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Dissipative Analysis and Stability of Nonlinear Stochastic State-Delayed Systems

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Abstract: In this paper, we extend the concept of dissipativeness developed for nondelay deterministic systems to stochastic state-delayed systems with Markov jump disturbances. We give necessary and sufficient conditions for the system to be dissipative and to have finite $\mathcal{L}_2$-gain also known as the bounded-real condition. Finally, we discuss the relationship between the dissipativeness of the system, its $\mathcal{L}_2$-gain, and its stochastic stability.

Keywords: *Nonlinear state-delayed system; Markov jump process; dissipative system; $\mathcal{L}_2$-gain; bounded-real lemma; stochastic stability.*

Mathematics Subject Classification (2000): 60H10, 93C10, 93D05.

1 Introduction

The important concept of dissipativity developed by Willems [14, 15], Hill and Moylan [5, 6] and Anderson [1], has been proven very successful in many feedback design synthesis problems [1, 11, 12, 14]. This concept which was originally inspired from electrical network considerations, in particular passive circuits, generalizes many other important concepts of physical systems such as positive realness, passivity, and losslessness. As such, many important mathematical relations of dynamical systems such as the bounded real lemma, positive real lemma, the existence of spectral factorization, and $\mathcal{L}_2$-gain of linear and nonlinear systems have been shown to be consequences of this important theory. Moreover, there has been renewed interest lately on this important concept as having been instrumental in the derivation of the solution of the nonlinear $\mathcal{H}_\infty$ control problem [12]. It has been shown that a sufficient condition for the solution to this problem is the existence of a solution to some dissipation inequalities.

However, the theory of dissipativeness more generally studied by Hill and Moylan [5, 6], Willems [14, 15] is purely from a deterministic setting. Many physical systems are however stochastic; for example, a control system is constantly perturbed by unwanted disturbances, a communication system is affected by noise while an aeroplane is frequently fluttered by air pockets. In addition, many physical systems are subject to random changes which may result from abrupt phenomenon such as component and interconnection failures. Hence fault-tolerant systems have been developed to ensure high reliability and performance in such situations.

Therefore, in this paper, we extend the theory of dissipativity to include stochastic state-delayed systems or systems that are subject to random disturbances. In particular, we consider a class of nonlinear stochastic systems with state-delay and random Markovian jump parameters or disturbances. This class of systems belongs to the class of hybrid systems with continuous state dynamics and discrete parameter variation. The control and filtering problems for this class of systems has been discussed by many authors [3, 9, 10]. In particular, Rishel [10] has derived the minimum principle for the general nonlinear case without state-delay and in which the adjoint equations are deterministic. While Ji and Chizeck [3, 7] have derived the structural properties, namely, controllability, observability and stability for the linear case. Furthermore, the problems of controller design for the linear case using LQ and LQG criteria have been discussed extensively in Mariton [9].

Thus, in this paper, we discuss additional structural (or internal) properties of this class of systems which are closely associated with their stability. We discuss the dissipative properties of this class of systems, which determine whether they absorb energy and conserve it, or dissipate it; and based on this property, what could we infer about the stability of such systems? We also give a fresh interpretation of the concept of dissipativity as both an input/output property and an internal property of a system. The closest work to the current one in this paper can be found in [4] for systems without state-delay.

The paper is organized as follows. In Section 2, we define the problem and discuss necessary and sufficient conditions for a nonlinear state-delayed system with Markov jump disturbances to be dissipative. We continue this discussion in Section 3 for the case of a quadratic supply rate and discuss the relationship between the dissipativity of the system and its $\mathcal{L}_2$-gain, which leads to the bounded-real lemma for this class of systems. Finally, in Section 4, we discuss the implications of dissipativity on the stability of the system. Conclusions are then given in Section 5.

2 Dissipativity of State-Delayed Nonlinear Stochastic Systems with Jumps

In this section, we define the concept of dissipativity of a state-delayed nonlinear system with jump Markov disturbances. The notation is standard except where specified otherwise. Moreover, R_+ is the positive real-line, R^n is the n-dimensional Euclidean space and $\|\cdot\|$ represents the Euclidean vector norm. The spaces $\mathcal{L}_{1,loc}((t_0,t_1),R)$, $\mathcal{L}_2([0,T],R^n)$ are the standard Lebesgue spaces of locally integrable on (t_0,t_1) and square integrable over $[0,T]$ vector functions on R^n respectively. While $\mathcal{L}_2([0,T],(\Omega,\mathcal{F},P))$ is the corresponding space over the probability space $(\Omega,\mathcal{F},P)$, in which Ω is the sample space, $\mathcal{F}$ is the σ-algebra generated by Ω and P is a probability measure over $\mathcal{F}$. Lastly, E will denote the mathematical expectation operator.

Let us at the outset consider the following piece-wise autonomous nonlinear state-delayed system defined over an open subset $\mathcal{X}\times\mathcal{S}$ of $R^n\times Z_+$ with $\mathcal{X}$ containing the origin,

$$\Sigma: \quad \dot{x}(t) = f(x(t),\, x(t-d),\, u(t),\, r(t)),$$

$$x(t) = \phi(t), \quad t \in [-d, 0], \quad x(t_0) = x_0 = \phi(t_0), \tag{1}$$

$$y(t) = h(x(t), r(t)), \tag{2}$$

where $x(t) \in \mathcal{X}$ is the state vector, $u(t) \in \mathcal{U} \subset R^p$ is the input function belonging to an input space $\mathcal{U}$, $d > 0$ is the delay, $y(t) \in \mathcal{Y} \subset R^m$ is the output function which belongs to the output space $\mathcal{Y} \subset R^m$, and $\phi(t) \in C[-d, 0]$ is the initial function. Besides the dependence on the input and initial conditions, the state of the system is also a function of the discrete parameter $r(t)$ which is a continuous-time homogeneous Markov process with finite discrete state-space $\mathcal{S} \triangleq \{1, 2, \dots, l\}$. We assume that the probabilities $P_t \triangleq (P_{1t}, \dots, P_{lt})$, with $P_{it} \triangleq P(r(t) = i)$, $i = 1, \dots, l$, satisfy the forward Kolmogorov equation

$$\frac{\partial P_t}{\partial t} = \Lambda P_t, \qquad P_0 = \bar{P}, \quad t \in [0, T],$$

where $\Lambda = [\lambda_{ij}]_{i,j \in \mathcal{S}}$ is the transition matrix, and λ_{ij} are real numbers such that for $i \neq j$, $\lambda_{ij} \geq 0$, and for all $i \in \mathcal{S}$, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. In other words, the transition probabilities are given by

$$P[r(t+h) = j,\ r(t) = i] = \begin{cases} \lambda_{ij} h + o(h) & \text{if} \quad j \neq i, \\ 1 + \lambda_{ii} h + o(h) & \text{if} \quad j = i, \end{cases}$$

where $o(h)$ are the remainder terms such that $\lim\limits_{h \to 0} \dfrac{o(h)}{h} = 0$.

The functions $f\colon \mathcal{X} \times \mathcal{X} \times \mathcal{U} \times \mathcal{S} \to X$, $h\colon \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ are real smooth functions of their arguments for each $r(t) \in \mathcal{S}$. We also assume the following.

Assumption 2.1 The system Σ is causal, time-invariant and finite-dimensional. Further, the functions $f(\cdot, \cdot, \cdot, r(t))$, $h(\cdot, r(t))$ for each value of $r(t) \in \mathcal{S}$ are smooth C^∞ functions of $x \in \mathcal{X}$ and $u \in \mathcal{U}$ such that the system (1) is well-defined; that is, for any initial state $x(t_0) \in \mathcal{X}$, initial mode $r(t_0) = r_0 \in \mathcal{S}$ and any admissible input, $u(t) \in \mathcal{U}$, there exists a unique solution $x(t, t_0, x_0, x_{t_0 - d}, r_0, u)$ to (1) on $[t_0, \infty)$ which continuously depends on the initial data.

Alternatively, the following assumptions are also sufficient to guarantee the existence and uniqueness of solutions to the system Σ [2].

Assumption 2.2 For all t, t_1, $t_2 \in [-d, \infty)$, $r(t) \in \mathcal{S}$,

(a) (Lipschitz condition)

$$\begin{gathered} \|f(x(t_2), x(t_2 - d),\, u(t_2),\, r(t))) - f(x(t_1),\, x(t_1 - d),\, u(t_1),\, r(t))\| \\ \leq K_1 \|x(t_2) - x(t_1)\| + K_2 \|x(t_2 - d) - x(t_1 - d)\| + K_3 \|u(t_2) - u(t_1)\| \\ \forall\, x(t_2),\, x(t_1),\, x(t_2 - d),\, x(t_1 - d) \in \mathcal{X}, \quad u(t_1),\, u(t_2) \in \mathcal{U}; \end{gathered}$$

(b) (Restriction on Growth)

$$\begin{gathered} \|f(x(t), x(t-d), u(t), r(t))\|^2 P \leq K_1^2 (1 + \|x(t)\|^2) + K_2^2 (1 + \|x(t-d)\|^2) \\ + K_3^2 (1 + \|u(t)\|^2), \quad \forall\, x(t), x(t-d) \in \mathcal{X}, \quad u \in \mathcal{U} \\ \|h(t, x(t), r(t))\| \leq K_4 (1 + \|x(t)\|^2), \quad \forall\, x(t) \in \mathcal{X}, \end{gathered}$$

where K_1, K_2, K_3, K_4 are positive constants.

Now let $\mathcal{F}_t$ be the σ-algebra generated by $r(t)$, $t \in [0,T]$. Then we take the input space $\mathcal{U}$ and output space $\mathcal{Y}$, to be $\mathcal{F}_t$-measurable, and piecewise continuous. Similarly, the functions $f(\cdot,\cdot,\cdot)$, $h(\cdot,\cdot)$ are also assumed to be $\mathcal{F}_t$ measurable by continuity with respect to $x \in \mathcal{X}$.

If the system Σ is viewed as a black box with only inputs and outputs, then in the above representation, the system Σ is a map $\Sigma\colon \mathcal{U} \times \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ which transforms inputs to outputs through state functions $x(t) \in \mathcal{X}$ for each $r(t) \in \mathcal{S}$. In view of this, if we assign an energy measure to both the inputs and outputs of the system, then it is possible to infer the internal behavior of the system by comparing these two quantities. This motivates the following definition of a supply rate to the system.

Definition 2.1 A function $s(u(t),y(t))\colon \mathcal{U} \times \mathcal{Y} \to R$ is a supply rate to the system Σ if $s(\cdot,\cdot)$ is piecewise continuous and locally integrable, i.e.,

$$E\left[\int_{t_0}^{t_1} |s(u(t),y(t))|\,dt\right] < \infty \tag{4}$$

or $s(\cdot,\cdot) \in \mathcal{L}_{1,loc}(t_0,t_1)$ for any $(t_0,t_1) \in R_+^2$, for all $u(t) \in \mathcal{U}$.

Remark 2.1 The supply rate $s(\cdot,\cdot)$ is a measure of the instantaneous power into the system. Part of this power is stored as internal energy and part of it is dissipated.

It follows from the above definition of supply rate that, to infer about the internal behavior of the system, it is sufficient to evaluate the expected total amount of energy expended by the system over a finite time interval. This leads us to the following definition.

Definition 2.2 The system Σ is dissipative with respect to (wrt) the supply rate $s(t) = s(u(t),y(t))$ if for all $u(t) \in \mathcal{U}$ and $t_0, t_1 \in R_+^2$,

$$E\left[\int_{t_0}^{t_1} s(u(t),y(t))\,dt\right] \geq 0; \quad \forall\, t_1 \geq t_0. \tag{5}$$

when evaluated along any trajectory of the system starting at t_0, $x(t) = 0$.

Remark 2.2 The above definition suggests that, the dissipativity of the system is an input-output property. This is also the notion put forward in [5]. Furthermore, it also raises the following question: Can every finite dimensional, time-invariant, causal system be rendered dissipative by a suitable choice of input? The answer to this question will be given in due course, but in short it is: yes and no!

The above Definition 2.2 being an inequality postulates the existence of a storage function and a possible dissipation rate for the system. It follows that if the system is assumed to have some stored energy which is measured by a function $\Psi\colon R_+ \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$, then for the system to be dissipative, it is necessary that in the transition from t_0 to t_1, the total amount of energy stored is less than the amount expended. This suggests the following alternative definition of dissipativity.

Definition 2.3 The system Σ is said to be dissipative with respect to a supply rate $s(u(t), y(t))$ if for all $(t_0, t_1) \in R_+^2$ there exist positive-semidefinite functions (storage functions) $\Psi\colon R_+ \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$, such that the inequality

$$E\Psi(t_1, x(t_1),\, x(t_1-d),\, r(t_1)) - \Psi(t_0, x(t_0),\, x(t_0-d),\, r(t_0)) \le E\left[\int_{t_0}^{t_1} s(u(t), y(t))\, dt\right] \quad (6)$$

is satisfied for all $t_1 \ge t_0$, modes $r(t_1), r(t_2) \in \mathcal{S}$ and initial states $x(t_0-d), x_0 \in \mathcal{X} \times \mathcal{X}$, where $x(t_1) = x(t_1, t_0, x_0, x_{t_0-d}, r_0, u)$.

In the sequel we shall also use the following notations $x(t_i) = x_{t_i} = x_i$, $x(t_i - d) = x_{t_i-d}$, $r(t_i) = r_i$, $i \in Z$.

Remark 2.3 The system is also said to be lossless if the above inequality (6) is satisfied as an equality.

The above inequality (6) can be converted to an equality by introducing the dissipation rate $d\colon \mathcal{M} \times \mathcal{U} \times \mathcal{S} \to R$ according to the following equation

$$\begin{gathered} E\Psi(t_1, x_{t_1}, x_{t_1-d}, r_1) - \Psi(t_0, x_0, x_{t_0-d}, r_0) = E\left[\int_{t_0}^{t_1} [s(t) + d(t)]\, dt\right], \\ \forall\, t_1 \ge t_0, \qquad \forall\, r_1, r_0 \in \mathcal{S}. \end{gathered} \quad (7)$$

Remark 2.4 The dissipation rate is nonnegative if the system is dissipative. Moreover, the dissipation rate uniquely determines the storage function $\Psi(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$ [15].

We now define the concept of available storage, the existence of which determines whether the system is dissipative or not.

Definition 2.4 The available storage $\Psi^a(t, x, r(t))$ for each $r(t) \in \mathcal{S}$ of the dynamical system Σ is the quantity:

$$\Psi^a(t, x(t),\, x(t-d),\, r(t)) = \sup_{x_0 = x,\, u \in \mathcal{U},\, t \ge 0} -E\left[\int_0^t s(u(\tau), y(\tau))\, d\tau\right], \quad (8)$$

where the supremum is taken over all possible inputs, $u \in \mathcal{U}$ starting at x and time $t_0 = 0$.

It follows that, if the system is dissipative, then the available storage is well-defined and finite in each state of the system x, and mode r_0. Moreover, it determines the maximum amount of energy which may be extracted from the system Σ. This is stated in the following theorem.

Theorem 2.1 *The available storage, $\Psi^a(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$, is finite if and only if (iff) the system is dissipative. Furthermore, any other storage function is lower bounded by $\Psi^a(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$, i.e., $0 \le \Psi^a(\cdot,\cdot,\cdot, r(t)) \le \Psi(\cdot,\cdot,\cdot, r(t))$, $r(t) \in \mathcal{S}$.*

Proof Notice that $\Psi^a(\cdot,\cdot,\cdot,\cdot) \ge 0$ since it is the supremum over a set with the zero element (at $t = 0$). Now assume that $\Psi^a(\cdot,\cdot,\cdot,\cdot) < \infty$. We have to show that the system is dissipative, i.e., for any $(t_0, t_1) \in R_+^2$

$$\Psi^a(t_0, x_0,\, x(t_0-d),\, r_0) + E\left[\int_{t_0}^{t_1} s(u(\tau), y(\tau))\, d\tau\right] \geq E\Psi^a(t_1, x_1,\, x(t_1-d),\, r_1), \quad \forall\, x_0, x_1 \in \mathcal{X}, \quad r_0, r_1 \in \mathcal{S}. \tag{9}$$

In this regard, notice that from (8)

$$E\Psi^a(t_1, x_1,\, x(t_1-d),\, r_1) - \Psi^a(t_0, x_0,\, x(t_0-d),\, r_0) = \sup_{x_0, u} E\left[-\int_{t_0}^{t_1} s(t)\, dt\right], \quad \forall\, r_0,\ r_1 \in \mathcal{S}. \tag{10}$$

This implies that

$$E\Psi^a(t_1, x_1,\, x(t_1-d),\, r_1) \geq \Psi^a(t_0, x_0,\, x(t_0-d),\, r_0) + E\left[\int_{t_0}^{t_1} s(t)\, dt\right], \tag{11}$$

and since all the above quantities are greater or equal to zero, it implies that $\Psi^a(\cdot,\cdot,\cdot, r(t))$ satisfies the dissipation inequality (6) for each $r(t)$.

Conversely, assume that Σ is dissipative. Then the dissipation inequality (6) implies that

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) + E\left[\int_{t_0}^{t_1} s(t)\, dt\right] \geq E\Psi(t_1, x_1, x_{t_1-d}, r_1) \geq 0; \quad \forall\, x_0, x_1 \in \mathcal{X}, \quad r_0, r_1 \in \mathcal{S}, \tag{12}$$

by definition. Therefore,

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) \geq -E\left[\int_0^{t_1} s(t)\, dt\right] + E\left[\int_0^{t_0} s(t)\, dt\right] \tag{13}$$

which implies that

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) \geq \sup_{x=x_0,\, u\in\mathcal{U},\, t\geq 0} E\left[-\int_0^{t_1} s(t)\, dt\right] = \Psi^a(t_0, x_0, x_{t_0-d}, r_0). \tag{14}$$

Hence $\Psi^a(t, x,\, x(t-d),\, r(t)) < \infty\ \ \forall\, x \in \mathcal{X},\ r(t) \in \mathcal{S}$.

Remark 2.5 The above theorem summarizes the answer to the question we raised above, that dissipativity is both an input/output property and an internal property. It suggests that a system that is not dissipative wrt one supply rate may be dissipative wrt to another. It therefore follows that the system must possess some internal structure such that, the available storage $\Psi^a(\cdot,\cdot,\cdot, r(t))$ is well-defined for each $r(t) \in \mathcal{S}$ and in each state of the system for a particular supply rate.

Remark 2.6 The importance of the above theorem in checking dissipativeness of the nonlinear system Σ cannot be overemphasized. It follows that, if the system is reachable from the origin $\{0\}$, then by an appropriate choice of an input $u(t)$ such that $\Psi^a(\cdot,\cdot,\cdot,r(t))$, $r(t)\in\mathcal{S}$ is finite, it can be rendered dissipative. However, evaluating $\Psi^a(\cdot,\cdot,\cdot,\cdot)$ is a difficult task without the output of the system specified a priori or solving the state equations. This therefore calls for an alternative approach for determining the dissipativeness of the system. This is discussed in the next section.

3 Relationship with $\mathcal{L}_2$-gain

In this section, we discuss the connection between the dissipativity of the nonlinear system Σ with its $\mathcal{L}_2$-gain. In the classical paper by Willems [14], the relationship between dissipativity and Linear Quadratic (LQ)-control has been shown and this relationship has been exploited to prove the existence of solutions to certain infinite-horizon LQ-control problems leading to the Algebraic-Ricatti equation (ARE). Similarly, we also discuss the relationship between the dissipativity of the nonlinear system with certain Hamilton-Jacobi equations arising in the $\mathcal{L}_2$-gain optimization of the nonlinear system. To this end and for the purpose of clearity, let us consider an affine representation Σ^a of the system Σ defined by:

$$\begin{aligned}
\Sigma^a:\quad & \dot{x}(t)=f(x(t),\,x(t-d),\,r(t))+g(x,r(t))u(t), && (15)\\
& x(t)=\phi(t),\quad t\in[-2d,0],\quad x(t_0)=x_0=\phi(t_0) &&\\
& y(t)=h(x(t),r(t)), && (16)
\end{aligned}$$

where $g(\cdot,\cdot)\in C^\infty(\mathcal{X}\times S)\in R^{n\times k}$. In this case, our existence and uniqueness Assumptions 2.2 take the following form:

Assumption 3.1 For all $t_1,t_2\in[-2d,\infty)$, $r(t)\in\mathcal{S}$,

(a) (Lipschitz condition)

$$\begin{gathered}
\|f(x(t_2),x(t_2-d),r(t))-f(x(t_1),x(t_1-d),r(t))\|+\|g(x(t_2),r(t))-g(x(t_1),r(t))\|\\
\le K_1\|x(t_2)-x(t_1)\|+K_2\|x(t_2-d)-x(t_1-d)\|+\|u(t_2)-u(t_1)\|,\\
\forall\, x(t_1),x(t_2)\in\mathcal{X},\quad u(t_1),u(t_2)\in\mathcal{U};
\end{gathered}$$

(b) (Restriction on growth)

$$\begin{gathered}
\|f(x(t),x(t-d),r(t))\|^2+\|g(x(t),r(t)\|^2\le K_1^2(1+\|x\|^2)+K_2^2(1+\|x(t-d)\|^2)\\
+K_3^2(1+\|u(t)\|^2),\quad \forall\, x(t),\,x(t-d)\in\mathcal{X},\quad u(t)\in\mathcal{U},
\end{gathered}$$

where K_1, K_2, K_3 are positive constants and $\|g\|^2=\mathrm{Tr}(gg^T)$ represents the matrix trace norm.

The question we wish to answer in this section is the following: If we restrict the input space $\mathcal{U}$ of the system to be the space $\mathcal{L}_2[-2d,\infty)$, then under what conditions is the system dissipative ? or can be rendered dissipative? To motivate the discussion, we expand the definition of $\mathcal{L}_2$-gain [12] as follows.

Definition 3.1 The system (15) is said to have $\mathcal{L}_2$-gain from $u(t)$ to $y(t)$ less than or equal to some number $\gamma' > 0$ if for all $(t_0, t_1) \in [-d, \infty)$, initial state vector $x_0 \in \mathcal{X}$, and mode $r_0 \in \mathcal{S}$, the response of the system $y(t)$ due to any $u(t) \in \mathcal{L}_2[0, \infty)$ satisfies

$$E\left[\int_{t_0}^{t_1} \|y(t)\|^2 \, dt\right] \le \frac{1}{2}\gamma'^2 \int_{t_0}^{t_1} (\|u(t)\|^2 + \|u(t-d)\|^2)\, dt + \beta(x_0, r_0); \quad \forall\, t_1 \ge t_0 \tag{17}$$

and some class $\mathcal{K}$ functions [13] $\beta\colon\ \mathcal{X} \times \mathcal{S} \to R_+,\ \beta(0, r(t)) = 0\ \ \forall\, r(t) \in \mathcal{S}$.

Remark 3.1 In the above definition, if $d = 0$, we recover the usual definition of $\mathcal{L}_2$-gain for non-delay systems. In this regard, right-hand side represents an average. Moreover, in the sequel we shall let $\gamma = \gamma'/\sqrt{2}$ and call γ the $\mathcal{L}_2$-gain of the system with a slight abuse of the definition.

Remark 3.2 It is also obvious from the definition of $\mathcal{L}_2$-gain and dissipativity of the nonlinear system (15) wrt to the supply rate $s(u(t), y(t))$, that, dissipativity of the system wrt the supply rate $s(u(t), y(t))$, implies finite $\mathcal{L}_2$-gain $\le \gamma$.

Furthermore, from the definition of dissipativity given in (6), if the function $\Psi(t, x(t), x(t-d), r(t))$ belongs to $C^1(R_+ \times \mathcal{X} \times \mathcal{X})$, it is possible to go from the integral version of the above dissipation inequality (6) to the differential form. This is stated in the following lemma. We shall also be particularly interested in the following supply rate $s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2,\ \gamma > 0$.

In the sequel, we shall also use the notation $r(t) = i$ and $r(t) = j,\ i, j \in \mathcal{S}$.

Lemma 3.1 *The nonlinear system Σ^a is dissipative wrt the supply rate*

$$s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2,$$

if there exist some C^1 nonnegative functions $\Psi\colon\ R \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$ such that the following differential dissipation inequality is satisfied for all $x(t) \in \mathcal{X},\ r(t) \in \mathcal{S}$:

$$\begin{aligned}
&\Psi_t(t, x_t, x_{t-d}, r(t)) + \Psi_{x_t}(t, x_t, x_{t-d}, r(t))[f(x_t, x_{t-d}, r(t)) + g(x_t, r(t))u] \\
&\qquad + \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d},\, r(t-d)) + g(x_{t-d},\, r(t-d))u(t-d)] \\
&+ \sum_{r(t)=j\in\mathcal{S}} \lambda_{ij}\Psi(t, x_t, x_{t-d}, j) - \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) + \frac{1}{2}\|y(t)\|^2 \le 0, \\
&\quad \Psi(t, 0, 0, r(t)) = 0 \quad \forall\, t \in R, \quad r(t) = i, \quad u(t),\, u(t-d) \in \mathcal{L}_2[-d, \infty),
\end{aligned} \tag{18}$$

where $\Psi_t(\cdot,\cdot,\cdot,\cdot)$, $\Psi_{x_t}(\cdot,\cdot,\cdot,\cdot)$ and $\Psi_{x_{t-d}}(\cdot,\cdot,\cdot)$ are the row vectors of partial derivatives of $\Psi(\cdot,\cdot,\cdot,\cdot)$ wrt t, x_t and x_{t-d} respectively.

Proof Without any lost of generality, we will take $t_0 = 0$ and $t_1 = T$. Now consider the following variation of the Dynkin's formula [8]:

$$\begin{aligned}
&E\Psi(T, x(T), x(T-d),\, r(T)) - \Psi(0, x_0, x_{-d}, r_0) \\
&= E\left[\int_0^T \mathcal{L}\Psi(t, x(t),\, x(t-d),\, r(t))\, dt\right] \quad \forall\, T > 0,
\end{aligned} \tag{19}$$

where $\mathcal{L}$ is the infinitesimal generator of the process $(x(t), r(t))$, $t \geq 0$ [8, 9]. Then using the above formula (19) in the dissipation inequality (6) and the fact that

$$\begin{aligned}\mathcal{L}\Psi(t, x_t, x_{t-d}, r(t)) &= \Psi_t(t, x_t, x_{t-d}, r(t)) \\ &+ \Psi_{x_t}(t, x_t, x_{t-d}, r(t))[f(x_t, x_{t-d}, r(t)) + g(x_t, r(t))u] \\ &+ \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d}, r(t-d)) + g(x_{t-d}, r(t-d))u(t-d)] \\ &+ \sum_{r(t)=j\in\mathcal{S}} \lambda_{ij}\Psi(x, r, j), \quad r(t) \in \mathcal{S},\end{aligned} \tag{20}$$

the result follows.

Remark 3.3 By virtue of the above lemma, we will henceforth consider only C^1 storage functions in this paper.

Lemma 3.2 *For the nonlinear system Σ^a, we have the following implications: $(a) \Leftrightarrow (b) \rightarrow (c)$*

(a) *the system Σ^a satisfies the dissipation inequality (18);*
(b) *the system Σ^a is dissipative wrt to the supply rate s(u(t),y(t));*
(c) *the system Σ^a has $\mathcal{L}_2$-gain from $u(t)$ to $y(t)$ less than or equal to γ.*

Proof (sketch) $(a) \Leftrightarrow (b)$ follows from Lemma 3.1 above, while (c) follows from (6),(17) and the fact that $E\Psi(\cdot,\cdot,\cdot,\cdot) \geq 0$ by Theorem 2.1.

We now state the main result of this section which is a consequence of Lemmas 3.1 and 3.2 above.

Theorem 3.1 *A necessary and sufficient condition for the nonlinear system (15) to be dissipative wrt the supply rate*

$$s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2$$

is that there exist a set of smooth positive-semidefinite solutions of the following stochastic Hamilton-Jacobi (HJ) inequality for each $r(t) \in \mathcal{S}$:

$$\begin{aligned}&\Psi_t(t, x_t, x_{t-d}, r(t)) + \Psi_{x_t}(t, x_t, x_{t-d}, r(t))f(x_t, x_{t-d}, r(t)) \\ &+ \Psi_{x_{t-d}}(t, x_t, x_{t-d}, r(t))f(x_{t-d}, x_{t-2d}, r(t)) + \frac{1}{2\gamma^2}\,\Psi_{x_t}g(x_t, r(t))g^T(x_t, r(t))\Psi_{x_t}^T \\ &+ \frac{1}{2\gamma^2}\,\Psi_{x_{t-d}}g(x_{t-d}, r(t-d))g^T(x_{t-d}, r(t-d))\Psi_{x_{t-d}}^T + \frac{1}{2}\,h^T(x_t, i)h(x_t, i) \\ &+ \sum_{r(t)=j\in\mathcal{S}} \lambda_{ij}\Psi(t, x_t, x_{t-d}, j) \leq 0, \quad \Psi(t, 0, 0, i) = 0 \quad \forall\, x \in \mathcal{X}, \quad r(t) = i \in \mathcal{S}.\end{aligned} \tag{21}$$

Proof (**Necessity**) Theorem 2.1 has shown that if the system Σ^a is dissipative, then there exists at least one set of solutions to the dissipation inequality (6) for each $r(t) \in \mathcal{S}$ which is given by the available storage, $\Psi^a(t, x_t, x_{t-d}, r(t))$, $r(t) \in \mathcal{S}$. We now show that any solution of the dissipation inequality (6) is also a solution to the HJ-inequality (21).

If the system is dissipative with storage function $\Psi(\cdot,\cdot,\cdot,)$, then along any trajectory of the system, the differential dissipation inequality (18) is satisfied. The left-hand-side (LHS) of this inequality is a quadratic function of u with maximum at

$$u^\star(t,x_t) = \frac{1}{\gamma^2}\, g^T(x_t,r(t))\Psi_{x_t}^T(x_t,r(t)). \tag{22}$$

The maximum value of the function corresponding to this stationary point, is given by

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)f(x_t,x_{t-d},i)\\
&+ \Psi_{x_{t-d}}(t,x_t,x_{t-d},r(t))f(x_{t-d},x_{t-2d},r(t)) + \frac{1}{2\gamma^2}\,\Psi_{x_t}g(x_t,i)g^T(x_t,i)\Psi_{x_t}^T\\
&+ \frac{1}{2\gamma^2}\,\Psi_{x_{t-d}}g(x_{t-d},r(t-d))g^T(x_{t-d},r(t-d))\Psi_{x_{t-d}}^T + \frac{1}{2}\,h^T(x,i)h(x,i)\\
&+ \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j) \quad \forall\, x\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned} \tag{23}$$

But the inequality (18) holds for all $u(t),\, u(t-d)\in\mathcal{L}_2[-d,\infty)$. Hence it must also hold for $u^\star(\cdot)$, and the result follows. This proves the necessity part of the theorem.

(**Sufficiency**) To prove sufficiency, we will show that, if there exists a solution to the HJ inequality (21), then the system is dissipative. Therefore, let $\Psi(\cdot,\cdot,\cdot)\geq 0$ satisfy (21), then completing the squares, we get

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)[f(x_t,x_{t-d},i) + g(x_t,i)u(t)]\\
&+ \Psi_{x_{t-d}}[f(x_{t-d},x_{t-2d},r(t-d)) + g(x_{t-d},i)u(t-d)] + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j)\\
&\leq \frac{\gamma^2}{2}\|u(t)\|^2 - \frac{1}{2}\|y(t)\|^2 - \frac{\gamma^2}{2}\|u(t) - \frac{1}{\gamma^2}\,g^T(x,i)\Psi_{x_t}^T(x,i)\|^2 + \frac{\gamma^2}{2}\|u(t-d)\|^2\\
&\quad - \frac{\gamma^2}{2}\|u(t-d) - \frac{1}{\gamma^2}\,g^T(x_{t-d},i)\Psi_{x_{t-d}}^T(t-d,x_{t-d},x_{t-2d},r(t-d))\|^2\\
&\qquad \forall\, x(t),x(t-d)\in\mathcal{X}, \quad i\in\mathcal{S},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)[f(x_t,x_{t-d},i) + g(x_t,i)u(t)]\\
&+ \Psi_{x_{t-d}}[f(x_{t-d},x_{t-2d},r(t-d)) + g(x_{t-d},i)u(t-d)] + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j)\\
&\leq \frac{\gamma^2}{2}(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2 \quad \forall\, x\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned}$$

Thus, the dissipation inequality (6) and (18) are satisfied, and hence the system is dissipative wrt to $s(u(t),y(t))$.

Remark 3.4 The inequality (21) is known as the bounded-real inequality or condition for the system Σ^a and Theorem 3.1 is the equivalent of the bounded-real lemma for linear systems.

Remark 3.5 The above theorem provides an alternative approach for determining dissipativeness wrt to the quadratic supply rate. It follows that, if the system possesses the structure such that there exist smooth solutions to the HJ inequality (21) for each mode of the system, then it guarantees the dissipativeness of the system.

4 Stability of Stochastic State-Delayed Jump Systems

In the previous two sections we have defined the concept of dissipativity of the state-delayed nonlinear Markovian jump stochastic system (1), and have derived necessary and sufficient conditions for the system to be dissipative wrt to any supply rate. We have also explored the relationship between the dissipativity of the system and its $\mathcal{L}_2$-gain which is expressed in terms of the bounded-real condition or a set of coupled HJ-inequalities. Finally in this section, we shall relate the three concepts of dissipativity, $\mathcal{L}_2$-gain and stability of the system Σ^a. The question we would like to answer is the following: under what conditions relating to the dissipativity of the system Σ^a is the equilibrium $x = \{0\}$ stable, asymptotically stable?

In the deterministic case, if we regard the storage functions $\Psi(\cdot,\cdot,\cdot,r(t))$, $r(t) \in \mathcal{S}$ as generalized energy functions similar to Lyapunov functions, then to investigate stability using these functions, we would require that they be positive-definite and their time derivatives along trajectories of the system are negative-definite. Such an approach can also be considered in the stochastic case with stability defined in a stochastic sense. Therefore, we begin by first considering the conditions under which the storage function $\Psi(\cdot,\cdot,\cdot)$ is positive definite. This leads us to the following definition.

Definition 4.1 The free system (15) (with $u(t) \equiv 0$) is said to be stochastically zero-state detectable if for any trajectory of the system such that $y(t) \equiv 0 \;\forall\, t \geq 0 \Rightarrow \lim_{t\to\infty} E\{\|x(t,0,x_0,x_{-d},r_0,0)\|^2\} = \{0\}$.

We now show that, if $\Psi(\cdot,\cdot,\cdot,\cdot) \geq 0 \;\forall\, x \in \mathcal{X}$, $r(t) \in \mathcal{S}$, satisfies the HJ-inequality (21) as in the above Theorem 3.1, and the free system is stochastically zero-state detectable, then the following lemma guarantees that $\Psi(\cdot,\cdot,\cdot) > 0 \;\forall\, x(t), x(t-d) \in \mathcal{X}$, $x(t) \neq 0$ or $x(t-d) \neq 0$, $r(t) \in \mathcal{S}$.

Lemma 4.1 *Suppose $\Psi(\cdot,\cdot,\cdot,\cdot) \geq 0 \;\; \forall\, x(t), x(t-d) \in \mathcal{X}$, $r(t) \in \mathcal{S}$, satisfies the HJ-inequality (21) and the system is dissipative as in Theorem 3.1 above, then if the free system is stochastically zero-state detectable, then $\Psi(\cdot,\cdot,\cdot,\cdot) > 0$ for all $x(t) \neq 0$ or $x(t-d) \neq 0$, $r(t) \in \mathcal{S}$.*

Proof The available storages given in equation (8) are strictly convex in u for each $r(t) \in \mathcal{S}$ and are the infima of all solutions of the HJ inequality (21). Any other set of solutions $\Psi(t, x(t), x(t-d), r(t))$, $\forall\, r(t) \in \mathcal{S}$ of the HJ inequality is lower bounded by $\Psi^a(\cdot,\cdot,\cdot,r(t))$, i.e.,

$$\begin{gathered}\Psi^a(t, x(t),\, x(t-d),\, r(t)) \leq \Psi(t, x(t),\, x(t-d),\, r(t)) \\ \forall\, x(t),\; x(t-d) \in \mathcal{X}, \qquad r(t) \in \mathcal{S}.\end{gathered} \tag{24}$$

We now show that, if the system (15) is reachable from the origin, then there exists a choice of input $u(x(t), r(t))$, such that $\Psi^a(t, x(t), x(t-d), r(t)) > 0 \;\; \forall\, x(t) \neq 0$,

$x(t-d) \neq 0,\ \forall\, r(t) \in \mathcal{S}$ and for $T > 0$

$$\Psi^a(t, x_t, x_{t-d}, r(t)) = \sup_{u \in \mathcal{U}} E\Bigg[-\frac{1}{2}\Bigg\{ \int_0^T (\gamma^2 \|u(t)\|^2 + \|u(t-d)\|^2) - \|y(t)\|^2 \Bigg\} dt \Bigg]. \quad (25)$$

It has been shown (Theorem 3.1) that for any solution $\Psi(\cdot,\cdot,\cdot,r(t)),\ r(t) \in \mathcal{S},$ of the dissipation inequality (18), the control $u^\star(\cdot,\cdot)$ attains the above supremum. Therefore,

$$\Psi^a(t, x_t, x_{t-d}, r(t)) = E\Bigg[-\frac{1}{2}\Bigg\{ \gamma^2 \int_0^T (\|u^\star(t)\|^2 + \|u^\star(t-d)\|) - \|y(t)\|^2 \Bigg\} dt \Bigg]. \quad (26)$$

Now using the HJ-inequality (21) or the dissipation inequality (18), we get

$$\begin{aligned}
\Psi^a(t, x_t, x_{t-d}, r(t)) &\geq -E\Bigg[\int_0^T \Bigg\{ \Psi_t(t, x_t, x_{t-d}, i) + \Psi_{x_t}(t, x_t, x_{t-d}, i)[f(x_t, x_{t-d}, i) \\
&\quad + g(x_t, i)u^\star(t)] + \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d}, r(t-d)) + g(x_{t-d}, r(t-d))u^\star(t-d)] \\
&\quad + \sum_{j\in\mathcal{S}} \lambda_{ij}\Psi(t, x_t, x_{t-d}, j) \Bigg\} dt \Bigg] \geq -E\Bigg[\int_0^T \mathcal{L}\Psi(t, x_t, x_{t-d}, r(t)) dt \Bigg] \\
&\geq \Psi(0, x_0, x_{-d}, r_0) - E\Psi(T, x(T), x(T-d), r(T)) \geq 0, \quad \forall\, T > 0
\end{aligned}$$

by dissipativity and Theorem 2.1. Now, from the above inequality, the condition when $\Psi^a(\cdot,\cdot,\cdot,0) = 0$ corresponds to

$$\Psi(0, x_0, x_{-d}, r_0) = E\Psi(T, x(T), x(T-d), r(T)) = 0,$$

and since this holds for all $T > 0$, it implies that $\Psi^a(\cdot,\cdot,\cdot,\cdot) \equiv \Psi(0, x_0, x_{-d}, r_0) \equiv E\Psi(T, x(T), x(T-d), r(T)) = 0$. This further implies that $y(t) \equiv 0,\ u(t) \equiv 0,$ which by stochastic zero-state detectability implies that $x_0 = x(T) = x(T-d) = \{0\}$. Since $T > 0$ is arbitrary, the result follows.

We are now in a position to exploit $\Psi(\cdot,\cdot,\cdot,\cdot)$ as a candidate Lyapunov function for the system Σ^a since any solution $\Psi(\cdot,\cdot,\cdot,r(t)),\ r(t) \in \mathcal{S},$ of the HJ-inequality is positive-definite and guarantees dissipativity of the system for all $r(t) \in \mathcal{S}$. To do this, we first define the following concept of stochastic stability.

Definition 4.2 The equilibrium point $x = 0$ of the nonlinear system (15) with $u(t) \equiv 0$ is stochastically stable, if for any initial state $x_0 \in \mathcal{X}$ and $r_0 \in \mathcal{S}$,

$$\int_0^\infty E\{\|x(t, t_0, x_0, x_{-d}, r_0, 0)\|^2\}\, dt < \infty. \quad (27)$$

However, the following definition of stochastic stability will be more appropriate for our application in this paper.

Definition 4.3 The equilibrium point $x = 0$ of the nonlinear system (15) with $u(t) \equiv 0$ is locally asymptotically mean-square stable, if for any initial state $x_0 \in \mathcal{X}$ and $r_0 \in \mathcal{S}$,

$$\lim_{t\to\infty} E\{\|x(t,t_0,x_0,x_{-d},r_0,0)\|^2\} = 0. \tag{28}$$

Remark 4.1 The above definition also implies that stochastic stability or asymptotic stability in the mean-square sense implies stochastic $\mathcal{L}_2$-stability [13].

Remark 4.2 It is also seen from the definition of $\mathcal{L}_2$-gain (Definition 3.1) that, if we take $(t_0,t_1) = (0,\infty)$, then if the $\mathcal{L}_2$-gain of the system is finite, then the system is stochastically $\mathcal{L}_2$-stable.

Furthermore, since the question of stability can only be addressed on the infinite-time horizon, the HJ-inequality (21) takes the following form:

$$\begin{aligned}
&\Psi_{x_t}(x_t,x_{t-d},i)f(x_t,x_{t-d},i) + \Psi_{x_{t-d}}(x_t,x_{t-d},r(t))f(x_{t-d},x_{t-2d},r(t))\\
&+\frac{1}{2\gamma^2}\Psi_{x_t}g(x_t,i)g^T(x_t,i)\Psi_{x_t}^T + \frac{1}{2\gamma^2}\Psi_{x_{t-d}}g(x_{t-d},r(t-d))g^T(x_{t-d},r(t-d))\Psi_{x_{t-d}}^T\\
&+\frac{1}{2}h^T(x,i)h(x,i) + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j) \le 0 \quad \forall\, x_t,x_{t-d}\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned} \tag{29}$$

We now state our main stability theorem.

Theorem 4.1 *Suppose Σ^a is dissipative wrt to the supply rate*

$$s(u(t),y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2,$$

then Σ^a satisfies HJ-inequality (23) for each $r(t)\in\mathcal{S}$ and the system has $\mathcal{L}_2$-gain less than or equal to γ. Moreover, if Σ^a is stochastically zero-state detectable, then the free system $\dot{x}(t) = f(x(t),x(t-d),r(t))$ is locally mean square asymptotically stable.

Proof The first part of the theorem has already been proved in Lemmas 3.1 and 3.2. For the second part, from Lemma 4.1, $\Psi(\cdot,\cdot,\cdot,r(t))$, $\forall\, r(t)\in\mathcal{S}$ is positive-definite. Since Σ^a is dissipative, the free system with $u(t) = u(t-d) = 0$ satisfies the following dissipation inequality:

$$\Psi(x(\infty),x(\infty),r(\infty)) + E\left[\frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt\right] \le \Psi(x_0,x_{-d},r_0)$$

for any initial conditions $x_0,x_{-d}\in\mathcal{X}$, $r_0\in\mathcal{S}$. This implies that

$$E\left[\frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt\right] \le \Psi(x_0,x_{-d},r_0), \quad \forall\, x_0,x_{-d}\in\mathcal{X}, \quad r_0\in\mathcal{S}$$

or $y(t)\in L_2((\Omega,\mathcal{F},P)[0,\infty))$, and therefore, $\lim\limits_{t\to\infty} E(\|y(t)\|^2) = 0$. By the assumption of stochastic zero-state detectability, we also get $\lim\limits_{t\to\infty} E(\|x(t)\|^2) = 0$.

Remark 4.3 Theorem 4.1 above gives the bounded-real [1] conditions for the nonlinear system Σ^a. In the special case of linear systems, it gives necessary and sufficient conditions for the $\mathcal{L}_2$-gain (or $\mathcal{H}_\infty$-norm) of the system to be less than or equal to γ and to be locally asymptotically stable [1].

Remark 4.4 As a final remark, we mention that, if the jump rates $\lambda_{ij},\ i,j \in \mathcal{S},$ are very small, then all the results derived in this paper will approach the deterministic case.

5 Conclusion

In this paper, we have extended the theory of dissipative system developed for deterministic systems to the case of stochastic state-delayed systems with jump Markov disturbances. We have derived necessary and sufficient conditions for the system to be dissipative and to have finite $\mathcal{L}_2$-gain or the bounded-real condition, and have given sufficient conditions for stochastic stability of the system.

This paper has clearly laid down a framework for studying the $\mathcal{H}_\infty$ control and filtering problems for such systems and the stability of feedback interconnections. Future work will concentrate on these issues.

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