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H_∞ Control for a Class of Nonlinear Stochastic Time-Delay Systems

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Abstract: This paper mainly deals with H_∞ Controller design for a class of nonlinear stochastic time-delay systems with state and control-dependent noise. Some locally (globally) robust H_∞ Controllable conditions are given in terms of matrix inequalities independent of delay length. As applications, some sufficient conditions for the existence of the static state feedback H_∞ control law are presented for linear and special nonlinear stochastic time-delay systems via linear matrix inequalities, respectively.

Keywords: *Stochastic systems; linear matrix inequality; H_∞ control; time-delay systems.*

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1 Introduction

Since the celebrated paper [6] appeared, H_∞ control and filtering problems based on state-space approach, have attracted much more researchers' attention. For example, [1, 11] and [13] treated of the nonlinear uncertain H_∞ control and filtering design, while the H_∞ for linear time-delay systems with norm-bounded uncertainties can be found in [8, 10, 14, 15] and the references therein. The aforementioned works are confined to deterministic systems. Up to date, there are few results on stochastic H_∞ about which the system equation is governed by Itô-type differential equation. Below, we summarize the recent development for stochastic H_∞ briefly.

It is fair to say that [4] is the first paper which systematically dealt with the linear stochastic H_∞ control for state and output feedback control, in which, a very useful stochastic bounded real lemma (SBRL) was also derived, which has been applied to H_∞ filtering design of the stationary continuous time linear stochastic systems [5]. [2] first studied linear stochastic H_2/H_∞ control, in which, necessary and sufficient conditions were given for both finite and infinite horizon H_2/H_∞ via coupled Riccati equations; [16] was on output feedback H_∞ control for linear stochastic systems with norm bounded uncertainty in a state matrix, moreover, an applicable algorithm for designing an H_∞ control law was presented based on linear matrix inequalities (LMIs). In [3], we discussed the general nonlinear stochastic H_∞ control based on dissipative system theory and an associated Hamilton-Jacobi equation, which can be viewed as an extension of the results of [1] in some sense. In conclusion, we can say that stochastic H_∞ has become an attractive topic in recent years.

In spite of deterministic systems or stochastic systems, time-delay phenomena are inevitable arising from many physical problems, which often cause instability of the systems (see [18, 19]). Therefore, the H_∞ control of time-delay systems has received much attention in the past years (e.g. [8, 12]). This paper is on robust H_∞ control for a class of continuous time stochastic time-delay systems with nonlinear perturbation. By imposing a loose limitation on the nonlinear term, a very general theorem is obtained via matrix inequalities, from which, for some special case, we derived many useful sufficient conditions for the existence of a desired H_∞ controller in terms of LMIs. More specifically, as corollary, we also improve the previous conclusions on stochastic stabilization.

The outline of the current paper is organized as follows. In Section 2, we first present a general theorem on local and global H_∞ control by means of matrix inequalities independent of the length of delays, respectively. As corollaries, for linear or nonlinearly perturbed stochastic time-delay systems ($D = 0$), we are in a position to design an LMI-based state-feedback H_∞ control law, which makes our results more applicable [10].

Section 3 presents two examples to illustrate the effectiveness of our developed theory.

Section 4 concludes this note by some remarks.

For convenience, we adopt the following notations: A' is the transpose of matrix A; $A \geq 0$ $(A > 0)$ is positive semi-definite (positive definite) matrix A; I is identity matrix; $\mathcal{L}^2_{\mathcal{F}}(R_+, R^l)$ is the space of non-anticipative stochastic processes $y(t) \in R^l$ with respect to an increasing σ-algebras $\mathcal{F}_t$ $(t \geq 0)$ satisfying

$$E\int_0^\infty \|y(t)\|^2 dt < \infty.$$

Here $\|\cdot\|$ denotes the standard Euclidean norm of a vector.

2 Main Results

In this section, we investigate the robust H_∞ state feedback control of the following stochastic time-delay system governed by Itô differential equations of the form

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_1u(t) + B_2v(t) + H_0(x(t),\, x(t-\tau),\, u(t)))\, dt \\ &\quad + (Cx(t) + Dx(t-\tau) + D_1u(t) + H_1(x(t), x(t-\tau), u(t)))dw(t), \\ z(t) &= C_2x(t) + D_2u(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \quad \tau > 0. \end{aligned} \tag{1}$$

In the above, $x(t) \in R^n$, $u(t) \in R^m$, $v(t) \in R^r$, and $z(t) \in R^s$ are called the system state, control input, disturbance input, controlled output, respectively. $w(t)$ is the standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ with an increasing filtration $\mathcal{F}_t$ satisfying the usual conditions. Without loss of generality, we can suppose $w(t)$ is one-dimensional, and $C_2' D_2 = 0$. Assume $u(t)$ and $v(t)$ to be adapted and measurable processes with respect to $\mathcal{F}_t$, $H_i(0, \cdot, \cdot) = 0$, $i = 0, 1$, i.e., $x \equiv 0$ is an equilibrium point of (1). A, B, B_1, B_2, C, C_2, D, D_1, and D_2 are constant matrices, $\tau > 0$ is an uncertain time-delay, where we refer the reader to [18] for the notion of $L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n))$. Under very mild conditions on $H_i(\cdot, \cdot, \cdot)$, $i = 0, 1$, (1) exists a unique global solution on $[0, T]$ for any $T > 0$ [18]. It should be pointed out that (1) can represent a class of more general nonlinear stochastic system via Taylor's series expansion at the origin. In what follows, we will show that, for a broader class of nonlinear functions $H_i(\cdot, \cdot, \cdot)$, $i = 0, 1$, LMI-based algorithms for robust H_∞ Control can be given, which is very efficient in practical computation by means of the existing LMI Toolbox [7]. Now, we first introduce the following definitions.

Definition 1 Stochastic time-delay differential system (1) with $v(t) \equiv 0$ is called *locally robustly stabilizable*, if there exists a constant state-feedback control law $u = Kx$, such that the equilibrium point of the closed-loop system

$$\begin{aligned} dx(t) &= ((A + B_1 K)x(t) + Bx(t-\tau) + H_0(x(t),\, x(t-\tau),\, Kx(t)))\, dt \\ &\quad + ((C + D_1 K)x(t) + Dx(t-\tau) + H_1(x(t),\, x(t-\tau),\, Kx(t)))\, dw, \\ x(t) &= \phi(t) \in L^2(\Omega,\, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{2}$$

is asymptotically stable in probability [9] for all $\tau > 0$. It is called globally robustly stabilizable, if the equilibrium point of (2) is asymptotically stable in the large [9] for all $\tau > 0$.

Definition 2 Stochastic time-delay differential system (1) with $\phi(t) \equiv 0$, $u(t) \equiv 0$, is said *to have an H_∞ performance level* $\gamma > 0$, if

$$\|z\|_2 < \gamma \|v\|_2, \quad \forall\, v \neq 0 \in \mathcal{L}^2_{\mathcal{F}}(R_+, R^r) \tag{3}$$

where

$$\|z\|_2^2 = E \int_0^\infty z'(t) z(t)\, dt.$$

Definition 3 Stochastic time-delay differential system (1) is called *locally (globally) robustly H_∞ controllable*, if there exists a constant state-feedback control law $u = Kx$, such that system (1) is locally (globally) stabilizable via state-feedback control law $u(t) = Kx$, and the corresponding closed-loop system has an H_∞ performance level $\gamma > 0$.

For robust stabilization of (1) $(B_2 = 0)$, a very general result is given as follows, which can be proved in the same way as Theorem 1 of [17], but for convenience, we would like to give its detailed proof here.

Lemma 1 *Suppose there exists $\epsilon \geq 0$, such that*

$$\sup_{y \in R^n} \|H_i(x, y, Kx)\| \leq \epsilon \|x\|, \quad i = 0, 1, \tag{4}$$

for all $x \in U$, where U is a neighborhood of the origin, $K \in R^{m\times n}$, $P > 0$ and $Q > 0$ are the solutions of the following matrix inequality

$$Z + Z_1 < 0, \tag{5}$$

then system (1) can be locally robustly stabilized by $u(t) = Kx(t)$. If U is replaced by R^n, then system (1) can be globally robustly stabilized by the same controller. In (5), Z and Z_1 are defined by

$$Z = \begin{bmatrix} \begin{matrix}\{P(A+B_1K)+(A+B_1K)'P+Q \\ +(C+D_1K)'P(C+D_1K)\}\end{matrix} & PB+(C+D_1K)'PD \\ B'P+D'P(C+D_1K) & D'PD-Q \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} (2\epsilon\|C\|+2\epsilon\|D_1\|\,\|K\|+\epsilon\|D\|+2\epsilon+\epsilon^2)\|P\|I & 0 \\ 0 & \epsilon\|D\|\,\|P\|I \end{bmatrix}.$$

Proof We construct the Lyapunov–Krasovskii functional as follows:

$$V(t,x) = x'Px + \int_0^\tau x'(t-s)Qx(t-s)\,ds$$

where $P > 0$ and $Q > 0$ are the solutions of (5). Let $\mathcal{L}_1$ be the infinitesimal generator of the closed-loop system (2) with K a solution to (5), then by Itô's formula, we have

$$\begin{aligned} \mathcal{L}_1V(t,x(t)) = {} & ((C+D_1K)x(t)+Dx(t-\tau)+H_1(x(t),\,x(t-\tau),\,Kx(t)))'P \\ & \times((C+D_1K)x(t)+Dx(t-\tau)+H_1(x(t),x(t-\tau),Kx(t)) \\ & +2[(A+B_1K)x(t)+Bx(t-\tau)+H_0(x(t),x(t-\tau),Kx(t))]'Px(t) \\ & +x'(t)Qx(t)-x'(t-\tau)Qx(t-\tau). \end{aligned} \tag{6}$$

Rearranging (6) yields

$$\begin{aligned} \mathcal{L}_1V(t,x(t)) = {} & x'(t)(P(A+B_1K)+(A+B_1K)'P+Q+(C+D_1K)'P(C+D_1K))x(t) \\ & +2x'(t)(PB+(C+D_1K)'PD)x(t-\tau)+x'(t-\tau)(D'PD-Q)x(t-\tau) \\ & +2H_0'(x(t),x(t-\tau),Kx(t))Px(t)+2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ & +H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)) \\ = {} & \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' Z \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} + 2H_0'(x(t),x(t-\tau),Kx(t))Px(t) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ & +H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)). \end{aligned} \tag{7}$$

In addition, by (4), we have

$$\begin{aligned} 2H_0'(x(t),x(t-\tau),Kx(t))Px(t) &+ 2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ &+ 2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ &+ H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)) \\ \le 2\epsilon\|P\|(\|C\| + \|D_1\|\,\|K\|)\|x(t)\|^2 &+ 2\epsilon\|D\|\,\|P\|\,\|x(t)\|\,\|x(t-\tau)\| \\ &+ \epsilon^2\|P\|\,\|x(t)\|^2 + 2\epsilon\|P\|\,\|x(t)\|^2). \end{aligned} \tag{8}$$

for $(t,x)\in\{t>0\}\times U$. By inequality $|ab|\le \dfrac{1}{2}(a^2+b^2)$, (8) follows

$$\begin{aligned} 2H_0'(x(t),x(t-\tau),Kx(t))Px &+ 2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ &+ 2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ &+ H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)) \\ \le (2\epsilon\|C\| + 2\epsilon\|D_1\|\,\|K\| &+ \epsilon\|D\| + 2\epsilon + \epsilon^2)\|P\|\,\|x(t)\|^2 \\ &+ \epsilon\|D\|\|P\|\,\|x(t-\tau)\|^2 \\ = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' Z_1 \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}&. \end{aligned} \tag{9}$$

Substituting (9) into (7), it follows

$$\mathcal{L}_1 V(t,x(t)) \le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' (Z+Z_1) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} < 0$$

due to (5). That is, $\mathcal{L}_1 V(t,x(t)) < 0$ in the domain $\{t>0\}\times U$ for $x\neq 0$. So the locally robust stabilization is obtained by Corollary 1 of [9] (page 168). By the same discussion, the globally robust stabilization can also be shown by Theorem 4.4 of [9].

Using Lemma 1, a sufficient condition for robust H_∞ control is obtained as follows.

Theorem 1 *Suppose there exists $\epsilon\ge 0$, such that (4) holds for all $x\in U$ with U a neighborhood of the origin, $K\in R^{m\times n}$, $P>0$ and $Q>0$ are the solutions to the following matrix inequality*

$$\Sigma = \begin{bmatrix} Z_{11} + C_2'C_2 + K'D_2'D_2K & Z_{12} & PB_2 \\ Z_{12}' & Z_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{10}$$

where

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}' & Z_{22} \end{bmatrix} = Z + Z_1.$$

Then system (1) is locally robustly H_∞ controlled by $u(t)=Kx(t)$. If U is replaced by R^n, then system (1) is globally robustly H_∞ controlled by the same controller.

Proof It is obvious that (5) can be derived from (10), i.e. system (1) is robustly stable. Therefore we only need to prove that the closed-loop system has H_∞ performance level

γ. For any $T > 0$, by (10), it follows

$$\begin{aligned}
&\|z\|^2_{2,[0,T]} - \gamma^2\|v\|^2_{2,[0,T]} = E\int_0^T [(z'(t)z(t) - \gamma^2 v'(t)v(t))\, dt \\
&= E\int_0^T [(x'(t)C_2'C_2x(t) + x'(t)K'D_2'D_2Kx(t) - \gamma^2 v'(t)v(t))\, dt + d(V(x(t))] - EV(x(T)) \\
&\leq E\int_0^T [(x'(t)C_2'C_2x(t) + x'(t)K'D_2'D_2Kx(t) - \gamma^2 v'(t)v(t))\, dt + d(V(x(t))] \qquad (11) \\
&\leq E\int_0^T \psi'(t)\Sigma\psi(t) < 0
\end{aligned}$$

for $\psi \neq 0$, where $\psi = [x'(t)\ \ x'(t-\tau))\ \ v'(t)]'$. Let $T \to \infty$ in (11), (3) is immediately obtained. Theorem 1 is proved.

Generally speaking, Theorem 1 cannot be directly used in practice, because the elements of Z_1 contain the norm of an unknown matrix P. However, from Theorem 1, we can derive some useful results, which can be expressed in terms of LMIs.

Corollary 1 *If the matrix inequality*

$$\begin{bmatrix} \overline{Z}_{11} + C_2'C_2 + K'D_2'D_2K & \overline{Z}_{12} & PB_2 \\ \overline{Z}_{12}' & \overline{Z}_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{12}$$

has solutions $P > 0$, $Q > 0$ *and* $K \in R^{m\times n}$, *and*

$$\lim_{\|x\|\to 0} \sup_{y\in R^n} \|H_i(x,y,Kx)\|/\|x\| = 0, \quad i = 0,1, \tag{13}$$

where

$$\begin{bmatrix} \overline{Z}_{11} & \overline{Z}_{12} \\ \overline{Z}_{12}' & \overline{Z}_{22} \end{bmatrix} = Z,$$

then system (1) can be locally robustly H_∞ *controlled by* $u(t) = Kx(t)$.

Corollary 2 *If* $H_i \equiv 0$, $i = 0,1$, *and the matrix inequality (12) has solutions* $P > 0$, $Q > 0$, *and* $K \in R^{m\times n}$, *then the linear stochastic time-delay system*

$$\begin{aligned}
dx(t) &= (Ax(t) + Bx(t-\tau) + B_1u(t) + B_2v(t))\, dt \\
&\quad + (Cx(t) + Dx(t-\tau) + D_1u(t))\, dw(t), \\
z(t) &= C_2x(t) + D_2u(t), \\
x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau,0], R^n)), \quad t \in [-\tau, 0],
\end{aligned} \tag{14}$$

is globally robustly H_∞ *controllable. Especially, if* $D = 0$, *and the following LMI*

$$\begin{bmatrix} A\widehat{P} + \widehat{P}A' + B_1Y + Y'B_1' + B\widehat{Q}B' & \widehat{P}C' + Y'D_1' & \widehat{P} & \widehat{P}C_2' & Y'D_2 & B_2 \\ C\widehat{P} + D_1Y & -\widehat{P} & 0 & 0 & 0 & 0 \\ \widehat{P} & 0 & -\widehat{Q} & 0 & 0 & 0 \\ C_2\widehat{P} & 0 & 0 & -I & 0 & 0 \\ D_2Y & 0 & 0 & 0 & -I & 0 \\ B_2' & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{15}$$

admits solutions $\widehat{P} > 0$, $\widehat{Q} > 0$ *and* $Y \in R^{m\times n}$, *then system (14) with* $D = 0$ *is globally robustly* H_∞ *controllable. In this case, the state feedback control law* $u(t) = Kx(t) = Y\widehat{P}^{-1}x(t)$.

Proof If $H_i(\cdot,\cdot,\cdot) \equiv 0$, $i = 0, 1$, we can take $\epsilon = 0$ in (4), then $\mathcal{L}_1 V(t, x(t)) < 0$ for $(t, x) \in \{t > 0\} \times R^n$, except possibly at $x = 0$, and $\Sigma < 0$. Thus, the first part of Corollary 2 is proved.

Furthermore, if $D = 0$, (10) degenerates into

$$\begin{bmatrix} \begin{matrix}\{P(A+B_1K)+(A+B_1K)'P+Q+ \\ (C+D_1K)'P(C+D_1K)+C_2'C_2+K'D_2'D_2K\}\end{matrix} & PB & PB_2 \\ B'P & -Q & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{16}$$

Pre- and postmultiply the above matrix inequality by $\operatorname{diag}(P^{-1}, I, I)$, and set $\widehat{P} = P^{-1}$, $Y = KP^{-1} = K\widehat{P}$, $\widehat{Q} = Q^{-1}$. Then by Schur's complement again, (16) is equivalent to (15). Thus the second part of Corollary 2 is also proved.

Corollary 3 *The unforced system*

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_2 v(t))\,dt + (Cx(t) + Dx(t-\tau))\,dw(t), \\ z(t) &= C_2 x(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{17}$$

is robustly stable and has H_∞ *performance level* γ, *if the following LMI*

$$\begin{bmatrix} PA + A'P + C'PC + Q + C_2'C_2 & PB + C'PD & PB_2 \\ B'P + D'PC & D'PD - Q & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{18}$$

has solutions $P > 0$, $Q > 0$.

Corollary 4 *The stochastic linear time-delay controlled system*

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_1 u(t)) + B_2 v(t))\,dt + (Cx(t) + Dx(t-\tau))\,dw(t), \\ z(t) &= C_2 x(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{19}$$

is globally robustly H_∞ *controlled, if the following LMI*

$$\begin{bmatrix} PA + A'P + C'PC + C_2'C_2 + Q & \sqrt{2}PB_1 & PB + C'PD & PB_2 \\ \sqrt{2}B_1'P & -Q & 0 & 0 \\ B'P + D'PC & 0 & D'PD - Q & 0 \\ B_2'P & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{20}$$

admitting solutions $P > 0$ *and* $Q > 0$. *Moreover, the feedback control law* $u(t) = Q^{-1}B_1'Px(t)$.

Proof Applying Theorem 1, this corollary is easily obtained.

Below, for $D = 0$, we give another sufficient condition for the local (global) H_∞ control of system (1) in terms of LMIs. Applying the well known inequality

$$X'Y + Y'X \le \varepsilon X'X + \varepsilon^{-1}Y'Y, \quad \forall \varepsilon > 0, \tag{21}$$

with $\varepsilon = 1$ for simplicity, we have (if $0 < P \le \dfrac{1}{\alpha} I$ for some $\alpha > 0$)

$$\begin{aligned} &2H_0'(x(t), x(t-\tau), Kx(t))Px(t) + 2H_1'(x(t), x(t-\tau), Kx(t))P(C + D_1K)x(t) \\ &\quad + H_1'(x(t), x(t-\tau), Kx(t))PH_1(x(t), x(t-\tau), Kx(t)) \\ &\quad \le \frac{3\epsilon^2}{\alpha}\|x(t)\|^2 + x'(t)Px(t) + x'(t)(C + D_1K)'P(C + D_1K)x(t). \end{aligned} \tag{22}$$

Substituting (22) into (7), it follows

$$\mathcal{L}_1 V(t, x(t)) \le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' \widehat{Z} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

where

$$\widehat{Z} = \begin{bmatrix} \begin{matrix} \{P(A + B_1K) + (A + B_1K)'P + Q + P + \\ \frac{3\epsilon^2}{\alpha} I + 2(C + D_1K)'P(C + D_1K)\} \end{matrix} & PB \\ B'P & -Q \end{bmatrix}.$$

So if (4) holds for all $x \in U$ ($x \in R^n$), and $\widehat{Z} < 0$, then system (1) can be locally (globally) robustly stabilized by $u(t) = Kx(t)$. Accordingly, (12) is equivalent to

$$\begin{bmatrix} \widehat{Z}_{11} + C_2'C_2 + K'D_2'D_2K & \widehat{Z}_{12} & PB_2 \\ \widehat{Z}_{12}' & \widehat{Z}_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{23}$$

admitting solutions $0 < P \le \dfrac{1}{\alpha} I$, $Q > 0$ and K, where

$$\begin{bmatrix} \widehat{Z}_{11} & \widehat{Z}_{12} \\ \widehat{Z}_{12}' & \widehat{Z}_{22} \end{bmatrix} = \widehat{Z}.$$

In analogy with the proof of Corollary 2, it is easy to show that (23) is equivalent to that the following LMIs

$$\begin{bmatrix} \begin{matrix} A\widehat{P} + \widehat{P}A' + B_1Y \\ +Y'B_1' + B\widehat{Q}B' + \widehat{P} \end{matrix} & \sqrt{2}(\widehat{P}C' + Y'D_1') & \widehat{P} & \widehat{P} & \widehat{P}C_2' & Y'D_2 & B_2 \\ \sqrt{2}(C\widehat{P} + D_1Y) & -\widehat{P} & 0 & 0 & 0 & 0 & 0 \\ \widehat{P} & 0 & -\widehat{Q} & 0 & 0 & 0 & 0 \\ \widehat{P} & 0 & 0 & -\frac{\alpha}{3\epsilon^2} I & 0 & 0 & 0 \\ C_2\widehat{P} & 0 & 0 & 0 & -I & 0 & 0 \\ D_2Y & 0 & 0 & 0 & 0 & -I & 0 \\ B_2' & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{24}$$

and

$$\widehat{P} \ge \alpha I \tag{25}$$

exist solutions $\widehat{P} > 0$, $\alpha > 0$, $\widehat{Q} > 0$ and $Y = KP^{-1} \in R^{m\times n}$, where $\widehat{P} = P^{-1}$, $Y = KP^{-1} = K\widehat{P}$, and $\widehat{Q} = Q^{-1}$.

Summarize the above discussion, we have the following result.

Theorem 2 *For $D = 0$ in (1), suppose (4) holds for all $x \in U$ $(x \in R^n)$. If LMIs (24) and (25) exist solutions $\widehat{P} > 0$, $\alpha > 0$, $\widehat{Q} > 0$ and $Y \in R^{m \times n}$, simultaneously, then system (1) can be locally (globally) robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t)$.*

Remark 1 All results obtained in this section can be extended without difficulty to systems with multiple delays and independent stochastic perturbations.

Remark 2 Following the same line adopted above, there is no any difficulty to generalize what we have obtained to delay-dependent results with time-varying delay. For instance, if we take $\tau(t)$ to be a time-varying bounded delay satisfying

$$0 < \tau(t) \le h, \dot{\tau}(t) \le d < 1$$

and take the Lyapunov–Krasovskii functional

$$V(x) = x'(t)Px(t) + \int_{t-\tau(t)}^{t} x'(\theta)Rx(\theta)\,d\theta + \int_{-\tau(t)}^{0}\int_{t+\beta}^{t} x'(s)Qx(s)\,ds\,d\beta,$$
$$P > 0, \quad R > 0, \quad Q > 0,$$

correspondingly, then the delay-dependent consequences can be obtained.

In (1), if we take $\tau = 0$, $B = D = B_2 = D_2 = C_2 = 0$, $\phi(0) = x(0)$, then for the system

$$\begin{aligned} dx(t) &= (Ax(t) + B_1u(t) + H_0(x(t), u(t)))\,dt \\ &\quad + (Cx(t) + D_1u(t) + H_1(x(t), u(t)))\,dw(t) \end{aligned} \tag{26}$$

a locally stabilizable condition is concluded by Theorem 2.

Corollary 5 *If for some $\widehat{R} > 0$, $\widehat{Q} > 0$, the following generalized algebraic Riccati equation (GARE)*

$$\widehat{P}A + A'\widehat{P} + C'\widehat{P}C - (\widehat{P}B_1 + C'\widehat{P}D_1)(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C) + \widehat{Q} = 0 \tag{27}$$

has a positive definite solution $\widehat{P} > 0$, and

$$\lim_{\|x\|\to 0} \|H_i(x, \widehat{K}x)\|/\|x\| = 0, \quad i = 0, 1, \tag{28}$$

holds for $\widehat{K} = -(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C)$, then system (27) is locally asymptotically stabilizable. In this case, $u(t) = \widehat{K}x(t) = -(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C)x(t)$ is a stabilizing control law.

It can be seen that Corollary 5 generalizes and improves Proposition 1 of [20].

Remark 3 There is something wrong in Proposition 1 of [20]. By checking its proof therein, we can find that the smallest eigenvalue of $\widehat{Q} \ge 0$ must be larger than zero, i.e., $\widehat{Q} > 0$. In other words, $(\widehat{Q}^{1/2}, A)$ being observable should be replaced by $\widehat{Q} > 0$.

3 Numerical Examples

Now, we present two examples to illustrate the validity of our developed theory in designing the H_∞ controller for nonlinear time-delay system (1).

Example 1 In (1), take $D = 0$, and

$$A = \begin{bmatrix} -4.12 & 1.23 \\ -0.36 & 1.15 \end{bmatrix}, \quad B = \begin{bmatrix} -0.13 & -0.91 \\ 0.22 & -0.76 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.25 \\ 3.48 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2 \\ 0.3 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.02 & -0.09 \\ 0.09 & -0.08 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.16 \\ 0.23 \end{bmatrix}, \quad C_2 = [\,0.1 \quad 0.02\,], \quad D_2 = [\,0.1\,],$$
$$H_0(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} \sin(u(t)x_2(t-\tau))x_1(t) \\ \cos(u(t)x_1(t-\tau))x_2(t) \end{bmatrix},$$
$$H_1(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} e^{-(u(t)+x_1(t-\tau)+x_2(t-\tau))^2} x_2(t) \\ e^{[-u^2(t)x_1^2(t-\tau)]} x_1(t) \end{bmatrix}, \quad \forall\, \tau > 0.$$

Obviously, (4) holds for all $x \in R^n$ with $\epsilon = 1$. Substituting all the above data into (24), and then solving the LMIs (24) and (25) by LMI Toolbox [7], we can obtain solutions, when $\gamma = 1$,

$$\widehat{P} = \begin{bmatrix} 0.3539 & -0.0042 \\ -0.0042 & 0.1263 \end{bmatrix} > 0, \quad \widehat{Q} = \begin{bmatrix} 1.1197 & 0.0008 \\ 0.0008 & 1.0076 \end{bmatrix} > 0,$$
$$Y = [\,-0.2930 \quad -1.3061\,], \quad \alpha = 1.1255 > 0.$$

So by Theorem 2, system (1) can be globally robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t) = -0.8566x_1(t) - 2.4518x_2(t)$.

Example 2 In Example 1, we take

$$H_0(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} (e^{x_1(t)} - 1)\sin u(t) \\ \sin x_2(t)\cos u(t) \end{bmatrix},$$
$$H_1(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} (\cos x_1(t) - 1)e^{-x_2^2(t-\tau)} \\ x_2(t)\sin u(t) \end{bmatrix}, \quad \forall\, \tau > 0.$$

Obviously, we have

$$\|H_0(\cdot,\cdot,\cdot)\| \le \sqrt{(e^{x_1(t)} - 1)^2 + \sin^2 x_2(t)}\,,$$
$$\|H_1(\cdot,\cdot,\cdot)\| \le \sqrt{(\cos x_1(t) - 1)^2 + x_2^2(t)}\,,$$

and

$$\lim_{x_1 \to 0} \frac{(e^{x_1} - 1)}{x_1} = 1, \quad \lim_{x_2 \to 0} \frac{\sin x_2}{x_2} = 1, \quad \lim_{x_1 \to 0} \frac{(\cos x_1 - 1)}{x_1} = 0.$$

So there exists a sufficient small neighborhood U of the origin, such that for all $x \in U$, (4) holds with $\epsilon = 1.05$. Substituting all coefficient matrices of Example 1 into (24) with $\epsilon = 1.05$, when $\gamma = 1$, via solving the LMIs (24) and (25), one has

$$\widehat{P} = \begin{bmatrix} 0.3527 & -0.0060 \\ -0.0060 & 0.1371 \end{bmatrix} > 0, \quad \widehat{Q} = \begin{bmatrix} 1.1064 & -0.0018 \\ -0.0018 & 1.0000 \end{bmatrix} > 0,$$
$$Y = [\,-0.2993 - 0.3101\,], \quad \alpha = 1.0995 > 0.$$

So by Theorem 2, system (1) can be locally robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t) = -0.8875x_1(t) - 2.3013x_2(t)$.

4 Conclusions

In the above sections, we have discussed the state feedback H_∞ control for a class of stochastic time-delay systems with nonlinear perturbations. By means of LMIs, some sufficient conditions are given for the existence of an H_∞ control law. Theorem 1 is a very general consequence, from which we derive some useful results for linear time-delay systems, delay-free systems or special nonlinearly perturbed time-delay systems. All consequences except Theorem 1 and Corollary 1 are expressed in terms of LMIs, which makes them more readily applied.

References

[1] van der Schaft, A.J. L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Trans. Automat. Contr.* **37** (1992) 770 – 784.

[2] Chen, B.S. and Zhang, W. Stochastic H_2/H_∞ control with state-dependent noise. *IEEE Trans. Automat. Contr.* **49** (2004) 45 – 574.

[3] Chen, B.S. and Zhang, W. State feedback H_∞ control of nonlinear stochastic systems. *SIAM J. Contr. Optim.*, (revised).

[4] Hinrichsen, D. and Pritchard, A.J. Stochastic H_∞. *SIAM J. Contr. Optim.* **36** (1998) 1504 –1538.

[5] Gershon, E., Limebeer, D.J.N., Shaked, U. and Yaesh, I. Robust H_∞ filtering of stationary continuous-time linear systems with stochastic uncertainties. *IEEE Trans. Automat. Contr.* **46** (2001) 1788 –1793.

[6] Doyle, J.C., Glover, K., Khargonekar, P.P. and Francis, B. State-space solutions to standard H_2 and H_∞ problems. *IEEE Trans. Automat. Contr.* **34** (1989) 831– 847.

[7] Gahinet, P., Nemirovski, A., Laub, A.J. and Chilali, M. *LMI Control Toolbox.* Math. Works, MA, 1995.

[8] Shi, P., Agarwal, R.K., Boukas, E.K. and Shue, S.P. Robust H_∞ state feedback control of discrete time-delay linear systems with norm-bounded uncertain. *Int. J. Systems Sciences* **31** (2000) 409 – 415.

[9] Has'minskii, R.Z. *Stochastic Stability of Differential Equations.* Sijtjoff and Noordhoff, Alphen, 1980.

[10] Boyd, S., Ghaoui, L.El, Feron, E. and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, PA, 1994.

[11] Nguang, S.K. Robust nonlinear H_∞-output feedback control. *IEEE Trans. Automat. Contr.* **41** (1996) 1003 –1007.

[12] Xu, S., Lam, J. and Chen, T. Robust H_∞ control for uncertain discrete stochastic time-delay systems. *Systems Control Lett.* **51** (2004) 203 – 215.

[13] Nguang, S.K. and Fu, M. Robust nonlinear H_∞ filtering. *Automatica* **32** (1996) 1195 – 1199.

[14] de Souza, C.E. and Li, X. Delay-dependent robust H_∞ control of uncertain linear state-delayed systems. *Automatica* **35** (1999) 1313 –1321.

[15] Li, X. and de Souza, C.E. Delay-dependent robust stability and stabilization of uncertain linear delay systems: An LMI Approach. *IEEE Trans. Automat. Contr.* **42** (1997) 1144 – 1149.

[16] Zhang, W., Li, Q. and Hua, Y. Quadratic stabilization and output feedback H_∞ control of stochastic uncertain systems. *Proceeding of the 5th World Congress on Intelligent Control and Automation*, Hangzhou, 2004, P.728 – 732.

[17] Zhang, W., Chen, B.S. and Li, Q. Feedback stabilization of nonlinear stochastic time-delay systems with state and control-dependent noise. *Proceeding of the American Control Conference*, Boston, 2004.

[18] Mao, X. *Stochastic Differential Equations and Applications.* Horwood, Chichester, UK, 1997.

[19] Niculescu, S.I. *Delay Effects on Stability: A Robust Control Approach.* Springer, 2001.

[20] Gao, Z.Y. and Ahmed, N.U. Feedback stabilizability of non-linear stochastic systems with state-dependent noise. *Int. J. Contr.* **45** (1987) 729 – 737.