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A Criterion for Stability of Nonlinear Time-Varying Dynamic System

Yongmao Wang[1], Yan Shi[2] and Hirofumi Sasaki[3]

[1] *School of Science, Yanshan University, Qinhuangdao, 066004, China*
[2] *School of Information Science, Kyushu Tokai University, Kumamoto, 862-8652, Japan*
[3] *School of Engineering, Kyushu Tokai University, Kumamoto, 862-8652, Japan*

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Abstract: In this paper, based on the assumption that both the leading principal submatrix of r-order and its complementary submatrix in $A(t)$ have eigenvalues with only negative real parts, we establish a criterion for the stability of a class of nonlinear time-varying dynamic system $dx/dt = A(t)x + f(t,x)$. Also a feasible method for decomposition and aggregation of large-scale system is provided. Moreover, we shall show the efficiency of the presented criterion by a numerical example.

Keywords: *Vector Liapunov function; nonlinear time-varying dynamic system; stability of system.*

Mathematics Subject Classification (2000): 93D30, 34D20, 93D05; 93A15.

1 Introduction

The problem of constructing Liapunov functions for non-autonomous systems in general case still remains open. The concept of vector Liapunov functions (see [1, 2]) in terms of differential inequalities (see Lakshmikantham, *et al.* [3]) allowed to express the existence conditions for certain dynamical properties of the initial system via the existence of the corresponding properties in the comparison system. This approach has been intensively developed in the stability investigation of large-scale systems (see [4−6]). For recent results of the direct Liapunov method development and some approaches to the problem of Liapunov functions construction see [7−10].

In this paper, based on the assumption that both the leading principal submatrix of matrix $A(t)$ and its complementary submatrix have eigenvalues with only negative real parts, we give a feasible method of constructing vector Liapunov function of dynamic system (1), and establish sufficient conditions for stability of the system

$$\frac{dx}{dt} = A(t)x + f(x,t), \tag{1}$$

where $A(t) = a_{ij}(t)_{n\times n}$, $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, $f(x,t) = (f_1(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n))^{\mathrm{T}}$, $a_{ij}(t)$ is differentiable and bounded on $[0, +\infty)$, $f(x,t)$ is continuous on field $t \geq 0$, $|x| \leq h$, $i = 1, 2, \dots, n$, and assume the system (1) have unique solution for any initial condition on the field.

Moreover, we also extend the result [11], and show that it is a special case of this paper for $r = 1$, $m = n - 1$. Finally, we give a numerical example to show the efficiency of the presented criterion.

2 Notations and Definitions

Let $A(t) = (a_{ij})_{n\times n}$, and partition $A(t)$ into the following:

$$A(t) = \begin{bmatrix} A_r & A_{r\times m} \\ A_{m\times r} & A_m \end{bmatrix}, \quad m = n - r, \quad 1 \leq r < n, \tag{2}$$

where A_r is a $r \times r$ matrix, which is called leading principal submatrix of order r and A_m is an $m \times m$ matrix, called complementary submatrix of A_r. The matrix $B(s,n)$ of order $(n-s+1)(n-s+2)/2$ is defined as

$$B(s,n) = \begin{bmatrix} a(ss,ss)+\delta_{ss} & \dots & a(sn,ss) & \dots & a(nn,ss) \\ \dots & \dots & \dots & \dots & \dots \\ a(ss,sn) & \dots & a(sn,sn)+\delta_{sn} & \dots & a(nn,sn) \\ a(ss,(s+1)(s+1)) & \dots & a(sn,(s+1)(s+1)) & \dots & a(nn,(s+1)(s+1)) \\ \dots & \dots & \dots & \dots & \dots \\ a(ss,nn) & \dots & a(sn,nn) & \dots & a(nn,nn)+\delta_{nn} \end{bmatrix}. \tag{3}$$

When $s = 1$ and $n = r$, let $B_r = B(1,r)$; when $s = r+1$ and $n = n$, let $B_m = B(r+1, n)$. Thus, B_r is a matrix of order $r(r+1)/2$, and B_m is a matrix of order $m(m+1)/2$, where $m = n - r$. The elements $a(ik, jl)$ in either matrix B_r or B_m satisfy the equalities $a(ik,jl) = a(ki,jl) = a(ki,lj)$ and

$$a(ik,jl) = \begin{cases} 0, & \text{if} \quad i \neq j, \quad k \neq l, \quad k \neq j, \quad j \neq l, \\ a_{kl}, & \text{if} \quad i = j, \quad k \neq l, \\ a_{ii} + a_{kk}, & \text{if} \quad i = j, \quad k = l, \quad i \neq k, \\ a_{ii}, & \text{if} \quad i = j = k = l, \end{cases}$$

where a_{ij} is an element either in A_r for $i, j = 1, 2, \dots, r$, or in A_m for $i, j = r+1, \dots, n$.

In matrix B_r, $\delta_{ik} = \alpha/2$ if $i = k$, and $\delta_{ik} = \alpha$ if $i \neq k$. In matrix B_m, $\delta_{ik} = \delta/2$ if $i = k$, and $\delta_{ik} = \delta$ if $i \neq k$. And $\alpha = \min(\inf|\operatorname{Re}\lambda_1|, \dots, \inf|\operatorname{Re}\lambda_r|)$ and $\delta = \min(\inf|\operatorname{Re}\mu_1|, \dots, \inf|\operatorname{Re}\mu_s|)$, where λ_i and μ_j are eigenvalues of A_r and A_m $(i = 1, \dots, r;\ j = 1, \dots, m)$, respectively.

Let p be an unknown variable, and $p_1, \dots, p_r$ be r roots of the equation

$$\left| \left(p - \frac{\alpha}{2}\right) E_r - A_r \right| = p^r + a_1 p^{r-1} + \dots + a_r = 0, \tag{4}$$

and let

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \dots, \quad \Delta_r = \begin{vmatrix} a_1 & a_3 & \dots & a_{2r-1} \\ a_0 & a_2 & \dots & a_{2r-2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_r \end{vmatrix},$$

where $a_0 = 1$, and $a_k = 0$ for $k > r$.

Let q be an unknown variable, and $q_1, \dots, q_m$ be m roots of the equation

$$\left|\left(q - \frac{\delta}{2}\right)E_m - A_m\right| = q^m + b_1 q^{m-1} + \dots + b_m = 0, \tag{5}$$

and let

$$\Delta_1^* = b_1, \quad \Delta_2^* = \begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix}, \quad \dots, \quad \Delta_r^* = \begin{vmatrix} b_1 & b_3 & \dots & b_{2m-1} \\ b_0 & b_2 & \dots & b_{2m-2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_m \end{vmatrix}$$

where $b_0 = 1$, and $b_k = 0$ for $k > m$.

The quadratic forms ω_1 and v_1 are respectively defined as

$$\omega_1 = -\Delta_1\Delta_2 \dots \Delta_r(x_1^2 + x_2^2 + \dots = +x_r^2), \tag{6}$$

$$v_1 = \frac{\prod_{i=1}^{r} \Delta_i}{\det B_r} \begin{vmatrix} 0 & x_1^2 \quad 2x_1x_2 \quad \dots \quad 2x_1x_r \quad x_2^2 \quad \dots \quad 2x_{r-1}x_r \quad x_r^2 \\ 1 & \\ 0 & \\ \vdots & \\ 1 & B_r \\ 0 & \\ \vdots & \\ 1 & \end{vmatrix} = c_1 \sum_{i,j=1}^{r} v_{ij}x_ix_j, \tag{7}$$

where $c_1 = \Delta_1\Delta_2 \dots \Delta_r/\det B_r$, and for $i, j = 1, 2, \dots, r$, v_{ij} and v_{ji} are both half the algebraic cofactor of the element $2x_ix_j$, while v_{ii} is the algebraic cofactor of x_i^2.

The quadratic forms ω_2 and v_2 are respectively defined as

$$\omega_2 = -\Delta_1^*\Delta_2^* \dots = \Delta_m^*(x_{r+1}^2 + x_{r+2}^2 + \dots + x_n^2), \tag{8}$$

$$v_2 = \frac{\prod_{i=1}^{m} \Delta_i^*}{\det B_m} \begin{vmatrix} 0 & x_{r+1}^2 \quad 2x_{r+1}x_{r+2} \quad \dots \quad 2x_{r+1}x_n \quad x_{r+2}^2 \quad \dots \quad 2x_{n-1}x_n \quad x_n^2 \\ 1 & \\ 0 & \\ \vdots & \\ 1 & B_m \\ 0 & \\ \vdots & \\ 1 & \end{vmatrix} \tag{9}$$

$$= c_2 \sum_{i,j=r+1}^{n} v_{ij}x_ix_j,$$

where $c_2 = \Delta_1^*\Delta_2^*\dots\Delta_m^*/\det B_m$ and for $i,j = r+1, r+2, \dots, n$, v_{ij} and v_{ji} are both half the algebraic cofactor of the element $2x_ix_j$, while v_{ij} is the algebraic cofactor of x_i^2.

For all $t \in [t_0, +\infty)$, the meanings of the letters v_1^*, v_2^*, Δ, Δ^*, b_r, b_m, M_1, M_2, β, γ, ε_1 and ε_2 are given by the following equalities, respectively:

$$v_1^* = \inf_{x_1^2+\dots+x_r^2=1} v_1(t, x_1, \dots, x_r), \quad v_2^* = \inf_{x_{r+1}^2+\dots+x_n^2=1} v_2(t, x_{r+1}, \dots, x_n),$$

$$\Delta = \sup(\Delta_1\Delta_2\dots\Delta_r), \quad \Delta^* = \sup(\Delta_1^*\Delta_2^*\dots\Delta_m^*),$$

$$b_r = \inf|\det B_r|, \quad b_m = \inf|\det B_m|,$$

$$M_1 = \sup(|v_{ij}|,\ i,j = 1,2,\dots,r), \quad M_2 = \sup(|v_{i=j}|,\ i,j = r+1, r+2, \dots, n),$$

$$\varepsilon_1 < \frac{b_r}{3r^2M_1}, \quad \varepsilon_2 < \frac{b_r}{3(n-r)^2M_2}, \quad \beta = r(n-r)M_1\left(\frac{\Delta r M_1 m_1^2}{b_r^2} + \frac{\Delta\varepsilon_1}{b_r}\right)\Big/ v_2^*,$$

$$\gamma = r(n-r)M_2\left(\frac{\Delta^*(n-r)M_2m_2^2}{b_m^2} + \frac{\Delta^*\varepsilon_2}{b_m}\right)\Big/ v_1^*,$$

where m_1 and m_2 are positive numbers.

3 Main Results

In the sequel, we shall give main results of this paper, that is, a criterion for stability of nonlinear time-varying dynamic system (1), and show the efficiency of the presented criterion by a numerical example.

3.1 A criterion for stability of nonlinear time-varying dynamic system

Theorem 3.1 *The trivial solution of (1) is asymptotically stable if*

(i) $\operatorname{Re}\lambda_i \le -\alpha < 0$, $\operatorname{Re}\mu_j \le -\delta < 0$, $i = 1,\dots,r$, $j = 1,\dots,m$;

(ii) *every* a_{ij} $(i,j = 1,\dots,n)$ *is differentiable and bounded on* $[t_0, +\infty)$, *especially, when* a_{ij} *is an element of* $A_{r\times m}$, $|a_{ij}| \le m_1$, *when* a_{ij} *is that of* $A_{m\times r}$, $|a_{ij}| \le m_2$;

(iii) $\alpha\delta - \beta\gamma > 0$, $|f_i(t, x_1, t_2, \dots, x_n)| \le \varepsilon(|x_1| + |x_2| + \dots + |x_n|)$, $i = 1,\dots,n$;

(iv) $\tilde{\lambda}_i < \left(1 - \dfrac{3r^2M_1\varepsilon}{b_r}\right)\Delta_1\dots\Delta_r$, $\tilde{\mu}_j < \left(1 - \dfrac{3(n-r)^2M_2\varepsilon}{b_m}\right)\Delta_1^*\dots\Delta_s^*$, *where* $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, $\tilde{\lambda}_i$ *and* $\tilde{\mu}_j$ *are eigenvalues of the matrixes* $\tilde{A}_r$ *and* $\tilde{A}_m$ *respectively, where*

$$\tilde{A}_r = ((c_1v_{ij})')_{r\times r}, \quad \tilde{A}_m = ((c_2v_{i=j})')_{m\times m}.$$

Proof Partition (1) into two correlative subsystems

$$\frac{d\zeta_1}{dt} = A_{11}(t)\zeta_1 + A_{12}(t)\zeta_2 + f^*(x,t), \tag{10}$$

$$\frac{d\zeta_2}{dt} = A_{21}(t)\zeta_1 + A_{22}(t)\zeta_2 + f^{**}(x,t), \tag{11}$$

where

$$A(t) = [a_{ij}(t)|_{n\times n} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}, \qquad i,j = 1,2,\dots,n,$$

$$A_{11} = \begin{bmatrix} a_{11}(t) & \dots & a_{1r}(t) \\ \dots & \dots & \dots \\ a_{r1}(t) & \dots & a_{rr}(t) \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} a_{r+1,r+1}(t) & \dots & a_{r+1,n}(t) \\ \dots & \dots & \dots \\ a_{n,r+1}(t) & \dots & a_{n,n}(t) \end{bmatrix},$$

$$f^* = [f_1(x,t),\dots,f_r(x,t)]^{\mathrm{T}}, \quad f^{**} = [f_{r+1}(x,t),\dots,f_n(x,t)]^{\mathrm{T}},$$

$$\zeta_1 = (x_1,\dots,x_r)^{\mathrm{T}}, \quad \zeta_2 = (x_{r+1},\dots,x_n)^{\mathrm{T}}.$$

Taking v_1 and v_2 as components of Liapunov function of systems (10) and (11) respectively, we have the following results:

$$\frac{dv_1}{dt}\bigg|_{(10)} = \nabla_x v_1(x,t)A_{11}(t)\zeta_1 + \nabla_x v_1(x,t)A_{12}(t)\zeta_2 + \frac{\partial v_1}{\partial t} + \nabla_x f^*, \tag{12}$$

$$\frac{dv_2}{dt}\bigg|_{(11)} = \nabla_x v_2(x,t)A_{21}(t)\zeta_1 + \nabla_x v_2(x,t)A_{22}(t)\zeta_2 + \frac{\partial v_2}{\partial t} + \nabla_x f^{**}. \tag{13}$$

Obviously, the eigenvalue λ_i of A_r, and the root p_i of (4) are related by expression $p_i = \lambda_i + \alpha/2$, which shows $\operatorname{Re} p_i \le -\alpha/2$ when $\operatorname{Re}\lambda_i \le -\alpha$ for $i = 1,\dots,r$. Hence, $\Delta_1 > 0$, $\Delta_2 > 0$, …, $\Delta_r > 0$. Moreover, $\Delta_2 \dots \Delta_r > k$, where k is such a positive as is decided by α, and not dependent on t.

Based on [12], we can prove that v_1 is positively definite function, and obtain the following result

$$\sum_{i=1}^{r} \frac{\partial v_1}{\partial x_i}\left[a_{i1}x_1 + \dots + \left(a_{ii} + \frac{\alpha}{2}\right)x_i + \dots + a_{ir}x_r\right] = 2\omega_1. \tag{14}$$

According to Barbashin formula [13], the v_1 is unique quadratic form that satisfies the equality (14). Therefore, v_1 should be in accordance with Liapunov function constructed in [12], that is,

$$v_1 = \Delta_2(t)\dots\Delta_r(t)\sum_{j=1}^{r} x_j^2 + \sum_{\sigma=1}^{r-1}\sum_{j=1}^{r}\prod_{s=1(\ne\sigma\pm1)}^{r} \Delta_s(t)\Delta_{\sigma j}^2(t)(x_1\dots x_r), \tag{15}$$

where the meaning of $\Delta_{\sigma j}$ is the same as in [12], if $a_{ii} + \alpha/2$ is substituted for a_{ii} in $\Delta_{\sigma j}$ from [12] for $i = 1,\dots,r$. Consequently, the following inequality holds

$$v_1 \ge \Delta_2\dots\Delta_r\sum_{j=1}^{r} x_j^2 \ge k\sum_{j=1}^{r} x_j^2$$

which means that v_1 is positive definite with respect to t, x_1, …, x_r. It can be proved similarly that quadratic form v_2 is positive definite with respect to t, x_{r+1}, …, x_n.

By means of Euler theorem on homogeneous function we can change (14) into

$$\sum_{i=1}^{r}\frac{\partial v_1}{\partial x_i}[a_{i1}x_1+\cdots+a_{ii}x_i+\cdots+a_{ir}x_r]=-\frac{\alpha}{2}\sum_{i=1}^{r}x_i\frac{\partial v_1}{\partial = x_i}+2\omega_1=-\alpha v_1+2\omega_1, \quad (16)$$

namely,

$$\nabla_x v_1(x,t)A_{11}(t)\zeta_1=-\alpha v_1+2\omega_1.$$

For the same reason, it can be done that

$$\nabla_x v_2(x,t)A_{22}(t)\zeta_2=-\frac{\delta}{2}\sum_{i=r+1}^{n}x_i\frac{\partial v_2}{\partial x_i}+2\omega_2=-\delta v_2+2\omega_2. \quad (17)$$

Calculating the second terms on the right-hand side of (12), we have

$$\begin{aligned}
&\nabla_x v_1(x,t)A_{12}(t)\zeta_2\\
&\quad=\left(a_{1,r+1}\frac{\partial v_1}{\partial x_1}+a_{2,r+1}\frac{\partial v_1}{\partial x_2}+\cdots+a_{r,r+1}\frac{\partial v_1}{\partial x_r}\right)x_{r+1}+\dots\\
&\quad\quad+\left(a_{1,n}\frac{\partial v_1}{\partial x_1}+a_{2,n}\frac{\partial v_1}{\partial x_2}+\cdots+a_{r,n}\frac{\partial v_1}{\partial x_r}\right)x_n\\
&\quad=2c_1\Bigg(a_{1,r+1}\sum_{j=1}^{r}v_{1j}x_j+a_{2,r+1}\sum_{j=1}^{r}v_{2j}x_j+\dots\\
&\quad\quad+a_{r,r+1}\sum_{j=1}^{r}v_{rj}x_j\Bigg)x_{r+1}+\dots\\
&\quad\quad+2c_1\left(a_{1,n}\sum_{j=1}^{r}v_{1j}x_j+a_{2,n}\sum_{j=1}^{r}v_{2j}x_j+\cdots+a_{r,n}\sum_{j=1}^{r}v_{rj}x_j\right)x_n\\
&\quad=2c_1x_1\left(x_{r+1}\sum_{i=1}^{r}a_{i,r+1}v_{i1}+\cdots+x_n\sum_{i=1}^{r}a_{i,n}v_{i1}\right)\\
&\quad\quad+2c_1x_2\left(x_{r+1}\sum_{i=1}^{r}a_{i,r+1}v_{i2}+\cdots+x_n\sum_{i=1}^{r}a_{i,n}v_{i2}\right)+\dots\\
&\quad\quad+2c_1x_r\left(x_{r+1}\sum_{i=1}^{r}a_{i,r+1}v_{ir}+\cdots+x_n\sum_{i=1}^{r}a_{i,n}v_{ir}\right). \qquad (18)
\end{aligned}$$

In order to reduce the sum on the right-hand side of (12) into the form of linear combination of v_1 and v_2, we set up the estimation, with the aid of condition (ii) and inequality $-az^2+bz\le -az^2/2+b^2/2a \ (a>0)$, as follows:

$$\begin{aligned}
&-2c_1x_1^2\det B_r+2c_1x_1\left(x_{r+1}\sum_{i=1}^{r}a_{i,r+1}v_{i1}+\cdots+x_n\sum_{i=1}^{r}a_{i,n}v_{i1}\right)\\
&\qquad\le -c_1x_1^2\det B_r+\frac{c_1}{\det B_r}\left(x_{r+1}\sum_{i=1}^{r}a_{i,r+1}v_{i1}+\cdots+x_n\sum_{i=1}^{r}a_{i,n}v_{i1}\right)^2
\end{aligned}$$

$$\leq -c_1 x_1^2 \det B_r + \frac{c_1 m_1^2}{\det B_r}\left(\sum_{i=1}^{r}|v_{i1}|\right)^2 (|x_{r+1}| + \dots + |x_n|)^2$$

. . . .

$$- 2c_1 x_r^2 \det B_r + 2c_1 x_r\left(x_{r+1}\sum_{i=1}^{r} a_{i,r+1}v_{ir} + \dots + x_n\sum_{i=1}^{r} a_{i,n}v_{ir}\right)$$
$$\leq -c_1 x_r^2 \det B_r + \frac{c_1}{\det B_r}\left(x_{r+1}\sum_{i=1}^{r} a_{i,r+1}v_{ir} + \dots + x_n\sum_{i=1}^{r} a_{i,n}v_{ir}\right)^2$$
$$\leq -c_1 x_r^2 \det B_r + \frac{c_1 m_1^2}{\det B_r}\left(\sum_{i=1}^{r}|v_{ir}|\right)^2 (|x_{r+1}| + \dots + |x_n|)^2.$$

The inequality obtained by adding corresponding terms on both sides of r inequalities above shows that

$$\begin{aligned}\left.\frac{dv_1}{dt}\right|_{(10)} &\leq -\alpha v_1 + \omega_1 + \frac{c_1 m_1^2}{\det B_r}(|x_{r+1}| + \dots + |x_n|)^2 \\ &\times\left[\left(\sum_{i=1}^{r}|v_{i1}|\right)^2 + \left(\sum_{i=1}^{r}|v_{i2}|\right)^2 + \dots + \left(\sum_{i=1}^{r}|v_{ir}|\right)^2\right] + \frac{\partial v_1}{\partial t} + |\nabla_x v_1 f^*| \\ &\leq -\alpha v_1 + \omega_1 + \frac{c_1 m_1^2}{\det B_r}(|x_{r+1}| + \dots + |x_n|)^2 \sum_{i,j=1}^{r} v_{ij}^2 + \frac{\partial v_1}{\partial t} + |\nabla_x v_1 f^*|.\end{aligned} \tag{19}$$

We estimate the last sum expression on the right-hand side of (19)

$$\begin{aligned}|\nabla_x v_1 f^*| &= \left|2c_1\sum_{i=1}^{r}\sum_{j=1}^{r} v_{ij}x_j f_i(t, x_1, \dots, x_n)\right| \\ &\leq 2\varepsilon|c_1|(|x_1| + |x_2| + \dots + |x_n|)\left(\sum_{i=1}^{r}\sum_{j=1}^{r}|v_{ij}|\,|x_j|\right) \\ &\leq 2\varepsilon|c_1|\, rM_1(|x_1| + |x_2| + \dots + |x_n|)(|x_1| + |x_2| + \dots + |x_r|) \\ &\leq 3r^2|c_1|M_1\varepsilon(x_1^2 + \dots + x_r^2) + r(n-r)|c_1|M_1\varepsilon(x_{r+1}^2 + \dots + x_n^2).\end{aligned} \tag{20}$$

Based on the deduction above, for (19) there is following estimation:

$$\begin{aligned}\left.\frac{dv_1}{dt}\right|_{(10)} &= -\alpha v_1 + (3r^2|c_1|M_1\varepsilon - \Delta_1\Delta_2\dots\Delta_r)(x_1^2 + \dots + x_r^2) \\ &\quad + (x_{r+1}^2 + \dots + x_n^2)\beta v_2^* + \frac{\partial v_1}{\partial t} \\ &\leq -\alpha v_1 + \beta v_2 + [(x_{r+1}^2 + \dots + x_n^2)\beta v_2^* - \beta v_2] \\ &\quad + \left[\Delta_1\Delta_2\dots\Delta_r\left(\frac{3r^2 M_1\varepsilon}{b_r} - 1\right)(x_1^2 + \dots + x_r^2) + \frac{\partial v_1}{\partial t}\right].\end{aligned} \tag{21}$$

In the same way, taking (17) into consideration, we can obtain

$$\left.\frac{dv_2}{dt}\right|_{(11)} = -\delta v_2 + 2\omega_2 + 2c_2 x_{r+1}\left(x_1\sum_{i=r+1}^{n} a_{i1}v_{i,r+1} + \dots + x_r\sum_{i=r+1}^{n} a_{ir}v_{i,r+1}\right)$$

$$+ 2c_2x_{r+2}\left(x_1\sum_{i=r+1}^{n}a_{i1}v_{i,r+2} + \dots + x_r\sum_{i=r+1}^{n}a_{ir}v_{i,r+2}\right) + \dots$$
$$+ 2c_2x_n\left(x_1\sum_{i=r+1}^{n}a_{i1}v_{i,n} + \dots + x_r\sum_{i=r+1}^{n}a_{ir}v_{i,n}\right) + \frac{\partial v_2}{\partial t} + |\nabla_x v_2 f^{**}|.$$

Similarly to the way of getting (21), we can estimate (13) as follows

$$\begin{aligned}\left.\frac{dv_2}{dt}\right|_{(11)} &= -\delta v_2 + 2\omega_2 + \frac{c_2m_2^2}{\det B_m}(|x_1| + \dots + |x_r|)^2 \\ &\quad\times\left[\left(\sum_{i=r+1}^{n}|v_{i,r+1}|\right)^2 + \dots + \left(\sum_{i=r+1}^{n}|v_{in}|\right)^2\right] + \frac{\partial v_2}{\partial t} + \nabla_x v_2 f^{**} \\ &\le -\delta v_2 + \omega_2 + \frac{c_2m_2^2}{\det B_m}(|x_1| + \dots + |x_r|)^2\sum_{i,j=r+1}^{n}v_{ij}^2 + \frac{\partial v_2}{\partial t} + \nabla_x v_2 f^{**} \\ &\le -\delta v_2 + \gamma v_1 + [\gamma v_1^*(x_1^2 + \dots + x_n^2) - \gamma v_1] \\ &\quad + \left[\Delta_1^*\Delta_2^*\dots\Delta_m^*\left(\frac{3(n-r)^2M_2\varepsilon}{b_m} - 1\right)(x_{r+1}^2 + \dots + x_n^2) + \frac{\partial v_2}{\partial t}\right].\end{aligned} \tag{22}$$

The simultaneous existence of (21) and (22) leads to the inequality system

$$\begin{aligned}\frac{dv_1}{dt} &\le -\alpha v_1 + \beta v_2 + [(x_1^2 + \dots + x_n^2)\beta v_2^* - \beta v_2] + \left[1 - \frac{3r^2M_1\varepsilon}{b_r}\right]\omega_1 + \frac{\partial v_1}{\partial t}, \\ \frac{dv_2}{dt} &\le \gamma v_1 - \delta v_2 + [(x_1^2 + \dots + x_r^2)\gamma v_1^* - \gamma v_1] + \left[1 - \frac{3(n-r)^2M_2\varepsilon}{b_m}\right]\omega_2 + \frac{\partial v_2}{\partial t}.\end{aligned} \tag{23}$$

Let $x_i = \rho_1\alpha_i$, where $i = r+1, r+2, \dots, n$, $\rho_1 = \sqrt{x_{r+1}^2 + \dots + x_n^2}$, then $\alpha_{r+1}^2 + \dots + \alpha_n^2 = 1$. It follows for arbitrary $t \in [t_0, +\infty)$ that

$$v_2^* = \inf_{x_{r+1}^2 + \dots + x_n^2 = 1} v_2(t, x_{r+1}, \dots, x_n) = \inf v_2(t, \alpha_{r+1}, \dots, \alpha_n) > 0.$$

The sum of the first expression in the system of inequalities (23) is

$$(x_{r+1}^2 + \dots + x_n^2)\beta v_2^* - v_2(t, x_{r+1}, \dots, x_n)\beta \le [v_2^* - v_2(t, \alpha_{r+1}, \dots, \alpha_n)]\beta\rho_1^2 = 0. \tag{24}$$

For the same reason, it follows that

$$(x_1^2 + \dots + x_r^2)\gamma v_1^* - v_1(t, x_1, \dots, x_r)\gamma \le [v_1^* - v_1(t, \alpha_1, \dots, \alpha_r)]\gamma\rho_2^2 = 0, \tag{25}$$

where $\rho_2 = \sqrt{x_1^2 + \dots + x_n^2}$.

Since $\tilde{A}_r$ and $\tilde{A}_m$ are real symmetric matrices, there exist orthogonal transformations

$$\zeta = P_r\eta \quad \text{and} \quad \zeta_1 = P_m\eta_1$$

to make the following equations to hold.

$$\frac{\partial v_1}{\partial t} = \zeta^{\mathrm{T}} \tilde{A}_r \zeta = \eta^{\mathrm{T}} P_r^{\mathrm{T}} \tilde{A}_r P_r \eta = \tilde{\lambda}_1 y_1^2 + \dots + \tilde{\lambda}_r y_r^2,$$
$$\frac{\partial v_2}{\partial t} = \zeta_1^{\mathrm{T}} \tilde{A}_m \zeta_1 = \eta_1^{\mathrm{T}} P_m^{\mathrm{T}} \tilde{A}_m P_m \eta_1 = \tilde{\mu}_1 y_{r+1}^2 + \dots + \tilde{\mu}_m y_n^2,$$

where

$$\eta = (y_1, \dots, y_r)^{\mathrm{T}}, \quad \eta_1 = (y_{r+1}, \dots, y_n)^{\mathrm{T}},$$

P_r and P_m are orthogonal matrices in correspondence with order r and m $(m = n - r)$, respectively. By using the orthogonal transformations above, we can change ω_1 and ω_2 into:

$$\begin{aligned} -\omega_1 &= \Delta_1 \Delta_2 \dots \Delta_r \zeta^{\mathrm{T}} E_r \zeta = \Delta_1 \Delta_2 \dots \Delta_r \eta^{\mathrm{T}} P_r^{\mathrm{T}} P_r \eta \\ &= \Delta_1 \Delta_2 \dots \Delta_r (y_1^2 + \dots + y_r^2), \end{aligned} \tag{26}$$

$$\begin{aligned} -\omega_2 &= \Delta_1^* \Delta_2^* \dots \Delta_m^* \zeta_1^{\mathrm{T}} E_m \zeta_1 = \Delta_1^* \Delta_2^* \dots \Delta_m^* \eta_1^{\mathrm{T}} P_m^{\mathrm{T}} P_m \eta_1 \\ &= \Delta_1^* \Delta_2^* \dots \Delta_m^* (y_{r+1}^2 + \dots + y_n^2). \end{aligned} \tag{27}$$

Taking (iv) into consideration, we can obtain

$$\begin{aligned} \left(1 - \frac{3r^2 M_1 \varepsilon}{b_r}\right)\omega_1 + \frac{\partial v_1}{\partial t} &= \left[\tilde{\lambda}_1 - \left(1 - \frac{3r^2 M_1 \varepsilon}{b_r}\right)\Delta_1 \dots \Delta_r\right] y_1^2 + \dots \\ &\quad + \left[\tilde{\lambda}_r - \left(1 - \frac{3r^2 M_1 \varepsilon}{b_r}\right)\Delta_1 \dots \Delta_r\right] y_r^2 \le 0; \\ \left(1 - \frac{3(n-r)^2 M_2 \varepsilon}{b_m}\right)\omega_1 + \frac{\partial v_2}{\partial t} &= \left[\tilde{\mu}_1 - \left(1 - \frac{3(n-r)^2 M_2 \varepsilon}{b_m}\right)\Delta_1^* \dots \Delta_m^*\right] y_{r+1}^2 + \dots \\ &\quad + \left[\tilde{\mu}_m - \left(1 - \frac{3(n-r)^2 M_2 \varepsilon}{b_m}\right)\Delta_1^* \dots \Delta_m^*\right] y_n^2 \le 0. \end{aligned}$$

The discussion above shows that (23) can take the form

$$\begin{aligned} \frac{dv_1}{dt} &\le -\alpha v_1 + \beta v_2, \\ \frac{dv_2}{dt} &\le \gamma v_1 - \delta v_2, \end{aligned} \tag{28}$$

where α, β, γ, δ are all positive number. Define vector Liapunov function $v = (v_1, v_2)^{\mathrm{T}}$, we rewrite inequality (28) as follows:

$$\frac{dv}{vt} \le Dv, \tag{29}$$

and establish differential equation system as

$$\frac{dX}{dt} = DX, \tag{30}$$

where D is a 2×2 order aggregation matrix

$$D = \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix}.$$

Let $v(t, v_0, t_0)$ and $X(t, X_0, t_0)$ be solution of (29) and (30), respectively. For $v_0 = X_0$, based on the result of [4], the following inequality holds for all $t \in [t_0, +\infty)$.

$$v(t, v_0, t_0) \leq X(t, v_0, t_0). \tag{31}$$

Because $\alpha\delta - \beta\gamma > 0,\ -\alpha < 0,\ -\delta > 0,$ we can conclude that zero solution of (30) is asymptotically stable, this means $\lim_{t \to +\infty} X(t, v_0, t_0) = 0$. By (31) and the positive definite character of v_1 and v_2, we have $t \to +\infty,\ v = (v_1, v_2)^{\mathrm{T}} \to (0, 0)^{\mathrm{T}},$ and the zero solution of the system (1) is asymptotically stable.

Remark 1 It should be noted that Theorem 3.1 is different from the approach proposed by Razumikhin (see, e.g., [14]). Especially, one can see this from the following numerical example.

3.2 Numerical example

Next, we give a numerical example to show the efficiency of the presented criterion. Consider the following nonlinear time-varying dynamic system:

$$\frac{dx}{dt} = A(t)x + f(x, t),$$

where

$$A(t) = \begin{bmatrix} -10 & 0 & 0 & -\frac{1}{20}\cos t \\ 0 & -8 & \frac{1}{20}\sin t & \frac{1}{50}e^{-t} \\ \frac{1}{60}e^{-t} & \frac{1}{40} & -6 & 0 \\ -\frac{1}{40}\cos^2 t & 0 & 0 & -10 \end{bmatrix}, \quad f(x, t) = \begin{bmatrix} \varepsilon x_3^2 \\ \varepsilon x_4^2 \\ \varepsilon x_1^2 \\ \varepsilon x_2^2 \end{bmatrix}.$$

According to (2), $A(t)$ can be partitioned into as follows:

$$A_r = \begin{bmatrix} -10 & 0 \\ 0 & -8 \end{bmatrix}, \qquad A_{r \times m} = \begin{bmatrix} 0 & -\frac{1}{20}\cos t \\ \frac{1}{20}\sin t & \frac{1}{50}e^{-t} \end{bmatrix},$$

$$A_m = \begin{bmatrix} -6 & 0 \\ 0 & -10 \end{bmatrix}, \qquad A_{m \times r} = \begin{bmatrix} \frac{1}{60}e^{-t} & \frac{1}{40} \\ -\frac{1}{40}\cos^2 t & 0 \end{bmatrix}.$$

The eigenpolynomial of A_r can be obtained as

$$\begin{bmatrix} \lambda + 10 & 0 \\ 0 & \lambda + 8 \end{bmatrix} = 0,$$

and consequently, the eigenvalues of A_r can be obtained as $\lambda_1 = -10,\ \lambda_2 = -8$. Make $\alpha = \min(\inf|\operatorname{Re}\lambda_1|, \inf|\operatorname{Re}\lambda_2|) = 8$, and substitute it into (3), we have

$$B_r = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -10-8+8 & 0 \\ 0 & 0 & -8+4 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -4 \end{bmatrix},$$

and, one obtains $\det B_r = -240$.

By using (4), we have

$$\begin{aligned} |(p-\alpha/2)E_r - A_r| &= |(p-4)E_r - A_r| \\ &= \begin{bmatrix} p-4+10 & 0 \\ 0 & p-4+8 \end{bmatrix} = p^2 + 10p + 24 = 0, \end{aligned}$$

and, one obtains $\alpha_0 = 1,\ \alpha_1 = 10,\ \alpha_2 = 24$.

Therefore, we have the following results

$$\Delta_1 = \alpha_1 = 10, \qquad \Delta_2 = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 1 & 24 \end{bmatrix} = 240,$$

and substitute them into (6), one obtains

$$\begin{aligned} v_1 &= \frac{\Delta_1\Delta_2}{\det B_r} \begin{bmatrix} 0 & x_1^2 & 2x_1x_2 & x_2^2 \\ 1 & & & \\ 0 & & B_r & \\ 1 & & & \end{bmatrix} \\ &= \frac{\Delta_1\Delta_2}{\det B_r} \begin{bmatrix} 0 & x_1^2 & 2x_1x_2 & x_2^2 \\ 1 & -6 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix} = 400x_1^2 + 600x_2^2, \end{aligned}$$

$$v_1^* = \inf_{x_1^2+x_2^2=1} v_1 = 400, \quad \Delta = \sup(\Delta_1\Delta_2) = 2400, \quad b_r = \inf|\det B_r| = 240,$$

$$M_1 = \sup(|v_{ij}|,\ \ i,j = 1,2) = 600,$$

$$\varepsilon_1 < \frac{b_r}{3r^2M_1} = \frac{240}{3\cdot 4\cdot 600} = \frac{1}{30}, \quad m_1 = \frac{1}{40},$$

the eigenpolynomial of A_m can be obtained as

$$\begin{bmatrix} \mu+6 & 0 \\ 0 & \mu+10 \end{bmatrix} = 0,$$

and consequently, the eigenvalues of A_m can be obtained as $\mu_1 = -10,\ \mu_2 = -6$. Make $\delta = \min(\inf|\operatorname{Re}\mu_1|,\ \inf|\operatorname{Re}\mu_2|) = 6$, and substitute it into (3), we have

$$B_m = \begin{bmatrix} -6+\dfrac{6}{2} & 0 & 0 \\ 0 & -6-10+6 & 0 \\ 0 & 0 & -10+\dfrac{6}{2} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -7 \end{bmatrix},$$

and, one obtains $\det B_m = -210$.

From (5), we have

$$|(q-\delta/2)E_m - A_m| = |(q-3)E_m - A_m| = \begin{bmatrix} q-3+6 & 0 \\ 0 & q-3+10 \end{bmatrix} = q^2 + 10q + 21 = 0,$$

and, one obtains $b_0 = 1,\ b_1 = 10,\ b_2 = 21$.

Therefore, we have the following results

$$\Delta_1^* = b_1 = 10, \qquad \Delta_2^* = \begin{bmatrix} b_1 & b_3 \\ b_0 & b_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 1 & 21 \end{bmatrix} = 210,$$

and substitute them into (7), we have

$$v_2 = \frac{\Delta_1^*\Delta_2^*}{\det B_m} \begin{bmatrix} 0 & x_3^2 & 2x_3x_4 & x_4^2 \\ 1 & & & \\ 0 & & B_m & \\ 1 & & & \end{bmatrix} = \frac{\Delta_1^*\Delta_2^*}{\det B_m} \begin{bmatrix} 0 & x_3^2 & 2x_3x_4 & x_4^2 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 1 & 0 & 0 & -7 \end{bmatrix} = 700x_3^2 + 300x_4^2,$$

$$v_2^* = \inf_{x_3^2+x_4^2=1} v_2 = 300, \quad \Delta^* = \sup(\Delta_1^*\Delta_2^*) = 2100, \quad b_m = \inf|\det B_m| = 210,$$

$$M_2 = \sup(|v_{ij}|,\ \ i,j = 3,4) = 700, \quad m_2 = \frac{1}{40},$$

$$\varepsilon_2 < \frac{b_m}{3(n-r)^2 M_2} = \frac{210}{3\cdot 4\cdot 700} = \frac{1}{40},$$

$$\gamma = r(n-r)M_2\left(\frac{\Delta^*(n-r)M_2m_2^2}{b_m^2} + \frac{\Delta^*\varepsilon_2}{b_m}\right)\Big/ v_1^* = \frac{49}{24},$$

$$\beta = r(n-r)M_2\left(\frac{\Delta^* M_2 m_2^2}{b_m^2} + \frac{\Delta^*\varepsilon_2}{b_m}\right)\Big/ v_2^* = \frac{35}{12}.$$

One can see that it satisfies the conditions (i)−(iv) of Theorem 3.1, that is, the zero solution of the system (1) is asymptotically stable, which shows that the proposed criterion is efficient for the stability of a class of nonlinear time-varying dynamic system.

4 Conclusions

In this paper, we have given a feasible method to construct Liapunov function of a dynamic system (1), and established some of sufficient conditions for stability of the system. It is shown that for any differentiable matrix $A(t)$, if there exist submatrices A_r and A_m in $A(t)$ such that their eigenvalues all have negative real parts, then it is always available to take $v = (v_1, v_2)^{\mathrm{T}}$ as a vector Liapunov function of the system $dx/dt = A(t)x + f(x,t)$, and based on this, the conditions to ensure stability of the system can be established. Also, the efficiency of the presented criterion has been confirmed by means of a numerical example.

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