (-1;1)$, $T_{ij} \in R^{n_i \times n_j}$, $G_i = \operatorname{diag}\{g_{i1},\ g_{i2}, \dots g_{in_i}\}$, $g_{ij} \in [-1;1]$, $C_i = \operatorname{diag}\{c_{i1}, c_{i2}, \dots\ c_{in_i}\}$, $c_{ij} \neq 0$ for all $i = 1,2$, $j = 1,2,\dots,n_i$. The functions s_{ij} are twice continuously differentiable functions, they are monotonically increasing and odd.

Together with the system (3.1) we decompose the uncertain system (3.2)

$$\begin{aligned} x_i(\tau+1) = (G_i + \Delta G_i)x_i(\tau) + (C_i + \Delta C_i)s_i\big((T_{i1} + \Delta T_{i1})x_1(\tau) \\ + (T_{i2} + \Delta T_{i2})x_2(\tau) + (I_i + \Delta I_i)\big). \end{aligned} \tag{3.4}$$

Here $\Delta G_i, \Delta C_i \in R^{n_i \times n_i}$, $\Delta T_{ij} \in R^{n_i \times n_j}$, $\Delta I_i \in R^{n_i}$ are uncertain matrices and vector.

Let $x_e = (x_{1e}^{\mathrm{T}}, x_{2e}^{\mathrm{T}})^{\mathrm{T}}$ denote the equilibrium state of (3.1), $s_i'(x_i) = \operatorname{diag}\{s_{i1}'(x_{i1}), s_{i2}'(x_{i2}), \dots, s_{in_i}'(x_{in_i})\}$, $s_i''(x_i) = \operatorname{diag}\{s_{i1}''(x_{i1}), s_{i2}''(x_{i2}), \dots, s_{in_i}''(x_{in_i})\}$, $L_{i1} = \sup_{x_i \in R^{n_i}} \|s_i'(x_i)\|$, $L_{i2} = \sup_{x_i \in R^{n_i}} \|s_i''(x_i)\|$.

All above assumptions concerning the matrices G_i, C_i, T_{ij}, the vectors I_i and the functions s_i are similar to the assumptions under which a scalar Lyapunov function is applied to the neural systems of (3.1) type in paper [5]. Further we need assumptions connected just with the decomposition of neural system.

Let us introduce the matrices

$$\begin{aligned} A_i &= G_i + C_i s_i'(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i)T_{ii}, \\ B_i &= C_i s_i'(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i)T_{ij}, \quad i,j = 1,2,\ i \neq j, \end{aligned} \tag{3.5}$$

and the following assumptions.

Assumption 3.1 *Assume that:*

(1) *for the matrices (3.5) the conditions (1) and (2) of Assumption 2.1 are satisfied;*
(2) x_e *is an equilibrium state of both (3.3) and (3.4).*

We set

$$\begin{aligned} \beta_i &= 1 + \big(\|C_i\| + \|T_{ii}\|\big)L_{i1} + \big(1 + \|x_{1e}\| + \|x_{2e}\|\big)\|C_i\|\|T_{ii}\|L_{i2}, \\ \delta_i &= \big(\|C_i\| + \|T_{ij}\|\big)L_{i1} + \big(1 + \|x_{1e}\| + \|x_{2e}\|\big)\|C_i\|\|T_{ij}\|L_{i2}, \\ K_i &= \min\left\{\frac{1}{2L_{i1}}\big((\beta^2 + 4(1-\gamma)\mu_i L_{i1})^{\frac12} - \beta_i\big),\ \frac{1}{2L_{i1}}\big((\delta^2 + 4\epsilon L_{i1})^{\frac12} - \delta_i\big)\right\} \end{aligned} \tag{3.6}$$

where $i,j = 1,2,\ i \neq j$, the constants μ_i, ϵ are computed by (2.5) for the matrices (3.5).

Theorem 3.1 *Let for the system (3.2) the decomposition (3.4) take place and all conditions of Assumption 3.1 be satisfied. If the inequalities*

$$\max\{\,\|\Delta G_i\|,\,\|\Delta C_i\|,\,\|\Delta T_{i1}\|,\,\|\Delta T_{i2}\|,\,\|\Delta I_i\|\,\} < K_i, \quad i = 1, 2, \tag{3.7}$$

are true, then the equilibrium x_e of (3.2) is global exponentially stable.

Proof We denote

$$\begin{aligned}
f_i(x) &= G_i x_i + C_i s_i\big(T_{i1}x_1 + T_{i2}x_2 + I_i\big),\\
h_i(x) &= \Delta G_i x_i + (C_i + \Delta C_i)s_i\big((T_{i1} + \Delta T_{i1})x_1 + (T_{i2} + \Delta T_{i2})x_2 + (I_i + \Delta I_i)\big)\\
&\qquad - C_i s_i\big(T_{i1}x_1 + T_{i2}x_2 + I_i\big).
\end{aligned}$$

As the functions f_i and h_i are twice continuously differentiable functions in the neighborhood of the equilibrium x_e, the equations (3.4) can be written in the equivalent form

$$\begin{aligned}
x_i(\tau+1) - x_e &= f_i(x_(\tau)) + h_i(x(\tau)) - f_i(x_e) - h_i(x_e)\\
&= \frac{\partial f_i(x_e)}{\partial x_i}(x_i(\tau) - x_{ie}) + \frac{\partial f_i(x_e)}{\partial x_j}(x_j(\tau) - x_{je})\\
&\quad + \frac{\partial h_i(x_e)}{\partial x_i}(x_i(\tau) - x_{ie}) + \frac{\partial h_i(x_e)}{\partial x_j}(x_j(\tau) - x_{je}) + g_i(x(\tau) - x_e),
\end{aligned} \tag{3.8}$$

where $g_i(x(\tau) - x_e)$ are the higher-order terms with respect to $(x(\tau) - x_e)$,

$$\frac{\partial f_i}{\partial x_i} = \begin{pmatrix} \frac{\partial f_{i1}}{\partial x_{i1}} & \frac{\partial f_{i1}}{\partial x_{i2}} & \cdots & \frac{\partial f_{i1}}{\partial x_{in_i}}\\ \frac{\partial f_{i2}}{\partial x_{i1}} & \frac{\partial f_{i2}}{\partial x_{i2}} & \cdots & \frac{\partial f_{i2}}{\partial x_{in_i}}\\ \hdotsfor{4}\\ \frac{\partial f_{in_i}}{\partial x_{i1}} & \frac{\partial f_{in_i}}{\partial x_{i2}} & \cdots & \frac{\partial f_{in_i}}{\partial x_{in_i}} \end{pmatrix}, \qquad \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial f_{i1}}{\partial x_{j1}} & \frac{\partial f_{i1}}{\partial x_{j2}} & \cdots & \frac{\partial f_{i1}}{\partial x_{jn_j}}\\ \frac{\partial f_{i2}}{\partial x_{i1}} & \frac{\partial f_{i2}}{\partial x_{j2}} & \cdots & \frac{\partial f_{i2}}{\partial x_{jn_j}}\\ \hdotsfor{4}\\ \frac{\partial f_{in_i}}{\partial x_{j1}} & \frac{\partial f_{in_i}}{\partial x_{j2}} & \cdots & \frac{\partial f_{in_i}}{\partial x_{jn_j}} \end{pmatrix},$$

and $\frac{\partial h_i}{\partial x_i}$, $\frac{\partial h_i}{\partial x_j}$ have an analogous form. We get

$$\begin{aligned}
\frac{\partial f_i(x_e)}{\partial x_i} &= G_i + C_i s_i'\big(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i\big)T_{ii} = A_i,\\
\frac{\partial f_i(x_e)}{\partial x_j} &= C_i s_i'\big(T_{i1}x_{1e} + T_{i2}x_{2e} + I_i\big)T_{ij} = B_i, \qquad i, j = 1, 2,\ i \neq j,
\end{aligned}$$

If we denote

$$\Delta A_i = \frac{\partial h_i(x_e)}{\partial x_i}, \qquad \Delta B_i = \frac{\partial h_i(x_e)}{\partial x_j}, \qquad y(\tau) = x(\tau) - x_e,$$

then the equations (3.8) are written in the form

$$y_i(\tau+1) = (A_i + \Delta A_i)\,y_i(\tau) + (B_i + \Delta B_i)\,y_j(\tau) + g_i(y(\tau)) \tag{3.9}$$

and the state $y = 0$ will be the equilibrium of the system (3.9). Letting

$$\begin{aligned}
z_i &= (T_{i1} + \Delta T_{i1})x_{1e} + (T_{i2} + \Delta T_{i2})x_{2e} + (I_i + \Delta I_i),\\
t_i &= T_{i1}x_{1e} + T_{i2}x_{2e} + I_i, \quad i = 1, 2.
\end{aligned}$$

we find

$$\begin{aligned}\Delta A_i &= \Delta G_i + (C_i + \Delta C_i)s_i'(z_i)(T_{ii} + \Delta T_{ii}) - C_i s_i'(t_i)T_{ii}\\ &= \Delta G_i + C_i s_i'(z_i)\Delta T_{ii} + \Delta C_i s_i'(z_i)(T_{ii} + \Delta T_{ii}) + C_i(s_i'(z_i) - s_i'(t_i))T_{ii}\\ &= \Delta G_i + C_i s_i'(z_i)\Delta T_{ii} + \Delta C_i s_i'(z_i)(T_{ii} + \Delta T_{ii}) + C_i Q_i(z_i, t_i)\Lambda_i(z_i - t_i)T_{ii}.\end{aligned}$$

Similarly to [5] here we have used the formula

$$f(a) - f(b) = (a-b)\int_0^1 f'(a + \xi(b-a))d\xi$$

for the functions $f = s_{ij}$,

$$Q_i(z_i,t_i) = \operatorname{diag}\left\{\int_0^1 s_{i1}''(z_i + \xi(t_i - z_i))\,d\xi, \int_0^1 s_{i2}''(z_i + \xi(t_i - z_i))\,d\xi, \dots, \int_0^1 s_{in_i}''(z_i + \xi(t_i - z_i))\,d\xi\right\},$$

$$\Lambda_i(z_i - t_i) = \operatorname{diag}\Big\{z_{i1} - t_{i1}, z_{i2} - t_{i2}, \dots, z_{in_i} - t_{in_i}\Big\}, \qquad i = 1, 2.$$

It is easy to see that

$$\begin{aligned}\|Q_i(z_i,t_i)\| &= \sup_{j=1,2,\dots,n_i}\left|\int_0^1 s_{ij}''(z_i + \xi(t_i - z_i))\,d\xi\right| \le L_{i2},\\ \|\Lambda_i(z_i - t_i)\| &\le \|z_i - t_i\| \le (1 + \|x_{1e}\| + \|x_{2e}\|)K_i.\end{aligned} \tag{3.10}$$

Using (3.6), (3.7) and (3.10), we get

$$\begin{aligned}\|\Delta A_i\| &< K_i + \|C_i\|L_{i1}K_i + L_{i1}(\|T_{ii}\| + K_i)K_i + \|C_i\|L_{i2}\|z_i - t_i\|\,\|T_{ii}\|\\ &\le L_{i1}K_i^2 + (1 + (\|C_i\| + \|T_{ii}\|)L_{i1} + \|C_i\|\|T_{ii}\|L_{i2}(1 + \|x_{1e}\| + \|x_{2e}\|))K_i\\ &= L_{i1}K_i^2 + \beta_i K_i \le (1-\gamma)\mu_i.\end{aligned} \tag{3.11}$$

Similarly for $i \ne j$

$$\begin{aligned}\Delta B_i &= (C_i + \Delta C_i)s_i'(z_i)(T_{ij} + \Delta T_{ij}) - C_i s_i'(t_i)T_{ij}\\ &= C_i s_i'(z_i)\Delta T_{ij} + \Delta C_i s_i'(z_i)(T_{ij} + \Delta T_{ij}) + C_i(s_i'(z_i)s_i'(t_i))T_{ij}\\ &= C_i s_i'(z_i)\Delta T_{ij} + \Delta C_i s_i'(z_i)(T_{ij} + \Delta T_{ij}) + C_i Q_i(z_i,t_i)\Lambda_i(z_i - t_i)T_{ij}\end{aligned}$$

and

$$\begin{aligned}\|\Delta B_i\| &< L_{i1}K_i^2 + ((\|C_i\| + \|T_{ij}\|)L_{i1} + \|C_i\|T_{ij}L_{i2}(1 + \|x_{1e}\| + \|x_{2e}\|))K_i\\ &= L_{i1}K_i^2 + \delta_i K_i \le \epsilon.\end{aligned} \tag{3.12}$$

It follows from (3.11), (3.12) and Assumption 3.1 that, for the system (3.9) all conditions of Theorem 2.1 are satisfied. Hence the equilibrium $y=0$ of the system (3.9) is global exponentially stable, and it is equivalent to global exponential stability of equilibrium x_e of the system (3.2). Theorem 3.1 is proved.

3.2 Hierarchical approach

In the framework of hierarchical approach we decompose each subsystem (3.3) into two interconnected components

$$
\begin{aligned}
x_{ij}(\tau+1) &= G_{ij}x_{ij}(\tau) + C_{ij}s_{ij}\Big(T^{i1}_{j1}x_{11}(\tau) + T^{i1}_{j2}x_{12}(\tau) \\
&\quad + T^{i2}_{j1}x_{21}(\tau) + T^{i2}_{j2}x_{22}(\tau) + I_{ij}\Big), \quad i,j=1,2.
\end{aligned} \tag{3.13}
$$

Here $x_i=(x^{\mathrm T}_{i1},x^{\mathrm T}_{i2})^{\mathrm T}$, $x_{ij}\in R^{n_{ij}}$, $R^{n_i}=R^{n_{i1}}\times R^{n_{i2}}$, $x_{ij}=(x_{ij1},x_{ij2},\dots,x_{ijn_{ij}})^{\mathrm T}$, x_{ijl} represents the state of the ijlth neuron, $x_{ijl}\in R$,

$$
T_{ij}=\begin{pmatrix} T^{ij}_{11} & T^{ij}_{12} \\ T^{ij}_{21} & T^{ij}_{22}\end{pmatrix},\; G_i=\begin{pmatrix} G_{i1} & 0 \\ 0 & G_{i2}\end{pmatrix}, C_i=\begin{pmatrix} C_{i1} & 0 \\ 0 & C_{i2}\end{pmatrix},\; I_i=\left(I^{\mathrm T}_{i1},I^{\mathrm T}_{i2}\right)^{\mathrm T},
$$

$T^{ip}_{jk}\in R^{n_{ij}\times n_{pk}}$, $I_{ij}\in R^{n_{ij}}$, $G_{ij}=\mathrm{diag}\,\{g_{ij1},g_{ij2},\dots g_{ijn_{ij}}\}$, $g_{ijl}\in[-1,1]$, $C_{ij}=\mathrm{diag}\,\{c_{ij1},\ c_{ij2},\dots c_{ijn_{ij}}\}$, $c_{ijl}\neq 0$, $s_i(x)=(s_{i1}(x_{i1})^{\mathrm T},\ s_{i2}(x_{i2})^{\mathrm T})^{\mathrm T}$, $s_{ij}\colon R^{n_{ij}}\to R^{n_{ij}}$, $s_{ij}(x_{ij})=(s_{ij1}(x_{ij1}),\,s_{ij2}(x_{ij2}),\dots,s_{ijn_{ij}}(x_{ijn_{ij}}))^{\mathrm T}$, the functions $s_{ijl}\colon\ R\to(-1,1)$, s_{ijl} are twice continuously differentiable, increasing and odd, $i,j,k,p=1,2,\ l=1,2,\dots,n_{ij}$.

Together with the system (3.3) we decompose the system (3.4) into interconnected components

$$
\begin{aligned}
x_{ij}(\tau+1) &= (G_{ij}+\Delta G_{ij})x_{ij}(\tau) + (C_{ij}+\Delta C_{ij})s_{ij}\Big((T^{i1}_{j1}+\Delta T^{i1}_{j1})x_{11}(\tau) \\
&\quad + (T^{i1}_{j2}+\Delta T^{i1}_{j2})x_{12}(\tau) + (T^{i2}_{j1}+\Delta T^{i2}_{j1})x_{21}(\tau) \\
&\quad + (T^{i2}_{j2}+\Delta T^{i2}_{j2})x_{22}(\tau) + (I_{ij}+\Delta I_{ij})\Big), \quad i,j=1,2.
\end{aligned} \tag{3.14}
$$

Here $\Delta G_{ij},\Delta C_{ij}$, ΔT^{ik}_{jp}, ΔI_{ij} are unknown matrices and a vector of corresponding dimensions. The only knowledge about it is that it lies in some known compact sets.

Let us denote by $x_e=(x^{e\,\mathrm T}_{11},x^{e\,\mathrm T}_{12},x^{e\,\mathrm T}_{21},x^{e\,\mathrm T}_{22})^{\mathrm T}$ the equilibrium of system (3.13), and

$$
\begin{aligned}
s'_{ij}(x_{ij}) &= \mathrm{diag}\,\{s'_{ij1}(x_{ij1}),s'_{ij2}(x_{ij2}),\dots,s'_{ijn_{ij}}(x_{ijn_{ij}})\},\\
s''_{ij}(x_{ij}) &= \mathrm{diag}\,\{s''_{ij1}(x_{ij1}),s''_{ij2}(x_{ij2}),\dots,s''_{ijn_{ij}}(x_{ijn_{ij}})\},\\
L^1_{ij} &= \sup_{x_{ij}\in R^{n_{ij}}}\|s''_{ij}(x_{ij})\|,\quad L^2_{ij}=\sup_{x_{ij}\in R^{n_{ij}}}\|s''_{ij}(x_{ij})\|,\\
t_{ij} &= T^{i1}_{j1}x_{11}+T^{i1}_{j2}x_{12}+T^{i2}_{j1}x_{21}+T^{i2}_{j2}x_{22}+I_{ij},\\
z_{ij} &= (T^{i1}_{j1}+\Delta T^{i1}_{j1})x_{11}+(T^{i1}_{j2}+\Delta T^{i1}_{j2})x_{12}\\
&\quad +(T^{i2}_{j1}+\Delta T^{i2}_{j1})x_{21}+(T^{i2}_{j2}+\Delta T^{i2}_{j2})x_{22}+(I_{ij}+\Delta I_{ij})\\
t^e_{ij} &= T^{i1}_{j1}x^e_{11}+T^{i1}_{j2}x^e_{12}+T^{i2}_{j1}x^e_{21}+T^{i2}_{j2}x^e_{22}+I_{ij},\\
z^e_{ij} &= (T^{i1}_{j1}+\Delta T^{i1}_{j1})x^e_{11}+(T^{i1}_{j2}+\Delta T^{i1}_{j2})x^e_{12}\\
&\quad +(T^{i2}_{j1}+\Delta T^{i2}_{j1})x^e_{21}+(T^{i2}_{j2}+\Delta T^{i2}_{j2})x^e_{22}+(I_{ij}+\Delta I_{ij}).
\end{aligned}
$$

and the matrices

$$\begin{aligned} A_{ij} &= G_{ij} + C_{ij}s'_{ij}(t^e_{ij})T^{ii}_{jj}, \\ B_{ij} &= C_{ij}s'_{ij}(t^e_{ij})T^{ii}_{jk}, \quad j \neq k, \\ M^{(i)}_{jl} &= C_{ij}s'_{ij}(t^e_{ij})T^{ip}_{jl}, \quad i \neq p. \end{aligned} \tag{3.15}$$

We need the following assumptions.

Assumption 3.2 *We assume that:*

(1) *all conditions of Assumption 2.2 are satisfied for the matrices (3.15);*
(2) *the state x_e is an equilibrium of both (3.13) and (3.14);*
(3) $\overline{m} = \max \|M^{(i)}_{jl}\| < m$, *where the constant m is computed by formula (2.13).*

Let us denote

$$\begin{gathered} \beta_{ij} = 1 + \big(\|C_{ij}\| + \|T^{ii}_{jj}\|\big)L^1_{ij} + \|C_{ij}\|\|T^{ii}_{jj}\|L^2_{ij}R_e, \\ \delta^{pk}_{ij} = \big(\|C_{ij}\| + \|T^{ip}_{jk}\|\big)L^1_{ij} + \|C_{ij}\|\|T^{ip}_{jk}\|L^2_{ij}R_e, \ i \neq p \text{ ff } \quad j \neq k, \\ R_e = 1 + \|x^e_{11}\| + \|x^e_{12}\| + \|x^e_{21}\| + \|x^e_{22}\|, \\ \alpha^1_{ij} = \big((\beta^2_{ij} + 4(1-\gamma_i)\mu_{ij}L^1_{ij})^{\frac{1}{2}} - \beta_{ij}\big)/2L^1_{ij}, \\ \alpha^2_{ij} = \big(((\delta^{ik}_{ij})^2 + 4\overline{\epsilon}_i L^1_{ij})^{\frac{1}{2}} - \delta^{ik}_{ij}\big)/2L^1_{ij}, \quad j \neq k, \\ \alpha^3_{ijl} = \big(((\delta^{pl}_{ij})^2 + 4\widetilde{m}L^1_{ij})^{\frac{1}{2}} - \delta^{pl}_{ij}\big)/2L^1_{ij}, \quad i \neq p, \\ K_{ij} = \min\{\alpha^1_{ij}, \alpha^2_{ij}, \alpha^3_{ij1}, \alpha^3_{ij2}\} \quad i,j,k,p,l = 1,2. \end{gathered}$$

Here $0 < \widetilde{m} < m - \overline{m}$ and $0 < \overline{\epsilon}_i < \epsilon_i$ the constants μ_{ij}, ϵ_i are computed for the matrices (3.15) by formula (2.12).

Theorem 3.2 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 3.2 be satisfied. If the inequalities*

$$\max_{p,k=1,2} \Big\{ \|\Delta G_{ij}\|, \|\Delta C_{ij}\|, \|\Delta T^{ip}_{jk}\|, \|\Delta I_{ij}\| \Big\} \leq K_{ij}, \quad i,j = 1,2,$$

are true, then the equilibrium x_e of (3.2) is global exponentially stable.

Proof We denote

$$\begin{aligned} f_{ij}(x) &= G_{ij}x_{ij} + C_{ij}s_{ij}\big(t_{ij}\big), \\ h_{ij}(x) &= \Delta G_{ij}x_{ij} + (C_{ij} + \Delta C_{ij})s_{ij}\big(z_{ij}\big) - C_{ij}s_{ij}\big(t_{ij}\big). \end{aligned}$$

For the functions f_{ij} we get

$$\frac{\partial f_{ij}(x_e)}{\partial x_{pk}} = \begin{cases} G_{ij} + C_{ij}s'_{ij}\big(t^e_{ij}\big)T^{ii}_{jj} = A_{ij}, & i = p, j = k, \\ C_{ij}s'_{ij}\big(t^e_{ij}\big)T^{ii}_{jk} = B_{ij}, & i = p, j \neq k, \\ C_{ij}s'_{ij}\big(t^e_{ij}\big)T^{ip}_{jk} = M^{(i)}_{jk}, & i \neq p. \end{cases}$$

Since the functions f_{ij} and h_{ij} are twice continuously differentiable in the neighborhood of the equilibrium x_e, the equations (3.14) can be written in the equivalent form

$$\begin{aligned} x_{ij}(\tau+1) - x_{ij}^e &= f_{ij}(x(\tau)) + h_{ij}(x(\tau)) - f_{ij}(x_e) - h_{ij}(x_e) = \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}(x_{ij}(\tau) - x_{ij}^e) \\ &+ \frac{\partial f_{ij}(x_e)}{\partial x_{ik}}(x_{ik}(\tau) - x_{ik}^e) + \frac{\partial f_{ij}(x_e)}{\partial x_{p1}}(x_{p1}(\tau) - x_{p1}^e) + \frac{\partial f_{ij}(x_e)}{\partial x_{p2}}(x_{p2}(\tau) - x_{p2}^e) \\ &+ \frac{\partial h_{ij}(x_e)}{\partial x_{ij}}(x_{ij}(\tau) - x_{ij}^e) + \frac{\partial h_{ij}(x_e)}{\partial x_{ik}}(x_{ik}(\tau) - x_{ik}^e) + \frac{\partial h_{ij}(x_e)}{\partial x_{p1}}(x_{p1}(\tau) - x_{p1}^e) \\ &+ \frac{\partial h_{ij}(x_e)}{\partial x_{p2}}(x_{p2}(\tau) - x_{p2}^e) + g_{ij}(x(\tau) - x_e), \qquad i,j,k,p = 1,2, \quad i \neq p, j \neq k, \end{aligned} \tag{3.16}$$

where $g_{ij}(x(\tau) - x_e)$ are the higher-order terms with respect to $x(\tau) - x_e$. If we denote

$$\Delta A_{ij} = \frac{\partial h_{ij}(x_e)}{\partial x_{ij}}, \quad \Delta B_{ij} = \frac{\partial h_{ij}(x_e)}{\partial x_{ik}}, \; y(\tau) = x(\tau) - x_e,$$

$$\Delta M_{jl}^{(i)} = \frac{\partial h_{ij}(x_e)}{\partial x_{pl}}, \quad i,j,k,p,l = 1,2, \quad i \neq p, \quad j \neq k,$$

the equations (3.16) are written in the form

$$\begin{aligned} &y_{ij}(\tau+1) = (A_{ij} + \Delta A_{ij})\, y_{ij}(\tau) + (B_{ij} + \Delta B_{ij})\, y_{ik}(\tau) \\ &+(M_{j1}^{(i)} + \Delta M_{j1}^{(i)})\, y_{p1}(\tau) + (M_{j2}^{(i)} + \Delta M_{j2}^{(i)})\, y_{p2}(\tau) + g_{ij}(y(\tau)), \end{aligned} \tag{3.17}$$

$i \neq p$, $j \neq k$, and the state $y = 0$ is an equilibrium of (3.17).

Then, as in proof of Theorem 3.1, we have

$$\begin{aligned} &\|\Delta A_{ij}\| \leq L_{ij}^1 K_{ij}^2 + \Big(1 + (\|C_{ij}\| + \|T_{jj}^{ii}\|)L_{ij}^1 \\ &+\|C_{ij}\|\|T_{jj}^{ii}\|L_{ij}^2 R_e\Big)K_{ij} = L_{ij}^1 K_{ij}^2 + \beta_{ij} K_{ij} \leq (1-\gamma_i)\mu_{ij}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \|\Delta B_{ij}\| &< L_{ij}^1 K_{ij}^2 + \Big((\|C_{ij}\| + \|T_{jk}^{ii}\|)L_{ij}^1 + \|C_{ij}\|T_{jk}^{ii} L_{ij}^2 R_e)\Big)K_{ij} \\ &= L_{ij}^1 K_{ij}^2 + \delta_{ij}^{ik} K_{ij} \leq \overline{\epsilon}_i, \quad j \neq k, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \|\Delta M_{jl}^{(i)}\| &< L_{ij}^1 K_{ij}^2 + \Big((\|C_{ij}\| + \|T_{jl}^{ip}\|)L_{ij}^1 + \|C_{ij}\|\|T_{jl}^{ip}\|L_{ij}^2 R_e)\Big)K_{ij} \\ &= L_{ij}^1 K_{ij}^2 + \delta_{ij}^{pl} K_{ij} \leq \widetilde{m} < m - \overline{m}, \quad i \neq p. \end{aligned} \tag{3.20}$$

It follows from (3.18) – (3.20) and conditions of Assumption 3.2, that for the system (3.17) all conditions of Theorem 2.2 are fulfilled. Hence, the equilibrium $y = 0$ of (3.17) is global exponentially stable, and it is equivalent to global exponential stability of the equilibrium x_e of (3.14). Theorem 3.2 is proved.

4 Neural System with Perturbed Equilibrium

The approach considered below may be used for the investigation of uncertain neural systems with perturbed equilibrium.

Assumption 4.1 *We assume that:*

(1) *for the matrices (3.5) the conditions (1) and (2) of Assumption 2.1 are satisfied;*
(2) x_e *is an equilibrium of (3.1),* $\overline{x}_e$ *is an equilibrium of (3.2),* $x_e \neq \overline{x}_e$.

We denote

$$\begin{aligned}
&\overline{K}_i = \min\left\{\frac{1}{2L_{i1}}\big((\beta_i^2 + 2(1-\gamma)\mu_i L_{i1})^{\frac{1}{2}} - \beta_i\big),\ \frac{1}{2L_{i1}}\big((\delta_i^2 + 2\epsilon L_{i1})^{\frac{1}{2}} - \delta_i\big)\right\},\\
&r_{ij} = L_{i2}\big(\|C_i\| + K_i\big)\big(\|T_{ij}\| + K_i\big)\big(\|T_{i1}\| + \|T_{i2}\| + 2K_i\big), \qquad i,j = 1,2, \qquad (4.1)\\
&\Delta < \min\left\{\frac{(1-\gamma)\mu_1}{2r_{11}},\ \frac{(1-\gamma)\mu_2}{2r_{22}},\ \frac{\epsilon}{2r_{12}},\ \frac{\epsilon}{2r_{21}}\right\},
\end{aligned}$$

where the constants μ_1, μ_2 and ϵ are computed by (2.5) for the matrices (3.5).

Theorem 4.1 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 4.1 be satisfied. If the inequalities*

$$\max\{\,\|\Delta G_i\|,\ \|\Delta C_i\|,\ \|\Delta T_{i1}\|,\ \|\Delta T_{i2}\|,\ \|\Delta I_i\|\,\} < \overline{K}_i,\ \ \|x_{ie} - \overline{x}_{ie}\| < \Delta,\ \ i = 1,2,$$

are true, then the equilibrium $\overline{x}_e$ *of (3.2) is global exponentially stable.*

Proof In the neighborhood $\overline{x}_e$ the equations (3.4) can be written in the equivalent form

$$\begin{aligned}
x_i(\tau+1) - \overline{x}_e &= f_i(x(\tau)) + h_i(x(\tau)) - f_i(\overline{x}_e) - h_i(\overline{x}_e)\\
&+ \left(\frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j}\right)(x_j(\tau) - \overline{x}_{je}) + g_i(x(\tau) - \overline{x}_e)\\
&= \left(\frac{\partial f_i(x_e)}{\partial x_i} + \frac{\partial f_i(\overline{x}_e)}{\partial x_i} + \frac{\partial h_i(\overline{x}_e)}{\partial x_i} - \frac{\partial f_i(x_e)}{\partial x_i}\right)(x_i(\tau) - \overline{x}_{ie})\\
&+ \left(\frac{\partial f_i(x_e)}{\partial x_j} + \frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j} - \frac{\partial f_i(x_e)}{\partial x_j}\right)(x_j(\tau) - \overline{x}_{je})\\
&+ g_i(x(\tau) - \overline{x}_e), \qquad i,j = 1,2,\ i \neq j,
\end{aligned}$$

where $g_{ij}(x(\tau) - \overline{x}_e)$ are higher-order terms. If we denote

$$\begin{gathered}
\Delta A_i = \frac{\partial f_i(\overline{x}_e)}{\partial x_i} + \frac{\partial h_i(\overline{x}_e)}{\partial x_i} - \frac{\partial f_i(x_e)}{\partial x_i}, \qquad \Delta B_i = \frac{\partial f_i(\overline{x}_e)}{\partial x_j} + \frac{\partial h_i(\overline{x}_e)}{\partial x_j} - \frac{\partial f_i(x_e)}{\partial x_j},\\
y(\tau) = x(\tau) - \overline{x}_e, \qquad i,j = 1,2,\ i \neq j,
\end{gathered}$$

then the equations (3.4) are written in the form (3.9) and further the proof of Theorem 4.1 is analogous to the proof of Theorem 3.1.

Assumption 4.2 *We assume that:*

(1) *for matrices (3.15) the conditions (1) and (2) of Assumption 2.2 are satisfied;*
(2) x_e *is an equilibrium state of (3.1),* $\overline{x}_e$ *is an equilibrium state of (3.2),* $x_e \neq \overline{x}_e$*;*
(3) $\overline{m} = \max\|M_{jk}^{(i)}\| < m$*, the constant* m *is computed by formula (2.13).*

Let us denote

$$\begin{aligned}
\alpha^1_{ij} &= \big((\beta^2_{ij} + 2(1-\gamma_i)\mu_{ij}L^1_{ij})^{\frac{1}{2}} - \beta_{ij}\big)/2L^1_{ij},\\
\alpha^2_{ij} &= \big(((\delta^{ik}_{ij})^2 + 2\overline{\epsilon}_i L^1_{ij})^{\frac{1}{2}} - \delta^{ik}_{ij}\big)/2L^1_{ij}, \quad j \ne k,\\
\alpha^3_{ijl} &= \big(((\delta^{pl}_{ij})^2 + 2\widetilde{m} L^1_{ij})^{\frac{1}{2}} - \delta^{pl}_{ij}\big)/2L^1_{ij}, \quad i \ne p,\\
\overline{K}_{ij} &= \min\{\alpha^1_{ij}, \alpha^2_{ij}, \alpha^3_{ij1}, \alpha^3_{ij2}\}, \quad i,j,k,p,l = 1,2. \qquad (4.2)\\
r^{ip}_{jl} &= L_{i2}\big(\|C_{ij}\| + \overline{K}_{ij}\big)\big(\|T^{ip}_{jl}\| + \overline{K}_{ij}\big)\left(\sum_{\nu,\xi=1}^{2} \|T^{i\nu}_{j\xi}\| + 4\overline{K}_i\right),\\
\Phi &= \min\left\{\frac{(1-\gamma_i)\mu_{ij}}{2r^{ii}_{jj}}, \frac{\overline{\epsilon}_i}{2r^{ii}_{jk}}, \frac{\widetilde{m}}{2r^{ip}_{jl}}\right\}, \quad i \ne p, \quad j \ne k,
\end{aligned}$$

where $0 < \widetilde{m} < m - \overline{m}$, $0 < \overline{\epsilon}_i < \epsilon_i$, the constants μ_{ij} and ϵ_i are computed by (2.12) for the matrices (3.15).

Theorem 4.2 *For the system (3.2) let the decomposition (3.4), (3.14) take place and all conditions of Assumption 4.2 be satisfied. If the inequalities*

$$\max_{p,k=1,2}\Big\{\|\Delta G_{ij}\|, \|\Delta C_{ij}\|, \|\Delta T^{ip}_{jk}\|, \|\Delta I_{ij}\|\Big\} \le \overline{K}_{ij}, \quad \|x^e_{ij} - \overline{x}^e_{ij}\| \le \Phi, \quad i,j = 1,2,$$

are true, then the equilibrium $\overline{x}_e$ of (3.2) is global exponentially stable.

Proof As in the proof of Theorem 4.5, the equations (3.14) are written in the equivalent form

$$\begin{aligned}
x_{ij}(\tau+1) - \overline{x}^e_{ij} &= f_{ij}(x(\tau)) + h_{ij}(x(\tau)) - f_{ij}(\overline{x}_e) - h_{ij}(\overline{x}_e)\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{ij}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ij}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}\right)(x_{ij}(\tau) - \overline{x}^e_{ij})\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{ik}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ik}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ik}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ik}}\right)(x_{ik}(\tau) - \overline{x}^e_{ik})\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{p1}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{p1}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{p1}}\right)(x_{p1}(\tau) - \overline{x}^e_{p1})\\
&= \left(\frac{\partial f_{ij}(x_e)}{\partial x_{p2}} + \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{p2}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{p2}} - \frac{\partial f_{ij}(x_e)}{\partial x_{p2}}\right)(x_{p2}(\tau) - \overline{x}^e_{p2})\\
&\quad + g_{ij}(x(\tau) - \overline{x}_e), \qquad i,j,k,p = 1,2, \quad i \ne p, \quad j \ne k.
\end{aligned}$$

If we denote

$$\Delta A_{ij} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ij}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ij}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ij}}, \quad \Delta B_{ij} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{ik}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{ik}} - \frac{\partial f_{ij}(x_e)}{\partial x_{ik}},$$

$$\Delta M^{(i)}_{jl} = \frac{\partial f_{ij}(\overline{x}_e)}{\partial x_{pl}} + \frac{\partial h_{ij}(\overline{x}_e)}{\partial x_{pl}} - \frac{\partial f_{ij}(x_e)}{\partial x_{pl}}, \quad y(\tau) = x(\tau) - \overline{x}_e,$$

$$i,j,k,p,l = 1,2, \quad i \ne p, \quad j \ne k.$$

then the equations (3.14) take the form (3.17) and further the proof is carried out analogously to the proof of Theorem 3.2.

5 Numerical Results

Example 5.1 Let us consider the system

$$\begin{cases} x_1(\tau+1) = \dfrac{4}{\pi}\arctan\left(x_1(\tau) + 0.005\,x_2(\tau) - 0.005\right), \\ x_2(\tau+1) = -0.75\,x_2(\tau) + \dfrac{7}{\pi}\arctan\left(\dfrac{1}{350}\,x_1(\tau) + x_2(\tau) - \dfrac{1}{350}\right), \end{cases}$$

with the equilibrium $x_e = (1;1)^{\mathrm{T}}$ and the perturbed system

$$\begin{cases} x_1(\tau+1) = \dfrac{4.04}{\pi}\arctan\left(1.02\,x_1(\tau) - 0.0052\,x_2(\tau) - 0.025\right), \\ x_2(\tau+1) = -0.75\,x_2(\tau) + \dfrac{7}{\pi}\arctan\left(\dfrac{1}{350}\,x_1(\tau) + 1.02\,x_2(\tau) - \dfrac{801}{35000}\right) \end{cases} \tag{5.1}$$

with the equilibrium $\overline{x}_e = (1.01;1)^{\mathrm{T}}$.

In the framework of scalar approach (see [5]), we have

$$K_1 = 0.0167, \quad \overline{\epsilon}_1 = 0.0942, \quad \Delta G = \operatorname{diag}\{0;0\}, \quad \Delta C = \operatorname{diag}\{0.02;0\},$$

$$\Delta I = \left(-\,0.02; -\frac{701}{35000}\right)^{\mathrm{T}}, \qquad \Delta T = \begin{pmatrix} 0.02 & -0.0102 \\ 0 & 0.02 \end{pmatrix}.$$

As $|\Delta T|_\infty = 0.0302 > K_1$, we can not make the conclusion about exponential stability of equilibrium state $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1).

In the framework of vector approach the constants computed by (4.1) are

$$K_1 = 0.0215, \quad K_2 = 0.0205, \quad \Phi = 0.1054.$$

Then

$$\max\{\,\|\Delta G_1\|,\, \|\Delta C_1\|,\, \|\Delta T_{11}\|,\, \|\Delta T_{12}\|,\, \|\Delta I_1\|\,\} = 0.02 < K_1,$$
$$\max\{\,\|\Delta G_2\|,\, \|\Delta C_2\|,\, \|\Delta T_{21}\|,\, \|\Delta T_{22}\|,\, \|\Delta I_2\|\,\} = 0.02002 < K_2,$$
$$\|\overline{x}_{1e} - x_{1e}\| = 0.01 < \Phi, \qquad \|\overline{x}_{2e} - x_{2e}\| = 0 < \Phi,$$

and, by Theorem 4.1, we can conclude that the equilibrium $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1) is global exponentially stable.

In the framework of vector approach the constants computed by (4.1) are

$$K_1 = 0.0215, \quad K_2 = 0.0205, \quad \Phi = 0.1054.$$

Then

$$\max\{\|\Delta G_1\|, \|\Delta C_1\|, \|\Delta T_{11}\|, \|\Delta T_{12}\|, \|\Delta I_1\|\} = 0.02 < K_1,$$
$$\max\{\|\Delta G_2\|, \|\Delta C_2\|, \|\Delta T_{21}\|, \|\Delta T_{22}\|, \|\Delta I_2\|\} = 0.02002 < K_2,$$
$$\|\overline{x}_{1e} - x_{1e}\| = 0.01 < \Phi, \qquad \|\overline{x}_{2e} - x_{2e}\| = 0 < \Phi,$$

and, by Theorem 4.1, we can conclude that the equilibrium $\overline{x}_e = (1.01;\,1)^{\mathrm{T}}$ of the system (5.1) is global exponentially stable.

Example 5.2 Let the system have the form

$$\begin{cases} x_{11}(\tau+1) = \dfrac{4}{\pi}\arctan\left(x_{11}(\tau) - 0.01\, x_{21}(\tau) - 0.01\right), \\ x_{12}(\tau+1) = -0.2\, x_{12}(\tau) + \dfrac{1.2}{\pi}\arctan(x_{12}(\tau) + 0.01x_{21}(\tau) + 0.01), \\ x_{21}(\tau+1) = 0.6 - \dfrac{1.6}{\pi}\arctan\left(0.01x_{11}(\tau) + x_{21}(\tau) - 1.99\right), \\ x_{22}(\tau+1) = x_{22}(\tau) - \dfrac{3}{\pi}\arctan(0.01x_{12}(\tau) + x_{22}(\tau)), \end{cases}$$

$x = (x_{11}, x_{12}, x_{21}, x_{22})^{\mathrm{T}} \in R^4$. $x_e = (1; 0; -1; 0)^{\mathrm{T}}$,

$$s(x) = \frac{2}{\pi}(\arctan x_{11},\ \arctan x_{12},\ \arctan x_{21},\ \arctan x_{22})^{\mathrm{T}},$$

$$G = \mathrm{diag}\,\{0;\ -0.2;\ 0.6;\ 1\},\ C = \mathrm{diag}\,\{2;\ 0.6;\ -0.8;\ -1.5\}, I = (-0.01;\ 0.01;\ 1.99;\ 0)^{\mathrm{T}},$$

$$T = \begin{pmatrix} 1 & 0 & -0.01 & 0 \\ 0 & 1 & 0.01 & 0 \\ 0.01 & 0 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \dfrac{2}{\pi} & 0 & -\dfrac{0.02}{\pi} & 0 \\ 0 & \dfrac{6-\pi}{5\pi} & \dfrac{0.012}{\pi} & 0 \\ -\dfrac{0.008}{\pi} & 0 & \dfrac{3\pi-4}{5\pi} & 0 \\ 0 & -\dfrac{0.03}{\pi} & 0 & \dfrac{\pi-3}{\pi} \end{pmatrix}.$$

As a result of hierarchical decomposition we get

$$A_{11} = \frac{2}{\pi}, \quad A_{12} = \frac{6-\pi}{5\pi}, \quad A_{21} = \frac{3\pi-4}{5\pi}, \quad A_{22} = \frac{\pi-3}{\pi},$$

$$B_{11} = B_{12} = B_{21} = B_{22} = 0,$$

$$M_{11}^{(1)} = -\frac{0.02}{\pi}, \quad M_{21}^{(1)} = \frac{0.012}{\pi}, \quad M_{11}^{(2)} = -\frac{0.008}{\pi}, \quad M_{22}^{(2)} = -\frac{0.03}{\pi},$$

$$M_{12}^{(1)} = M_{22}^{(1)} = M_{12}^{(2)} = M_{21}^{(2)} = 0,$$

$$R_e = 3, \quad L_{jk}^i = \frac{2}{\pi}, \quad \overline{m} = 0.0095.$$

We get $\gamma_1 = \gamma_2 = 0.5$, $\overline{\epsilon}_1 = \overline{\epsilon}_2 = 0.1$ and do relevant computations

$$\begin{array}{llll} P_{11} = 1.6814, & P_{12} = 1.0342, & P_{21} = 1.1354, & P_{22} = 1.0020, \\ \mu_{11} = 0.3633, & \mu_{12} = 0.8180, & \mu_{21} = 0.6546, & \mu_{22} = 0.9549, \\ a_1 = 1.3187, & c_1 = 0.0980, & a_2 = 1.0666, & c_2 = 0.1667, \\ \epsilon_1 = 0.2726, & \epsilon_2 = 0.3953, & m = 0.0504. & \end{array}$$

If $\overline{\epsilon}_1 = \overline{\epsilon}_2 = 0.1$, $\widetilde{m} = m - \overline{m} = 0.0409$ then we get the robust bounds

$$\overline{K}_{11} = 0.0269, \quad \overline{K}_{12} = 0.0407, \quad \overline{K}_{21} = 0.0304, \quad \overline{K}_{22} = 0.0393.$$

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Asymptotic Behavior in Some Classes of Functional Differential Equations

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Abstract: In this paper we shall investigate the asymptotic behavior (at $+\infty$) of certain classes of functional differential equations, involving causal (abstract Volterra) operators. Vast literature exists on this subject, mainly in the case of ordinary differential equations, delay equations and integro-differential equations. We mention here the book, non-linear differential equations, by G. Sansone and R. Conti for classical results. More recent contributions can be found in the books by C. Corduneanu, with pertinent references. See also the papers by A.R. Aftabizadeh, and C. Corduneanu and M. Mahdavi.

Keywords: *Functional differential equations; causal operators; asymptotic behavior.*

Mathematics Subject Classification (2000): 65F10, 65L05, 65L20, 65F15, 15A18.

1 A Result of Global Existence

Let us consider the first order functional differential equation

$$\dot{x}(t) + (Lx)(t) = (Mx)(t), \tag{1}$$

on the positive half-axis $R_+ = [0, \infty)$. In equation (1), L and M stand for causal operators acting on convenient function spaces (to be specified below), with L assumed to be linear and continuous, while M is in general nonlinear. As we know (see [3], Chapter 3), a local solution does exist for (1), under suitable conditions, satisfying the initial condition

$$x(0) = x_0, \tag{2}$$

and being defined on some interval $[0, T)$, $T \leq \infty$. For instance, if we also assume the linearity and continuity of M on the underlying space, then necessarily $T = \infty$ (see [3]).

We shall now obtain an upper estimate for the solution of (1), (2). This estimate will allow us to conclude that all solutions of (1), (2) are defined on the whole half-axis R_+.

In view of obtaining the estimate, we shall formulate and utilize certain assumptions on the data. Also, we need to choose the space on which the causal operators L and M are acting.

The assumptions are:

(H_1) The operators L and M in (1) are causal operators on the space $C(R_+, R^n)$, with L linear and continuous, while M is continuous and nonlinear.

(H_2) The operator L satisfies the condition

$$\int_0^t \langle (Lx)(s), x(s)\rangle \, ds \geq \lambda(t) \int_0^t |x(s)|^2 \, ds, \tag{3}$$

for each $t \in R_+$, with $\lambda(t)$ a non-increasing function on R_+.

(H_3) There exists a function $m \in L^2(R_+, R)$, such that

$$|(Mx)(t)| \leq m(t), \quad a.e. \quad \text{on} \quad R_+, \tag{4}$$

for every $x \in C(R_+, R^n)$.

Remark 1 Condition (H_3) is certainly very restrictive, and we shall use it in obtaining an estimate for the solution of (1), (2). It is particularly useful with regard to the boundedness of solutions. As we know, there are nonlinear maps/operators, such as

$$(M_1 x)(t) = \exp\{-|x(t)|\}, \quad x \in C(R_+, R),$$

or

$$(M_2 x)(t) = \tan^{-1} x(t), \quad x \in C(R_+, R),$$

which can be easily used to get operators satisfying (4). For instance, in the case $n = 1$, one can take $(Mx)(t) = m(t)(M_1 x)(t)$, with $m \in L^2(R_+, R)$, i.e.,

$$\int_0^\infty m^2(t) \, dt < \infty. \tag{5}$$

Let us now consider equation (1), under hypotheses $(H_1) - (H_3)$ and let $x(t)$ be a local solution of (1), (2). The existence of such a solution is guaranteed by our hypotheses (see [3], Chapter 3). On the existence interval, we multiply scalarly (in R^n) both sides of (1) by $x(t)$:

$$\langle x(t), \dot{x}(t)\rangle \; + \; \langle (Lx)(t), x(t)\rangle \; = \; \langle (Mx)(t), x(t)\rangle \, . \tag{6}$$

Integrating (6) from 0 to t, we obtain

$$\int_0^t \langle x(s), \dot{x}(s)\rangle \, ds + \int_0^t \langle (Lx)(s), x(s)\rangle \, ds = \int_0^t \langle (Mx)(s), x(s)\rangle \, ds. \tag{7}$$

Taking into account (3) and (4), and noticing that for each $\epsilon > 0$,

$$\left| \int_0^t \langle (Mx)(s), x(s) \rangle \, ds \right| \le \frac{1}{2\epsilon} \int_0^t m^2(s) \; ds + \frac{\epsilon}{2} \int_0^t |x(s)|^2 \; ds. \tag{8}$$

We derive from (7) the inequality:

$$\frac{1}{2}(|x(t)|^2 - |x_0|^2) + \lambda(t) \int_0^t |x(s)|^2 \, ds \le \frac{1}{2\epsilon} \int_0^t m^2(s) \; ds + \frac{\epsilon}{2} \int_0^t |x(s)|^2 \, ds. \tag{9}$$

From (9) we easily derive the Gronwall's type inequality

$$|x(t)|^2 + [2\lambda(t) - \epsilon] \int_0^t |x(s)|^2 \, ds \le |x_0|^2 + \frac{1}{\epsilon} \int_0^t m^2(s) \, ds. \tag{10}$$

The inequality (10) provides, in the usual way, an estimate for $x(t)$ on the interval of existence. In particular, under our assumptions, we have the fact that each solution of (1), (2) remains bounded on the interval of (local) existence, which implies that all solutions of (1), (2) are defined on the whole positive half-axis R_+.

The conclusion of the above carried discussion can be formulated as follows:

Theorem 1 *Consider the initial value problem (1), (2), under hypotheses (H_1) – (H_3), and for arbitrary $x_0 \in R^n$. Then, there exists a solution $x(t)$ of this problem, defined on the whole positive half-axis R_+.*

Remark 2 If one looks at the inequality (10), it is easily seen that the conclusion of Theorem 1 remains valid if (H_3) is substituted by the weaker condition

$$m \in L^2_{loc}(R_+, R). \tag{11}$$

Indeed, in this case $\int_0^t m^2(s) \, ds$ is bounded for any $t \in R_+$, which means the right hand side in (10) remains bounded on any finite interval of R_+.

2 Dissipativity Conditions for System (1)

We shall now try to exploit further the inequality (10), under our hypotheses. Let us assume that a strengthened version of the inequality (3), in the hypothesis (H_2), is valid. Namely,

(H_4) There exists $\lambda_0 > 0$, such that

$$\int_0^t \langle (Lx)(s), x(s) \rangle \, ds \ \ge \lambda_0 \int_0^t |x(s)|^2 ds, \tag{12}$$

for any $t \in R_+$, and $x \in C(R_+, R^n)$.

Returning now to the inequality (10), we notice it becomes

$$|x(t)|^2 + (2\lambda_0 - \epsilon)\int_0^t |x(s)|^2\,ds \le |x_0|^2 + \frac{1}{\epsilon}\int_0^\infty m^2(s)\,ds, \tag{13}$$

taking into account (H_3). This is true as far as $x(t)$ is defined.

Let us now consider the problem (1), (2), under hypotheses (H_1), (H_3), and (H_4). Since $\epsilon > 0$ is arbitrary in (13), we will choose it small enough, such that

$$2\lambda_0 - \epsilon > 0. \tag{14}$$

The right hand side in (13) is a constant, which means that the left hand side of (13) must be bounded on R_+. This means

$$|x(t)| \le C_1, \quad t \in R_+, \tag{15}$$

and

$$\int_0^t |x(s)|^2\,ds \le C_2, \quad t \in R_+,$$

which actually means $x \in L^\infty(R_+, R^n) \bigcap L^2(R_+, R^n)$, or

$$\int_0^\infty |x(s)|^2 ds \le C_2. \tag{16}$$

From (15) and (16) we shall derive

$$\lim |x(t)| = 0, \quad \text{as} \quad t \to \infty, \tag{17}$$

which proves the dissipativity of the system (1), under our hypotheses.

We need, yet, one more condition in order to derive (17). We shall formulate this condition as hypothesis:

(H_5) The space $L^\infty(R_+, R^n)$ is invariant for the operator L in (1).

Under hypotheses (H_1), (H_3), (H_4), and (H_5), the equation (1) shows that $\dot{x}(t)$ can be represented as a sum of two terms, one in $L^\infty(R_+, R^n)$ and the second in $L^2(R_+, R^n)$. Say, $\dot{x}(t) = u(t) + v(t)$, with $u \in L^\infty$ and $v \in L^2$. Since for any $t,\, s \in R_+$ we have

$$|x(t) - x(s)| = \left|\int_s^t \dot{x}(\tau)\,d\tau\right| \le \left|\int_s^t u(\tau)\,d\tau\right| + \left|\int_s^t v(\tau)\,d\tau\right|, \tag{18}$$

we can write

$$\left|\int_s^t u(\tau)\,d\tau\right| \le k\,|t - s|, \tag{19}$$

for $k = ess - \sup |u(t)|,\ t \in R_+$, and

$$\left| \int\limits_s^t v(\tau) d\tau \right| \le |t-s|^{\frac{1}{2}} \left(\int\limits_0^\infty |v(\tau)|^2 d\tau \right)^{\frac{1}{2}}. \tag{20}$$

The inequalities (18) – (20) lead to

$$|x(t) - x(s)| \le k\,|t-s| + \left(\int\limits_0^\infty |v(\tau)|^2 d\tau \right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}, \tag{21}$$

which proves the uniform continuity of $x(t)$ on R_+. Moreover, since $x(t) \in L^\infty(R_+, R^n)$, we see that $|x(t)|^2$ is also uniformly continuous on R_+.

Now using a fact which is known as Barbalat's lemma (see, for instance, [4]), the property (17) is proven.

Summing up the above discussion, we can formulate

Theorem 2 *If hypotheses* (H_1), (H_3), (H_4), *and* (H_5) *are satisfied, then any solution of the problem (1), (2) tends to zero at infinity.*

Remark 3 We cannot term the result in Theorem 2 as a stability result of the zero solution, because (1) may not admit the zero solution. It is rather a result of dissipativity of the system (1).

With further assumptions, we can estimate $|x(t) - y(t)|$ in terms of $|x_0 - y_0|$, and other data of the system (1). We shall not pursue this problem here.

3 A Second Order Functional Equation

Results similar to those given in Theorems 1 and 2, but for second order functional differential equations, can be found in the references [1] and [6]. We shall now consider a second order equation, namely

$$\ddot{x}(t) + (L\dot{x})(t) = (Vx)(t), \tag{22}$$

where L stands for a linear causal operator, continuous on the space $C(R_+, R^n)$, and V is also causal on $C(R_+, R^n)$, generally nonlinear.

The initial conditions for (22) will be the classical Cauchy conditions, namely

$$x(0) = x^0, \quad \dot{x}(0) = v^0, \tag{23}$$

with $x^0,\ v^0 \in R^n$.

Unlike the procedure in [4], at this time, we shall transform the problem (22), (23) into an integral functional equation.

First, we notice that (22), under the second initial condition (23), is equivalent to the first order equation

$$\dot{x}(t) = X(t,0)\,v^0 + \int\limits_0^t X(t,s)\,(Vx)(s)\,ds, \tag{24}$$

where $X(t,s)$ is the Cauchy matrix associated with the linear operator L. This is obtained by using the variation of parameters formula for equations of the form $\dot{y}(t) = (Ly)(t) + f(t)$, see [2], or [3].

Integrating both sides of (24) from 0 to t, $t \in R_+$, and considering the first condition (23), we obtain the functional integral equation

$$x(t) = x^0 + \int_0^t X(s,0)\, v^0\, ds + \int_0^t\int_0^s X(s,u)\,(Vx)(u)\, du\, ds. \tag{25}$$

The equation (25) can be processed in a similar manner to that used in [5]. In order to obtain local or global existence, it suffices to impose a growth condition on the operator V. In [5], the following condition has been used:

$$\int_0^t \left(\int_0^s |(Vx)(u)|\, du \right)^2 ds \le \lambda(t) \int_0^t |x(s)|^2\, ds + \mu(t), \tag{26}$$

with λ and μ non-decreasing on R_+. Condition (26) is also sufficient for assuring the local existence of solutions to (22), under conditions (23). We shall not pursue this direction here. Instead, we will consider the problem of obtaining an upper estimate for the norms of the solution to (22), (23).

In order to obtain this upper estimate, we need the following assumptions on the data:

(i) There exists $M > 0$, such that

$$\int_0^t |X(s,0)| ds \le M, \qquad \int_0^t |X(t,s)| ds \le M, \tag{27}$$

for $0 \le s \le t < \infty$.

(ii)
$$|(Vx)(t)| \le \lambda(t) \sup_{0 \le s \le t} |x(s)|, \quad t \in R_+, \tag{28}$$

where $\lambda(t)$ is a non-negative non-decreasing function on R_+.

Now from (25) we derive

$$|x(t)| \le |x^0| + \int_0^t |X(s,0)|\, |v^0| ds + \int_0^t \int_0^s |X(s,u)|\, |(Vx)(u)|\, du\, ds, \tag{29}$$

on the interval of existence for $x(t)$.

The inequality (29) leads to

$$|x(t)| \le (|x^0| + M|v^0|) + M \int_0^t \sup_{0 \le u \le s} |(Vx)(u)|\, ds, \tag{30}$$

and taking into account (28), we obtain

$$|x(t)| \leq (|x^0| + M|v^0|) + M \int_0^t \lambda(s) \sup_{0 \leq u \leq s} |x(u)|\, ds. \tag{31}$$

Let us denote,

$$X(t) = \sup |x(s)|, \quad 0 \leq s \leq t. \tag{32}$$

Since the right hand side of (31) is non-decreasing in t, (31) and (32) imply

$$X(t) \leq (|x^0| + M|v^0|) + M \int_0^t \lambda(s)\, X(s)\, ds, \tag{33}$$

for all $t > 0$ for which $x(t)$ is defined.

The inequality (33) is a Gronwall type integral inequality, and implies

$$X(t) \leq (|x^0| + M|v^0|) \exp\left(M \int_0^t \lambda(s)\, ds \right), \tag{34}$$

which means, on behalf of (32),

$$|x(t)| \leq (|x^0| + M|v^0|) \exp\left(M \int_0^t \lambda(s) ds \right).$$

The estimate (35) is actually valid on R_+, because it also implies the possibility of continuing a local solution to (25) on the semi-axis R_+.

Let us point out the fact that the inequality (29) for $|x(t)|$ could be also exploited in obtaining other estimates for $|x(t)|$.

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On H_∞ Control Design for Singular Continuous-Time Delay Systems with Parametric Uncertainties

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Abstract: In this paper we study the problem of H_∞ control of singular linear continuous-time systems with parametric uncertainty. The system under consideration is subjected to time-delay in state, and norm-bounded parametric uncertainty entering all matrices of the system and output equations. First, the problem of robust stabilization of the underlying system is investigated. Next, we address the problem of robust H_∞ state feedback control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainty and time-delay. It is shown that the above control problem can be solved in terms of solutions of some linear matrix inequalities.

Keywords: *Singular continuous-time systems; parameter uncertainty; time-delay; linear matrix inequality (LMI).*

Mathematics Subject Classification (2000): 37N35, 55N10.

1 Introduction

Time delay is commonly encountered in various engineering systems, which often occurs in the transmission of information or material between different parts of a system and is frequently a source of instability and poor performance (Malek-Zavarei and Jamshidi [15]. Transportation systems, communications systems, chemical process, power systems are typical examples of time-delay systems. During the past years, the study of time-delay systems has received considerable interest, see, e.g., Suh and Bien [30]. In the work of Gutman and Palmor [8], nonlinear state feedback controllers have been considered whereas Basher, *et al.* [9] has focused on memoryless linear state feedback. Recently, memoryless stabilization and H_∞ control of uncertain continuous-time delay systems have been extensively investigated. For some representative prior work on this general topic, we refer the reader to Shen, *et al.* [21], Lee, *et al.* [12], Mahmoud and Al-Muthairi [14], Nguang [18], Benjelloun, *et al.* [1], Kim, *et al.* [10], Moheimani and Petersen [16], Li and de Souza [13] and the very recent book by Boukas and Liu [2]. The problem of robust stabilization for a class of time varying delay systems with both Lipschitz and non-Lipschitz bounded uncertainties has been studied by Nguang [18] via Riccati equation approach, and a memoryless state feedback controller is designed. In the research conducted by Mahmoud and Al-Muthairi [14], quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties has been considered. More recently, optimal quadratic guaranteed cost control for a class of uncertain linear time-delay systems with norm-bounded uncertainty has been designed by Moheimani and Petersen [16]. The issue of delay-dependent robust stability and stabilization of uncertain linear delay systems has been tackled by Li and de Souza [13] via a linear matrix inequality approach. However, to the best of authors' knowledge, the problems of robust stability and H_∞ control of singular continuous-time delay uncertain systems has not been fully investigated yet. In this paper, the problems of robust stability and control of a class of singular uncertain systems with unknown time delays in both system state and output equations are addressed. We consider uncertain systems with norm-bounded time-varying parameter uncertainty in all system matrices. We deal with the problems of robust stabilization and robust H_∞ control, where in the latter the controller is required to guarantee both the robust stability and a prescribed robust H_∞ performance, irrespective of the uncertainty and unknown time delay.

Notation. The notation in this paper is quite standard. $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix of appropriate dimension. $L_2[0,\infty)$ is the space of square integrable functions over $[0,\infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm.

2 Problem Formulation and Preliminaries

The system considered in this paper is assumed to be a state-space model as follows:

$$
\begin{aligned}
E\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_d + \Delta A_d(t)]x(t - d_1(t)) \\
&\quad + [B + \Delta B(t)]u(t) + [B_w + \Delta B_w(t)]w(t), \\
z(t) &= [C + \Delta C(t)]x(t) + [C_d + \Delta C_d(t)]x(t - d_1(t)) \\
&\quad + [D + \Delta D(t)]u(t) + [D_w + \Delta D_w(t)]w(t), \\
x(t) &= \phi_1(t), \ \forall t \in [-d_1(t), 0],
\end{aligned}
\tag{2.1}
$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control, $w(t) \in \mathbf{R}^p$ is the disturbance from $L_2[0,\infty)$, i.e., square-integrable, $z(t) \in \mathbf{R}^q$ is the controlled output, A, A_d, B, B_w, C, C_d, D and D_w are real-valued constant matrices of appropriate dimensions that describe the nominal system, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_w(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$, and $\Delta D_w(t)$ are real time-varying matrix functions representing parameter uncertainties, and the matrix E is a singular matrix with $\operatorname{rank}(E) = r \le n$. $d_1(t) \ge 0$ is an unknown time-varying time delay in state, $\phi_1(t)$, $t \in [-d_1(t), 0]$, is continuous vector valued initial function. $d_1(t)$ satisfies the following condition:

$$0 \le d_1(t) < \infty, \qquad \dot{d}_1(t) \le \beta_1 < 1. \tag{2.2}$$

The *admissible parameter uncertainties* in this paper is assumed to be modeled as

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_w(t) \\ \Delta C(t) & \Delta C_d(t) & \Delta D(t) & \Delta D_w(t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)[E_1 \;\; E_2 \;\; E_3 \;\; E_4], \tag{2.3}$$

where H_1, H_2, E_1, E_2, E_3, E_4 and E_5 are known real constant matrices, and $F(t)$ is an unknown time-varying matrix function satisfying

$$\|F(t)\| \le 1, \qquad \forall\, t \in [0, \infty). \tag{2.4}$$

Remark 2.1 It should be noted that (2.1) encompasses many state space models of delay systems and can be used to represent many important physical systems; for example, power systems [29], singular space perturbation theory [31], circuits theory [17], and also cold rolling mills, wind tunnel and water resources systems (see, e.g., [15] and the references therein).

Remark 2.2 The parameter uncertainty structure as in (2.3) and (2.4) is an extension of the so-called "matching condition" which has been widely used in the problems of robust control and robust filtering of uncertain systems (see, e.g., [3, 19, 22−27, 33] and the references therein), and many practical systems possess parameter uncertainties which can be either exactly modeled, or overbounded by (2.4). The matrices H_1, H_2, E_1, E_2, E_3 and E_4 specify how the uncertain parameters in $F(t)$ affect the nominal matrices of system (2.1). Observe that the unknown matrix $F(t)$ in (2.3) can even be allowed to be state-dependent, i.e., $F(t) = F(t, x(t))$, as long as (2.4) is satisfied. It also should be noted that the unit overbound for $F(t)$ does not cause any loss of generality. Indeed, $F(t)$ can be always normalized, in the sense of (2.4), by appropriately choosing the matrices H_1, H_2, E_1, E_2, E_3 and E_4. Furthermore, we may consider the more general structure of the uncertainties in system (2.1), that is,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + A_d x(t - d_1(t)) + B_w w(t) + \Delta_1(t, x, u), \\ z(t) &= Cx(t) + Du(t) + C_d x(t - d_1(t)) + D_w w(t) + \Delta_2(t, x, u), \end{aligned}$$

where

$$\begin{gathered} \|\Delta_i(t, x, u)\| \le a_i\|x\| + b_i\|u\|, \\ i = 1, 2, \quad \forall\, t \in [0, \infty), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \end{gathered} \tag{2.5}$$

where $a_i \ge 0$ and $b_i \ge 0$, $i = 1, 2$, are known constant numbers. In the work of Shi and Shue [28], it has been shown that the set of the uncertainties satisfying (2.3) and (2.4) is equivalent to the set of the uncertainties satisfying (2.5) after appropriately choosing the constants a_i, b_i and the matrices H_1, H_2, E_1, E_2, E_3 and E_4.

Definition 2.1 For any given two matrices $E \in \mathbf{R}^{n \times n}$ and $A \in \mathbf{R}^{n \times n}$, the *pencil* (E, A) is said *regular* if there exists a constant number α such that $|\alpha E + A| \ne 0$ or the polynomial $|sE - A| \ne 0$.

In this paper, we assume that the nominal system (2.1) is regular, i.e., the pair $(E, A + A_d e^{-sd_1})$ is regular, where $d_1 = \max_t d_1(t)$. This condition will guarantee the existence and uniqueness of the solution for the nominal system (2.1). In addition, we assume that the nominal system (2.1) is impulse free, which ensures the delay system has no infinite poles. Throughout this paper, it is also assumed that the state is measurable for feedback. In this paper, we are concerned with the problem of robust state feedback control for the singular uncertain time-delay system (2.1) for all admissible uncertainties. Our attention is to design a state feedback controller $\mathcal{G}$:

$$u(t) = Kx(t), \tag{2.6}$$

such that for a given scalar $\gamma > 0$, for all non-zero $w(t) \in L_2[0, \infty)$ and for all parameter uncertainties satisfying (2.3) and (2.4)

$$\sup_{0 \neq w \in L_2[0,\infty)} \left(\frac{\|z\|_2}{\|w\|_2} \right) < \gamma. \tag{2.7}$$

In this situation, the system of (2.1) with the controller (2.6) is said to have a *robust* H_∞ *performance* (2.7). More specifically, our objective is to *design a state feedback controller* $\mathcal{G}$ *such that: the system of (2.1) with* $\mathcal{G}$ *is robustly stable and has a robust* H_∞ *performance (2.7)*. Here, *robustly stable* means that the uncertain system (2.1) is asymptotically stable about the origin for all admissible uncertainties. In the remainder of this section, we will establish stability and H_∞ control results associated with the nominal system of (2.1), i.e., the case when $F(t) = 0$.

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_d x(t - d_1(t)) + Bu(t) + B_w w(t), \\ \quad z(t) = Cx(t) + Du(t) + C_d x(t - d_1(t)) + D_w w(t), \\ \quad x(t) = \phi(t), \quad \forall\, t \in [-d_1(t), 0]. \end{cases} \tag{2.8}$$

First we recall the following lemma.

Lemma 2.1 (Schur Complements) *Given constant matrices* M*,* L *and* Q *of appropriate dimensions with* M *and* Q *are symmetric and* $Q > 0$*, then* $M + L^T Q L < 0$ *if and only if*

$$\begin{bmatrix} M & L^T \\ L & -Q^{-1} \end{bmatrix} < 0 \qquad or \qquad \begin{bmatrix} -Q^{-1} & L \\ L^T & M \end{bmatrix} < 0.$$

Lemma 2.2 *Let* $T_0, \cdots, T_p \in R^{n\times n}$ *be symmetric matrices. If there exists* $0 \le \tau_i$*,* $1 \le \tau \le p$ *such that*

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0,$$

then we have

$$\xi^\top T_0 \xi > 0 \tag{2.9}$$

holds for all $\xi \neq 0$ *satisfying* $T_0 - \sum_{i=1}^{p} \tau_i T_i > 0$.

Lemma 2.3 *Let H be a symmetric matrix and D, E be matrices with appropriate dimensions. Then, $H + DF(t)E + E^T F^T(t)D^T < 0$ holds for any $F^T(t)F(t) \le I$ if and only if there exists a constant scalar $\varepsilon > 0$ satisfying $H + \varepsilon DD^T + \frac{1}{\varepsilon}E^T E < 0$.*

Lemma 2.4 *Let G, U, V be given matrices with G being symmetric. Then there exists matrix X such that*

$$G + UXV^\top + VX^\top U^\top > 0 \tag{2.10}$$

if and only if

$$U_\perp^\top G U_\perp > 0, \qquad V_\perp^\top G V_\perp > 0 \tag{2.11}$$

hold, where $U_\perp$, $V_\perp$ are orthogonal complements of U and V respectively. $U_\perp^\top G U_\perp > 0$ holds if and only if there exists a scalar σ such that

$$G - \sigma UU^\top > 0.$$

Throughout this paper we will make the following assumption.

Assumption 2.1 There exist a positive scalar ρ such that

$$\|x(t)\|^2 \le \rho \|x(t - d_1(t))\|^2. \tag{2.12}$$

Theorem 2.1 *Consider the singular time-delay system (2.8) with all uncertainties disturbance input $w(t) = 0$. Under Assumption 2.1, system (2.1) is asymptotically stable for all $d_1(t) \ge 0$ satisfying (2.2), if there exist matrices $P > 0$ and $R_1 > 0$ such that the following inequality holds*

$$\begin{bmatrix} E^T PA + A^T PE + R_1 - \rho\tau I & E^T PA_d \\ A_d^T PE & -\tilde{R}_1 + \tau I \end{bmatrix} < 0, \tag{2.13}$$

where $\tilde{R}_1 = (1 - \beta_1)R_1 > 0$.

Remark 2.3 Before proving Theorem 2.1, we have the following observations:

1) If system (2.8) is stable, then (2.12) is satisfied with $\rho = 1$. If system (2.8) is instable, assumption (2.12) means that increase rate of the system trajectory can not be greater than ρ.
2) With no loss of generality, we can assume E is of form $E = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$, where I is the identity and N is a nilpotent. In this case, $E^T PA + A^\top PE + R_1$ is always infeasible. The purpose of introducing Assumption 2.1 is to overcome this infeasibility.

Proof of Theorem 2.1 Let $\mathbf{x}_t \in C[-d_1, 0]$ be defined by $\mathbf{x}_t(s) = x(t+s)$, $s \in [-d_1, 0]$. Let us consider a Lyapunov functional candidate as

$$V(\mathbf{x}_t) \stackrel{\Delta}{=} x^T(t)E^T PEx(t) + \int_{t-d_1(t)}^{t} x^T(\tau)R_1 x(\tau)\, d\tau. \tag{2.14}$$

The derivative of the Lyapunov functional (2.14) along the trajectory of (2.1) is

$$\begin{aligned}\dot{V}(\mathbf{x}_t) &= \dot{x}^T(t)E^TPEx(t) + x^T(t)E^TPE\dot{x}(t)\\ &\quad + x^T(t)R_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t))\\ &= [Ax(t) + A_dx(t-d_1(t))]^TPEx(t) + x^T(t)E^TP[Ax(t) + A_dx(t-d_1(t))]\\ &\quad + x^T(t)R_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t))\\ &\le \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}^T \begin{bmatrix} E^TPA + A^TPE + R_1 & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}.\end{aligned}$$

Using Lemma 2.2, together with (2.13), implies that

$$\begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}^T \begin{bmatrix} E^TPA + A^TPE + R_1 & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix} < 0,$$

holds for any $x(t) \neq 0$ satisfying (2.12), which concludes the stability of system (2.8).

Remark 2.4 Theorem 2.1 provides a delay independent stability criteria since inequality (2.13) does not include the unknown delay $d_1(t)$. However, we have used the information β_1 on $d_1(t)$, which seems the best we can do. It should be noted that when $E = I$, Theorem 2.1 will reduce to the result in [4]. Furthermore, when $A_d = 0$ and $E = I$, the inequality (2.13) becomes the standard necessary and sufficient condition of stability for the non-singular systems without time-delay.

Next, we conduct the H_∞ analysis for the nominal system (2.1) (setting $F(t) = 0$).

Theorem 2.2 *Consider the singular time-delay system (2.1) with all uncertainties being zero. Under Assumption 2.1, for a given constant $\gamma > 0$, system (2.1) is asymptotically stable and has an H_∞ performance γ for all $d_1(t) \ge 0$ satisfying (2.2), if there exist matrices $P > 0$, $R_1 > 0$ and a scalar $\tau > 0$ such that the following inequality holds*

$$\begin{bmatrix} E^TPA + A^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C & C_d & D_w & -I \end{bmatrix} < 0, \tag{2.15}$$

where $\tilde{R}_1 = (1-\beta_1)R_1 > 0$.

Proof We first show that the stability of the closed-loop system (2.1) under the condition of (2.15). Again, let us define a Lyapunov functional candidate as

$$V(\mathbf{x}_t) \triangleq x^T(t)E^TPEx(t) + \int_{t-d_1(t)}^{t} x^T(\tau)R_1x(\tau)\,d\tau. \tag{2.16}$$

Note that the negativeness of (2.15) implies

$$\begin{bmatrix} E^TPA + A^TPE + R_1 - \rho\tau I & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 + \tau I \end{bmatrix} < 0, \tag{2.17}$$

which combined with Theorem 2.1 implies that the system is internally asymptotically stable, i.e., system (2.8) is asymptotically stable with $w(t) \equiv 0$. Next, we analyze the H_∞ performance of the closed-loop system (2.1). Without loss of generality, we assume the system has a zero initial condition. Taking the derivative of the Lyapunov functional (2.16) along the trajectory of (2.1), we have

$$\begin{aligned}\dot{V}(\mathbf{x}_t) &= \dot{x}^T(t)E^T PEx(t) + x^T(t)E^T PE\dot{x}(t) + x^T(t)R_1x(t) \\ &\quad - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t)) \\ &= [Ax(t) + A_dx(t-d_1(t)) + B_ww(t)]^T PEx(t) \\ &\quad + x^T(t)E^T P[Ax(t) + A_dx(t-d_1(t)) + B_ww(t)] \\ &\quad + x^T(t)R_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t)) \\ &\le [Ax(t) + A_dx(t-d_1(t)) + B_ww(t)]^T PEx(t) \\ &\quad + x^T(t)E^T P[Ax(t) + A_dx(t-d_1(t)) + B_ww(t)] \\ &\quad + x^T(t)R_1x(t) - x^T(t-d_1(t))\tilde{R}_1x(t-d_1(t)) \triangleq \dot{\overline{V}}(\mathbf{x}_t).\end{aligned}$$

Let us define performance function

$$J = \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]\,dt. \tag{2.18}$$

Then for any $0 \not\equiv w(t) \in L_2[0,\infty)$, one has

$$\begin{aligned}J &\le \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(\mathbf{x}_t)]\,dt \\ &\le \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{\overline{V}}(x(t))]\,dt.\end{aligned} \tag{2.19}$$

Substituting $\dot{\overline{V}}(x(t))$ into (2.19), we obtain

$$J \le \int_0^\infty \xi^T(t)Z\xi(t)\,dt,$$

where

$$\xi(t) = [x^T(t) \;\; x^T(t-d_1(t)) \;\; w^T(t)]^T$$

$$Z = \begin{bmatrix} H & E^T PA_d + C^T C_d & E^T PB_w + C^T D_w \\ A_d^T PE + C_d^T C & C_d^T C_d - (1-\beta_1)R_1 & C_d^T D_w \\ B_w^T PE + D_w^T C & D_w^T C_d & -\gamma^2 I + D_w^T D_w \end{bmatrix},$$

where

$$H = E^T PA + A^T PE + C^T C + R_1.$$

Therefore, using Lemma 2.1, (2.15) implies $J < 0$, that is, $\|z(t)\|_2 < \gamma \|w(t)\|_2$. Therefore, system (2.8) is internally asymptotically stable and has an H_∞ disturbance attenuation γ. The proof ends.

3 Robust Controller Design

Substituting (2.6) into (2.1) yields the dynamics of the closed-loop system as follows:

$$\begin{cases} E\dot{x}(t) = A_c(t)x(t) + A_d(t)x(t-d_1(t)) + [B_w + \Delta B_w(t)]w(t), \\ \quad z(t) = C_c(t)x(t) + [C_d + \Delta C_d(t)]x(t-d_1(t)) + [D_w + \Delta D_w(t)]w(t), \end{cases} \tag{3.1}$$

where $A_c(t) = A_c + H_1F(t)E_c$, $C_c(t) = C_c + H_2F(t)E_c$ with $A_c = A + BK$, $C_c = C + DK$, $E_c = E_1 + E_3K$. By the same arguments as in the proof of Theorem 2.2, we have the following result.

Proposition 3.1 *Consider the singular time-delay system (3.1) with all uncertainties being zero. Under Assumption 2.1, for a given constant $\gamma > 0$, system (3.1) is asymptotically stable and has an H_∞ performance γ for all $d_1(t) \geq 0$ satisfying (2.2), if there exist matrices $P > 0$, $R_1 > 0$ and a scalar $\tau > 0$ such that the following inequality holds*

$$\begin{bmatrix} E^TPA_c(t) + A_c^T(t)PE + R_1 - \rho\tau I & E^TPA_d(t) & E^TPB_w(t) & C_c^T(t) \\ A_d^T(t)PE & -\tilde{R}_1 + \tau I & 0 & C_d^T(t) \\ B_w^T(t)PE & 0 & -\gamma^2 I & D_w^T(t) \\ C_c(t) & C_d(t) & D_w(t) & -I \end{bmatrix} < 0. \tag{3.2}$$

Noting that the left hand side of (3.2) can be rewritten as

$$\begin{bmatrix} E^TPA_c + A_c^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C_c^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C_c & C_d & D_w & -I \end{bmatrix}$$

$$+ \begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix} F(t)\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix}$$

$$+ \left[\begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix} F(t)\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix}\right]^T.$$

Using Lemma 2.3, we conclude that (3.2) holds if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} E^TPA_c + A_c^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C_c^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C_c & C_d & D_w & -I \end{bmatrix}$$
$$+\varepsilon \begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix} \begin{pmatrix} H_1^TPE & 0 & 0 & H_2^T \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} E_c^T \\ E_2^T \\ E_4^T \\ 0 \end{pmatrix}\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix} < 0. \tag{3.3}$$

We therefore get the following proposition.

Proposition 3.2 *For a given matrix K, if μ_0 is the solution of the following optimization problem*

$$\min_{K,\,P>0} \mu, \tag{3.4}$$

$$s.t.\ (3.3) \text{ with } \gamma^2 \text{ replaced by } \mu, \tag{3.5}$$

then controller (2.6) robustly stabilizes system (2.1) and the closed-loop system has noise attenuation level $\sqrt{\mu_0}$.

Since (3.2) in nonlinear with respect to design parameters K, P, it can not be used to design a controller directly. To solve (3.3) using LMI toolbox, we will use an iterative algorithm. For this purpose, let's give the following equivalent forms of (3.3). Using Schur complement, (3.3) hold if and only if

$$\Phi_1 \triangleq \begin{bmatrix} J_1 & E^TPA_d + \varepsilon E_c^T E_2 & E^TPB_w + \varepsilon E_c^T E_4 & C_c^T & E^TPH_1 \\ A_d^TPE + \varepsilon E_2^T E_c & -\tilde{R}_1 + \tau I + \varepsilon E_2^T E_2 & \varepsilon E_2^T E_4 & C_d^T & 0 \\ B_w^TPE + \varepsilon E_4^T E_c & \varepsilon E_4^T E_2 & -\mu I + \varepsilon E_4^T E_4 & D_w^T & 0 \\ C_c & C_d & D_w & -I & H_2 \\ H_1^TPE & 0 & 0 & H_2^T & -\varepsilon I \end{bmatrix} < 0, \tag{3.6}$$

where $J_1 = E^TPA_c + A_c^TPE + R_1 - \rho\tau I + \varepsilon E_c^T E_c$, ε and τ are positive scalars or

$$\Phi_2 = \begin{bmatrix} J_2 & E^TPA_d & E^TPB_w & C_c^T + \eta E^TPH_1H_2^T & E_c^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C_c + \eta H_2 H_1^T PE & C_d & D_w & -I + \eta H_2 H_2^T & 0 \\ E_c & E_2 & E_4 & 0 & -\eta I \end{bmatrix} < 0, \tag{3.7}$$

where $J_2 = E^TPA_c + A_c^TPE + R_1 - \rho\tau I + \eta E^TPH_1H_1^TPE$, and η is a positive scalar. Since Φ_2 can be rewritten as

$$\Phi_2 = \begin{bmatrix} \tilde{J}_2 & E^TPA_d & E^TPB_w & C_c^T + \eta E^TPH_1H_2^T & E^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C + \eta H_2 H_1^T PE & C_d & D_w & -I + \eta H_2 H_2^T & 0 \\ E_1 & E_2 & E_4 & 0 & -\eta I \end{bmatrix}$$

$$+ \begin{pmatrix} E^TPB \\ 0 \\ 0 \\ B \\ E_3 \end{pmatrix} K \begin{pmatrix} I & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} K^T \begin{pmatrix} B^TPE & 0 & 0 & B^T & E_3^T \end{pmatrix},$$

where $\tilde{J}_2 = E^TPA + A^TPE + R_1 - \rho\tau I + \eta E^TPH_1H_1^TPE$, it follows from Lemma 2.4 that (3.2) is equivalent to

$$\begin{bmatrix} J_3 & E^TPA_d & E^TPB_w & C_c^T + \eta E^TPH_1H_2^T & E^T - \varepsilon_1 E^TPBE_3^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C + \eta H_2 H_1^T PE & C_d & D_w & -I + \eta H_2 H_2^T & 0 \\ E_1 - \varepsilon_1 E_3 B^T PE & E_2 & E_4 & 0 & -\eta I - \varepsilon_1 E_3 E_3^T \end{bmatrix} < 0, \tag{3.8}$$

where $J_3 = E^T PA + A^T PE + R_1 - \rho\tau I + \eta E^T PH_1 H_1^T PE - \varepsilon_1 E^T PBB^T PE$, and ε_1 is a positive scalar. Using Proposition 3.1 and noting that (3.2) is equivalent to (3.7), we obtain the following theorem.

Theorem 3.1 *If there exist matrix $P > 0$ positive scalars η, ε_1, τ, μ satisfying (3.8), then there exists gain matrix K such that controller (2.6) internally stabilizes system (2.1) and guarantees that the closed-loop system verifies noise attenuation level $\sqrt{\mu}$.*

This theorem shows that (3.8) provides a LMI for the existence of linear memoryless state feedback controller (2.6) that internally stabilizes system (2.1) and guarantees the closed-loop system verifies noise attenuation level $\sqrt{\mu}$. However, since the present of E the conventional method to solve LMI can not be directly used here. The following algorithm establishes an iterative algorithm to handle the controller design problem.

Algorithm 3.1 (Robust Controller Design Algorithm)

Step 1 Set an error bound $\varrho_0 > 0$ and give an initial $P_0 > 0$.

Step 2 With P given, solve the following optimization problem K and denote the optimal v by v_1,

$$\min_{\mu>0,\,\eta>0,\,\tau>0,\,K} v, \qquad s.t. \quad \Phi_2 < vI;$$

Step 3 With K obtained in Step 2, solve the following optimization problem to get P and denote the optimal performance by v_2

$$\min_{\varepsilon>0,\,\tau>0,\,P>0} v, \qquad s.t. \quad \Phi_1 < vI.$$

If $\|v_1 - v_2\| < \varrho_0$ and $v_1 < 0$, $v_2 < 0$, stop, else go to Step 2.

4 Illustrative Example

To illustrate the validness of the algorithm developed in previous section, this section gives a numerical example. Let us consider a system described by (2.1) with the following system parameters:

$$R_1 = I, \qquad \rho = 2, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 2 & -0.1 & 0.1 & 0 \\ 0 & 0.1 & 1 & 0.1 \\ 0.1 & 0 & -1 & 0.1 \\ 0.2 & 0.1 & -0.1 & -1 \end{pmatrix}, \qquad A_d = \begin{pmatrix} 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & -0.1 & 0 \\ -0.1 & 0 & 0 & 0.1 \\ 0 & 0 & -0.1 & -0.1 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \\ -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \qquad B_w = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_w = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0 & -0.1 & 0 & 0.1 \end{pmatrix}^T, \qquad H_2 = -0.1,$$

$$E_1 = \begin{pmatrix} 0.1 & 0.1 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0.1 & 0 & 0 & 0.1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0.1 & -0.1 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0.1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}, \qquad C_d = \begin{pmatrix} 0.4 & 0 & -0.1 & 0 \end{pmatrix},$$

$$D = (\,0 \quad -0.1\,), \qquad D_w = (\,0.4 \quad -0.1\,),$$

$$\beta = 0.2, \qquad \eta = 1, \qquad \tau = 0.1, \qquad \mu = 2.$$

With this set of data and choosing initial $P = 0.4 * I$, $\varepsilon_0 = 0.01$, using Algorithm 3.1 yields $K = \begin{pmatrix} 15.9056 & 3.1213 & 9.7273 & 4.2990 \\ 12.0251 & 2.5436 & 9.5698 & -0.0953 \end{pmatrix}$, then the corresponding controller (2.6) stabilizes system (2.1) with a guaranteed disturbance attanuation $\sqrt{\mu}$.

5 Conclusion

This paper dealt with the class of singular continuous-time systems with delay. Under the norm bounded uncertainties, the problems of asymptotic stability, stabilizability, H_∞ control and their robustness have been studied. Delay independent sufficient conditions provided to solve all the problems. *These conditions are in some sense restrictive.* Presently we are working on the more general delay-dependent conditions for the above problems.

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Effects of Substantial Mass Loss on the Attitude Motions of a Rocket-Type Variable Mass System

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Abstract: This study uses a relatively complex model to analyze the influence of various system parameters on the attitude behavior of a rocket-type variable mass system moving in a torque-free environment. Some of the parameters studied include the system's size, the nozzle expansion ratio, and the location of the propellant within the system's casing. Results obtained indicate that the spin rate as well as the transverse rate can increase, decrease, or stay constant depending on the choice of system parameters. Dramatic changes in the character of these variables can result from relatively minor changes in a rocket's nozzle expansion ratio.

Keywords: *Variable mass processes; rockets.*

Mathematics Subject Classification (2000): 70P05, 70M20, 34C60.

1 Introduction

The behavior of mechanical systems with varying mass is of interest to scientists and engineers because of the vast array of physical systems (both natural and engineered) that exhibit variable mass characteristics. Aerospace systems have high visibility as variable mass systems, and are the main focus of this study.

One of the earliest studies of the dynamics of variable mass systems was performed by Buquoy [1]. He developed his "motion formula" for these systems, and presented solutions to a large number of examples in this area. Another early and major contributor to the field is Meshcherskii [8, 9], who essentially laid the foundation for the theoretical study of variable mass systems in mechanics. The focus of practically all the early work in this area was on the translational motion of such systems. This paper investigates the rotational dynamics of rocket-type variable mass systems in a torque-free environment.

Attitude dynamics studies of rotating bodies usually involve the derivation of the equations of motion of the system of interest, followed by some attempt at extracting useful motion information from these equations. Strategies for the development of equations of motion of mechanical systems, especially those with a solid base, have been presented by a number of authors (see, for example, Kovalev [5], Eke and Wang [3]). For variable mass systems in general, and rocket systems in particular, equation derivation can proceed along one of at least two different paths — the discrete model approach, and the control volume method. When used in the study of rocket-type systems, the discrete model approach introduces simplifying assumptions very early in the equation derivation process. For example, it is common in this approach to consider that all the particles leaving the system during a propellant burn, do so at the same velocity relative to the main rocket body, and that this relative velocity is always parallel to the rocket axis. This method was popularized by Thomson in the 1960's [11 − 13], and was effectively used by him and others in the analysis of rocket motion.

The control volume approach starts by viewing the system, in a general way, as consisting of solid and fluid phases contained in a region that is delimited by a closed surface of constant or variable internal volume. Equations of motion for such a general variable mass system are then derived using well-known methods of fluid and classical dynamics. At this stage, the resulting equations are generally very complex, containing several surface and volume integrals. They are then reduced to tractable forms by specializing them to the specific system under study. Thus, unlike the discrete model approach, the control volume method introduces most of its simplifying assumptions at the end of the equation derivation process. Equations of motion derived in this way are now readily available in the literature (see, for example, references [2, 3, 7]).

There are three basic types of physical model that have been used in the study of the dynamics of rocket systems: the variable mass cylinder, the general axisymmetric model, and the two-body axisymmetric system. The variable mass cylinder models a typical rocket system as a simple right circular cylinder. Wang, Eke, and Mao [4, 6] exploited such a simple-minded model to great advantage. Its main merit is its simplicity; yet, surprisingly, it does capture a great deal of the important features of a real rocket. The disadvantage is that it does not permit certain refinements in the study. For example, the model does not include a nozzle, and so, nozzle effects cannot be explored. Nor is it possible to study the effect of the geometric location of the propellant grain within a rocket system, since the model normally views the whole of the cylinder as combustible.

The general axisymmetric model represents a rocket as an axisymmetric body (not just a cylinder) of diminishing mass and inertia [7]. In this case, the manner in which the system's inertia properties vary with mass depletion is not known and cannot be precisely determined. Because of this shortcoming, it is difficult to push analytical studies of the system's motion to their limit; one is thus limited to qualitative inferences in this case. Some authors have tried to circumvent this difficulty by assigning simple, decreasing functions of time to the mass, as well as to the axial and the transverse inertia. The difficulty with such a strategy is that the transverse and axial inertia scalars vary in a dependent manner as the system's mass decreases. It is thus next to impossible to make correct guesses for all the inertia functions as well as the system mass. This problem is discussed in some detail in reference [6].

The two-body axisymmetric model is the most versatile of the three models mentioned above. It separates the system into two parts — a constant mass part, and a variable mass portion. In a rocket system for example, the fuel or propellant would represent the variable mass part, and the other parts of the system outside the fuel would be the

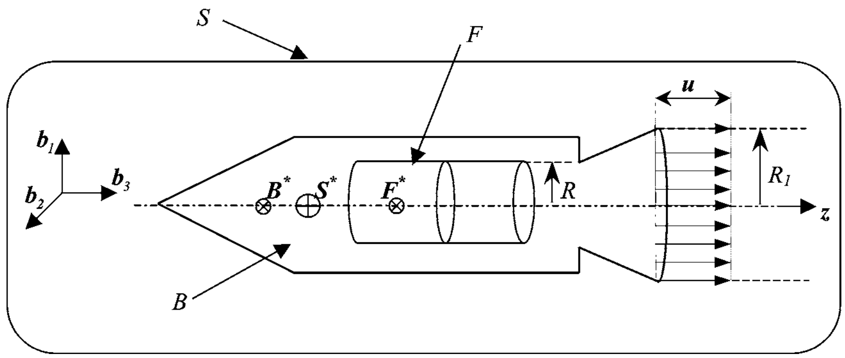

Figure 2.1. Two-body axisymmetric model.

constant mass section. Mass loss comes from the burning and expulsion of particles of the propellant. The model can provide for the existence of a nozzle; there is some flexibility in the geometric location of the fuel within the system; and various propellant depletion geometries can be explored with this model. The goal of this study is to use this two body axisymmetric model to examine how the attitude motion of a rocket system is influenced by substantial mass loss and by changes in various system parameters, and to compare the outcome with results obtained in previous studies that used much simpler models. In particular, we wish to explore the effect of the nozzle on attitude motion — a study that has not as yet been done, and that could not really be done with a less sophisticated model than the one employed here.

2 Equations of Attitude Motion

As stated earlier, the model used for this study is the two-body axisymmetric system as shown in Figure 2.1. B represents the entire system with the exception of the propellant, and it is assumed to constitute a constant mass, axisymmetric rigid body. For the purposes of this study, the fuel F is also assumed to be a rigid body, whose shape at ignition is that of a uniform, right, circular cylinder, with its axis aligned with that of the main body B. F is shown in a partially burned state in the figure; it is assumed to burn in such a way that its unburned part is always axisymmetric with respect to the longitudinal axis z of the original, unburned, cylindrical fuel. The overall system, that is, the combination of B and F is designated S and has its mass center at S^*. B^* and F^* are the mass centers of B and F respectively.

One version of the vector equation of attitude motion for the type of variable mass system under study here is given as equation (1) below. This equation comes from Eke and Wang [3], and is, in its simplified form, equivalent to versions of rotational equations derived by other authors [2, 7, 12]

$$
\begin{aligned}
\mathbf{M} = \mathbf{I}\cdot\alpha + \omega\times\mathbf{I}\cdot\omega + \left(\frac{d\mathbf{I}}{dt}\right)\cdot\omega + \int_C \rho[\mathbf{p}\times(\omega\times\mathbf{p})](\mathbf{v}\cdot\mathbf{n})\,dS \\
+ \int_C \rho\omega\times(\mathbf{p}\times\mathbf{v})\,dV + \frac{d}{dt}\int_C \rho(\mathbf{p}\times\mathbf{v})\,dV + \int_C \rho(\mathbf{p}\times\mathbf{v})(\mathbf{v}\cdot\mathbf{n})\,dS.
\end{aligned}
\tag{1}
$$

In this equation, $\mathbf{M}$ is the sum of the moments of external forces on the system S about the system mass center, S^*; $\mathbf{I}$ is the system's instantaneous central inertia dyadic; $\mathbf{p}$ is the position vector from S^* to a generic particle of the system; ρ is the mass density; $\mathbf{v}$ is the velocity of a generic particle relative to B; C is a fictitious outer shell that encloses the whole system; $\mathbf{n}$ is a unit vector that is normal to, and pointing outwards of the surface C; and ω and α are, respectively, the inertial angular velocity and angular acceleration of the main body B of the system. All the vector and dyadic time derivatives are taken in the reference frame of the rigid body B.

There are at least two arguments that can be used to bring equation (1) down to a manageable form. First, one can exploit the symmetry of the system and assume that at steady state, the motion of gas particles inside the system's combustion chamber is symmetric relative to the z-axis, and that whirling motion (helical motion) of these particles relative to B is negligible. Because of these assumptions, the last three terms on the right hand side of equation (1) vanish (see details in [3]), and the equation reduces to

$$\mathbf{M} = \mathbf{I}\cdot\alpha + \omega\times\mathbf{I}\cdot\omega + \left(\frac{d\mathbf{I}}{dt}\cdot\omega\right) + \int_C \rho\,[\mathbf{p}\times(\omega\times\mathbf{p})](\mathbf{v}\cdot\mathbf{n})\,dS. \tag{2}$$

A second argument that can be used to obtain the same result is based on the fact that there are situations where the velocity $\mathbf{v}$ of the gas particles can be considered negligible within the system's boundary but not at an exit from the boundary, such as the nozzle. An example is an inflated balloon with a small hole. Velocities of gas particles within the balloon are negligible in magnitude compared to those of gas particles leaving through the hole. Whenever $\mathbf{v}$ can be considered negligible within a system's boundary, but is finite and forms a symmetric field at each exit from the system, equation (1) reduces once more to equation (2). It is reasonable to assume that such is approximately the situation for the system under study.

Equation (2) can be broken down into three scalar components by expressing each of the terms on the right hand side of the equation in terms of the unit vectors $\mathbf{b}_i$ $(i = 1, 2, 3)$ shown in Figure 2.1. Assuming that the velocity distribution of the gas particles as they leave the nozzle exit plane is uniform as shown in Figure 2.1, we have that

$$\mathbf{v}\cdot\mathbf{n} = u = \text{const} \tag{3}$$

over the nozzle exit. If we then restrict the study to the case of zero external moment $(\mathbf{M} = \mathbf{0})$, equation (2) takes the scalar forms

$$I\dot{\omega}_1 + \left[\dot{I} - \dot{m}\left(z_e^2 + \frac{R_1^2}{4}\right)\right]\omega_1 + [(J-I)\omega_3]\,\omega_2 = 0, \tag{4}$$

$$I\dot{\omega}_2 + \left[\dot{I} - \dot{m}\left(z_e^2 + \frac{R_1^2}{4}\right)\right]\omega_2 - [(J-I)\omega_3]\,\omega_1 = 0, \tag{5}$$

$$J\dot{\omega}_3 + \left(\dot{J} - \frac{\dot{m}R_1^2}{2}\right)\omega_3 = 0, \tag{6}$$

where m represents the instantaneous mass of the system, I and J are, respectively, the central transverse and spin moment of inertia for the system, z_e is the distance from S^* to the nozzle exit plane, R_1 is the radius of the circular nozzle exit area, and ω_i $(i = 1, 2, 3)$ are the scalar components of the inertial angular velocity of B in the $\mathbf{b}_i$ $(i = 1, 2, 3)$ unit vector basis. We note here that the unit vectors $\mathbf{b}_i$ are assumed fixed to the body B. Details of the transition from equation (2) to (4), (5), and (6) can be found in several places, including Morris [10].

Equations (4) through (6) can be non-dimensionalized as follows. First, we note that the rate, m_r, of mass depletion from the system can be written as a surface integral

$$M_r = -\dot{m} = \int (\mathbf{v}\cdot\mathbf{n})\rho\, dS = \pi\rho u R_1^2 = \text{const}, \tag{7}$$

where ρ is the mass density of the fluid products of combustion — considered constant over the nozzle exit plane. The time, t_b, taken for the mass m_F of the propellant to go from its initial value, m_{F0}, to the final value of zero (that is, burnout) can be expressed as

$$t_b = m_{F0}/m_r. \tag{8}$$

We choose as dimensionless time variable, the quantity τ given by

$$\tau = t/t_b = (m_r/m_{F0})t, \tag{9}$$

where t is time measured from the beginning of the propellant burn (ignition). We then note that $\tau = 0$ at propellant ignition, and $\tau = 1$ at burnout. Furthermore,

$$\frac{d\tau}{dt} = \frac{1}{t_b}, \tag{10}$$

so that the time derivative of any quantity can be written as

$$(\dot{\ \cdot\ }) = \frac{d(\,\cdot\,)}{dt} = \frac{d(\,\cdot\,)}{d\tau}\cdot\frac{d\tau}{dt} = \frac{1}{t_b}\cdot\frac{d(\,\cdot\,)}{d\tau} = \frac{1}{t_b}(\,\cdot\,)', \tag{11}$$

where a prime ($'$) indicates derivative with respect to τ. We define other dimensionless quantities as

$$\overline{m} = m/m_{F0}, \quad \bar{I} = I/m_{F0}R^2, \quad \bar{J} = J/m_{F0}R^2, \quad \text{and} \quad \overline{\omega}_i = \omega_i t_b \quad (i = 1,2,3) \tag{12}$$

and use these to convert equations (4)–(6) to

$$\bar{I}\overline{\omega}_1' + \left\{\bar{I}' - \overline{m}'\left[\left(\frac{z_e}{R}\right)^2 + \frac{\beta^2}{4}\right]\right\}\overline{\omega}_1 + [(\bar{J}-\bar{I})\overline{\omega}_3]\overline{\omega}_2 = 0, \tag{13}$$

$$\bar{I}\overline{\omega}_2' + \left\{\bar{I}' - \overline{m}'\left[\left(\frac{z_e}{R}\right)^2 + \frac{\beta^2}{4}\right]\right\}\overline{\omega}_2 + [(\bar{J}-\bar{I})\overline{\omega}_3]\overline{\omega}_1 = 0, \tag{14}$$

and

$$\bar{J}\overline{\omega}_3' + \left(\bar{J}' - \overline{m}'\frac{\beta^2}{2}\right)\overline{\omega}_3 = 0. \tag{15}$$

In these equations, R is the external radius of the cylindrical propellant, and β is the ratio R_1/R.

From equation (15),

$$\frac{\overline{\omega}_3(\tau)}{\overline{\omega}_3(0)} = \exp\left[-\int_0^{\tau}\frac{\psi(\tau)}{\bar{J}}\,d\tau\right], \tag{16}$$

where

$$\psi(\tau) = \bar{J}' - \overline{m}' \frac{\beta^2}{2}. \tag{17}$$

By defining a dimensionless, complex angular velocity

$$\overline{\omega}_T = \overline{\omega}_1 + i\overline{\omega}_2, \tag{18}$$

where $i = \sqrt{-1}$, equations (13) and (14) are combined to give

$$\frac{\overline{\omega}_T(\tau)}{\overline{\omega}_T(0)} = \left\{ \exp\left[-\int_0^\tau \frac{\varphi(\tau)}{\bar{I}}\, d\tau \right] \right\} \cdot \left[\exp\left(i \int_o^\tau \Theta\, d\tau \right) \right], \tag{19}$$

where

$$\varphi(\tau) = \bar{I}' - \overline{m}' \left[\left(\frac{z_e}{R} \right)^2 + \frac{\beta^2}{4} \right] \tag{20}$$

and

$$\Theta = [(\bar{J}/\bar{I}) - 1]\, \overline{\omega}_3. \tag{21}$$

The function $\varphi(\tau)$ determines the magnitude of the transverse angular velocity vector, $\Theta(\tau)$ governs the frequency, and $\psi(\tau)$ [see equation (17)] tells us whether the spin rate will increase or decrease with τ. We will limit this study to an examination of how the spin rate and transverse angular velocity magnitude vary with propellant burn.

3 Spin Motion

We now take an in depth look at the spin rate, to see how it is affected by mass loss or propellant burn. It is clear from equations (16) and (17) that expressions for the system's mass and inertia as functions of the dimensionless time variable τ are needed in order to make progress with the study of the spin rate. On the other hand, these functions can only be determined if a propellant depletion geometry is stipulated. For this study, we choose to examine the case of a burn that is an idealization of a common burn pattern in solid rocket motors. In this burn, which is often referred to as radial burn, it is imagined that the propellant has the shape of a hollow cylinder at ignition. The interior surface is ignited, and the fuel then burns radially outwards in a uniform manner, so that the interior surface always remains cylindrical. Figure 3.1 shows an intermediate shape for the propellant during such a burn. We now proceed to determine the elements that are needed to express the function ψ in equation (17) in terms of the variable τ.

From Figure 3.1, the mass of the fuel just before ignition is

$$m_{F0} = \rho_F \pi L (R^2 - r_0^2) \tag{22}$$

and the mass m_F at a general instant during the burn is

$$m_F = \rho_F \pi L (R^2 - r^2). \tag{23}$$

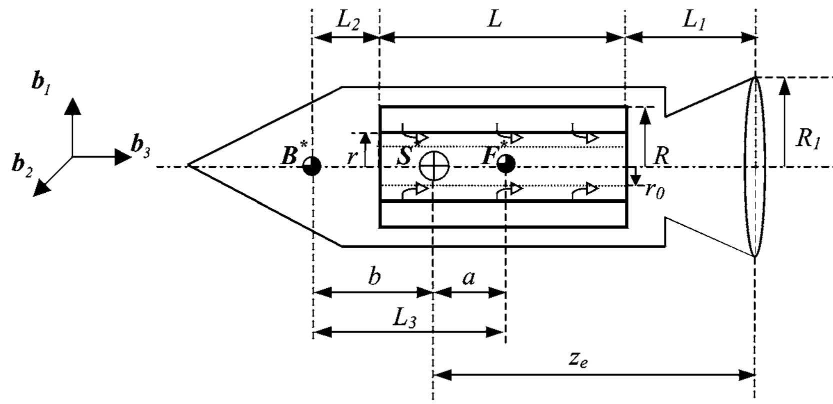

Figure 3.1. Rocket with radially burning propellant.

Here, L is the length of the cylindrical fuel, ρ_F is the mass density of F, r_0 is the initial internal radius of F, and r is the internal radius at some general instant after ignition. From equations (8), (22), and (23),

$$t_b = \frac{m_{F0}}{m_r} = \frac{m_{F0}}{-\dot{m}_F} = \frac{R^2 - r_0^2}{\dfrac{d}{dt}(r^2)}. \tag{24}$$

Equation (24) is integrated to give

$$r^2(t) = r_0^2 + \frac{R^2 - r_0^2}{t_b}\, t \tag{25}$$

so that

$$\left(\frac{r}{R}\right)^2 = \left(\frac{r_0}{R}\right)^2 = \left[1 - \left(\frac{r_0}{R}\right)^2\right]\tau = \gamma^2 + (1-\gamma^2)\tau, \tag{26}$$

where γ is the ratio r_0/R. We thus have from equations (22), (23), and (26), that the non-dimensional mass $\overline{m}_F$ for the propellant is

$$\overline{m}_F = \frac{m_F}{m_{F0}} = \frac{\rho_F \pi L(R^2 - r^2)}{\rho_F \pi L(R^2 - r_0^2)} = \frac{1 - (r/R)^2}{1 - (r_0/R)^2} = 1 - \tau, \tag{27}$$

which yields

$$\overline{m}' = \overline{m}'_F = -1. \tag{28}$$

In a similar way, we use equations (26) and (27) to show that the dimensionless axial inertia of the propellant F is given by

$$\bar{J}_F = \frac{J_F}{m_{F0}R^2} = \frac{\overline{m}_F}{2}\left[1 + \left(\frac{r}{R}\right)^2\right] = \left[\frac{1-\tau}{2}\right][1 + \gamma^2 + (1-\gamma^2)\tau]. \tag{29}$$

The overall system axial moment of inertia is thus given, in dimensionless form, as

$$\bar{J} = \bar{J}_B + \bar{J}_F = \bar{J}_B + \frac{1+\gamma^2}{2} - \gamma^2\tau - \frac{1-\gamma^2}{2}\tau^2. \tag{30}$$

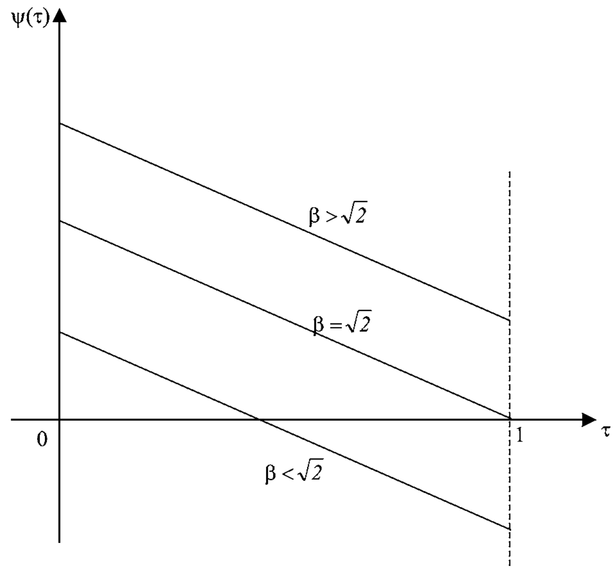

Figure 3.2. Possible shapes for $\psi(\tau)$.

Equations (28), (30), and (17) lead to

$$\psi(\tau) = \left[\frac{\beta^2}{2} - \gamma^2\right] + (\gamma^2 - 1)\tau. \tag{31}$$

3.1 Qualitative discussion

The function $\psi(\tau)$ is linear in τ with slope (γ^2-1), which is negative for the burn we have selected. Hence $\psi(\tau)$ decays linearly with τ, and $\psi(0) = \beta^2/2-\gamma^2$, with $\psi(1) = \beta^2/2-1$. $\psi(0)$ is almost certain to be positive for real rocket systems, since, for these systems one would expect $\gamma \ll 1$ and $\beta \geq 1$. For example, picking such conservative numbers as $\gamma = 0.5$, and $\beta = 1$ still results in $\psi(0) > 0$. On the other hand, $\psi(1)$ can, conceivably, take on a value that is either positive or negative depending on whether or not the quantity β, which we shall refer to in the remainder of this paper as the nozzle expansion ratio, is greater than or less than $\sqrt{2}$. Figure 3.2 summarizes the behavior of the function $\psi(\tau)$ for three values of β. We conclude from this figure that the spin rate will always decrease initially. This trend will continue all the way to burnout if the nozzle expansion ratio is greater than some threshold value, β_L ($\sqrt{2}$ for radial burn). If β happens to be less than β_L, then, the decreasing trend in the spin rate will be reversed at some point during the burn, and the spin rate will increase for the remainder of the burn. The point at which this change in trend occurs is ($\psi = 0$)

$$\tau = (\beta^2/2 - \gamma^2)/(1 - \gamma^2). \tag{32}$$

For a variable mass cylinder with no nozzle, $\beta = 1$, and only one scenario ($\beta < \beta_L$) is possible. As explained above for this case, the spin rate will decrease initially until τ attains a value given by equation (32). After this, the spin rate increases till burnout.

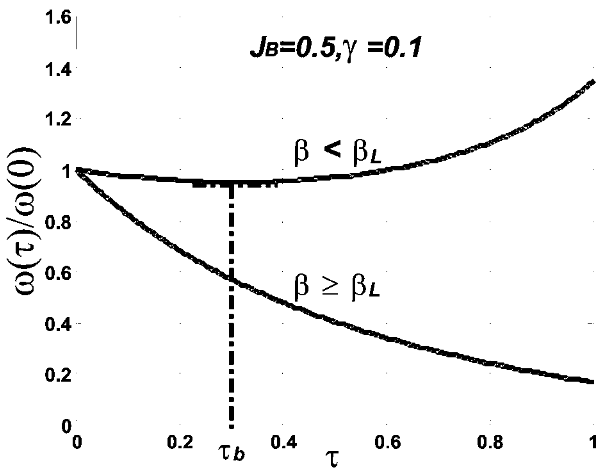

Figure 3.3. Spin rate for radial burn.

By substituting $\beta = 1$ into equation (32), and putting the resulting value for τ into equation (26), we find that the minimum value of the spin rate will occur when the ratio $r/R = 1/\sqrt{2}$. This is the same result obtained in [6] for the simple variable mass cylinder.

3.2 Closed form solution

If equations (30) and (31) are substituted into equation (16), a closed form solution can be obtained for equation (16) as follows:

$$\frac{\overline{\omega}_3(\tau)}{\overline{\omega}_3(0)} = \left[\frac{\Pi^2 - \gamma^4}{\Pi^2 - [\gamma^2 + (1-\gamma^2)\tau]^2}\right] \times \exp\left\{\frac{-\beta^2}{\Pi}\left[\tanh^{-1}\frac{[\gamma^2 + (1-\gamma^2)\tau]}{\Pi} - \tanh^{-1}\frac{\gamma^2}{\Pi}\right]\right\}, \tag{33}$$

where

$$\Pi = \sqrt{2\bar{J}_B(1 - \gamma^2 + 1)}. \tag{34}$$

The curves shown in Figure 3.3 come from equation (33), and they confirm the initial negative slope of the spin rate, and the fact that the spin rate can change from a decreasing to an increasing function of τ during a propellant burn, for small values of the nozzle expansion ratio. It would appear, from equation (33) that besides the parameters γ and β, the axial inertia $\bar{J}_B$ can also play a role in the behavior of the spin rate. Figure 3.4 is obtained from equation (33), and shows how a change in axial inertia $\bar{J}_B$ for the main body B affects the spin rate. The smaller the value of $\bar{J}_B$ the more the spin rate curve deviates from that which is expected from a constant mass rigid body; that is, a constant. This is certainly in agreement with engineering intuition. However, the basic character of the spin rate curve is not affected by a change in $\bar{J}_B$. It turns out that a change in the ratio γ of the initial internal radius to external radius of the fuel without a change in $\bar{J}_B$ has minimal effect on the spin rate.

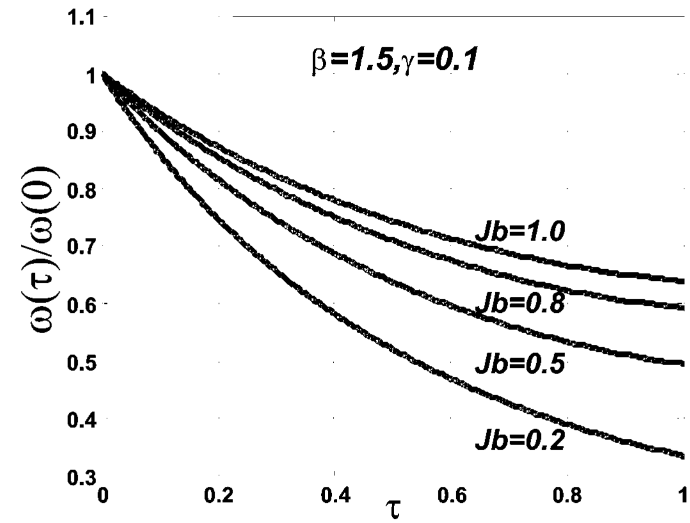

Figure 3.4. Influence of spin interia on the spin rate.

3.3 Stable spin

It is not desirable to have the spin rate grow substantially, nor is it acceptable for the spin rate to decrease excessively during a propellant burn. In one case, the system's structural integrity can become impaired, and in the other case, there is loss of spin rigidity. It is therefore useful to find ways to limit variations in spin rate during a burn. One way to accomplish this is to force the spin rate at the end of the burn to be the same as that at ignition; that is, $\varpi_3(1) = \varpi_3(0)$. From equation (33), we have

$$\frac{\varpi_3(1)}{\varpi_3(0)} = \left[\frac{\Pi^2 - \gamma^4}{\Pi^2 - 1}\right] \cdot \exp\left\{\frac{-\beta^2}{\Pi}\left[\tanh^{-1}\frac{1}{\Pi} - \tanh^{-1}\frac{\gamma^2}{\Pi}\right]\right\} = 1, \tag{35}$$

which leads eventually to

$$\beta_b = \sqrt{\frac{\Pi \cdot \ln[(\Pi^2 - \gamma^4)/(\Pi^2 - 1)]}{\tanh^{-1}(1/\Pi) - \tanh^{-1}(\gamma^2/\Pi)}}. \tag{36}$$

Equation (36) gives the value of the nozzle expansion ratio that will bring the spin rate at burnout back to its value at ignition, and in so doing, limit the overall variation in the spin rate. Figure 3.5 shows that the necessary nozzle expansion ratio is sensitive to the axial inertia $\bar{J}_B$ of the main rocket body, especially at low values of $\bar{J}_B$. Figure 3.6 gives an indication of the level of sensitivity of the spin rate to deviations in the choice of β. This figure also shows that when β is taken to be β_b, the difference between the extreme values of the spin rate is not as large as when $\beta \neq \beta_b$.

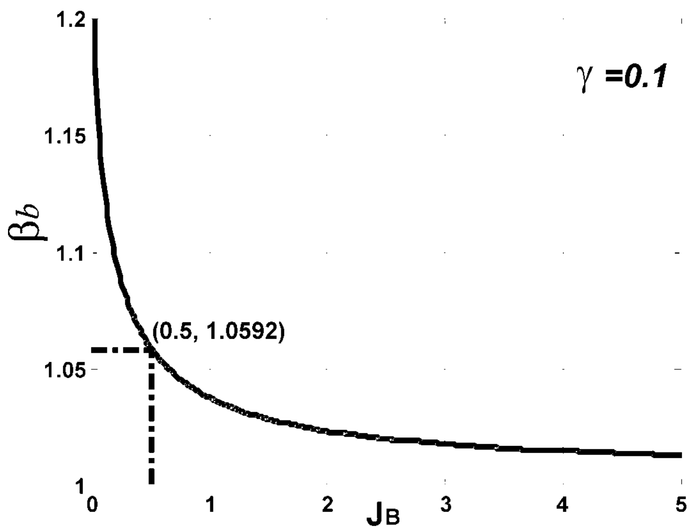

Figure 3.5. Relationship between $\bar{J}_B$ and β_b.

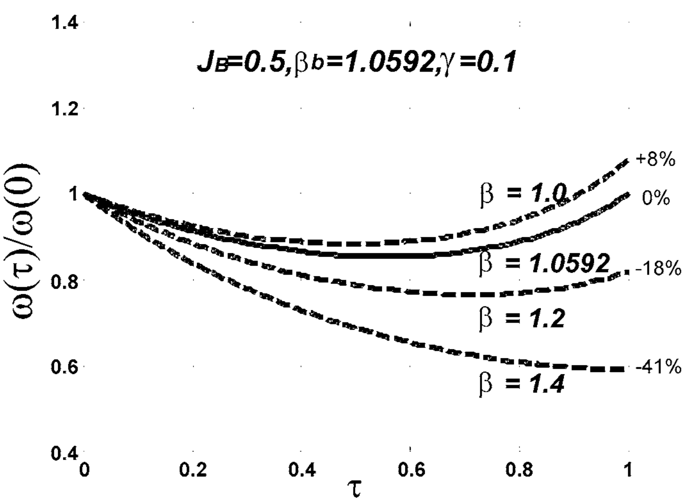

Figure 3.6. Spin rate deviations for $\beta \neq \beta_b$.

4 Transverse Angular Speed

The system's central transverse moment of inertia can be written, in non-dimensional form as

$$\bar{I} = \bar{I}_B + \bar{I}_F + (m_B b^2 = m_F a^2)/m_{F0}R^2, \tag{37}$$

where the subscripts B and F refer to the main body and the fuel respectively, and the distances a and b as well as other distances such as L, L_i $(i = 1, 2, 3)$ are as shown in Figure 3.1. The transverse inertia of B is $\bar{I}_B = I_B/m_{F0}R^2$. Keeping in mind that we are assuming a radial burn for the fuel,

$$\bar{I}_F = \frac{I_F}{m_{F0}R^2} = \overline{m}_F\left[\frac{1}{4}+\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{12}\left(\frac{L}{R}\right)^2\right] = (1-\tau)\left[\frac{1+\gamma^2+(1-\gamma^2)\tau}{4}+\frac{\delta^2}{12}\right]. \tag{38}$$

The distances a and b can be expressed as

$$a = \frac{m_B L_3}{m_B + m_F} \tag{39}$$

and

$$b = \frac{m_F L_3}{m_B + m_F}. \tag{40}$$

Substituting equations (38), (39), and (40) into (37), we obtain, after some algebra,

$$\bar{I} = \bar{I}_B + (1-\tau)\left[\frac{1+\gamma^2+(1-\gamma^2)\tau}{4} + \frac{\delta^2}{12}\right] + \frac{\overline{m}_B(1-\tau)\delta_3^2}{\overline{m}_B + 1 - \tau} \tag{41}$$

so that

$$\bar{I}' = -\left\{\frac{\gamma^2+(1-\gamma^2)\tau}{2} + \frac{\delta^2}{12} + \left(\frac{\overline{m}_B\delta_3}{\overline{m}_B + 1 - \tau}\right)^2\right\}. \tag{42}$$

The distance

$$z_e = L_1 + \frac{L}{2} + a. \tag{43}$$

Hence, equations (43), (39), and (27) give

$$\frac{z_e}{R} = \frac{(\overline{m}_B + \overline{m}_F)(\delta_1 + \delta/2) + \overline{m}_B\delta_3}{\overline{m}_B + \overline{m}_F} = \frac{(\overline{m}_B + 1 - \tau)(\delta_1 + \delta/2) + \overline{m}_B\delta_3}{\overline{m}_B + 1 - \tau}, \tag{44}$$

where $\delta = L/R$, and $\delta_i = L_i/R$ $(i = 1, 2, 3)$. From equations (20), (28), (42), and (44),

$$\varphi(\tau) = -\left[\frac{\gamma^2+(1-\gamma^2)\tau}{2}\right] + \frac{\delta^2}{6} + \delta\delta_1 + \delta_1^2 + \frac{\beta^2}{4} + \frac{2\overline{m}_B\delta_3}{\overline{m}_B + 1 - \tau}\left(\frac{\delta}{2} + \delta_1\right). \tag{45}$$

4.1 Qualitative discussion of transverse motion

The transverse angular speed is given by [see equation (19)]

$$\left|\frac{\overline{\omega}_T(\tau)}{\overline{\omega}_T(0)}\right| = \exp\left[-\int_0^\tau \frac{\varphi(\tau)}{\bar{I}}\, d\tau\right]. \tag{46}$$

The function $\bar{I}$ decreases with τ but is always positive. Hence, the sign of $\varphi(\tau)$ determines whether the transverse rate increases or decreases with the burn. We can rewrite equation (45) as

$$\varphi(\tau) = \varphi_1(\tau) + \varphi_2(\tau) \tag{47}$$

where

$$\varphi_1(\tau) = -\left[\frac{\gamma^2 + (1-\gamma^2)\tau}{2}\right] \tag{48}$$

and

$$\varphi_2(\tau) = \frac{\delta^2}{6} + \delta\delta_1 + \delta_1^2 + \frac{\beta^2}{4} + \frac{2\overline{m}_B\delta_3}{\overline{m}_B + 1 - \tau}\left(\frac{\delta}{2} + \delta_1\right). \tag{49}$$

The function φ_1 is clearly negative, since $\gamma < 1$. On the other hand, φ_2 is positive because δ, δ_1, δ_3, β and $\overline{m}_B$ are all positive quantities for real rocket systems, and $0 \le \tau \le 1$. At $\tau = 0$, $\varphi_1 = -\gamma^2/2$; and, since $\gamma = r_0/R$ would be expected to be less than 1/2 for a real system, $|\varphi_1| \le 1/8$. Similarly, $\varphi_2 > \beta^2/4 > 1/4$ in practice. Hence, $\varphi(0)$ is likely to be positive. As τ varies between 0 and 1, φ_1 decreases linearly with τ while φ_2 increases with τ. It appears unlikely that φ_2 will ever become less than φ_1 in absolute value, or that φ will change sign from positive to negative during the propellant burn. Thus, the transverse angular speed is likely to decrease between ignition and burnout.

There are several factors that can change this state of affair for the transverse angular speed. They include small values of what may be referred to as the propellant aspect ratio $\delta = L/R$, low values of the nozzle expansion ratio β, proximity of the propellant grain to the nozzle (δ_1), and closeness of the propellant center of mass to that of the rocket's main body. These can lower the value of φ_2 to the point where $|\varphi_2| < |\varphi_1|$ and $\varphi < 0$. Of these, the parameters that a rocket designer has most control over are δ and β.

We note here that relatively recent studies [4, 6] that used the cylinder model came to the conclusion that a "fat and short" propellant grain (i.e. low δ) can cause the transverse angular speed to grow without bounds for a radial burn. This paper arrives at the same result, but adds another component that the nozzle expansion ratio can also have an important damping influence on the transverse angular speed. In fact, a large enough expansion ratio can, single-handedly, reverse the potential for a runaway growth in transverse rate. For instance, the maximum value that φ_1 can have during a radial burn is [see equation (48)] $-1/2$, and this occurs at $\tau = 1$. Hence, it suffices that $\beta \ge \sqrt{2}$ [see equation (49)] to guarantee that $\varphi > 0$ and that $\overline{\omega}_T$ will decrease with the burn.

Figure 4.1 is obtained by numerical integration of equation (46), and shows the transverse angular speed as a function of the time variable τ. This figure confirms that a low value of δ, coupled with a small β can indeed cause the transverse rate to reverse its initial decreasing trend sometime during the burn, and increase continuously through burnout.

5 Modified Radial Burn

So far, what has been presented is a general study of the radial burn. We now move briefly to the case where the system mass center, S^*, does not shift relative to the rocket's main body B. In other words, we assume that, prior to ignition, the mass centers B^* of the rocket body and F^* of the solid fuel are coincident, so that S^* is also located at

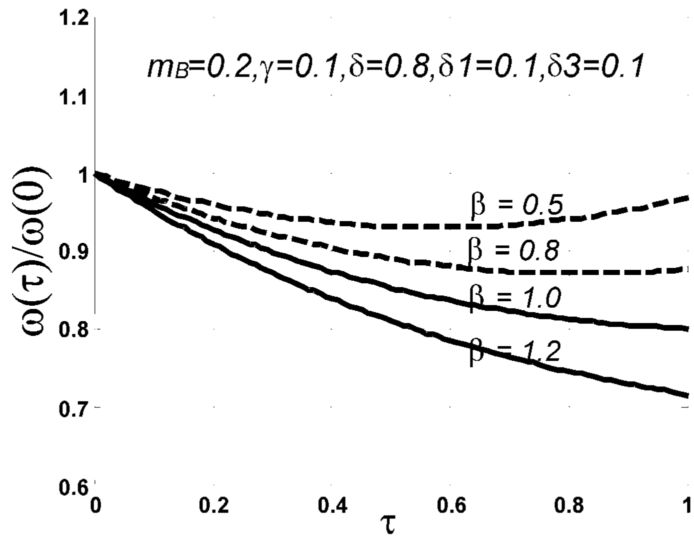

Figure 4.1. Transverse angular speed.

the same point. During a radial burn, F^* does not shift relative to B. Hence, all three points F^*, B^* and S^* remain coincident throughout the burn. This can be accomplished in a real system by balancing the system in such a way that S^* and F^* (or F^* and B^*) coincide prior to propellant ignition. Our interest here is to see whether there is anything to be gained, from the point of view of attitude dynamics, in balancing the system in this way.

As before, the focus here is again on the functions $\varphi(\tau)$ and $\psi(\tau)$ given in equations (45) and (31) respectively. In this case, the parameter δ_3 becomes zero. The function φ_1 is unaffected, but the last term of φ_2 (a positive term) drops. Thus, $\varphi(0)$ is reduced somewhat but remains positive. It is now slightly easier for $\varphi(1)$ to become negative, leading to a possibility of transverse rate increase during a propellant burn. The expression for $\psi(\tau)$ remains as given in equation (31), so the spin rate is unaffected by the proposed change. Overall, we can state that there is no advantage whatsoever in balancing the system in such a way as to avoid system mass center shift. In fact, such an action renders the system more sensitive to divergence in transverse angular speed.

6 Conclusion

This study deals with the dynamic behavior of spinning bodies of the rocket type, that lose mass while moving in a torque-free environment. The attitude behavior of systems of this type is known to be influenced by the manner in which mass loss affects the geometry of the system. One specific mass loss scenario — the radial burn — was studied. This scenario assumes that the propellant of the rocket system is a hollow cylindrical solid whose internal radius grows uniformly as the propellant burns. This appears to restrict the study to rockets with solid propellants. However, the results of this study can in fact be applied to some systems with liquid propellant. When liquid propellant is used in rocket systems that spin, it is generally distributed in several tanks positioned symmetrically with respect to the spin axis. As the solid portion of the system spins, centrifugal effects cause the liquid propellant to move outward and the overall behavior of the fuel system becomes very similar to that of a solid propellant undergoing radial burn.

Results obtained indicate that the spin rate always begins by decreasing with propellant burn. If the ratio, β, of the nozzle exit radius to the external radius of the propellant grain is greater than $\sqrt{2}$, then the spin rate will continue to decrease until propellant burnout. If, however, the value of β is less than $\sqrt{2}$, the spin rate attains a minimum value during the burn, begins to increase as the burn proceeds, and continues this trend through burnout. The value of the nozzle expansion ratio thus plays a pivotal role in determining the character of the spin rate curve.

The transverse angular speed will normally decrease with propellant burn. However, there are circumstances under which growth in transverse angular speed becomes possible. Such a situation can arise if the ratio of the length of the propellant grain to its radius is very small at the same time that the nozzle expansion ratio is also small. In this case, the curve of the transverse rate as a function of propellant burn decreases initially, but flattens out sometime during the burn, and then rises for the remainder of the burn. This study brings out the important role that the nozzle expansion ratio can play in determining how both the spin rate and the transverse angular speed evolve with propellant burn.

Balancing a rocket system so that its mass center does not shift during propellant burn actually renders the system more prone to growth in transverse rate if the nozzle expansion ratio is low. Another way of viewing this is that studies that assume no shift in system mass center will in general produce conservative results.

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Development of Industrial Servo Control System for Elevator-Door Mechanism Actuated by Direct-Drive Induction Machine

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Abstract: In this study, an industrial sliding-mode servo control system is developed for the motion control of a direct-drive-type elevator-door servomechanism. The mechanical structure and dynamic analyses of an elevator-door mechanism with an indirect field-oriented induction servomotor drive is described initially. Moreover, a newly designed total sliding-mode control (TSMC) system, which is insensitive to uncertainties in the whole control process, is introduced. In addition, numerical simulation and experimental results due to specific position and velocity profiles are provided to verify the effectiveness of the proposed control scheme with regard to parameter variations and external disturbance. Furthermore, the merits of the TSMC system are exhibited in comparison with computed torque control (CTC) and conventional sliding-mode control (CSMC). The salient features of this study are 1) the controlled system has a total sliding motion without a reaching phase and no chattering torque, and 2) this simple control strategy is easily implemented by hardware/software to an industrial servo controller.

Keywords: *Sliding-mode control; computed torque control; indirect field-oriented; induction servomotor drive; elevator door.*

Mathematics Subject Classification (2000): 70B15, 68T40, 93C85.

1 Introduction

Most nonlinear mechanism systems comprise driven motors, coupling gears and the nonlinear mechanism. Therefore, complex modeling procedures are usually required to design a suitable control scheme. Besides, there are many uncertainties such as system parameter variations, external disturbance, friction force and unmodelled dynamics to influence the prior-designed control characteristics in industrial applications. Though many modern control techniques have been designed to control the nonlinear mechanism systems by using complex control laws with high control efforts, degraded control performances are often resulted due to the existence of uncertainty [1]. Thus it is natural to explore other nonlinear controls that can circumvent the problem of uncertainties and achieve better compensation and global stability [2] – [4].

Sliding-mode control (SMC) has been demonstrated to be an effective nonlinear robust control approach for controlling electric drive systems since it provides system dynamics with an invariance property to uncertainties once the system dynamics are controlled in the sliding mode [2] – [3]. It offers a fast dynamic response, a stable control system and an easy hardware/software implementation. However, this control strategy produces some drawbacks associated with the large torque chattering that may excite mechanical resonance and unstable dynamics. Besides, the insensitivity of the controlled system to uncertainties only exists in the sliding mode, but not during the reaching phase. Thus the system dynamic in the reaching phase is still influenced by uncertainties. To keep robustness in the whole sliding- mode control system, several researchers have focused on eliminating the effect of the reaching phase [5] – [8]. A newly-designed sliding curve, that is chosen as close as possible to time-optimal trajectory, was proposed in Harashima, *et al.* [5] and Hashimoto, *et al.* [6] to keep robustness from the initial point to final point. Gao and Hung [7] partially shaped the reaching law to specify the system dynamics in the reaching phase. However, the system dynamics are still subjected to uncertainties. Therefore, this study adopts the idea of total sliding-mode control [8] to get a sliding motion through the entire state trajectory. In other words, no reaching phase exists in the control process. Thus the controlled system during the whole control process is insensitive to the occurrence of uncertainties.

In the past several decades, dc motors have been widely used in factory automation as high-performance drives. However, the mechanical commutators and brush assembly make dc motors much more expensive than ac motors. Besides, the use of mechanical commutators may produce undesired sparks that are not allowed in some applications. As compared with dc motor, an induction motor (IM) is robust, cheap and easily maintained. These characteristics make it desirable to employ them in variable-speed or servo system. However, its control characteristics are more complicated than the dc motors. In the scalar control techniques, the transient dipping of flux reduces the torque sensitivity with slip and lengthens the response time. In order to overcome the foregoing limitation, the field-oriented control technique has been widely used in industry for high-performance IM drive [9] – [10]. With the field-oriented control approach, the dynamic behavior of an IM is rather similar to that of a separately excited dc motor. Thus, the IM has been adopted widely as a driver in the elevator system recently [11]. However, a traditional open-loop scalar (constant V/f ratio) controller with a gear transmission is always utilized in the control of elevator car or automatic door. The motivation of this study is to develop a total sliding-mode servo controller for a gearless elevator-door mechanism actuated with direct-drive-type IM [12] – [13].

2 Mechanical Structure and Dynamic Analyses

The mechanical structure and drive system of an elevator-door servomechanism is depicted in Figure 2.1, which is composed of a direct-drive-type induction servomotor, single-side-opened elevator door, belting and weighting mechanisms. In Figure 2.1, the

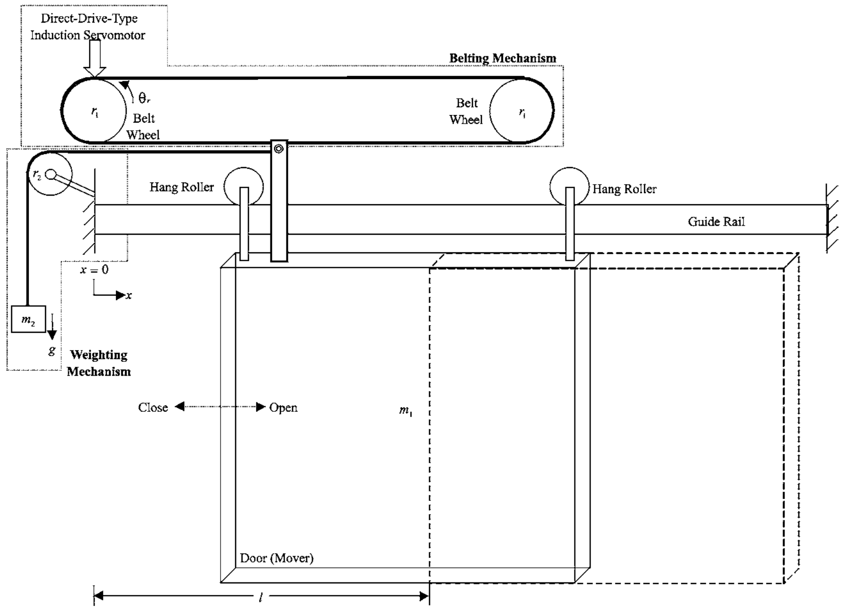

Figure 2.1. Mechanical structure and drive system of elevator-door servomechanism.

symbols m_1 and m_2 represent the masses of door and counterweight, respectively; r_1 and r_2 denote the radiuses of belting and weighting wheels, respectively; θ_r is the rotor position of the induction servomotor; x is the moving position of the door, l is the total length of the moving path, and g is the gravity acceleration.

A Direct-Drive-Type Induction Servomotor

The vocabulary "direct-drive" means that the transmittal mechanism is passed through belts directly without a gearbox. The configuration of an indirect field-oriented induction servomotor drive system is depicted in Figure 2.2(a) [8]. It consists of an induction servomotor coupled with a mechanism, a ramp comparison current-controlled pulse-width-modulation (PWM) voltage source inverter (VSI), a unit vector generator (where θ_e is the position of rotor flux), a coordinate translator, a speed control loop and a position control loop. The induction servomotor used in this drive system is a three-phase Y-connected eight-pole 150W 60Hz 220V/3.3A type. The current-controlled VSI is implemented by insulated gate bipolar transistor (IGBT) switching components with a switching frequency of 15kHz. The mechanical equation of an induction servomotor drive can be represented as

$$J\ddot{\theta}_r(t) + B\dot{\theta}_r(t) + T_L = T_e, \tag{1}$$

where J is the moment of inertia; B is the damping coefficient; T_L represents the load torque and external disturbance; T_e denotes the electric torque. With the implementation of field-oriented control [9] – [10], the induction servomotor drive system can be simplified to a control system block diagram as shown in Figure 2.2(b), in which the electric torque can be represented as

$$T_e = K_t i_{qs}^* \tag{2}$$

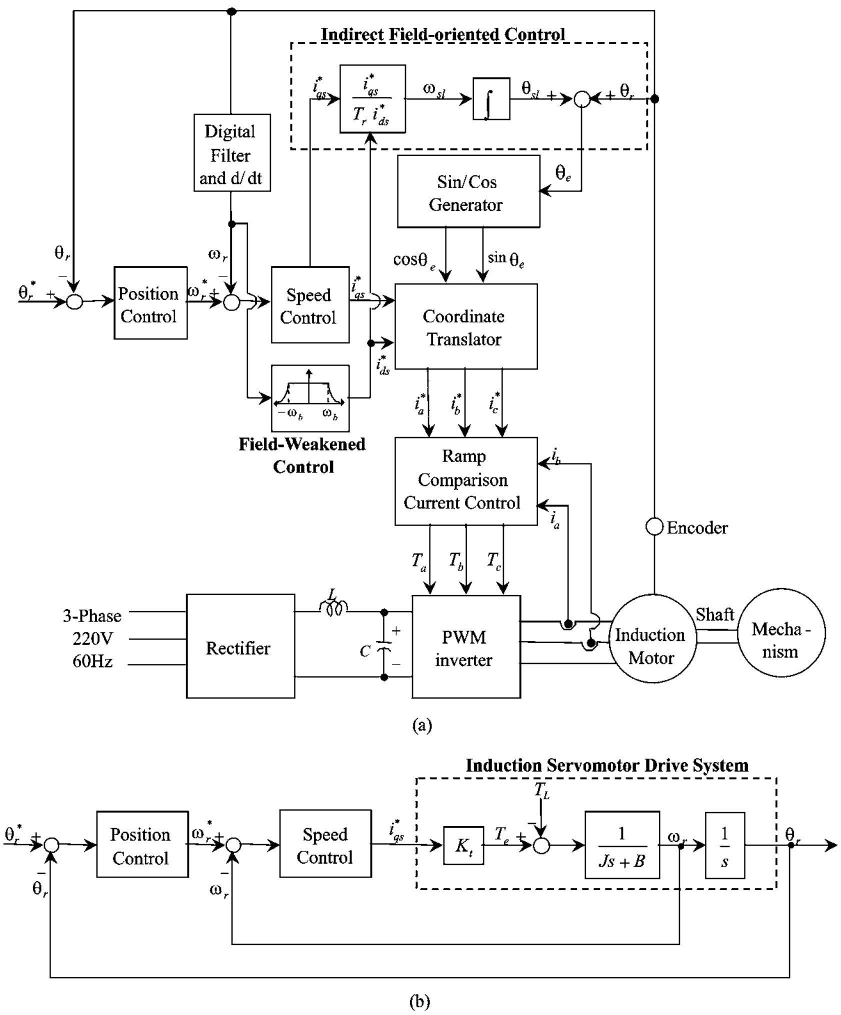

Figure 2.2. (a) Indirect field-oriented induction servomotor drive. (b) Simplified control system.

with

$$K_t = \frac{3n_p}{2} \frac{L_m^2}{L_r} i_{ds}^*, \tag{3}$$

where K_t is the torque constant; i_{qs}^* is the torque current command; i_{ds}^* is the flux current command; n_p is the number of pole pairs; L_m is the magnetizing inductance per phase; L_r is the rotor inductance per phase; ω_r is the rotor speed; θ_r^* and ω_r^* are the rotor position and speed commands.

B Dynamic Model of Elevator-Door Servomechanism

The Newtonian motion law is utilized to derive the dynamic equation of the elevator-door servomechanism in this subsection. Since the belt makes the electric torque convert into the linear driving force and the control object is the door position, the conversion relationship between the rotor position and door position is

$$\theta_r(t) = x(t)/r_1. \tag{4}$$

Substituting (4) into (1) and using (2), one can obtain

$$J\ddot{x}(t) + B\dot{x}(t) = r_1 K_t i_{qs}^* - r_1 T_L. \tag{5}$$

Consider (5), the in the elevator-door servomechanism is

$$T_L = T_i + T_w + T_f + T_d, \tag{6}$$

where $T_i = (m_1+m_2)r_1\ddot{x}$ represents the load inertia torque; $T_w = m_2 g r_1 \mathrm{sgn}\,(\dot{x})$ denotes the weighting torque, in which $\mathrm{sgn}\,(\cdot)$ is a sign function; $T_f = \mu m_1 g$ exhibits the friction torque between the hang roller and guide rail, in which μ is the friction coefficient; T_d is the external disturbance torque. Combined (5) with (6), the complete dynamic equation of the elevator-door servomechanism can be obtained as

$$\ddot{x}(t) \equiv A_p\dot{x}(t) + B_p U(t) + C_p + D_p, \tag{7}$$

where $U(t) = i_{qs}^*(t)$ is the control input, and

$$\begin{aligned}
A_p &= -B[J + r_1^2(m_1+m_2)]^{-1},\\
B_p &= r_1 K_t [J + r_1^2(m_1+m_2)]^{-1} > 0,\\
C_p &= -[J + r_1^2(m_1+m_2)]^{-1}[r_1 m_2 \mathrm{sgn}\,(\dot{x}) + \mu m_1] r_1 g,\\
D_p &= -r_1[J + r_1^2(m_1+m_2)]^{-1} T_d,
\end{aligned}$$

Note that, the unmodelled dynamics, e.g. the friction existing in the belting mechanism, can be considered as the external disturbance torque. The most important parameters that affect the control performance of the elevator-door servomechanism are the external disturbance torque T_d and the variations of motor parameters.

3 Total Sliding-Mode Control

Consider the system parameters in nominal conditions without external disturbance torque, rewriting (7) as follows can represent the nominal model of the elevator-door servomechanism:

$$\ddot{x}(t) = A_{pn}\dot{x}(t) + B_{pn}U(t) + C_{pn}, \tag{8}$$

where A_{pn}, B_{pn} and C_{pn} are the nominal values of A_p, B_p and C_p, respectively. Consider (8) parametric variation, external disturbance and unpredictable uncertainties for the actual elevator-door servomechanism

$$\begin{aligned}
\ddot{x}(t) &= (A_{pn} + \Delta A)\dot{x}(t) + (B_{pn} + \Delta B)U(t) + (C_{pn} + \Delta C) + D_p + \beta\\
&\equiv A_{pn}\dot{x}(t) + B_{pn}U(t) + C_{pn} + W(t),
\end{aligned} \tag{9}$$

where ΔA, ΔB and ΔC denote the uncertainties introduced by the variations of motor parameters; β represents the unstructured uncertainty due to nonideal field orientation in

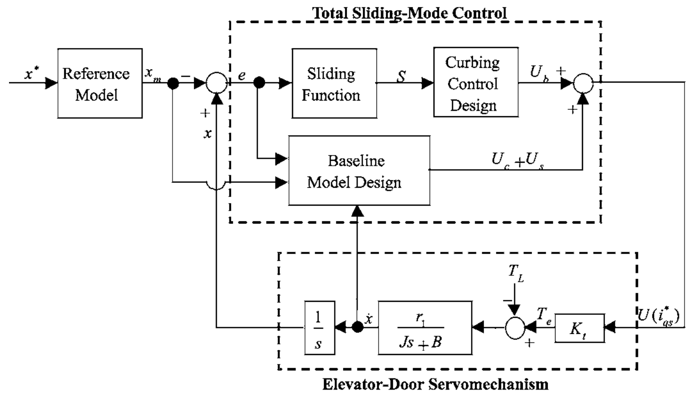

Figure 3.1. Block diagram of TSMC system.

transient state, and the unpredictable dynamics in practical applications; $W(t)$ is called the lumped uncertainty and is defined as

$$W(t) = \Delta A\dot{x}(t) + \Delta BU(t) + \Delta C + D_p + \beta. \tag{10}$$

Here the bound of the lumped uncertainty is assumed to be given; that is,

$$|W(t)| < \rho, \tag{11}$$

where ρ is a given positive constant. The control problem is to find a control law so that the state χ can track specific desired trajectories in the presence of uncertainties. To achieve this control objective, define the tracking error $e = x - x_m$, in which x_m represents a desired position specified by a reference model. The presentation of TSMC for the elevator-door servomechanism is divided into two main parts and is depicted in Figure 3.1. The first part addresses performance design. The object is to specify the desired performance in terms of the nominal model, and it is referred to as baseline model design. Following the baseline model design, the second part is the curbing controller design to totally eliminate the unpredictable perturbation effect from the parameter variations and external disturbance so that the baseline model design performance can be exactly assured. Define a sliding function $S(t)$ as follows [8]:

$$S(t) = C(\boldsymbol{E}) - C(\boldsymbol{E}_0) - \int_0^t \frac{\partial C}{\partial \boldsymbol{E}^T} \boldsymbol{AE}\, d\tau, \tag{12}$$

where $C(\boldsymbol{E})$ is a scalar variable designed as $\dfrac{\partial C}{\partial \boldsymbol{E}^T} = [0 \quad B_{pn}^{-1}]$; $\boldsymbol{E}_0$ is the initial state of $\boldsymbol{E}(t)$, and

$$\boldsymbol{E} = [e \quad \dot{e}]^T \qquad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -K_p & -K_\nu \end{pmatrix} \tag{13}$$

in which K_p and K_ν are positive constants. Note that, since the function $S(t) = 0$ when $t = 0$, there is no reaching phase as in the traditional sliding-mode control [2]–[3]. Then, the TSMC law is assumed to take the following form:

$$U(t) = U_c(t) + U_s(t) + U_b(t) \tag{14}$$

with

$$U_c(t) = -B_{pn}^{-1}[A_{pn}\dot{x}(t) + C_{pn}], \tag{15}$$

$$U_s(t) = B_{pn}^{-1}[\ddot{\chi}(t) - K_p e(t) - K_\nu \dot{e}(t)], \tag{16}$$

$$U_b(t) = -KS(t) - \rho B_{pn}^{-1}\operatorname{sgn}(S(t)), \tag{17}$$

where K is a positive constant. The first controller, U_c, is used to compensate for the nonlinear effects and attempts to cancel the nonlinear terms in the model. After the nonlinear model is linearized, the second controller, U_s, is used to specify the desired system performance. The objective of the third controller U_b is to keep the controlled system dynamics on the surface $S(t) = 0$. That is, curb the system dynamics onto $S(t) = 0$ for all time. Thus U_b is called a curbing controller, which is a constant plus proportional rate control scheme providing a measure for the reduction of chattering [14].

Substitute (14), (15) and (16) into (9), the state variable form can be obtained as follows:

$$\dot{\boldsymbol{E}} = \boldsymbol{A}\boldsymbol{E} + \boldsymbol{B}_m[U_b(t) + B_{pn}^{-1}W(t)], \tag{18}$$

where $\boldsymbol{B}_m = [0 \quad B_{pn}]^T$. Now $S(t) = 0$ when $t = 0$. To maintain the state on the surface $S(t) = 0$ for all time, one only needs to show that

$$S(t)\dot{S}(t) < 0, \quad \text{if} \quad S(t) \neq 0. \tag{19}$$

Differentiating $S(t)$ shown in (12) with respect to time and using error dynamics shown in (19) yields

$$\begin{aligned}\dot{S}(t) = \frac{\partial C}{\partial \boldsymbol{E}^T}\,\dot{\boldsymbol{E}} - \frac{\partial C}{\partial \boldsymbol{E}^T}\,\boldsymbol{A}\boldsymbol{E} &= \frac{\partial C}{\partial \boldsymbol{E}^T}\,\{\boldsymbol{A}\boldsymbol{E} + \boldsymbol{B}_m[U_b(t) + B_{pn}^{-1}W(t)] - \boldsymbol{A}\boldsymbol{E}\} \\ &= U_b(t) + B_{pn}^{-1}W(t).\end{aligned} \tag{20}$$

Multiplying $S(t)$ by (20) and inserting control U_b shown in (17) into (20) yields

$$\begin{aligned}&S(t)\dot{S}(t) = S(t)[U_b(t) + B_{pn}^{-1}W(t)] \le S(t)U_b(t) + B_{pn}^{-1}|S(t)|\,|W(t)| \\ &= -KS^2(t) - \rho B_{pn}^{-1}|S(t)| + B_{pn}^{-1}|S(t)|\,|W(t)| < -KS^2(t) < 0.\end{aligned} \tag{21}$$

Thus the sliding mode can be assured throughout the whole control period. Wai [8] presented an adaptive sliding-mode control system to control the position of an induction servo motor drive, where a simple adaptive algorithm was utilized to estimate the bound of uncertainties in the curbing controller of total sliding-mode control system for reducing the chattering torque. However, the adaptation law for the bound of uncertainties is always positive and tracking error introduced by any uncertainty will cause the estimated bound growth. It implies that the curbing controller will result in large chattering with time gradually. This results that the IM will eventually be saturated and the system may be unstable. Wai [15] described the dynamic responses of a recurrent-fuzzy-neural-network (RFNN) sliding- mode controlled permanent magnet synchronous servomotor, where a RFNN bound observer was utilized to adjust the uncertainty bounds in the curbing controller of total sliding-mode control system. Although it can solve the problem of parameter divergence, this control scheme seems to be more complicate such that it is difficult to implement in practical applications. Compared the modified control strategy used in this study with our previous works [8, 15], it can reduce effectively the chattering phenomena without any auxiliary algorithms such that this simple control scheme can be easily implemented in industrial applications. The effectiveness of the proposed TSMC system is verified by the following numerical simulation and experimental results.

4 Numerical Simulation and Experimental Results

For numerical simulations, the parameters of the elevator-door mechanism are designed as follows:

$$\begin{aligned} m_1 = 20\text{kg}, \quad m_2 = 1.5\text{kg}, \quad g = 9.8, \quad \mu = 0.1, \\ r_1 = 1.417 \times 10^{-2} m, \quad r_2 = 4.2 \times 10^{-2} m, \quad l = 1.2m. \end{aligned} \tag{22}$$

Moreover, the parameters of the direct-drive-type induction servomotor system are

$$K_t = 0.4851\text{Nm/A}, \quad \bar{J} = 4.78 \times 10^{-3}\text{Nms}^2/\text{rad}, \quad \bar{B} = 5.34 \times 10^{-3}\text{Nms/rad}, \tag{23}$$

where the overbar symbol represents the system parameter in the nominal condition. In addition, the gains of the proposed TSMC control system are given as

$$K_\nu = 14, \quad K_p = 49, \quad \rho = 0.1, \quad K = 80. \tag{24}$$

Properly choosing the values of K_ν and K_p, the desired nominal system dynamics such as rise time, overshoot, and settling time can be easily designed by a second-order system, $\ddot{e} + K_\nu \dot{e} + K_p e = 0$. Moreover, the fixed bound ρ and the constant gain K in the curbing controller are determined roughly to achieve the superior transient control performance in both simulation and experimentation considering the requirement of stability and the possible operating conditions. Note that, introducing the constant gain K into the curbing controller can tune the convergent speed of the tracking performance and ensure the stability as the selection of a small value ρ for reducing the chattering phenomena induced by the sign function in the curbing controller. Two simulation cases including motor parameter variations and external disturbance torque in the mechanism are addressed as follows to verify the robust characteristic of the TSMC system: Case 1: $J = \bar{J},\ B = \bar{B},\ T_L = 1\text{Nm}$ occurring between 14s-16s; Case 2: $J = 3 \times \bar{J},\ B = 3 \times \bar{B}$, $T_L = 1\text{Nm}$ occurring between 14s-16s. The control objective is to make the door position and velocity follow the specific reference profiles under the occurrence of uncertainties. The door position command is obtained based on the velocity profile of the elevator door. The door is moving from the left/right side to right/left side with $\pm 0.2\text{m/s}$ reference velocity.

In the simulation, firstly, the computed torque control (CTC) system that is equal to baseline model design $(U_c + U_s)$ is demonstrated for comparison. The simulated results of CTC system at Case 1 and Case 2 are depicted in Figure 4.1. The position tracking are depicted in Figures 4.1(a) and 4.1(c), and the associated velocity response are depicted in Figures 4.1(b) and 4.1(d). From the simulated results, favorable tracking responses shown in the beginning of Figures 4.1(a) and 4.1(c) only can be obtained at the nominal condition, and poor tracking responses are resulted owing to the motor parameter variations and external disturbance torque. Though large control gains (K_ν and K_p) may solve the problem of delay or degenerate tracking responses, it will result

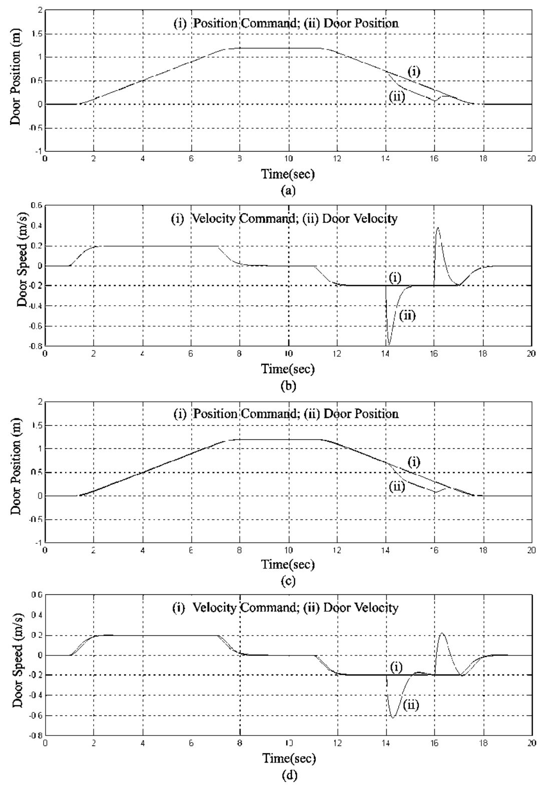

Figure 4.1. Simulated results of CTC system: (a) position tracking at Case 1; (b) velocity response at Case 1; (c) position tracking at Case 2; (d) velocity response at Case 2.

in impractical large control efforts. Therefore, the control gains are difficult to determine due to the unknown uncertainties in practical applications, and are ordinarily chosen as a compromise between the stability and control performance.

Secondly, the conventional sliding-model control (CSMC) designed by Slotine and Li [2] as follows is introduced to compare the control performance of the TSMC system:

$$U = U_{eq} + U_r \tag{25}$$

in which $U_{eq} = U_c + U_s$ is the equivalent control and $U_r = \alpha B_{pn}^{-1}\mathrm{sgn}\,(S_L)$ is the hitting control, where $S_L = \dot{e} + \lambda e$; λ is a positive constant; α is the hitting gain that is selected to satisfy the sliding condition. The simulated results of CSMC system ($\lambda = 5$ and $\alpha = 8$) at Case 1 and Case 2 are depicted in Figure 4.2. The position tracking are depicted in Figures 4.2(a) and 4.2(c), and the associated velocity response are depicted in Figures 4.2(b) and 4.2(d). Compared Figure 4.2 with Figure 4.1, the CSMC system got the better control performance than the CTC system, especially under the occurrence of uncertainties. However, the chattering velocity responses show in Figures 4.2(b) and 4.2(d) are caused by the large selection of hitting-control gain, α. Although the chattering phenomena can be reduced with small hitting-control gain, it will result in degenerate control performance.

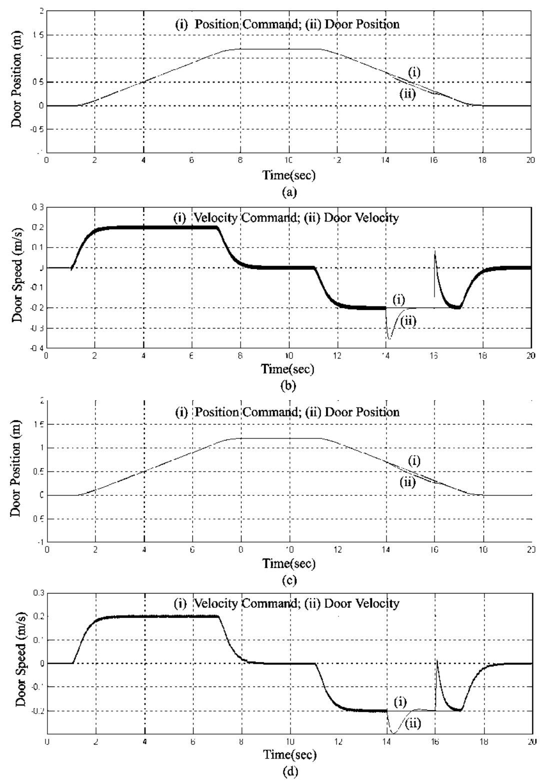

Figure 4.2. Simulated results of CSMC system: (a) position tracking at Case 1; (b) velocity response at Case 1; (c) position tracking at Case 2; (d) velocity response at Case 2.

Now, the designed TSMC system is simulated under the same cases, and its results are depicted in Figure 4.3. The chattering phenomenon does not exist in the velocity response of the TSMC system as shown in Figures 4.3(b) and 4.3(d). Moreover, the robust control performance of the TSMC system, both in the conditions of motor parameter variations and external disturbance torque, are obvious as shown in Figures 4.3(a) and 4.3(c). According to the above numerical simulation, the TSMC system yields the superior control performance than the CTC and CSMC systems.

The experimentation of the CTC, CSMC and TSMC systems for an actual elevator-door servomechanism are provided here to further demonstrate the advantage of the proposed TSMC control system. The PIC16F877 single chip micro-controller produced by Microchip company is used as a main CPU which has a 2ms control loop to implement the TSMC algorithm and a 0.2ms interrupt loop to execute current control inner loop with field-oriented mechanism. In the experimentation, a braking machine is driven by a current source drive to provide a disturbance torque, and an iron disk is coupled to the rotor shaft of an induction servomotor taking as an inertia varying mechanism.

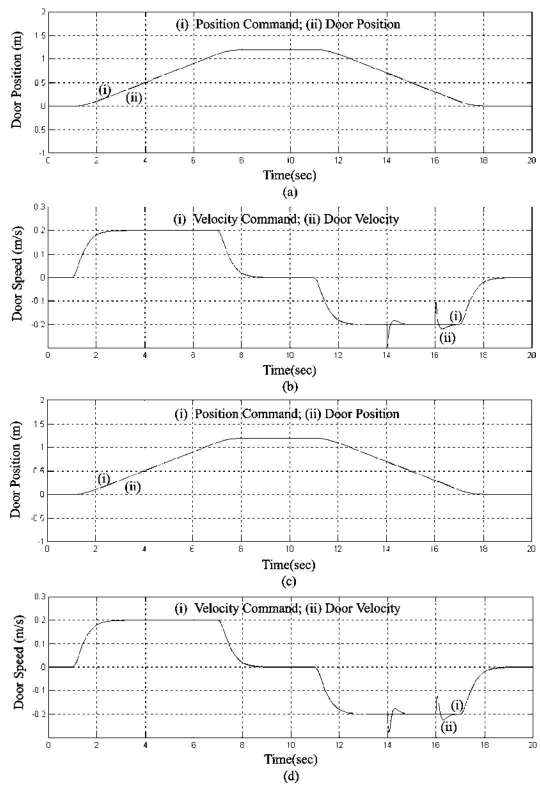

Figure 4.3. Simulated results of TSMC system: (a) position tracking at Case 1; (b) velocity response at Case 1; (c) position tracking at Case 2; (d) velocity response at Case 2.

Two experimental conditions are given to verify the robust control performance. One is the disturbance condition that is the nominal motor inertia with 1Nm disturbance torque occurring between 14s-16s. The other is the perturbed condition that is the increasing of the motor inertia to approximately three times the nominal value with 1Nm disturbance torque occurring between 14s-16s. The experimental results of CTC, CSMC and TSMC systems at disturbance and perturbed conditions are depicted in Figures 4.4 and 4.5, respectively. Note that, there are slight difference between the numerical simulation and experimental results due to the existence of unpredictable uncertainties in practical applications. It can be seen from the experimental results that the TSMC system tracks well with the specific position and velocity profiles during the whole operation. Consequently, the proposed TSMC system is more suitable to control the elevator-door servomechanism considering the existence of uncertainties.

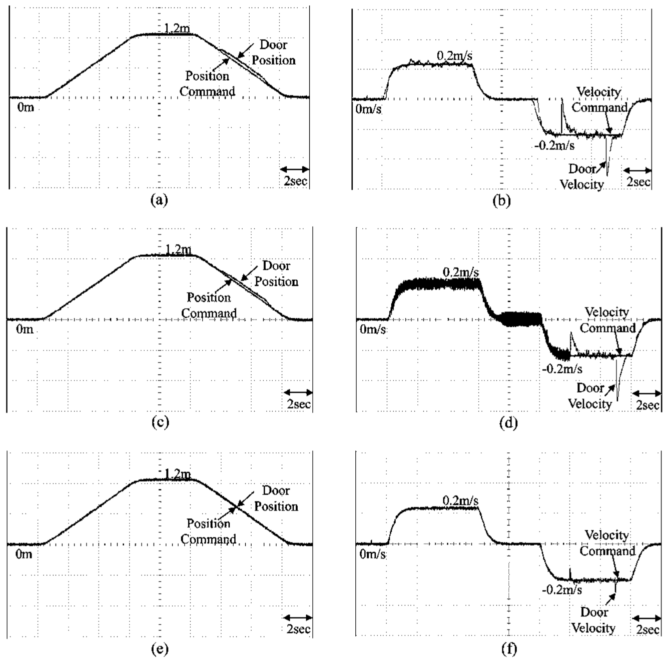

Figure 4.4. Experimental results of CTC, CSMC, TSMC systems at disturbance condition: (a) position tracking of CTC system; (b) velocity response of CTC system; (c) position tracking of CSMC system; (d) velocity response of CSMC system; (e) position tracking of TSMC system; (f) velocity response of TSMC system.

5 Conclusions

This study has successfully demonstrated the application of a total sliding-mode control system to control the motion of an elevator-door mechanism with an indirect field-oriented induction servomotor drive directly. First, the mechanical structure and dynamic analyses of an elevator-door servomechanism was introduced. Moreover, the theoretical bases and stability analyses of the proposed TSMC systems were described in detail. In addition, simulation and experimentation were carried out using a specific reference profile to verify the effectiveness of the proposed control strategy. Compared with the CTC and CSMC systems, the TSMC system results in reduced chattering with robust control performance. The major contributions of this study are the successful development of a TSMC system, which has a total sliding motion without a reaching phase, and the successful application of the proposed TSMC system to control the motion of the elevator-door servomechanism considering the existence of uncertainties.

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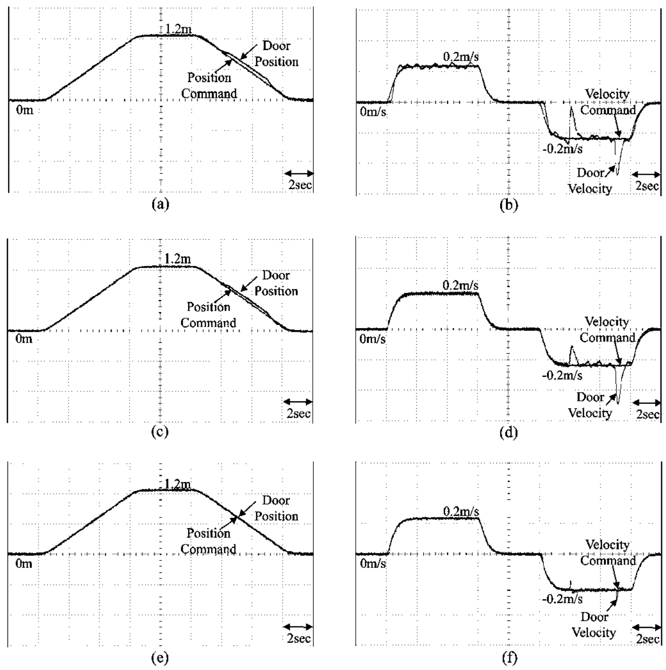

Figure 4.5. Experimental results of CTC, CSMC, TSMC systems at perturbed condition: (a) position tracking of CTC system; (b) velocity response of CTC system; (c) position tracking of CSMC system; (d) velocity response of CSMC system; (e) position tracking of TSMC system; (f) velocity response of TSMC system.

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Stability and $\mathcal{L}_2$ Gain Analysis for a Class of Switched Symmetric Systems

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Abstract: In this paper, we study stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of a finite number of linear time-invariant symmetric subsystems. We focus our attention mainly on discrete-time systems. When all subsystems are Schur stable, we show that the switched system is exponentially stable under arbitrary switching. Furthermore, when all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we show that the switched system is exponentially stable and has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching. The key idea for both stability and $\mathcal{L}_2$ gain analysis in this paper is to establish a general Lyapunov function for all subsystems in the switched system.

Keywords: *Switched symmetric system; exponential stability; $\mathcal{L}_2$ gain; arbitrary switching; general Lyapunov function; linear matrix inequality (LMI).*

Mathematics Subject Classification (2000): 93C10, 93C55, 93D20, 93D25, 93D30.

1 Introduction

By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time or discrete-time subsystems and as a rule orchestrating the switching among the subsystems. In the last two decades, there has been increasing interest in the stability analysis and controller design for such switched systems. The motivation for studying switched systems is from the fact that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors [1], and that the methods of intelligent control design are based on the idea of switching among different controllers [2 − 5]. For recent progress and perspectives in the field of switched systems, see the survey papers [3, 6] and the references cited therein.

As also pointed out in [3, 6], there are three basic problems in stability and design of switched systems: (i) find conditions for stabilizability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching signals; and (iii) construct a stabilizing switching signal. There are many existing works concerning Problem (ii) and (iii). For example, references [7 − 10] considered Problem (ii) using piecewise Lyapunov functions, and references [11 − 13] considered Problem (ii) for switched systems with pairwise commutation or Lie-algebraic properties. References [14 − 15] considered Problem (iii) by dividing the state space associated with appropriate switching depending on state, and references [16, 17] considered quadratic stabilization, which belongs to Problem (iii), for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. However, we see very few dealing with the first problem, though it is desirable to require arbitrary switching in many real applications. Reference [18] showed that when all subsystems are stable and commutative pairwise, the switched system is stable under arbitrary switching. There are some other results concerning general Lyapunov functions for the subsystems in a switched system, but we do not find any explicit answer to Problem (i) except [18]. In this paper, rather than considering for a given switched system the condition for stabilizability under arbitrary switching, we are interested in the following question: *What kind of switched systems are stable under arbitrary switching? Specifically, is there a switched system whose subsystems are not commutative pairwise, yet it is stable under arbitrary switching?*

For switched systems, there are a few results concerning $\mathcal{L}_2$ gain analysis. Hespanha considered such a problem in his Ph.D. dissertation [19], by using a piecewise Lyapunov function approach. In [20], a modified approach has been proposed for more general switched systems and more exact results have been obtained. In that context, it has been shown that when all continuous-time subsystems are Hurwitz stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ_0, the switched system under an average dwell time scheme [7] achieves a weighted $\mathcal{L}_2$ gain γ_0, and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain if the average dwell time is chosen sufficiently large. However, the results obtained in [19] and [20] are conservative, and it is supposed that the main reason is in the use of piecewise Lyapunov functions. Recently, reference [21] considered the computation of $\mathcal{L}_2$ gain for switched linear systems with large dwell time, and gave an algorithm by considering the separation between the stabilizing and antistabilizing solutions to a set of algebraic Riccati equations. Noticing that these papers deal with the class of switching signals with (average) dwell time, we are motivated to ask the following question: *Is there a switched system that preserves its subsystems' $\mathcal{L}_2$ gain properties under arbitrary switching?*

For the above questions concerning stability and $\mathcal{L}_2$ gain, we give a clear (though not complete) answer in this paper. More exactly, we will show that a class of switched systems, which are composed of a finite number of linear time-invariant symmetric subsystems and called shortly *switched symmetric systems*, will preserve their subsystems' stability and $\mathcal{L}_2$ gain properties under arbitrary switching. We take such symmetric systems into consideration since they appear quite often in many engineering disciplines (for example, RC or RL electrical networks, viscoelastic materials and chemical reactions) [22], and thus belong to an important class in control engineering.

Though our discussion is applicable also to continuous-time switched systems with some modifications, we focus our attention mainly on the following discrete-time switched system

$$\begin{aligned} x[k+1] &= A_{\sigma(k)}x[k] + B_{\sigma(k)}w[k], \quad x[k_0] = x_0, \\ z[k] &= C_{\sigma(k)}x[k] + D_{\sigma(k)}w[k], \end{aligned} \tag{1.1}$$

where $x[k] \in R^n$ is the state, $w[k] \in R^m$ is the input, $z[k] \in R^p$ is the output, $k_0 \geq 0$ is the initial point and x_0 is the initial state. $\sigma(k)\colon \mathcal{I}_+ \to \mathcal{I}_N = \{1, 2, \dots, N\}$ is a piecewise constant function, called a *switching signal*, which is assumed to be arbitrary. Here, $\mathcal{I}_+$ denotes the set of all nonnegative integers not less than k_0, and A_i, B_i, C_i, D_i $(i \in \mathcal{I}_N)$ are constant matrices of appropriate dimensions denoting the subsystems, $N > 1$ is the number of subsystems. Throughout this paper, we assume that all subsystems in (1.1) are symmetric in the sense of satisfying

$$A_i = A_i^{\mathrm{T}}, \quad B_i = C_i^{\mathrm{T}}, \quad D_i = D_i^{\mathrm{T}}, \quad \forall\, i \in \mathcal{I}_N\,. \tag{1.2}$$

It should be noted here that the assumption (1.2) does not cover all symmetric subsystems, which are usually defined in the form of transfer functions, and a more general definition is that $T_iA_i = A_i^{\mathrm{T}}T_i$, $T_iB_i = C_i^{\mathrm{T}}$, $D_i = D_i^{\mathrm{T}}$ holds for some nonsingular symmetric matrix T_i [22–24]. However, (1.2) represents an interesting class of symmetric systems [25], and for the benefit of this paper we are concentrated on such kind of symmetric systems.

We will say the switched system (1.1) is *exponentially stable* if $\|x[k]\| \leq \mu^{k-k_0}\|x_0\|$ with $0 < \mu < 1$ holds for any $k > k_0$ and any initial state x_0, and will say the switched system (1.1) *has an $\mathcal{L}_2$ gain* γ if $\sum\limits_{j=k_0}^{k} z^{\mathrm{T}}[j]z[j] \leq \gamma^2 \sum\limits_{j=k_0}^{k} w^{\mathrm{T}}[j]w[j]$ holds for any integer $k > k_0$ when $x_0 = 0$. These definitions are also valid for all the subsystems in (1.1).

This paper is organized as follows. In Section 2, assuming that all subsystems are Schur stable, we show that there exists a general Lyapunov function for all subsystems, and that the switched system is exponentially stable under arbitrary switching. In Section 3, assuming that all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we prove that there exists a general Lyapunov function for all subsystems in the sense of $\mathcal{L}_2$ gain, and that the switched system has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching. Finally we give some concluding remarks in Section 4.

2 Stability Analysis

In this section, we set $w[k] \equiv 0$ in the switched system (1.1) to consider stability of the system under arbitrary switching. We first give a preliminary result.

Lemma 2.1 *Consider the discrete-time symmetric system*

$$x[k+1] = Ax[k], \tag{2.1}$$

where $x[k] \in R^n$ is the state and A is a constant symmetric matrix. The system (2.1) is Schur stable if and only if

$$A^2 < I. \tag{2.2}$$

Proof Since A is a symmetric matrix, there exists a nonsingular matrix Q satisfying $Q^{\mathrm{T}} = Q^{-1}$ such that

$$Q^{\mathrm{T}}AQ = \mathrm{diag}\{\lambda_1,\, \lambda_2,\, \dots,\, \lambda_n\}, \tag{2.3}$$

where $\lambda_i,\ i = 1, 2, \dots, n,$ are A's real eigenvalues (noticing that the symmetric matrix A has only real eigenvalues), and thus

$$Q^{\mathrm{T}}A^2Q = \mathrm{diag}\{\lambda_1^2,\, \lambda_2^2,\, \dots,\, \lambda_n^2\}. \tag{2.4}$$

The discrete-time system (2.1) is Schur stable if and only if $|\lambda_i| < 1,\ i = 1, 2, \dots, n,$ which is equivalent to

$$Q^{\mathrm{T}}A^2Q < I \tag{2.5}$$

according to (2.4). Since $QQ^{\mathrm{T}} = I,$ the inequality (2.5) is equivalent to (2.2). This completes the proof.

Remark 2.1 Lemma 2.1 implies that if all A_i's in (1.1) are Schur stable, there exists a general Lyapunov matrix $P = I$ for all A_i's, satisfying the LMI

$$A_i^{\mathrm{T}}PA_i - P < 0, \quad \forall\, i \in \mathcal{I}_N. \tag{2.6}$$

Hence, $V(x) = x^{\mathrm{T}}x$ serves as a general Lyapunov function for all subsystems in (1.1).

Now we state and prove the main result in this section.

Theorem 2.1 *When all subsystems in (1.1) are Schur stable, the switched symmetric system (1.1) is exponentially stable under arbitrary switching.*

Proof Since all subsystems in (1.1) are Schur stable, according to Lemma 2.1, the matrix inequality

$$A_i^2 < I \tag{2.7}$$

holds for all $i \in \mathcal{I}_N,$ and thus there exists a scalar $\epsilon \in (0,\, 1)$ such that

$$A_i^2 < (1-\epsilon)I, \quad \forall i \in \mathcal{I}_N. \tag{2.8}$$

Now, we consider the Lyapunov function candidate

$$V(x) = x^{\mathrm{T}}x. \tag{2.9}$$

According to (2.8), we obtain for any integer $k > k_0$ that

$$\begin{aligned} V(x[k]) &= x^{\mathrm{T}}[k]\,x[k] \le (1-\epsilon)x^{\mathrm{T}}[k-1]x[k-1] \\ &= (1-\epsilon)V(x[k-1]) \end{aligned} \tag{2.10}$$

holds under arbitrary switching, and thus

$$V(x[k]) \le (1-\epsilon)^{k-k_0}V(x[k_0]) \tag{2.11}$$

which means

$$\|x[k]\| \le (\sqrt{1-\epsilon})^{k-k_0}\|x_0\|. \tag{2.12}$$

Since this inequality holds for any initial state x_0, the switched system (1.1) is exponentially stable.

Remark 2.2 For the continuous-time switched symmetric system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \tag{2.13}$$

where $A_i = A_i^{\mathrm{T}},\ i \in \mathcal{I}_N,$ are constant Hurwitz stable matrices, we easily see that $A_i < 0$ holds for all i, which implies that the general Lyapunov matrix $P = I$ satisfies the LMI

$$A_i^{\mathrm{T}} P + PA_i < 0, \quad \forall\, i \in \mathcal{I}_N, \tag{2.14}$$

and thus the switched system (2.13) is exponentially stable under arbitrary switching.

3 $\mathcal{L}_2$ Gain Analysis

In this section, we assume $x_0 = 0$ in the switched symmetric system (1.1) to study the $\mathcal{L}_2$ gain property of the system under arbitrary switching. First, we state and prove a lemma which plays an important role in the discussion of this section. We note that the idea of this lemma and its proof is motivated by Lemma 2 of [25], where continuous-time symmetric systems are dealt with.

Lemma 3.1 *Consider the discrete-time symmetric system*

$$\begin{aligned} x[k+1] &= Ax[k] + Bw[k] \\ z[k] &= Cx[k] + Dw[k], \end{aligned} \tag{3.1}$$

where $x[k] \in R^n$ is the state, $w[k] \in R^m$ is the input, $z[k] \in R^p$ is the output, and A, B, C, D are constant matrices of appropriate dimensions, satisfying $A = A^{\mathrm{T}},\ B = C^{\mathrm{T}},$ $D = D^{\mathrm{T}}$. The system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ if and only if

$$\begin{bmatrix} -I & A & B & 0 \\ A & -I & 0 & B \\ B^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & B^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.2}$$

Proof **Sufficiency** The condition (3.2) means that the matrix inequality

$$\begin{bmatrix} -P & PA & PB & 0 \\ A^{\mathrm{T}}P & -P & 0 & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & 0 & -\gamma I & D^{\mathrm{T}} \\ 0 & C & D & -\gamma I \end{bmatrix} < 0 \tag{3.3}$$

is satisfied with $P = I$. Hence, according to the Bounded Real Lemma [26] for discrete-time LTI system, the system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ.

Necessity Suppose that the system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ . Then, there exists a matrix $P_0 > 0$ such that

$$\begin{bmatrix} -P_0 & P_0A & P_0B & 0 \\ AP_0 & -P_0 & 0 & B \\ B^{\mathrm{T}}P_0 & 0 & -\gamma I & D \\ 0 & B^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.4}$$

where C was replaced by B^{T}.

Now, we prove that $P = I$ is also a solution of the above matrix inequality (i.e., (3.4) holds when replacing P_0 with I). Since $P_0 > 0$, there always exists a nonsingular matrix U satisfying $U^{\mathrm{T}} = U^{-1}$ such that

$$\begin{gathered} U^{\mathrm{T}} P_0 U = \Sigma_0 = \mathrm{diag}\,\{\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_n\}, \\ \sigma_i > 0, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{3.5}$$

Pre- and post-multiplying (3.4) by $\mathrm{diag}\,\{U^{\mathrm{T}}, U^{\mathrm{T}}, I, I\}$ and $\mathrm{diag}\,\{U, U, I, I\}$, respectively, we obtain

$$\begin{bmatrix} -\Sigma_0 & \Sigma_0 \bar{A} & \Sigma_0 \overline{B} & 0 \\ \bar{A}\Sigma_0 & -\Sigma_0 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}} \Sigma_0 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.6}$$

where $\bar{A} = U^{\mathrm{T}} A U$, $\overline{B} = U^{\mathrm{T}} B$. Furthermore, pre- and post-multiplying the first and second rows and columns in (3.6) by Σ_0^{-1} leads to

$$\begin{bmatrix} -\Sigma_0^{-1} & \bar{A}\Sigma_0^{-1} & \overline{B} & 0 \\ \Sigma_0^{-1}\bar{A} & -\Sigma_0^{-1} & 0 & \Sigma_0^{-1}\overline{B} \\ \overline{B}^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}}\Sigma_0^{-1} & D & -\gamma I \end{bmatrix} < 0. \tag{3.7}$$

In (3.7), we exchange the first and second rows and columns, and then exchange the third and fourth rows and columns, to obtain

$$\begin{bmatrix} -\Sigma_0^{-1} & \Sigma_0^{-1}\bar{A} & \Sigma_0^{-1}\overline{B} & 0 \\ \bar{A}\Sigma_0^{-1} & -\Sigma_0^{-1} & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_0^{-1} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.8}$$

Since $\sigma_1 > 0$, there always exists a scalar λ_1 such that

$$0 < \lambda_1 < 1, \quad \lambda_1 \sigma_1 + (1 - \lambda_1)\sigma_1^{-1} = 1. \tag{3.9}$$

Then, by computing $\lambda_1 \times (3.6) + (1 - \lambda_1) \times (3.8)$, we obtain

$$\begin{bmatrix} -\Sigma_1 & \Sigma_1 \bar{A} & \Sigma_1 \overline{B} & 0 \\ \bar{A}\Sigma_1 & -\Sigma_1 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}} \Sigma_1 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.10}$$

where

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}\,\{\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1}, \lambda_1\sigma_2 + (1-\lambda_1)\sigma_2^{-1}, \ldots, \lambda_1\sigma_n + (1-\lambda_1)\sigma_n^{-1}\} \\ &\triangleq \mathrm{diag}\,\{1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n\} > 0. \end{aligned} \tag{3.11}$$

In the similar way to obtain (3.8), we can obtain from (3.10) that

$$\begin{bmatrix} -\Sigma_1^{-1} & \Sigma_1^{-1}\bar{A} & \Sigma_1^{-1}\overline{B} & 0 \\ \bar{A}\Sigma_1^{-1} & -\Sigma_1^{-1} & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_1^{-1} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.12}$$

Since $\bar{\sigma}_2 > 0$, there exists a scalar λ_2 such that

$$0 < \lambda_2 < 1, \quad \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1} = 1. \tag{3.13}$$

Then, the linear combination $\lambda_2 \times (3.10) + (1-\lambda_2) \times (3.12)$ results in

$$\begin{bmatrix} -\Sigma_2 & \Sigma_2\bar{A} & \Sigma_2\overline{B} & 0 \\ \bar{A}\Sigma_2 & -\Sigma_2 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_2 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.14}$$

where

$$\begin{aligned} \Sigma_2 &= \mathrm{diag}\,\{1,\, \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1},\, \dots,\, \lambda_2\bar{\sigma}_n + (1-\lambda_2)\bar{\sigma}_n^{-1}\} \\ &\triangleq \mathrm{diag}\,\{1, 1, \dots, \widetilde{\sigma}_n\} > 0. \end{aligned} \tag{3.15}$$

By repeating this process, we see that $\Sigma_n = I$ also satisfies (3.6), i.e.,

$$\begin{bmatrix} -I & \bar{A} & \overline{B} & 0 \\ \bar{A} & -I & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.16}$$

Pre- and post-multiplying this matrix inequality by

$$\mathrm{diag}\,\{U, U, I, I\} \quad \text{and} \quad \mathrm{diag}\,\{U^{\mathrm{T}}, U^{\mathrm{T}}, I, I\},$$

respectively, we obtain (3.2). This completes the proof.

Now, we assume that all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ. Then, according to Lemma 3.1, we have

$$\begin{bmatrix} -I & A_i & B_i & 0 \\ A_i & -I & 0 & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}} & 0 & -\gamma I & D_i \\ 0 & C_i & D_i & -\gamma I \end{bmatrix} < 0 \tag{3.17}$$

for all $i \in \mathcal{I}_N$, which is equivalent to

$$\begin{bmatrix} A_i^2 + \dfrac{1}{\gamma}C_i^{\mathrm{T}}C_i - I & A_iB_i + \dfrac{1}{\gamma}C_i^{\mathrm{T}}D_i \\ B_i^{\mathrm{T}}A_i + \dfrac{1}{\gamma}D_iC_i & B_i^{\mathrm{T}}B_i + \dfrac{1}{\gamma}D_i^2 - \gamma I \end{bmatrix} < 0. \tag{3.18}$$

We compute the difference of the Lyapunov function candidate $V(x) = x^{\mathrm{T}}x$ along the trajectory of any subsystem to obtain

$$\begin{aligned} V(x[k+1]) - V(x[k]) &= x^{\mathrm{T}}[k+1]x[k+1] - x^{\mathrm{T}}[k]x[k] \\ &= (A_i x[k] + B_i w[k])^{\mathrm{T}}(A_i x[k] + B_i w[k]) - x^{\mathrm{T}}[k]\,x[k] \\ &= [\, x^{\mathrm{T}}[k] \;\; w^{\mathrm{T}}[k]\,] \begin{bmatrix} A_i^2 - I & A_i^{\mathrm{T}} B_i \\ B_i^{\mathrm{T}} A_i & B_i^{\mathrm{T}} B_i \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \\ &< -[\, x^{\mathrm{T}}[k] \;\; w^{\mathrm{T}}[k]\,] \begin{bmatrix} \frac{1}{\gamma} C_i^{\mathrm{T}} C_i & \frac{1}{\gamma} C_i^{\mathrm{T}} D_i \\ \frac{1}{\gamma} D_i C_i & \frac{1}{\gamma} D_i^2 - \gamma I \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \\ &= -\frac{1}{\gamma}\left(z^{\mathrm{T}}[k]z[k] - \gamma^2 w^{\mathrm{T}}[k]w[k]\right), \end{aligned} \tag{3.19}$$

where (3.18) was used to obtain the inequality, and the trivial case of $x[k] = 0,\ w[k] = 0$ is excluded here.

For any piecewise constant switching signal and any given integer $k > k_0$, we let $k_1, \ldots, k_r$ $(r \geq 1)$ denote the switching points of $\sigma(k)$ over the interval $[k_0, k)$. Then, using the difference inequality (3.19), we obtain

$$\begin{aligned} V(x[k]) - V(x[k_r]) &< -\frac{1}{\gamma}\sum_{j=k_r}^{k-1}(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j]) \\ V(x[k_r]) - V(x[k_{r-1}]) &< -\frac{1}{\gamma}\sum_{j=k_{r-1}}^{k_r-1}\left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j]\right) \\ &\cdots\cdots\cdots \\ V(x[k_1]) - V(x[k_0]) &< -\frac{1}{\gamma}\sum_{j=k_0}^{k_1-1}\left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j]\right). \end{aligned} \tag{3.20}$$

We add all the above inequalities to get to

$$V(x[k]) - V(x[k_0]) < -\frac{1}{\gamma}\sum_{j=k_0}^{k-1}\left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j]\right). \tag{3.21}$$

Then, we use the assumption that $x[k_0] = x_0 = 0$ and the fact of $V(x[k]) \geq 0$ to obtain

$$\sum_{j=k_0}^{k-1} z^{\mathrm{T}}[j]z[j] < \gamma^2 \sum_{j=k_0}^{k-1} w^{\mathrm{T}}[j]w[j]. \tag{3.22}$$

We note that the above inequality holds for any $k > k_0$ including the case of $k \to \infty$, and that we did not add any limitation on the switching signal up to now.

We summarize the above discussion in the following theorem.

Theorem 3.2 *When all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ, the switched symmetric system (1.1) is exponentially stable and has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching.*

Remark 3.1 From Lemma 3.1 and the proof of Theorem 3.1, we see that if all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ, then there exists a general Lyapunov matrix $P = I$ for all subsystems, satisfying the LMI

$$\begin{bmatrix} -P & PA_i & PB_i & 0 \\ A_i^{\mathrm{T}}P & -P & 0 & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}}P & 0 & -\gamma I & D_i^{\mathrm{T}} \\ 0 & C_i & D_i & -\gamma I \end{bmatrix} < 0. \tag{3.23}$$

Hence, $V(x) = x^{\mathrm{T}}x$ serves as a general Lyapunov function for all subsystems in the sense of $\mathcal{L}_2$ gain.

Remark 3.2 Consider the continuous-time switched symmetric system

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t) \\ z(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t), \end{aligned} \tag{3.24}$$

where all the notations have the same meanings as in (1.1) except that the vectors $x(t)$, $w(t)$, $z(t)$ and the switching signal $\sigma(t)$ are with respect to the continuous time t . We assume that all subsystems in (3.24) are Hurwitz stable and have $\mathcal{L}_2$ gains smaller than γ. Then, the LMI

$$\begin{bmatrix} A_i^{\mathrm{T}}P + PA_i & PB_i & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}}P & -\gamma I & D_i^{\mathrm{T}} \\ C_i & D_i & -\gamma I \end{bmatrix} < 0 \tag{3.25}$$

has a general solution $P = I$ for all $i \in \mathcal{I}_N$. Using the same technique as in the proof of Theorem 3.1, we can prove that the switched symmetric system (3.24) is exponentially stable and has an $\mathcal{L}_2$ gain smaller than γ under arbitrary switching.

4 Concluding Remarks

In this paper, we have studied stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of a finite number of linear time-invariant symmetric subsystems. Assuming that all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we have shown for both stability and $\mathcal{L}_2$ gain analysis that there exists a general Lyapunov function $V(x) = x^{\mathrm{T}}x$ for all subsystems, and that the switched system is exponentially stable and achieves an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching.

We note finally that the result of the present paper can be extended to the switched symmetric systems in a more general sense. More precisely, if the equations $TA_i = A_i^{\mathrm{T}}T$, $TB_i = C_i^{\mathrm{T}}$, $D_i = D_i^{\mathrm{T}}$ are satisfied for a constant matrix $T > 0$, then we consider the similarity transformation $A_{\star i} = T^{1/2}A_iT^{-1/2}$, $B_{\star i} = T^{1/2}B_i$, $C_{\star i} = C_iT^{-1/2}$, $D_{\star i} = D_i$. Since stability and $\mathcal{L}_2$ gain properties of the system in this transformation do not change and we can easily confirm that $A_{\star i} = A_{\star i}^{\mathrm{T}}$, $B_{\star i} = C_{\star i}^{\mathrm{T}}$, we can apply the result we have obtained up to now for the system represented by the quadruplet $(A_{\star i}, B_{\star i}, C_{\star i}, D_{\star i})$ and thus derive corresponding result for the original switched system under arbitrary switching.

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Explicit Solutions to a Class of Linear Partial Difference Equations

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Abstract: In this paper, we present explicit solutions of a class of linear partial difference equations with constant coefficients, and two kinds of linear partial difference equations with constant coefficients are discussed and their explicit solutions are obtained. As an application we give examples to show the efficiency of the solutions.

Keywords: *Partial difference equation; combinatorial enumeration; induction.*

Mathematics Subject Classification (2000): 35D05, 35E05.

1 Introduction

Difference equations often appear in the study of numerical methods, combinatorial enumeration and system analysis [5, 6]. There are many methods for solving the linear difference equations of one argument with constant coefficients such as method of generating functions, method of Z transformation and that similar to the methods for solving the linear differential equations [1, 2]. But there are few papers on the difference equations of two variables or partial difference equations. In this paper two kinds of linear partial difference equations with constant coefficients are discussed and their explicit solutions are obtained.

2 Definitions and a Lemma

Definition 1 The following *difference equation* is called *first order linear partial difference equation with constant coefficients*

$$\alpha u(t,s) + \beta u(t,s-1) + \gamma u(t-1,s) + \delta u(t-1,s-1) + \lambda = 0, \tag{1}$$

where α, β, γ and δ are all constants, t and s are positive integers, and $\lambda = \lambda(t,s)$ is a given function.

For convenience, we will classify equations (1) as follows:
If $\alpha \neq 0,\ \gamma \neq 0$, then (1) can be written as

$$u(t,s) = -\frac{\alpha}{\beta}\, u(t,s-1) - \frac{\gamma}{\alpha}\, u(t-1,s) - \frac{\delta}{\alpha}\, u(t-1,s-1) - \frac{\lambda}{\alpha}. \tag{2}$$

If $\alpha \neq 0,\ \gamma = 0$, then (1) can be written as

$$u(t,s) = -\frac{\beta}{\alpha}\, u(t,s-1) - \frac{\delta}{\alpha}\, u(t-1,s-1) - \frac{\lambda}{\alpha}. \tag{3}$$

If $\alpha = 0,\ \beta \neq 0,\ \gamma \neq 0$, then (1) can be written as

$$u(t,s-1) = -\frac{\gamma}{\beta} u(t-1,s) - \frac{\delta}{\beta} u(t-1,s-1) - \frac{\lambda}{\beta}. \tag{4}$$

Equation (1) also can be expressed by

$$u(t,s) = -\frac{\gamma}{\beta}\, u(t-1,s+1) - \frac{\delta}{\beta}\, u(t-1,s) - \frac{\lambda}{\beta}. \tag{5}$$

If the term $u(t-1,s)$ of (4) is placed on the left-hand side, the other terms are moved to the right-hand side, and $t-1$ is replaced by t, then (4) can be written as

$$u(t,s) = -\frac{\beta}{\gamma} u(t+1,s-1) - \frac{\delta}{\gamma} u(t,s-1) - \frac{\lambda}{\gamma}. \tag{6}$$

If $\alpha = 0,\ \beta \neq 0,\ \gamma = 0$ or if $\alpha = 0,\ \beta = 0,\ \gamma \neq 0$, the equations (1) are both actually difference equations of one variable, which is not discussed here. If $\alpha = \beta = \gamma = 0$, $\delta \neq 0$, then the equation (1) is trivial one, which is also not considered here.

Without loss of generality the equations (2) − (6) can be classified in following four types:

$$\text{(A)} \qquad \begin{aligned} &u(s,t) = au(t,s-1) + bu(t-1,s) + cu(t-1,s-1) + d(t,s),\\ &u(t,0) = F(t), \quad u(0,s) = E(s). \end{aligned} \tag{7}$$

$$\text{(B)} \qquad \begin{aligned} &u(s,t) = au(t,s-1) + bu(t-1,s-1) + d(t,s),\\ &u(t,0) = F(t). \end{aligned} \tag{8}$$

$$\text{(C)} \qquad \begin{aligned} &u(s,t) = au(t-1,s+1) + bu(t-1,s) + d(t,s),\\ &u(0,s) = E(s). \end{aligned} \tag{9}$$

$$\text{(D)} \qquad \begin{aligned} &u(s,t) = au(t+1,s-1) + bu(t,s-1) + d(t,s),\\ &u(t,0) = F(t). \end{aligned} \tag{10}$$

where a, b, c are constants and $d(t,s)$ is a given function of t and s.

Definition 2 The following *difference equation* is called *second order linear partial difference equation with constant coefficients*

$$au(t,s)+bu(t,s-1)+cu(t,s-2)+d(t-1,s-1)+eu(t-1,s)+fu(t-2,s)+g(t,s) = 0. \tag{11}$$

where a, b, c, d, e, f are all constants, $t \geq 2$, $s \geq 2$ and $g = g(t,s)$ is a given function.

In this paper we only discuss equation of type (E)

$$\text{(E)} \qquad \begin{aligned} &u(t,s) = au(t-2,s) + bu(t,s-2) + g(t,s),\\ &u(t,0) = F_0(t), \quad u(t,1) = F_1(t), \quad u(0,s) = E_0(s), \quad u(1,s) = E_1(s). \end{aligned} \tag{12}$$

Lemma *For the given constant a, b, c and nonnegative integers k and n, we set*

$$R(n,k) = \sum_{\substack{i_1+i_2+\dots+i_n=k \\ i_r\ge 0\ (r=1,2\dots,n)}} \prod_{r=1}^{n} (H * b^{i_r-1})$$

where

$$R(n,k) = \begin{cases} 0, & k>0; \\ 1, & k=0. \end{cases}, \quad H = ab + c, \quad H * b^{m-1} = \begin{cases} a, & m=0; \\ Hb^{m-1}, & m\ge 1. \end{cases}$$

Then for the given function $f(t)$, $R(n,k)$ satisfies the following equation

$$\sum_{k=0}^{t} R(m,k) \sum_{j=0}^{t-k} R(1,j) f(t-k-j) = \sum_{k=0}^{t} R(m+1,\, k) f(t-k).$$

The lemma is easy to verify by induction, so the proof is omitted.

3 Solution of Explicit Expressions of the Partial Difference Equations

In the sequel, we shall give main results of this paper.

Theorem 1 *For $t\ge 1$ and $s\ge 1$,*

(1) *the solution of type* (A) *is*

$$u(t,s) = \sum_{k=0}^{t-1} R(s,k) F(t-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) M(t-k,\, s-j),$$

where

$$M(t,s) = b^{t-1}[cu(0,\, s-1) + bu(0,s)] + \sum_{i=0}^{t-1} b^i d(t-i,\, s)$$
$$= b^{t-1}[cE(s-1) + bE(s)] + \sum_{i=0}^{t-1} b^i d(t-i,\, s);$$

(2) *the solution of type* (B) *is*

$$u(t,s) = \sum_{k=0}^{s} C_s^k a^{s-k} b^k F(t-k) + \sum_{k=0}^{s-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-k,\, s-j);$$

(3) *the solution of type* (C) *is*

$$u(t,s) = \sum_{k=0}^{t} C_t^k a^{t-k} b^k E(t+s-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-k,\, s+k-j);$$

(4) *the solution of type* (D) *is*

$$u(t,s) = \sum_{k=0}^{s} C_s^k a^{s-k} b^k F(t+s-k) + \sum_{k=0}^{s-1}\sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t+k-j,\, s-k).$$

Proof We only prove (1) and (3), the other proofs are similar to each other, hence omitted.

(1) For the problem (A), $b \neq 0$. Let us set $N(t,s) = au(t,\, s-1) + cu(t-1,\, s-1) + d(t,s)$, then

$$\begin{aligned} u(t,s) &= N(t,s) + bu(t-1,s) = N(t,s) + b[N(t-1,\, s) + bu(t-2,\, s)] \\ &= N(t,s) + bN(t-1,\, s) + b^2 u(t-2,\, s) = \dots \end{aligned} \tag{13}$$

By induction we have

$$\begin{aligned} u(t,s) &= \sum_{k=0}^{t-1} b^k N(t-k,s) + b^t u(0,s) = \sum_{k=0}^{t-1} H * b^{k-1} u(t-k, s-1) + M(t,s) \\ &= \sum_{k=0}^{t-1} R(1,k) u(t-k, s-1) + M(t,s). \end{aligned} \tag{14}$$

For any positive integer i $(1 \leq i \leq s)$, it is easy to be tested by induction also that

$$u(t,s) = \sum_{k=0}^{t-1} R(i,k) u(t-k,\, s-i) + \sum_{j=0}^{i-1} \sum_{k=0}^{t-1} R(j,k) M(t-k,\, s-j), \tag{15}$$

where $i = s$ (15) is the solution to (A).

(3) The situation "$\gamma = 0$ in (1)" is equivalent to the situation "$b = 0$ in (A)". But the proof (1) is under the condition $b \neq 0$, so the solution of problem (B) can be not the solution of (A) by letting $b = 0$. Now we will prove that for any positive integer i $(1 \leq i \leq s)$ the following holds:

$$u(t,s) = \sum_{k=0}^{i} C_i^k a^{i-k} b^k u(t-k,\, s-i) + \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k). \tag{16}$$

In fact, while $i = 1$, (16) becomes (8). Suppose (16) holds for i, then for $i+1$, substituting $u(t-k,\, s-i)$ in (16) by (8) we have

$$\begin{aligned} u(t,s) = &\sum_{k=0}^{i} C_i^k a^{i-k} b^k [au(t-k,\, s-i-1) + bu(t-k-1,\, s-i-1) + d(t-k, s-i)] \\ &+ \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k) \\ = &\sum_{k=0}^{i} C_i^k a^{i+1-k} b^k u(t-k,\, s-i-1) + \sum_{k=0}^{i} C_i^k a^{i-k} b^{k+1} u(t-k-1,\, s-i-1) \\ &+ \sum_{k=0}^{i} C_i^k a^{i-k} b^k d(t-k, s-i) + \sum_{k=0}^{i-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j, s-k). \end{aligned} \tag{17}$$

The first two terms of (17) can be merged into one term by using formula $C_i^k + C_i^{k-1} = C_{i+1}^k$, the third term can be merged into the last term by substituting k by j and i by k respectively, hence

$$u(t,s) = \sum_{k=0}^{i+1} C_{i+1}^k a^{i+1-k} b^k u(t-k,\, s-i-1) + \sum_{k=0}^{(i+1)-1} \sum_{j=0}^{k} C_k^j a^{k-j} b^j d(t-j,\, s-k),$$

which shows that (16) is proved and by letting $i = s$ (16) becomes the solution of (2).

Theorem 2 *The solution to* (E) *is*

$$\begin{aligned} u(t,s) = a^{t_1} \sum_{k=0}^{s_1-1} C_{k+t_1-1}^k b^k u(\delta(t),\, s-2k) + b^{s_1} \sum_{j=0}^{t_1-1} C_{j+s_1-1}^j a^j u(t-2j,\, \delta(s)) \\ + \sum_{k=0}^{s_1-1} \sum_{j=0}^{t_1-1} C_{k+j}^k a^j b^k g(t-2j,\, s-2k), \end{aligned} \tag{18}$$

where $t_1 = \left[\dfrac{t-1}{2}\right]$, $s_1 = \left[\dfrac{s-1}{2}\right]$, $[x]$ *expresses the minimum integer which is greater than or equals to* x,

$$\delta(x) = \begin{cases} 1, & \text{if } x \text{ is odd;} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

Proof If $b = 0$ in (12), then the equation becomes the second order difference equation of one argument t. The solution can be easily calculated as follows:

$$u(t,s) = a^{t_1} u(\delta(t), s) + \sum_{k=0}^{t_1-1} a^k g(t-2k,\, s). \tag{19}$$

If $b \neq 0$ in (12), taking $u(t, s-2)$ in (12) as an iterative term calculated successively one obtains

$$\begin{aligned} u(t,s) &= au(t-2,\, s) + bu(t,\, s-2) + g(t,s) \\ &= au(t-2,\, s) + b[au(t-2,\, s-2) + bu(t,\, s-4) + g(t,\, s-2)] + g(t,s) \\ &= au(t-2,\, s) + abu(t-2,\, s-2) + b^2[au(t-2,\, s-4) \\ &\quad + bu(t,\, s-6) + g(t,\, s-4)] + bg(t,\, s-2) + g(t,s) \\ &= au(t-2,\, s) + abu(t-2,\, s-2) + ab^2u(t-2,\, s-4) + b^3u(t,\, s-6) \\ &\quad + b^2g(t, s-4) + bg(t, s-2) + g(t,s) \end{aligned}$$

Hence one can get the following by induction

$$u(t,s) = a \sum_{k=0}^{s_1-1} b^k u(t-2,\, s-2k) + b^{s_1} u(t, \delta(s)) + \sum_{k=0}^{s_1-1} b^k g(t,\, s-2k). \tag{20}$$

Now we will prove by induction again that for any positive integer i $(1 \le i \le t_1)$

$$u(t,s) = a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k u(t-2i,\, s-2k) + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s)) + \sum_{k=0}^{s_1-1}\sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k). \tag{21}$$

In fact, when $i = 1$ (21) becomes (20). Suppose (21) holds for i, then for $i+1$, substituting the term $u(t-2i,\, s-2k)$ in (21) by (20) one can get:

$$\begin{aligned} u(t,s) = a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \Bigg[a \sum_{j=0}^{\left[\frac{s-2k-1}{2}\right]-1} b^j u(t-2i-2,\, s-2k-2j) \\ + b^{\left[\frac{s-2k-1}{2}\right]-1} u(t-2i,\, \delta(s-2k)) + \sum_{j=0}^{\left[\frac{s-2k-1}{2}\right]-1} b^j g(t-2i,\, s-2k-2j) \Bigg] \\ + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s)) + \sum_{k=0}^{s_1-1}\sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k). \end{aligned} \tag{22}$$

Because

$$\left[\frac{s-2k-1}{2}\right] = \left[\frac{s-1}{2} - k\right] = \left[\frac{s-1}{2}\right] - k = s_1 - k, \quad \delta(s-2k) = \delta(s),$$

$$\sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j = \sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k, \qquad \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} = C_{s_1+i-1}^{i-1},$$

hence (22) can be written as

$$\begin{aligned} u(t,s) &= a^{i+1} \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j u(t-2i-2,\, s-2k-2j) \\ &\quad + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^{s_1} u(t-2i,\, \delta s) + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{i-1} b^k \sum_{j=0}^{s_1-k-1} b^j g(t-2i, s-2k-2j) \\ &\quad + \sum_{k=0}^{s_1-1}\sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j, s-2k) + b^{s_1} \sum_{j=0}^{i-1} C_{s_1+j-1}^{j} a^j u(t-2j, \delta s) \\ &= a^{i+1} \sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k u(t-2(i+1),\, s-2k) + b^{s_1} \sum_{j=0}^{(i+1)-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta s) \\ &\quad + a^i \sum_{k=0}^{s_1-1} C_{k+i-1}^{(i+1)-1} b^k g(t-2i,\, s-2k) + \sum_{k=0}^{s_1-1}\sum_{j=0}^{i-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k) \end{aligned}$$

$$= a^{i+1}\sum_{k=0}^{s_1-1} C_{k+(i+1)-1}^{(i+1)-1} b^k u(t-2(i+1),\, s-2k) + b^{s_1}\sum_{j=0}^{(i+1)-1} C_{s_1+j-1}^{j} a^j u(t-2j,\, \delta(s))$$
$$+\sum_{k=0}^{s_1-1}\sum_{j=0}^{(i+1)-1} C_{k+j}^{j} a^j b^k g(t-2j,\, s-2k).$$

One obtains (21). With $i = t_1 = \left[\dfrac{t-1}{2}\right]$, (21) becomes (18). It is easy to know that (19) is the exception of (18), thus the proof of Theorem 2 is completed.

4 Examples

Example 1 Find the numbers of the shortest lattice paths with diagonal steps [3].

On the coordinate plane, the number of the shortest lattice paths with diagonal steps is called Delannoy number [3], which satisfies the difference equation:

$$\begin{gathered} D(t,s) = D(t,\, s-1)D(t-1,\, s) + D(t-1,\, s-1), \\ D(t,0) = D(0,s) = 1. \end{gathered} \tag{23}$$

Problem (23) is the type of (A), where $a = b = c = 1$, $d = 0$ and its solution is

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k)F(t-k) + \sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k)M(t-k,\, s-j),$$

where $F(t) = D(t,0) = 1$, $M(t,s) = D(0,\, s-1) + D(0,s) = 2$, from which one gets

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k) + 2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k). \tag{24}$$

Because

$$H = ab + c = 2, \quad H * b^{m-1} = \begin{cases} 1, & m = 0; \\ 2, & m \ge 1; \end{cases}$$

$$R(n,k) = \sum_{\substack{i_1+i_2+\dots+i_n=k,\\ i_r\ge 0,\ (r=1,2\dots,n)}} \prod_{r=1}^{n}(H * b^{i_r-1}) = \sum_{m=0}^{n} C_n^m \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} \prod_{r=1}^{n-m}(H * b^{i_r-1}) \right)$$
$$= \sum_{m=0}^{n} C_n^m \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} 2^{n-m} \right) = \sum_{m=0}^{n} C_n^m 2^{n-m} \left(\sum_{\substack{i_1+i_2+\dots+i_{n-m}=k,\\ i_r\ge 1,\ (r=1,2\dots,n-m)}} 1 \right).$$

From the literature [4] we know that the number of the natural number solution to the equation $x_1 + x_2 + \dots + x_{n-m} = k$ is C_{k-1}^{n-m-1}, hence

$$R(n,k) = \sum_{m=0}^{n} 2^{n-m} C_n^m C_{k-1}^{n-m-1}.$$

It follows that

$$\sum_{k=0}^{t-1} R(s,k) = \sum_{k=0}^{t-1}\sum_{m=0}^{s} 2^{s-m}C_s^m C_{k-1}^{s-m-1} = \sum_{m=0}^{s} 2^{s-m}C_s^m\left(\sum_{k=0}^{t-1} C_{k-1}^{s-m-1}\right)$$
$$= \sum_{m=0}^{s} 2^{s-m}C_s^m\left(\sum_{k=s-m}^{t-1} C_{k-1}^{s-m-1}\right) = \sum_{m=0}^{s} 2^{s-m}C_s^m C_{t-1}^{s-m} = \sum_{m=0}^{s} 2^m C_s^m C_{t-1}^m,$$

where

$$\sum_{k=s-m}^{t-1} C_{k-1}^{s-m-1} = C_{t-1}^{s-m}.$$

Calculating shows that

$$2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) = 2\sum_{j=0}^{s-1}\left(\sum_{k=0}^{t-1} R(j,k)\right) = 2\sum_{j=0}^{s-1}\sum_{m=0}^{j} 2^m C_j^m C_{t-1}^m$$
$$= 2\sum_{m=0}^{s-1}\sum_{j=m}^{s-1} 2^m C_j^m C_{t-1}^m = 2\sum_{m=0}^{s-1} 2^m C_{t-1}^m\left(\sum_{j=m}^{s-1} C_j^m\right)$$
$$= 2\sum_{m=0}^{s-1} 2^m C_{t-1}^m C_s^{m+1} = \sum_{m=0}^{s-1} 2^{m+1} C_{t-1}^m C_s^{m+1},$$

where

$$\sum_{j=m}^{s-1} C_j^m = C_s^{m+1}.$$

With the above result one gets

$$D(t,s) = \sum_{k=0}^{t-1} R(s,k) + 2\sum_{k=0}^{t-1}\sum_{j=0}^{s-1} R(j,k) = \sum_{m=0}^{s} 2^m C_s^m C_{t-1}^m + \sum_{m=0}^{s-1} 2^{m+1} C_s^{m+1} C_{t-1}^m$$
$$= \sum_{m=0}^{s} 2^m C_s^m (C_{t-1}^m + C_{t-1}^{m-1}) = \sum_{m=0}^{s} 2^m C_s^m C_t^m.$$

This is the very result given out of [3].

Example 2 Find all eigenvalues of the following matrix B

$$B = \begin{pmatrix} \alpha & & & \gamma & & \\ & \ddots & & & & \\ & & \alpha & & & \beta \\ & & & \ddots & & \\ & \gamma & & & \ddots & \\ & & \beta & & & \alpha \end{pmatrix},$$

where the matrix B has the following characteristics:

1. All the main diagonal elements of B are α.

2. All the skew diagonal elements of the downright square submatrix of B are β. If the number of the submatrix's order is odd, then the cross-element of the main diagonal line of the submatrix and its skew diagonal line is still α (the line connecting the upright corner of a square matrix with its downleft corner is called skew diagonal line).

3. The remained elements of B are all γ.

To find the eigenvalues of the matrix B is to find the roots of equation $\det(\lambda I - B) = 0$, where I is a unit matrix of order equivalent to B. Let us first calculate $\det B$. In fact, the matrix B is similar to matrix $A(t,s)$ where

$$A(t,s) = \begin{pmatrix} \alpha & \beta & & & & & & & & \\ \beta & \alpha & & & & & & & & \\ & & \alpha & \beta & & & & & & \\ & & \beta & \alpha & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \alpha & \beta & & & \\ & & & & & \beta & \alpha & & & \\ & & & & & & & \alpha & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \alpha \end{pmatrix}_{(t+s)\times(t+s)} \tag{25}$$

$(t \ge s \ge 0)$ is a matrix of order $(t+s)\times(t+s)$, where there are s submatrixes of order 2: $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, and a submatrix of order $(t-s)$: $\begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix}$, the unlisted elements are all γ, which is shown above.

It is easy to know that

$$\det B = \det A(t,s) = 2(\alpha-\beta)\det A(t,s-1) - (\alpha-\beta)^2 \det A(t-1,\, s-1), \qquad t \ge s \ge 1. \tag{26}$$

The initial condition is that

$$\det A(t,0) = (\alpha-\gamma)^{t-1}[\alpha + (t-1)\gamma], \quad \det A(0,0) = 1.$$

This is the very type of problem (B), where

$$a = 2(\alpha-\beta), \quad b = -(\alpha-\beta)^2, \quad c = 0, \quad F(t) = (\alpha-\gamma)^{t-1}[\alpha+(t-1)\gamma].$$

The solution of it can be calculated as follows

$$\begin{aligned} \det A(t,s) = {} & (\alpha-\beta)^s(\alpha+\beta-2\gamma)^{s-1}(\alpha-\gamma)^{t-s-1} \\ & \times \{(\alpha+\beta-2\gamma)[\alpha+(t-1)\gamma] + s\gamma(\alpha-\beta)\}. \end{aligned} \tag{27}$$

Substituting elements α, β and γ of matrix B by $\lambda-\alpha$, $-\beta$ and $-\gamma$ respectively, then we can get immediately eigenvalues of B:

(i) If $t > s > 1$, then

$\lambda_1 = (\alpha - \beta)$ is a root of s-multiplicity;

$\lambda_2 = (\alpha - \gamma)$ is a root of $(t - s - 1)$-multiplicity;

$\lambda_3 = (\alpha + \beta - 2\gamma)$ is a root of $(s - 1)$-multiplicity;

$$\lambda_4 = \frac{1}{2}\Big\{[(\alpha + \beta - 2\gamma) + (\alpha + (t-1)\gamma) + s\gamma] + \sqrt{[(\alpha + \beta - 2\gamma) + (\alpha + (t+s-1)\gamma)]^2 - 4[(\alpha + \beta - 2\gamma)(\alpha + (t-1)\gamma + s\gamma(\alpha - \beta))]}\Big\},$$

$$\lambda_5 = \frac{1}{2}\Big\{[(\alpha + \beta - 2\gamma) + (\alpha + (t-1)\gamma) + s\gamma] - \sqrt{[(\alpha + \beta - 2\gamma) + (\alpha + (t+s-1)\gamma)]^2 - 4[(\alpha + \beta - 2\gamma)(\alpha + (t-1)\gamma + s\gamma(\alpha - \beta))]}\Big\}$$

are two single roots.

(ii) If $t = s \geq 1$, then

$$\det A(t,s) = (\alpha - \beta)^s(\alpha + \beta - 2\gamma)^{s-1}(\alpha + \beta + (s-2)\gamma),$$

$\lambda_1 = (\alpha - \beta)$ is a root of s-multiplicity;
$\lambda_2 = (\alpha + \beta - 2\gamma)$ is a root of $(s - 1)$-multiplicity;
$\lambda_3 = (\alpha + \beta - 2\gamma)$ is a single root.

(iii) If $t > s = 0$, then

$$\det A(t,s) = (\alpha - \gamma)^{t-1}(\alpha + (t-1)\gamma);$$

$\lambda_1 = \alpha - \gamma$ is a root of $(t - 1)$-multiplicity;
$\lambda_2 = \alpha + (t - 1)\gamma$ is a single root.

5 Conclusions

This paper has been focused on the study of the solution's explicit expressions of some kind of partial difference equations. The method is very simple, but the results can be used in the study of some kind of combinatorial enumerations and some other related fields.

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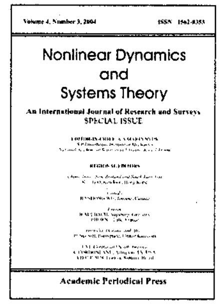

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