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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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ABSTRACTING INFORMATION

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Editorial

Stochastic nonlinear differential equations have been widely used to model physical systems that have abrupt variations in their structures. These abrupt variations may result from component and interconnection failures or repairs, parameters shifting, sudden environmental disturbances, abrupt variations of the operating point, etc. Stochastic nonlinear differential equations typically consist of both continuous and discrete states, which are, respectively, modelled by nonlinear differential equations and stochastic processes. Time-varying engineering systems such as electrical networks, economic systems, manufacturing systems, communication systems, and so forth have the characteristics of time-delay. In general, the existence of time delays degrades the control performance and may make the closed-loop stabilization very difficult.

Over the past two decades, considerable researches have been done on the analysis and synthesis of time-delay stochastic linear systems (TDSLS). Delay-independent methodologies for TDSLS which guarantee stability and prescribed performance level have been obtained. Recently, delay-dependent methodologies for TDSLS have been developed to reduce the conservativeness of the delay-independent methodologies. To the best of our knowledge, stability analysis and synthesis for time-delay stochastic nonlinear systems (TDSNS) have not been thoroughly invesigated yet. It was an inspiration to organize a special issue of this journal on:

Stability Analysis and Synthesis
for Time Delay Stochastic Nonlinear Systems

This special issue is composed of the invited papers written by leading researchers in the field of control systems science and engineering. Various novel methodologies have been proposed for TDSNS. The stabilization problem for TDSNS is addressed in three papers. Four papers extend $H_{\inf}$ design methodologies for TDSLS to TDSNS. One paper generalizes the concept of dissipativeness deloped for non-delay deterministic systems to TDSNS and one paper studies the problem of adaptive control for a class of TDSNS.

We would like to thank Professor A.A.Martynyuk, Editor-in-Chief of the Journal, for providing us the opportunity to organize the special issue. Finally, we would like to sincerely thank the contributors and the referees for all their hard works.

We hope that the Journal readers will share our evaluation and that the issue will be welcome by a broad scientific community and will become a long-standing reference.

Sing Kiong Nguang[1] and Peng Shi[2] – Guest Editors

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Dissipative Analysis and Stability of Nonlinear Stochastic State-Delayed Systems

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Abstract: In this paper, we extend the concept of dissipativeness developed for nondelay deterministic systems to stochastic state-delayed systems with Markov jump disturbances. We give necessary and sufficient conditions for the system to be dissipative and to have finite $\mathcal{L}_2$-gain also known as the bounded-real condition. Finally, we discuss the relationship between the dissipativeness of the system, its $\mathcal{L}_2$-gain, and its stochastic stability.

Keywords: *Nonlinear state-delayed system; Markov jump process; dissipative system; $\mathcal{L}_2$-gain; bounded-real lemma; stochastic stability.*

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1 Introduction

The important concept of dissipativity developed by Willems [14, 15], Hill and Moylan [5, 6] and Anderson [1], has been proven very successful in many feedback design synthesis problems [1, 11, 12, 14]. This concept which was originally inspired from electrical network considerations, in particular passive circuits, generalizes many other important concepts of physical systems such as positive realness, passivity, and losslessness. As such, many important mathematical relations of dynamical systems such as the bounded real lemma, positive real lemma, the existence of spectral factorization, and $\mathcal{L}_2$-gain of linear and nonlinear systems have been shown to be consequences of this important theory. Moreover, there has been renewed interest lately on this important concept as having been instrumental in the derivation of the solution of the nonlinear $\mathcal{H}_\infty$ control problem [12]. It has been shown that a sufficient condition for the solution to this problem is the existence of a solution to some dissipation inequalities.

However, the theory of dissipativeness more generally studied by Hill and Moylan [5, 6], Willems [14, 15] is purely from a deterministic setting. Many physical systems are however stochastic; for example, a control system is constantly perturbed by unwanted disturbances, a communication system is affected by noise while an aeroplane is frequently fluttered by air pockets. In addition, many physical systems are subject to random changes which may result from abrupt phenomenon such as component and interconnection failures. Hence fault-tolerant systems have been developed to ensure high reliability and performance in such situations.

Therefore, in this paper, we extend the theory of dissipativity to include stochastic state-delayed systems or systems that are subject to random disturbances. In particular, we consider a class of nonlinear stochastic systems with state-delay and random Markovian jump parameters or disturbances. This class of systems belongs to the class of hybrid systems with continuous state dynamics and discrete parameter variation. The control and filtering problems for this class of systems has been discussed by many authors [3, 9, 10]. In particular, Rishel [10] has derived the minimum principle for the general nonlinear case without state-delay and in which the adjoint equations are deterministic. While Ji and Chizeck [3, 7] have derived the structural properties, namely, controllability, observability and stability for the linear case. Furthermore, the problems of controller design for the linear case using LQ and LQG criteria have been discussed extensively in Mariton [9].

Thus, in this paper, we discuss additional structural (or internal) properties of this class of systems which are closely associated with their stability. We discuss the dissipative properties of this class of systems, which determine whether they absorb energy and conserve it, or dissipate it; and based on this property, what could we infer about the stability of such systems? We also give a fresh interpretation of the concept of dissipativity as both an input/output property and an internal property of a system. The closest work to the current one in this paper can be found in [4] for systems without state-delay.

The paper is organized as follows. In Section 2, we define the problem and discuss necessary and sufficient conditions for a nonlinear state-delayed system with Markov jump disturbances to be dissipative. We continue this discussion in Section 3 for the case of a quadratic supply rate and discuss the relationship between the dissipativity of the system and its $\mathcal{L}_2$-gain, which leads to the bounded-real lemma for this class of systems. Finally, in Section 4, we discuss the implications of dissipativity on the stability of the system. Conclusions are then given in Section 5.

2 Dissipativity of State-Delayed Nonlinear Stochastic Systems with Jumps

In this section, we define the concept of dissipativity of a state-delayed nonlinear system with jump Markov disturbances. The notation is standard except where specified otherwise. Moreover, R_+ is the positive real-line, R^n is the n-dimensional Euclidean space and $\|\cdot\|$ represents the Euclidean vector norm. The spaces $\mathcal{L}_{1,loc}((t_0,t_1),R)$, $\mathcal{L}_2([0,T],R^n)$ are the standard Lebesgue spaces of locally integrable on (t_0,t_1) and square integrable over $[0,T]$ vector functions on R^n respectively. While $\mathcal{L}_2([0,T],(\Omega,\mathcal{F},P))$ is the corresponding space over the probability space $(\Omega,\mathcal{F},P)$, in which Ω is the sample space, $\mathcal{F}$ is the σ-algebra generated by Ω and P is a probability measure over $\mathcal{F}$. Lastly, E will denote the mathematical expectation operator.

Let us at the outset consider the following piece-wise autonomous nonlinear state-delayed system defined over an open subset $\mathcal{X}\times\mathcal{S}$ of $R^n\times Z_+$ with $\mathcal{X}$ containing the origin,

$$\Sigma:\quad \dot{x}(t) = f(x(t),\, x(t-d),\, u(t),\, r(t)),$$
$$x(t) = \phi(t), \quad t \in [-d, 0], \quad x(t_0) = x_0 = \phi(t_0), \tag{1}$$
$$y(t) = h(x(t), r(t)), \tag{2}$$

where $x(t) \in \mathcal{X}$ is the state vector, $u(t) \in \mathcal{U} \subset R^p$ is the input function belonging to an input space $\mathcal{U}$, $d > 0$ is the delay, $y(t) \in \mathcal{Y} \subset R^m$ is the output function which belongs to the output space $\mathcal{Y} \subset R^m$, and $\phi(t) \in C[-d, 0]$ is the initial function. Besides the dependence on the input and initial conditions, the state of the system is also a function of the discrete parameter $r(t)$ which is a continuous-time homogeneous Markov process with finite discrete state-space $\mathcal{S} \triangleq \{1, 2, \dots, l\}$. We assume that the probabilities $P_t \triangleq (P_{1t}, \dots, P_{lt})$, with $P_{it} \triangleq P(r(t) = i)$, $i = 1, \dots, l$, satisfy the forward Kolmogorov equation

$$\frac{\partial P_t}{\partial t} = \Lambda P_t, \qquad P_0 = \bar{P}, \quad t \in [0, T],$$

where $\Lambda = [\lambda_{ij}]_{i,j \in \mathcal{S}}$ is the transition matrix, and λ_{ij} are real numbers such that for $i \neq j$, $\lambda_{ij} \geq 0$, and for all $i \in \mathcal{S}$, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. In other words, the transition probabilities are given by

$$P[r(t+h) = j,\ r(t) = i] = \begin{cases} \lambda_{ij} h + o(h) & \text{if} \quad j \neq i, \\ 1 + \lambda_{ii} h + o(h) & \text{if} \quad j = i, \end{cases}$$

where $o(h)$ are the remainder terms such that $\lim\limits_{h \to 0} \dfrac{o(h)}{h} = 0$.

The functions $f\colon \mathcal{X} \times \mathcal{X} \times \mathcal{U} \times \mathcal{S} \to X$, $h\colon \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ are real smooth functions of their arguments for each $r(t) \in \mathcal{S}$. We also assume the following.

Assumption 2.1 The system Σ is causal, time-invariant and finite-dimensional. Further, the functions $f(\cdot, \cdot, \cdot, r(t))$, $h(\cdot, r(t))$ for each value of $r(t) \in \mathcal{S}$ are smooth C^∞ functions of $x \in \mathcal{X}$ and $u \in \mathcal{U}$ such that the system (1) is well-defined; that is, for any initial state $x(t_0) \in \mathcal{X}$, initial mode $r(t_0) = r_0 \in \mathcal{S}$ and any admissible input, $u(t) \in \mathcal{U}$, there exists a unique solution $x(t, t_0, x_0, x_{t_0 - d}, r_0, u)$ to (1) on $[t_0, \infty)$ which continuously depends on the initial data.

Alternatively, the following assumptions are also sufficient to guarantee the existence and uniqueness of solutions to the system Σ [2].

Assumption 2.2 For all $t,\, t_1,\, t_2 \in [-d, \infty)$, $r(t) \in \mathcal{S}$,

(a) (Lipschitz condition)

$$\begin{gathered} \|f(x(t_2), x(t_2 - d),\, u(t_2),\, r(t))) - f(x(t_1),\, x(t_1 - d),\, u(t_1),\, r(t))\| \\ \leq K_1 \|x(t_2) - x(t_1)\| + K_2 \|x(t_2 - d) - x(t_1 - d)\| + K_3 \|u(t_2) - u(t_1)\| \\ \forall\, x(t_2),\, x(t_1),\, x(t_2 - d),\, x(t_1 - d) \in \mathcal{X}, \quad u(t_1),\, u(t_2) \in \mathcal{U}; \end{gathered}$$

(b) (Restriction on Growth)

$$\begin{gathered} \|f(x(t), x(t-d), u(t), r(t))\|^2 P \leq K_1^2 (1 + \|x(t)\|^2) + K_2^2 (1 + \|x(t-d)\|^2) \\ + K_3^2 (1 + \|u(t)\|^2), \quad \forall\, x(t), x(t-d) \in \mathcal{X}, \quad u \in \mathcal{U} \\ \|h(t, x(t), r(t))\| \leq K_4 (1 + \|x(t)\|^2), \quad \forall\, x(t) \in \mathcal{X}, \end{gathered}$$

where K_1, K_2, K_3, K_4 are positive constants.

Now let $\mathcal{F}_t$ be the σ-algebra generated by $r(t)$, $t \in [0,T]$. Then we take the input space $\mathcal{U}$ and output space $\mathcal{Y}$, to be $\mathcal{F}_t$-measurable, and piecewise continuous. Similarly, the functions $f(\cdot,\cdot,\cdot)$, $h(\cdot,\cdot)$ are also assumed to be $\mathcal{F}_t$ measurable by continuity with respect to $x \in \mathcal{X}$.

If the system Σ is viewed as a black box with only inputs and outputs, then in the above representation, the system Σ is a map $\Sigma\colon \mathcal{U} \times \mathcal{X} \times \mathcal{S} \to \mathcal{Y}$ which transforms inputs to outputs through state functions $x(t) \in \mathcal{X}$ for each $r(t) \in \mathcal{S}$. In view of this, if we assign an energy measure to both the inputs and outputs of the system, then it is possible to infer the internal behavior of the system by comparing these two quantities. This motivates the following definition of a supply rate to the system.

Definition 2.1 A function $s(u(t), y(t))\colon \mathcal{U} \times \mathcal{Y} \to R$ is a supply rate to the system Σ if $s(\cdot,\cdot)$ is piecewise continuous and locally integrable, i.e.,

$$E\left[\int_{t_0}^{t_1} |s(u(t), y(t))|\, dt\right] < \infty \tag{4}$$

or $s(\cdot,\cdot) \in \mathcal{L}_{1,loc}(t_0, t_1)$ for any $(t_0, t_1) \in R_+^2$, for all $u(t) \in \mathcal{U}$.

Remark 2.1 The supply rate $s(\cdot,\cdot)$ is a measure of the instantaneous power into the system. Part of this power is stored as internal energy and part of it is dissipated.

It follows from the above definition of supply rate that, to infer about the internal behavior of the system, it is sufficient to evaluate the expected total amount of energy expended by the system over a finite time interval. This leads us to the following definition.

Definition 2.2 The system Σ is dissipative with respect to (wrt) the supply rate $s(t) = s(u(t), y(t))$ if for all $u(t) \in \mathcal{U}$ and $t_0, t_1 \in R_+^2$,

$$E\left[\int_{t_0}^{t_1} s(u(t), y(t))\, dt\right] \geq 0; \quad \forall\, t_1 \geq t_0. \tag{5}$$

when evaluated along any trajectory of the system starting at t_0, $x(t) = 0$.

Remark 2.2 The above definition suggests that, the dissipativity of the system is an input-output property. This is also the notion put forward in [5]. Furthermore, it also raises the following question: Can every finite dimensional, time-invariant, causal system be rendered dissipative by a suitable choice of input? The answer to this question will be given in due course, but in short it is: yes and no!

The above Definition 2.2 being an inequality postulates the existence of a storage function and a possible dissipation rate for the system. It follows that if the system is assumed to have some stored energy which is measured by a function $\Psi\colon R_+ \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$, then for the system to be dissipative, it is necessary that in the transition from t_0 to t_1, the total amount of energy stored is less than the amount expended. This suggests the following alternative definition of dissipativity.

Definition 2.3 The system Σ is said to be dissipative with respect to a supply rate $s(u(t), y(t))$ if for all $(t_0, t_1) \in R_+^2$ there exist positive-semidefinite functions (storage functions) $\Psi\colon R_+ \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$, such that the inequality

$$E\Psi(t_1, x(t_1),\, x(t_1-d),\, r(t_1)) - \Psi(t_0, x(t_0),\, x(t_0-d),\, r(t_0)) \le E\left[\int_{t_0}^{t_1} s(u(t), y(t))\, dt\right] \quad (6)$$

is satisfied for all $t_1 \ge t_0$, modes $r(t_1), r(t_2) \in \mathcal{S}$ and initial states $x(t_0-d), x_0 \in \mathcal{X} \times \mathcal{X}$, where $x(t_1) = x(t_1, t_0, x_0, x_{t_0-d}, r_0, u)$.

In the sequel we shall also use the following notations $x(t_i) = x_{t_i} = x_i$, $x(t_i - d) = x_{t_i-d}$, $r(t_i) = r_i$, $i \in Z$.

Remark 2.3 The system is also said to be lossless if the above inequality (6) is satisfied as an equality.

The above inequality (6) can be converted to an equality by introducing the dissipation rate $d\colon \mathcal{M} \times \mathcal{U} \times \mathcal{S} \to R$ according to the following equation

$$\begin{gathered} E\Psi(t_1, x_{t_1}, x_{t_1-d}, r_1) - \Psi(t_0, x_0, x_{t_0-d}, r_0) = E\left[\int_{t_0}^{t_1} [s(t) + d(t)]\, dt\right], \\ \forall\, t_1 \ge t_0, \qquad \forall\, r_1, r_0 \in \mathcal{S}. \end{gathered} \quad (7)$$

Remark 2.4 The dissipation rate is nonnegative if the system is dissipative. Moreover, the dissipation rate uniquely determines the storage function $\Psi(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$ [15].

We now define the concept of available storage, the existence of which determines whether the system is dissipative or not.

Definition 2.4 The available storage $\Psi^a(t, x, r(t))$ for each $r(t) \in \mathcal{S}$ of the dynamical system Σ is the quantity:

$$\Psi^a(t, x(t),\, x(t-d),\, r(t)) = \sup_{x_0 = x,\, u \in \mathcal{U},\, t \ge 0} -E\left[\int_0^t s(u(\tau), y(\tau))\, d\tau\right], \quad (8)$$

where the supremum is taken over all possible inputs, $u \in \mathcal{U}$ starting at x and time $t_0 = 0$.

It follows that, if the system is dissipative, then the available storage is well-defined and finite in each state of the system x, and mode r_0. Moreover, it determines the maximum amount of energy which may be extracted from the system Σ. This is stated in the following theorem.

Theorem 2.1 *The available storage, $\Psi^a(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$, is finite if and only if (iff) the system is dissipative. Furthermore, any other storage function is lower bounded by $\Psi^a(\cdot,\cdot,\cdot, r(t))$ for each $r(t) \in \mathcal{S}$, i.e., $0 \le \Psi^a(\cdot,\cdot,\cdot, r(t)) \le \Psi(\cdot,\cdot,\cdot, r(t))$, $r(t) \in \mathcal{S}$.*

Proof Notice that $\Psi^a(\cdot,\cdot,\cdot,\cdot) \ge 0$ since it is the supremum over a set with the zero element (at $t = 0$). Now assume that $\Psi^a(\cdot,\cdot,\cdot,\cdot) < \infty$. We have to show that the system is dissipative, i.e., for any $(t_0, t_1) \in R_+^2$

$$\Psi^a(t_0, x_0, x(t_0-d), r_0) + E\left[\int_{t_0}^{t_1} s(u(\tau), y(\tau))\, d\tau\right] \geq E\Psi^a(t_1, x_1, x(t_1-d), r_1), \quad \forall\, x_0, x_1 \in \mathcal{X}, \quad r_0, r_1 \in \mathcal{S}. \tag{9}$$

In this regard, notice that from (8)

$$E\Psi^a(t_1, x_1, x(t_1-d), r_1) - \Psi^a(t_0, x_0, x(t_0-d), r_0) = \sup_{x_0, u} E\left[-\int_{t_0}^{t_1} s(t)\, dt\right], \quad \forall\, r_0,\ r_1 \in \mathcal{S}. \tag{10}$$

This implies that

$$E\Psi^a(t_1, x_1, x(t_1-d), r_1) \geq \Psi^a(t_0, x_0, x(t_0-d), r_0) + E\left[\int_{t_0}^{t_1} s(t)\, dt\right], \tag{11}$$

and since all the above quantities are greater or equal to zero, it implies that $\Psi^a(\cdot, \cdot, \cdot, r(t))$ satisfies the dissipation inequality (6) for each $r(t)$.

Conversely, assume that Σ is dissipative. Then the dissipation inequality (6) implies that

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) + E\left[\int_{t_0}^{t_1} s(t)\, dt\right] \geq E\Psi(t_1, x_1, x_{t_1-d}, r_1) \geq 0; \quad \forall\, x_0, x_1 \in \mathcal{X}, \quad r_0, r_1 \in \mathcal{S}, \tag{12}$$

by definition. Therefore,

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) \geq -E\left[\int_0^{t_1} s(t)\, dt\right] + E\left[\int_0^{t_0} s(t)\, dt\right] \tag{13}$$

which implies that

$$\Psi(t_0, x_0, x_{t_0-d}, r_0) \geq \sup_{x=x_0,\, u\in\mathcal{U},\, t\geq 0} E\left[-\int_0^{t_1} s(t)\, dt\right] = \Psi^a(t_0, x_0, x_{t_0-d}, r_0). \tag{14}$$

Hence $\Psi^a(t, x, x(t-d), r(t)) < \infty\ \forall\, x \in \mathcal{X},\ r(t) \in \mathcal{S}$.

Remark 2.5 The above theorem summarizes the answer to the question we raised above, that dissipativity is both an input/output property and an internal property. It suggests that a system that is not dissipative wrt one supply rate may be dissipative wrt to another. It therefore follows that the system must possess some internal structure such that, the available storage $\Psi^a(\cdot, \cdot, \cdot, r(t))$ is well-defined for each $r(t) \in \mathcal{S}$ and in each state of the system for a particular supply rate.

Remark 2.6 The importance of the above theorem in checking dissipativeness of the nonlinear system Σ cannot be overemphasized. It follows that, if the system is reachable from the origin $\{0\}$, then by an appropriate choice of an input $u(t)$ such that $\Psi^a(\cdot,\cdot,\cdot,r(t))$, $r(t)\in\mathcal{S}$ is finite, it can be rendered dissipative. However, evaluating $\Psi^a(\cdot,\cdot,\cdot,\cdot)$ is a difficult task without the output of the system specified a priori or solving the state equations. This therefore calls for an alternative approach for determining the dissipativeness of the system. This is discussed in the next section.

3 Relationship with $\mathcal{L}_2$-gain

In this section, we discuss the connection between the dissipativity of the nonlinear system Σ with its $\mathcal{L}_2$-gain. In the classical paper by Willems [14], the relationship between dissipativity and Linear Quadratic (LQ)-control has been shown and this relationship has been exploited to prove the existence of solutions to certain infinite-horizon LQ-control problems leading to the Algebraic-Ricatti equation (ARE). Similarly, we also discuss the relationship between the dissipativity of the nonlinear system with certain Hamilton-Jacobi equations arising in the $\mathcal{L}_2$-gain optimization of the nonlinear system. To this end and for the purpose of clearity, let us consider an affine representation Σ^a of the system Σ defined by:

$$\Sigma^a:\quad \dot{x}(t)=f(x(t),\,x(t-d),\,r(t))+g(x,r(t))u(t), \tag{15}$$

$$x(t)=\phi(t),\quad t\in[-2d,0],\quad x(t_0)=x_0=\phi(t_0)$$

$$y(t)=h(x(t),r(t)), \tag{16}$$

where $g(\cdot,\cdot)\in C^\infty(\mathcal{X}\times S)\in R^{n\times k}$. In this case, our existence and uniqueness Assumptions 2.2 take the following form:

Assumption 3.1 For all $t_1,t_2\in[-2d,\infty)$, $r(t)\in\mathcal{S}$,

(a) (Lipschitz condition)

$$\begin{aligned}\|f(x(t_2),x(t_2-d),r(t))-f(x(t_1),x(t_1-d),r(t))\|+\|g(x(t_2),r(t))-g(x(t_1),r(t))\|\\ \le K_1\|x(t_2)-x(t_1)\|+K_2\|x(t_2-d)-x(t_1-d)\|+\|u(t_2)-u(t_1)\|,\\ \forall\, x(t_1),x(t_2)\in\mathcal{X},\quad u(t_1),u(t_2)\in\mathcal{U};\end{aligned}$$

(b) (Restriction on growth)

$$\begin{aligned}\|f(x(t),x(t-d),r(t))\|^2+\|g(x(t),r(t)\|^2\le K_1^2(1+\|x\|^2)+K_2^2(1+\|x(t-d)\|^2)\\ +K_3^2(1+\|u(t)\|^2),\quad \forall\, x(t),\,x(t-d)\in\mathcal{X},\quad u(t)\in\mathcal{U},\end{aligned}$$

where K_1, K_2, K_3 are positive constants and $\|g\|^2=\mathrm{Tr}(gg^T)$ represents the matrix trace norm.

The question we wish to answer in this section is the following: If we restrict the input space $\mathcal{U}$ of the system to be the space $\mathcal{L}_2[-2d,\infty)$, then under what conditions is the system dissipative ? or can be rendered dissipative? To motivate the discussion, we expand the definition of $\mathcal{L}_2$-gain [12] as follows.

Definition 3.1 The system (15) is said to have $\mathcal{L}_2$-gain from $u(t)$ to $y(t)$ less than or equal to some number $\gamma' > 0$ if for all $(t_0, t_1) \in [-d, \infty)$, initial state vector $x_0 \in \mathcal{X}$, and mode $r_0 \in \mathcal{S}$, the response of the system $y(t)$ due to any $u(t) \in \mathcal{L}_2[0, \infty)$ satisfies

$$E\left[\int_{t_0}^{t_1} \|y(t)\|^2 \, dt\right] \leq \frac{1}{2} \gamma'^2 \int_{t_0}^{t_1} (\|u(t)\|^2 + \|u(t-d)\|^2) \, dt + \beta(x_0, r_0); \quad \forall \, t_1 \geq t_0 \tag{17}$$

and some class $\mathcal{K}$ functions [13] $\beta \colon \mathcal{X} \times \mathcal{S} \to R_+$, $\beta(0, r(t)) = 0 \ \forall \, r(t) \in \mathcal{S}$.

Remark 3.1 In the above definition, if $d = 0$, we recover the usual definition of $\mathcal{L}_2$-gain for non-delay systems. In this regard, right-hand side represents an average. Moreover, in the sequel we shall let $\gamma = \gamma'/\sqrt{2}$ and call γ the $\mathcal{L}_2$-gain of the system with a slight abuse of the definition.

Remark 3.2 It is also obvious from the definition of $\mathcal{L}_2$-gain and dissipativity of the nonlinear system (15) wrt to the supply rate $s(u(t), y(t))$, that, dissipativity of the system wrt the supply rate $s(u(t), y(t))$, implies finite $\mathcal{L}_2$-gain $\leq \gamma$.

Furthermore, from the definition of dissipativity given in (6), if the function $\Psi(t, x(t), x(t-d), r(t))$ belongs to $C^1(R_+ \times \mathcal{X} \times \mathcal{X})$, it is possible to go from the integral version of the above dissipation inequality (6) to the differential form. This is stated in the following lemma. We shall also be particularly interested in the following supply rate $s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2$, $\gamma > 0$.

In the sequel, we shall also use the notation $r(t) = i$ and $r(t) = j$, $i, j \in \mathcal{S}$.

Lemma 3.1 *The nonlinear system Σ^a is dissipative wrt the supply rate*

$$s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2,$$

if there exist some C^1 nonnegative functions $\Psi \colon R \times \mathcal{X} \times \mathcal{X} \times \mathcal{S} \to R_+$ such that the following differential dissipation inequality is satisfied for all $x(t) \in \mathcal{X}$, $r(t) \in \mathcal{S}$:

$$\begin{aligned}
&\Psi_t(t, x_t, x_{t-d}, r(t)) + \Psi_{x_t}(t, x_t, x_{t-d}, r(t))[f(x_t, x_{t-d}, r(t)) + g(x_t, r(t))u] \\
&\quad + \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d}, \, r(t-d)) + g(x_{t-d}, \, r(t-d))u(t-d)] \\
&+ \sum_{r(t)=j \in \mathcal{S}} \lambda_{ij} \Psi(t, x_t, x_{t-d}, j) - \frac{1}{2} \gamma^2 (\|u(t)\|^2 + \|u(t-d)\|^2) + \frac{1}{2}\|y(t)\|^2 \leq 0, \\
&\quad \Psi(t, 0, 0, r(t)) = 0 \quad \forall \, t \in R, \quad r(t) = i, \quad u(t), \, u(t-d) \in \mathcal{L}_2[-d, \infty),
\end{aligned} \tag{18}$$

where $\Psi_t(\cdot,\cdot,\cdot,\cdot)$, $\Psi_{x_t}(\cdot,\cdot,\cdot,\cdot)$ and $\Psi_{x_{t-d}}(\cdot,\cdot,\cdot)$ are the row vectors of partial derivatives of $\Psi(\cdot,\cdot,\cdot,\cdot)$ wrt t, x_t and x_{t-d} respectively.

Proof Without any lost of generality, we will take $t_0 = 0$ and $t_1 = T$. Now consider the following variation of the Dynkin's formula [8]:

$$\begin{aligned}
&E\Psi(T, x(T), x(T-d), \, r(T)) - \Psi(0, x_0, x_{-d}, r_0) \\
&= E\left[\int_0^T \mathcal{L}\Psi(t, x(t), \, x(t-d), \, r(t)) \, dt\right] \quad \forall \, T > 0,
\end{aligned} \tag{19}$$

where $\mathcal{L}$ is the infinitesimal generator of the process $(x(t), r(t))$, $t \geq 0$ [8,9]. Then using the above formula (19) in the dissipation inequality (6) and the fact that

$$\begin{aligned}\mathcal{L}\Psi(t, x_t, x_{t-d}, r(t)) &= \Psi_t(t, x_t, x_{t-d}, r(t)) \\ &+ \Psi_{x_t}(t, x_t, x_{t-d}, r(t))[f(x_t, x_{t-d}, r(t)) + g(x_t, r(t))u] \\ &+ \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d}, r(t-d)) + g(x_{t-d}, r(t-d))u(t-d)] \\ &+ \sum_{r(t)=j\in\mathcal{S}} \lambda_{ij}\Psi(x, r, j), \quad r(t) \in \mathcal{S},\end{aligned} \tag{20}$$

the result follows.

Remark 3.3 By virtue of the above lemma, we will henceforth consider only C^1 storage functions in this paper.

Lemma 3.2 *For the nonlinear system Σ^a, we have the following implications:* $(a) \Leftrightarrow (b) \rightarrow (c)$

(a) *the system Σ^a satisfies the dissipation inequality (18);*
(b) *the system Σ^a is dissipative wrt to the supply rate s(u(t),y(t));*
(c) *the system Σ^a has $\mathcal{L}_2$-gain from $u(t)$ to $y(t)$ less than or equal to γ.*

Proof (sketch) $(a) \Leftrightarrow (b)$ follows from Lemma 3.1 above, while (c) follows from (6),(17) and the fact that $E\Psi(\cdot,\cdot,\cdot,\cdot) \geq 0$ by Theorem 2.1.

We now state the main result of this section which is a consequence of Lemmas 3.1 and 3.2 above.

Theorem 3.1 *A necessary and sufficient condition for the nonlinear system (15) to be dissipative wrt the supply rate*

$$s(u(t), y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2$$

is that there exist a set of smooth positive-semidefinite solutions of the following stochastic Hamilton-Jacobi (HJ) inequality for each $r(t) \in \mathcal{S}$:

$$\begin{aligned}&\Psi_t(t, x_t, x_{t-d}, r(t)) + \Psi_{x_t}(t, x_t, x_{t-d}, r(t))f(x_t, x_{t-d}, r(t)) \\ &+ \Psi_{x_{t-d}}(t, x_t, x_{t-d}, r(t))f(x_{t-d}, x_{t-2d}, r(t)) + \frac{1}{2\gamma^2}\Psi_{x_t}g(x_t, r(t))g^T(x_t, r(t))\Psi_{x_t}^T \\ &+ \frac{1}{2\gamma^2}\Psi_{x_{t-d}}g(x_{t-d}, r(t-d))g^T(x_{t-d}, r(t-d))\Psi_{x_{t-d}}^T + \frac{1}{2}h^T(x_t, i)h(x_t, i) \\ &+ \sum_{r(t)=j\in\mathcal{S}} \lambda_{ij}\Psi(t, x_t, x_{t-d}, j) \leq 0, \quad \Psi(t, 0, 0, i) = 0 \quad \forall\, x \in \mathcal{X}, \quad r(t) = i \in \mathcal{S}.\end{aligned} \tag{21}$$

Proof (**Necessity**) Theorem 2.1 has shown that if the system Σ^a is dissipative, then there exists at least one set of solutions to the dissipation inequality (6) for each $r(t) \in \mathcal{S}$ which is given by the available storage, $\Psi^a(t, x_t, x_{t-d}, r(t))$, $r(t) \in \mathcal{S}$. We now show that any solution of the dissipation inequality (6) is also a solution to the HJ-inequality (21).

If the system is dissipative with storage function $\Psi(\cdot,\cdot,\cdot,)$, then along any trajectory of the system, the differential dissipation inequality (18) is satisfied. The left-hand-side (LHS) of this inequality is a quadratic function of u with maximum at

$$u^\star(t,x_t) = \frac{1}{\gamma^2}\, g^T(x_t,r(t))\Psi^T_{x_t}(x_t,r(t)). \tag{22}$$

The maximum value of the function corresponding to this stationary point, is given by

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)f(x_t,x_{t-d},i)\\
&+ \Psi_{x_{t-d}}(t,x_t,x_{t-d},r(t))f(x_{t-d},x_{t-2d},r(t)) + \frac{1}{2\gamma^2}\,\Psi_{x_t}g(x_t,i)g^T(x_t,i)\Psi^T_{x_t}\\
&+ \frac{1}{2\gamma^2}\,\Psi_{x_{t-d}}g(x_{t-d},r(t-d))g^T(x_{t-d},r(t-d))\Psi^T_{x_{t-d}} + \frac{1}{2}\,h^T(x,i)h(x,i)\\
&+ \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j) \quad \forall\, x\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned} \tag{23}$$

But the inequality (18) holds for all $u(t),\ u(t-d)\in\mathcal{L}_2[-d,\infty)$. Hence it must also hold for $u^\star(\cdot)$, and the result follows. This proves the necessity part of the theorem.

(**Sufficiency**) To prove sufficiency, we will show that, if there exists a solution to the HJ inequality (21), then the system is dissipative. Therefore, let $\Psi(\cdot,\cdot,\cdot)\geq 0$ satisfy (21), then completing the squares, we get

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)[f(x_t,x_{t-d},i) + g(x_t,i)u(t)]\\
&+ \Psi_{x_{t-d}}[f(x_{t-d},x_{t-2d},r(t-d)) + g(x_{t-d},i)u(t-d)] + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j)\\
&\leq \frac{\gamma^2}{2}\|u(t)\|^2 - \frac{1}{2}\|y(t)\|^2 - \frac{\gamma^2}{2}\|u(t) - \frac{1}{\gamma^2}\,g^T(x,i)\Psi^T_{x_t}(x,i)\|^2 + \frac{\gamma^2}{2}\|u(t-d)\|^2\\
&\quad - \frac{\gamma^2}{2}\|u(t-d) - \frac{1}{\gamma^2}\,g^T(x_{t-d},i)\Psi^T_{x_{t-d}}(t-d,x_{t-d},x_{t-2d},r(t-d))\|^2\\
&\qquad \forall\, x(t),x(t-d)\in\mathcal{X}, \quad i\in\mathcal{S},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Psi_t(t,x_t,x_{t-d},i) + \Psi_{x_t}(t,x_t,x_{t-d},i)[f(x_t,x_{t-d},i) + g(x_t,i)u(t)]\\
&+ \Psi_{x_{t-d}}[f(x_{t-d},x_{t-2d},r(t-d)) + g(x_{t-d},i)u(t-d)] + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j)\\
&\leq \frac{\gamma^2}{2}(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2 \quad \forall\, x\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned}$$

Thus, the dissipation inequality (6) and (18) are satisfied, and hence the system is dissipative wrt to $s(u(t),y(t))$.

Remark 3.4 The inequality (21) is known as the bounded-real inequality or condition for the system Σ^a and Theorem 3.1 is the equivalent of the bounded-real lemma for linear systems.

Remark 3.5 The above theorem provides an alternative approach for determining dissipativeness wrt to the quadratic supply rate. It follows that, if the system possesses the structure such that there exist smooth solutions to the HJ inequality (21) for each mode of the system, then it guarantees the dissipativeness of the system.

4 Stability of Stochastic State-Delayed Jump Systems

In the previous two sections we have defined the concept of dissipativity of the state-delayed nonlinear Markovian jump stochastic system (1), and have derived necessary and sufficient conditions for the system to be dissipative wrt to any supply rate. We have also explored the relationship between the dissipativity of the system and its $\mathcal{L}_2$-gain which is expressed in terms of the bounded-real condition or a set of coupled HJ-inequalities. Finally in this section, we shall relate the three concepts of dissipativity, $\mathcal{L}_2$-gain and stability of the system Σ^a. The question we would like to answer is the following: under what conditions relating to the dissipativity of the system Σ^a is the equilibrium $x = \{0\}$ stable, asymptotically stable?

In the deterministic case, if we regard the storage functions $\Psi(\cdot,\cdot,\cdot,r(t))$, $r(t)\in\mathcal{S}$ as generalized energy functions similar to Lyapunov functions, then to investigate stability using these functions, we would require that they be positive-definite and their time derivatives along trajectories of the system are negative-definite. Such an approach can also be considered in the stochastic case with stability defined in a stochastic sense. Therefore, we begin by first considering the conditions under which the storage function $\Psi(\cdot,\cdot,\cdot)$ is positive definite. This leads us to the following definition.

Definition 4.1 The free system (15) (with $u(t)\equiv 0$) is said to be stochastically zero-state detectable if for any trajectory of the system such that $y(t)\equiv 0\ \forall\, t\ge 0 \Rightarrow \lim\limits_{t\to\infty} E\{\|x(t,0,x_0,x_{-d},r_0,0)\|^2\} = \{0\}$.

We now show that, if $\Psi(\cdot,\cdot,\cdot,\cdot)\ge 0\ \forall\, x\in\mathcal{X},\ r(t)\in\mathcal{S}$, satisfies the HJ-inequality (21) as in the above Theorem 3.1, and the free system is stochastically zero-state detectable, then the following lemma guarantees that $\Psi(\cdot,\cdot,\cdot)>0\ \forall\, x(t),x(t-d)\in\mathcal{X},\ x(t)\ne 0$ or $x(t-d)\ne 0,\ r(t)\in\mathcal{S}$.

Lemma 4.1 *Suppose* $\Psi(\cdot,\cdot,\cdot,\cdot)\ge 0\ \ \forall\, x(t),x(t-d)\in\mathcal{X},\ r(t)\in\mathcal{S}$, *satisfies the HJ-inequality (21) and the system is dissipative as in Theorem 3.1 above, then if the free system is stochastically zero-state detectable, then* $\Psi(\cdot,\cdot,\cdot,\cdot)>0$ *for all* $x(t)\ne 0$ *or* $x(t-d)\ne 0,\ r(t)\in\mathcal{S}$.

Proof The available storages given in equation (8) are strictly convex in u for each $r(t)\in\mathcal{S}$ and are the infima of all solutions of the HJ inequality (21). Any other set of solutions $\Psi(t,x(t),x(t-d),r(t)),\ \forall\, r(t)\in\mathcal{S}$ of the HJ inequality is lower bounded by $\Psi^a(\cdot,\cdot,\cdot,r(t))$, i.e.,

$$\begin{gathered}\Psi^a(t,x(t),\,x(t-d),\,r(t))\le\Psi(t,x(t),\,x(t-d),\,r(t))\\ \forall\, x(t),\ x(t-d)\in\mathcal{X},\qquad r(t)\in\mathcal{S}.\end{gathered}\tag{24}$$

We now show that, if the system (15) is reachable from the origin, then there exists a choice of input $u(x(t),r(t))$, such that $\Psi^a(t,x(t),x(t-d),r(t)) > 0\ \ \forall\, x(t)\ne 0$,

$x(t-d) \neq 0,\ \forall\, r(t) \in \mathcal{S}$ and for $T > 0$

$$\Psi^a(t, x_t, x_{t-d}, r(t)) = \sup_{u\in\mathcal{U}} E\Bigg[-\frac{1}{2}\Bigg\{\int_0^T (\gamma^2\|u(t)\|^2 + \|u(t-d)\|^2) - \|y(t)\|^2\Bigg\}dt\Bigg]. \quad (25)$$

It has been shown (Theorem 3.1) that for any solution $\Psi(\cdot,\cdot,\cdot,r(t)),\ r(t) \in \mathcal{S},$ of the dissipation inequality (18), the control $u^\star(\cdot,\cdot)$ attains the above supremum. Therefore,

$$\Psi^a(t, x_t, x_{t-d}, r(t)) = E\Bigg[-\frac{1}{2}\Bigg\{\gamma^2\int_0^T (\|u^\star(t)\|^2 + \|u^\star(t-d)\|) - \|y(t)\|^2\Bigg\}dt\Bigg]. \quad (26)$$

Now using the HJ-inequality (21) or the dissipation inequality (18), we get

$$\begin{aligned}
\Psi^a(t, x_t, x_{t-d}, r(t)) \geq -E\Bigg[\int_0^T \Bigg\{&\Psi_t(t, x_t, x_{t-d}, i) + \Psi_{x_t}(t, x_t, x_{t-d}, i)[f(x_t, x_{t-d}, i) \\
&+ g(x_t, i)u^\star(t)] + \Psi_{x_{t-d}}[f(x_{t-d}, x_{t-2d}, r(t-d)) + g(x_{t-d}, r(t-d))u^\star(t-d)] \\
&+ \sum_{j\in\mathcal{S}} \lambda_{ij}\Psi(t, x_t, x_{t-d}, j)\Bigg\}dt\Bigg] \geq -E\Bigg[\int_0^T \mathcal{L}\Psi(t, x_t, x_{t-d}, r(t))dt\Bigg] \\
&\geq \Psi(0, x_0, x_{-d}, r_0) - E\Psi(T, x(T), x(T-d), r(T)) \geq 0, \quad \forall\, T > 0
\end{aligned}$$

by dissipativity and Theorem 2.1. Now, from the above inequality, the condition when $\Psi^a(\cdot,\cdot,\cdot,0) = 0$ corresponds to

$$\Psi(0, x_0, x_{-d}, r_0) = E\Psi(T, x(T), x(T-d), r(T)) = 0,$$

and since this holds for all $T > 0$, it implies that $\Psi^a(\cdot,\cdot,\cdot,\cdot) \equiv \Psi(0, x_0, x_{-d}, r_0) \equiv E\Psi(T, x(T), x(T-d), r(T)) = 0$. This further implies that $y(t) \equiv 0,\ u(t) \equiv 0,$ which by stochastic zero-state detectability implies that $x_0 = x(T) = x(T-d) = \{0\}$. Since $T > 0$ is arbitrary, the result follows.

We are now in a position to exploit $\Psi(\cdot,\cdot,\cdot,\cdot)$ as a candidate Lyapunov function for the system Σ^a since any solution $\Psi(\cdot,\cdot,\cdot,r(t)),\ r(t) \in \mathcal{S},$ of the HJ-inequality is positive-definite and guarantees dissipativity of the system for all $r(t) \in \mathcal{S}$. To do this, we first define the following concept of stochastic stability.

Definition 4.2 The equilibrium point $x = 0$ of the nonlinear system (15) with $u(t) \equiv 0$ is stochastically stable, if for any initial state $x_0 \in \mathcal{X}$ and $r_0 \in \mathcal{S}$,

$$\int_0^\infty E\{\|x(t, t_0, x_0, x_{-d}, r_0, 0)\|^2\}\, dt < \infty. \quad (27)$$

However, the following definition of stochastic stability will be more appropriate for our application in this paper.

Definition 4.3 The equilibrium point $x = 0$ of the nonlinear system (15) with $u(t) \equiv 0$ is locally asymptotically mean-square stable, if for any initial state $x_0 \in \mathcal{X}$ and $r_0 \in \mathcal{S}$,

$$\lim_{t\to\infty} E\{\|x(t,t_0,x_0,x_{-d},r_0,0)\|^2\} = 0. \tag{28}$$

Remark 4.1 The above definition also implies that stochastic stability or asymptotic stability in the mean-square sense implies stochastic $\mathcal{L}_2$-stability [13].

Remark 4.2 It is also seen from the definition of $\mathcal{L}_2$-gain (Definition 3.1) that, if we take $(t_0,t_1) = (0,\infty)$, then if the $\mathcal{L}_2$-gain of the system is finite, then the system is stochastically $\mathcal{L}_2$-stable.

Furthermore, since the question of stability can only be addressed on the infinite-time horizon, the HJ-inequality (21) takes the following form:

$$\begin{aligned}
&\Psi_{x_t}(x_t,x_{t-d},i)f(x_t,x_{t-d},i) + \Psi_{x_{t-d}}(x_t,x_{t-d},r(t))f(x_{t-d},x_{t-2d},r(t))\\
&+ \frac{1}{2\gamma^2}\Psi_{x_t}g(x_t,i)g^T(x_t,i)\Psi_{x_t}^T + \frac{1}{2\gamma^2}\Psi_{x_{t-d}}g(x_{t-d},r(t-d))g^T(x_{t-d},r(t-d))\Psi_{x_{t-d}}^T\\
&+ \frac{1}{2}h^T(x,i)h(x,i) + \sum_{j\in\mathcal{S}}\lambda_{ij}\Psi(t,x_t,x_{t-d},j) \le 0 \quad \forall\, x_t,x_{t-d}\in\mathcal{X}, \quad i\in\mathcal{S}.
\end{aligned} \tag{29}$$

We now state our main stability theorem.

Theorem 4.1 *Suppose Σ^a is dissipative wrt to the supply rate*

$$s(u(t),y(t)) = \frac{1}{2}\gamma^2(\|u(t)\|^2 + \|u(t-d)\|^2) - \frac{1}{2}\|y(t)\|^2,$$

then Σ^a satisfies HJ-inequality (23) for each $r(t)\in\mathcal{S}$ and the system has $\mathcal{L}_2$-gain less than or equal to γ. Moreover, if Σ^a is stochastically zero-state detectable, then the free system $\dot{x}(t) = f(x(t),x(t-d),r(t))$ is locally mean square asymptotically stable.

Proof The first part of the theorem has already been proved in Lemmas 3.1 and 3.2. For the second part, from Lemma 4.1, $\Psi(\cdot,\cdot,\cdot,r(t))$, $\forall\, r(t)\in\mathcal{S}$ is positive-definite. Since Σ^a is dissipative, the free system with $u(t) = u(t-d) = 0$ satisfies the following dissipation inequality:

$$\Psi(x(\infty),x(\infty),r(\infty)) + E\left[\frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt\right] \le \Psi(x_0,x_{-d},r_0)$$

for any initial conditions $x_0,x_{-d}\in\mathcal{X}$, $r_0\in\mathcal{S}$. This implies that

$$E\left[\frac{1}{2}\int_0^\infty \|y(t)\|^2\,dt\right] \le \Psi(x_0,x_{-d},r_0), \quad \forall\, x_0,x_{-d}\in\mathcal{X}, \quad r_0\in\mathcal{S}$$

or $y(t)\in L_2((\Omega,\mathcal{F},P)[0,\infty))$, and therefore, $\lim_{t\to\infty} E(\|y(t)\|^2) = 0$. By the assumption of stochastic zero-state detectability, we also get $\lim_{t\to\infty} E(\|x(t)\|^2) = 0$.

Remark 4.3 Theorem 4.1 above gives the bounded-real [1] conditions for the nonlinear system Σ^a. In the special case of linear systems, it gives necessary and sufficient conditions for the $\mathcal{L}_2$-gain (or $\mathcal{H}_\infty$-norm) of the system to be less than or equal to γ and to be locally asymptotically stable [1].

Remark 4.4 As a final remark, we mention that, if the jump rates $\lambda_{ij},\ i,j \in \mathcal{S},$ are very small, then all the results derived in this paper will approach the deterministic case.

5 Conclusion

In this paper, we have extended the theory of dissipative system developed for deterministic systems to the case of stochastic state-delayed systems with jump Markov disturbances. We have derived necessary and sufficient conditions for the system to be dissipative and to have finite $\mathcal{L}_2$-gain or the bounded-real condition, and have given sufficient conditions for stochastic stability of the system.

This paper has clearly laid down a framework for studying the $\mathcal{H}_\infty$ control and filtering problems for such systems and the stability of feedback interconnections. Future work will concentrate on these issues.

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Robust $\mathcal{H}_\infty$ Fuzzy Control Design for Time Delay Nonlinear Markovian Jump Systems: An LMI Approach

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Abstract: This paper considers the problem of designing a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller for a class of time delay nonlinear Markovian jump systems. The proposed controller guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than some prescribed value. Solutions to the problem are provided in terms of linear matrix inequalities. To illustrate the effectiveness of the design developed in this paper, a numerical example is also provided.

Keywords: *$\mathcal{H}_\infty$ fuzzy control; Takagi–Sugeno (TS) fuzzy model; linear matrix inequalities (LMIs); Markovian jump parameters; time-varying delay.*

Mathematics Subject Classification (2000): 93C23, 93D09, 93E15.

1 Introduction

Markovian jump systems are also called hybrid systems, that is, the state space of a system contains both continuous (differential equation) and discrete states (Markov process). The Markovian jump system has been widely used to describe a physical system that changes abruptly from one mode to another mode. These abrupt changes may be caused by environmental disturbances, component and interconnection failures, parameters shifting, tracking, and fast variations in the operating point of the system. Over the past few decades, the Markovian jump system has been extensively studied by many researchers (see [1 – 7]).

It is a well known fact that engineering processes frequently contain time delays. Stability and control synthesis for time delay systems have been one of the most significant issues in control engineering applications. Linear systems with Markovian jumps and time delays have been addressed by a number of researchers (see, for example, [9 − 11]). In [11], the delay-dependent robust stability and the $\mathcal{H}_\infty$ control of time delay linear Markovian jump systems have been investigated. Although many researchers have studied the control design for time delay linear Markovian jump systems for many years, the control design for time delay nonlinear Markovian jump systems remains as an open area.

In the past two decades, the $\mathcal{H}_\infty$ control design for a class of nonlinear systems described by a Takagi-Sugeno (TS) fuzzy model has been studied by a number of researchers (see [12 − 25]). In this TS fuzzy model, local dynamics in different state space regions are represented by local linear systems. The overall model of the system is obtained by "blending" of these linear models through nonlinear membership functions. In other words, a TS fuzzy model is essentially a multi-model approach in which simple sub-models are combined to represent the global behavior of the system. Recently, the design of fuzzy $\mathcal{H}_\infty$ control for a class of nonlinear systems without delays has been significantly considered and many results have been reported (e.g., [12 − 14]). Furthermore, there have been also some attempts in [18 − 23] in which robust fuzzy control analysis and synthesis for nonlinear time-delay systems have been examined. To the best of our knowledge, the global robust $\mathcal{H}_\infty$ fuzzy state-feedback control problem for a class of uncertain nonlinear Markovian jump systems with time-varying delay via an LMI approach has not yet been considered in the literature.

The main contribution of this paper is to design an $\mathcal{H}_\infty$ fuzzy state-feedback controller for a class of time delay nonlinear Markovian jump systems described by a Takagi-Sugeno (TS) fuzzy model. Based on an LMI approach, we develop a state-feedback controller that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than a prescribed value. The solutions are given in terms of a family of linear matrix inequalities.

This paper is organized as follows. In Section 2, system description and definition are presented. In Section 3, based on an LMI approach we develop a technique for designing a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than a prescribed value. The validity of this approach is demonstrated by an example from the literature in Section 4. Finally in Section 5, the conclusion is given.

2 System Description and Definition

The class of time delay uncertain nonlinear Markovian jump system under consideration is described by the following TS fuzzy models:

Plant Rule i: If $\nu_1(t)$ is M_{i1} and $\cdots$ and $\nu_\vartheta(t)$ is $M_{i\vartheta}$ then

$$\begin{aligned}
\dot{x}(t) &= [A_i(\eta(t)) + \Delta A_i(\eta(t))]x(t) + A_{d_i}(\eta(t))x(t-\tau(t)) \\
&\quad + B_{1_i}(\eta(t))w(t) + [B_{2_i}(\eta(t)) + \Delta B_{2_i}(\eta(t))]u(t), \quad x(0) = 0, \\
z(t) &= [C_{1_i}(\eta(t)) + \Delta C_{1_i}(\eta(t))]x(t) + [D_{12_i}(\eta(t)) + \Delta D_{12_i}(\eta(t))]u(t) \\
x(t) &= \psi(t), \quad t \in [-\tau, 0], \quad \tau(t) \le \tau
\end{aligned} \tag{2.1}$$

where M_{iq} $(j = 1, 2, \ldots, \vartheta)$ is fuzzy sets q for rule i, $\nu_i(t)$ are the premise variables, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, $w(t) \in R^p$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, $z(t) \in R^s$ is the controlled output, the matrices $A_i(\eta(t))$, $A_{d_i}(\eta(t))$, $B_{1_i}(\eta(t))$, $B_{2_i}(\eta(t))$, $C_{1_i}(\eta(t))$ and $D_{12_i}(\eta(t))$ are of appropriate dimensions, r is the number of IF-THEN rules, $\tau(t)$ is the bounded time-varying delay in the state with the following assumption

$$0 \le \tau(t) \le \tau \quad \text{and} \quad \dot{\tau}(t) \le \beta < 1$$

and $\psi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau, 0]$. $\{\eta(t)\}$, $t \ge 0$ is a continuous-time discrete-state homogenous Markov process taking values on a finite set $\mathcal{S} = \{1, 2, \dots, s\}$ with transition probability matrix $Pr = \{P_{\imath k}(t)\}$ given by

$$\begin{aligned} P_{\imath k}(t) &= Pr(\eta(t+\Delta) = k \mid \eta(t) = \imath) \\ &= \begin{cases} \lambda_{\imath k}\Delta + O(\Delta) & \text{if} \quad \imath \ne k, \\ 1 + \lambda_{\imath\imath}\Delta + O(\Delta) & \text{if} \quad \imath = k, \end{cases} \end{aligned} \tag{2.2}$$

and $\sum\limits_{k=1}^{s} P_{\imath k}(t) = 1$, where $\Delta > 0$; $\lim\limits_{\Delta \to 0} \dfrac{O(\Delta)}{\Delta} = 0$; $\lambda_{\imath k} \ge 0$, $\imath \ne k$ is the transition rate from mode $\imath$ to mode k; $\lambda_{\imath\imath} = -\sum\limits_{k=1,\, k\ne\imath}^{s} \lambda_{\imath k}$, $\imath, k \in \mathcal{S}$ gives the infinitesimal generator of the Markov process $\{\eta(t),\, t \ge 0\}$.

The matrices $\Delta A_i(\eta(t))$, $\Delta B_{2_i}(\eta(t))$, $\Delta C_{1_i}(\eta(t))$ and $\Delta D_{12_i}(\eta(t))$ represent the uncertainties in the system and satisfy the following assumption.

Assumption 2.1 Following equalities take place

$$\begin{aligned} \Delta A_i(\eta(t)) &= E_{1_i}(\eta(t))F(x(t),\eta(t),t)H_{1_i}(\eta(t)), \\ \Delta B_{2_i}(\eta(t)) &= E_{2_i}(\eta(t))F(x(t),\eta(t),t)H_{2_i}(\eta(t)), \\ \Delta C_{1_i}(\eta(t)) &= E_{3_i}(\eta(t))F(x(t),\eta(t),t)H_{3_i}(\eta(t)), \\ \Delta D_{12_i}(\eta(t)) &= E_{4_i}(\eta(t))F(x(t),\eta(t),t)H_{4_i}(\eta(t)), \end{aligned}$$

where $E_{j_i}(\eta(t))$ and $H_{j_i}(\eta(t))$, $j = 1, 2, \dots, 4$, are known matrix functions which characterize the structure of the uncertainties. Furthermore, the following inequality holds:

$$\|F(x(t),\eta(t),t)\| \le \rho(\eta(t)) \tag{2.3}$$

for any known positive constant $\rho(\eta(t))$.

Let

$$\varpi_i(\nu(t)) = \prod_{q=1}^{n} M_{iq}(\nu_q(t)), \quad \text{and} \quad \mu_i(\nu(t)) = \frac{\varpi_i(\nu(t))}{\sum\limits_{i=1}^{r} \varpi_i(\nu(t))},$$

where $M_{iq}(\nu_q(t))$ is the grade of membership of $\nu_q(t)$ in M_{iq}. It is assumed in this paper that

$$\varpi_i(\nu(t)) \ge 0, \quad i = 1, 2, \dots, n, \quad \text{and} \quad \sum_{i=1}^{r} \varpi_i(\nu(t)) > 0,$$

where r are the number of local plant rules, for all t. Therefore,

$$\mu_i(\nu(t)) \ge 0, \quad i = 1, 2, \dots, n, \quad \text{and} \quad \sum_{i=1}^{r} \mu_i(\nu(t)) = 1$$

for all t. For the convenience of notations, let $\varpi_i = \varpi_i(\nu(t))$, $\mu_i = \mu_i(\nu(t))$, $\eta = \eta(t)$ and any matrix $N(\mu, \eta(t) = \imath) = N(\mu, \imath)$.

The resulting fuzzy system model is inferred as the weighted average of the local models of the form

$$\begin{aligned} \dot{x}(t) &= [A(\mu,\imath) + \Delta A(\mu,\imath)]x(t) + A_d(\mu,\imath)\, x(t-\tau(t)) \\ &\quad + B_1(\mu,\imath)w(t) + [B_2(\mu,\imath) + \Delta B_2(\mu,\imath)]u(t), \quad x(0) = 0, \\ z(t) &= [C_1(\mu,\imath) + \Delta C_1(\mu,\imath)]\, x(t) + [D_{12}(\mu,\imath) + \Delta D_{12}(\mu,\imath)]\, u(t), \end{aligned} \tag{2.4}$$

where

$$A(\mu,\imath) = \sum_{i=1}^{r} \mu_i A_i(\imath), \quad A_d(\mu,\imath) = \sum_{i=1}^{r} \mu_i A_{d_i}(\imath), \quad B_1(\mu,\imath) = \sum_{i=1}^{r} \mu_i B_{1_i}(\imath),$$

$$B_2(\mu,\imath) = \sum_{i=1}^{r} \mu_i B_{2_i}(\imath), \quad C_1(\mu,\imath) = \sum_{i=1}^{r} \mu_i C_{1_i}(\imath), \quad D_{12}(\mu,\imath) = \sum_{i=1}^{r} \mu_i D_{12_i}(\imath),$$

$$\Delta A(\mu,\imath) = \sum_{i=1}^{r} \mu_i \Delta A_i(\imath) = E_1(\mu,\imath) F(x(t),\imath,t) H_1(\mu,\imath),$$

$$\Delta B_2(\mu,\imath) = \sum_{i=1}^{r} \mu_i \Delta B_{2_i}(\imath) = E_2(\mu,\imath) F(x(t),\imath,t) H_2(\mu,\imath),$$

$$\Delta C_1(\mu,\imath) = \sum_{i=1}^{r} \mu_i \Delta C_{1_i}(\imath) = E_3(\mu,\imath) F(x(t),\imath,t) H_3(\mu,\imath),$$

$$\Delta D_{12}(\mu,\imath) = \sum_{i=1}^{r} \mu_i \Delta D_{12_i}(\imath) = E_4(\mu,\imath) F(x(t),\imath,t) H_4(\mu,\imath)$$

with

$$E_1(\mu,\imath) = \sum_{i=1}^{r} \mu_i E_{1_i}(\imath), \quad E_2(\mu,\imath) = \sum_{i=1}^{r} \mu_i E_{2_i}(\imath), \quad E_3(\mu,\imath) = \sum_{i=1}^{r} \mu_i E_{3_i}(\imath),$$

$$E_4(\mu,\imath) = \sum_{i=1}^{r} \mu_i E_{4_i}(\imath), \quad H_1(\mu,\imath) = \sum_{i=1}^{r} \mu_i H_{1_i}(\imath), \quad H_2(\mu,\imath) = \sum_{i=1}^{r} \mu_i H_{2_i}(\imath),$$

$$H_3(\mu,\imath) = \sum_{i=1}^{r} \mu_i H_{3_i}(\imath), \quad H_4(\mu,\imath) = \sum_{i=1}^{r} \mu_i H_{4_i}(\imath).$$

Definition 2.1 Suppose γ is a given positive real number. A system of the form (2.4) is said to have $\mathcal{L}_2[0, T_f]$ gain less than or equal to γ if

$$E\left[\int_0^{T_f} \{ z^{\mathrm{T}}(t) z(t) - \gamma w^{\mathrm{T}}(t) w(t) \}\, dt \right] < 0, \tag{2.5}$$

where $E[\,\cdot\,]$ denotes as the expectation operator.

In this paper, we consider the following $\mathcal{H}_\infty$ fuzzy state-feedback which is inferred as the weighted average of the local models of the form:

$$u(t) = K(\mu, \imath)x(t), \tag{2.6}$$

where $K(\mu, \imath) = \sum_{j=1}^{r} \mu_j K_j(\imath)$. Before ending this section, we describe the problem under our study as follows.

Problem Formulation Given the system (2.4), design an $\mathcal{H}_\infty$ fuzzy state-feedback controller of the form (2.6) such that the $\mathcal{L}_2$ gain γ-performance (2.5) is guaranteed.

3 Main Result

First, let us consider the closed-loop state space form of the fuzzy system model (2.4) with the controller (2.6) which is given by

$$\begin{aligned} \dot{x}(t) &= [A(\mu, \imath) + B_2(\mu, \imath)K(\mu, \imath)]x(t) + A_d(\mu, \imath)x(t - \tau(t)) \\ &\quad + [\Delta A(\mu, \imath) + \Delta B_2(\mu, \imath)K(\mu, \imath)]x(t) + B_1(\mu, \imath)w(t), \quad x(0) = 0, \end{aligned} \tag{3.1}$$

or in a more compact form

$$\begin{aligned} &\dot{x}(t) = [A(\mu, \imath) + B_2(\mu, \imath)K(\mu, \imath)]x(t) + A_d(\mu, \imath)x(t - \tau(t)) + \widetilde{B}_1(\mu, \imath)\widetilde{w}(t), \\ &x(0) = 0, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \widetilde{B}_1(\mu, \imath) &= [\, E_1(\mu, \imath) \quad E_2(\mu, \imath) \quad B_1(\mu, \imath) \quad 0 \quad 0\,], \\ \widetilde{w}(t) &= \begin{bmatrix} F(x(t), \imath, t)H_1(\mu, \imath)x(t) \\ F(x(t), \imath, t)H_2(\mu, \imath)K(\mu, \imath)x(t) \\ w(t) \\ F(x(t), \imath, t)H_3(\mu, \imath)x(t) \\ F(x(t), \imath, t)H_4(\mu, \imath)K(\mu, \imath)x(t) \end{bmatrix}. \end{aligned} \tag{3.3}$$

To provide LMI-based solutions to the problem of designing a robust $\mathcal{H}_\infty$ controller that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than some prescribed value for a class of time delay uncertainty nonlinear Markovian jump systems, the following theorem is given.

Theorem 3.1 *Given the system (2.4), the inequality (2.5) holds if there exist a prescribed $\mathcal{H}_\infty$ performance $\gamma > 0$, positive definite symmetric matrices $P(\imath)$ and $W(\imath)$ for $\imath = 1, 2, \dots, s$, such that the following conditions hold:*

$$\Omega_{ii}(\imath) < 0, \quad i = 1, 2, \dots, r, \tag{3.5}$$

$$\Omega_{ij}(\imath) + \Omega_{ji}(\imath) < 0, \quad i < j \le r, \tag{3.6}$$

where

$$\Omega_{ij}(\imath) = \begin{pmatrix} \Psi_{ij}(\imath) & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} \\ \mathcal{B}_{ij}(\imath) & -\mathcal{M} + \widetilde{E}_i^{\mathrm{T}}(\imath)\widetilde{E}_j(\imath) & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} \\ W(\imath)A_{d_i}(\imath) & 0 & -(1-\beta)W(\imath) & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} \\ P(\imath) & 0 & 0 & -W(\imath) & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} \\ \Gamma_{ij}(\imath) & 0 & 0 & 0 & -I & (*)^{\mathrm{T}} & (*)^{\mathrm{T}} \\ \Upsilon_{ij}(\imath) & 0 & 0 & 0 & 0 & -I & (*)^{\mathrm{T}} \\ \mathcal{Z}^{\mathrm{T}}(\imath) & 0 & 0 & 0 & 0 & 0 & -\mathcal{P}(\imath) \end{pmatrix}, \tag{3.7}$$

$$\Psi_{ij}(\imath) = A_i(\imath)P(\imath) + P(\imath)A_i^{\mathrm{T}}(\imath) + B_{2_i}(\imath)Y_j(\imath) + Y_j^{\mathrm{T}}(\imath)B_{2_i}^{\mathrm{T}}(\imath) + \lambda_{\imath\imath}P(\imath), \tag{3.8}$$

$$\mathcal{B}_{ij}(\imath) = \widetilde{B}_{1_i}^{\mathrm{T}}(\imath) + \widetilde{E}_i^{\mathrm{T}}(\imath)C_{1_i}(\imath)P(\imath) + \widetilde{E}_i^{\mathrm{T}}(\imath)D_{12_i}(\imath)Y_j(\imath), \tag{3.9}$$

$$\Gamma_{ij}(\imath) = C_{1_i}(\imath)P(\imath) + D_{12_i}(\imath)Y_j(\imath), \tag{3.10}$$

$$\Upsilon_{ij}(\imath) = \widetilde{C}_i(\imath)P(\imath) + \widetilde{D}_i(\imath)Y_j(\imath), \tag{3.11}$$

$$\mathcal{M} = \mathrm{diag}\{I, I, \gamma I, I, I\}, \tag{3.12}$$

$$\mathcal{Z}(\imath) = \left(\sqrt{\lambda_{\imath 1}}P(\imath)\ldots\sqrt{\lambda_{\imath(\imath-1)}}P(\imath)\sqrt{\lambda_{\imath(\imath+1)}}P(\imath)\ldots\sqrt{\lambda_{\imath s}}P(\imath)\right), \tag{3.13}$$

$$\mathcal{P}(\imath) = \mathrm{diag}\left\{P(1), \ldots, P(\imath-1), P(\imath+1), \ldots, P(s)\right\}, \tag{3.14}$$

with

$$\widetilde{B}_{1_i}(\imath) = [\, E_{1_i}(\imath) \quad E_{2_i}(\imath) \quad B_{1_i}(\imath) \quad 0 \quad 0\,], \tag{3.15}$$

$$\widetilde{C}_i(\imath) = [\, \rho(\imath)H_{1_i}^{\mathrm{T}}(\imath) \quad \rho(\imath)H_{3_i}^{\mathrm{T}}(\imath) \quad 0 \quad 0\,]^{\mathrm{T}}, \tag{3.16}$$

$$\widetilde{D}_i(\imath) = [\,0 \quad 0 \quad \rho(\imath)H_{2_i}^{\mathrm{T}}(\imath) \quad \rho(\imath)H_{4_i}^{\mathrm{T}}(\imath)\,]^{\mathrm{T}}, \tag{3.17}$$

$$\widetilde{E}_i(\imath) = [\,0 \quad 0 \quad 0 \quad E_{3_i}(\imath) \quad E_{4_i}(\imath)\,]]. \tag{3.18}$$

Furthermore, a suitable choice of the fuzzy controller is

$$u(t) = \sum_{j=1}^{r} \mu_j K_j(\imath)x(t) \tag{3.19}$$

where

$$K_j(\imath) = Y_j(\imath)(P(\imath))^{-1}. \tag{3.20}$$

Proof Consider a Lyapunov-Krasovskii functional candidate as follows:

$$V(x(t),\imath) = x^{\mathrm{T}}(t)Q(\imath)x(t) + \int\limits_{t-\tau(t)}^{t} x^{\mathrm{T}}(v)G(\imath)x(v)\,dv, \quad \forall\, \imath \in \mathcal{S}, \tag{3.21}$$

where $Q(\imath) > 0$ and $G(\imath) > 0$. Now let us consider the weak infinitesimal operator $\widetilde{\Delta}$ of the joint process $\{(x(t),\imath),\ t \geq 0\}$, which is the stochastic analog of the deterministic derivative [28]. $\{(x(t),\imath),\ t \geq 0\}$ is a Markov process with infinitesimal operator given by [3]

$$\begin{aligned}\widetilde{\Delta}V(x(t),\imath) = {} & x^{\mathrm{T}}(t)[Q(\imath)(A(\mu,\imath) + B_2(\mu,\imath)K(\mu,\imath)) + (A(\mu,\imath) + B_2(\mu,\imath)K(\mu,\imath))^{\mathrm{T}}Q(\imath) \\ & + G(\imath)]x(t) + x^{\mathrm{T}}(t)Q(\imath)\widetilde{B}_1(\mu,\imath)\widetilde{w}(t) + \widetilde{w}^{\mathrm{T}}(t)\widetilde{B}_1^{\mathrm{T}}(\mu,\imath)Q(\imath)x(t) \\ & + x^{\mathrm{T}}(t)\sum_{k=1}^{s}\lambda_{\imath k}Q(k)x(t) - (1-\dot{\tau})x^{\mathrm{T}}(t-\tau(t))G(\imath)x(t-\tau(t)) \\ & + x^{\mathrm{T}}(t)Q(\imath)A_d(\mu,\imath)x(t-\tau(t)) + x^{\mathrm{T}}(t-\tau(t))A_d^{\mathrm{T}}(\mu,\imath)Q(\imath)x(t).\end{aligned} \tag{3.22}$$

Using the fact that for any vectors $x(t)$ and $x(t-\tau(t))$

$$\begin{aligned}
&x^{\mathrm T}(t)Q(\imath)A_d(\mu,\imath)x(t-\tau(t)) + x^{\mathrm T}(t-\tau(t))A_d^{\mathrm T}(\mu,\imath)Q(\imath)x(t)\\
&\quad\le \frac{1}{(1-\beta)}x^{\mathrm T}(t)Q(\imath)A_d(\mu,\imath)G^{-1}(\imath)A_d^{\mathrm T}(\mu,\imath)Q(\imath)x(t)\\
&\qquad + (1-\beta)x^{\mathrm T}(t-\tau(t))G(\imath)x(t-\tau(t)),
\end{aligned}$$

(3.22) becomes

$$\begin{aligned}
\widetilde{\Delta}V(x(t),\imath) \le x^{\mathrm T}(t)\Big[&Q(\imath)(A(\mu,\imath)+B_2(\mu,\imath)K(\mu,\imath)) + (A(\mu,\imath)+B_2(\mu,\imath)K(\mu,\imath))^{\mathrm T}Q(\imath)\\
&+\frac{1}{(1-\beta)}\,Q(\imath)A_d(\mu,\imath)G^{-1}(\imath)A_d^{\mathrm T}(\mu,\imath)Q(\imath) + G(\imath) + \sum_{k=1}^{s}\lambda_{\imath k}Q(k)\Big]x(t)\\
&+x^{\mathrm T}(t)Q(\imath)\widetilde{B}_1(\mu,\imath)\widetilde{w}(t) + \widetilde{w}^{\mathrm T}(t)\widetilde{B}_1^{\mathrm T}(\mu,\imath)Q(\imath)x(t).
\end{aligned} \tag{3.23}$$

Adding and subtracting $-z^{\mathrm T}(t)z(t) + \widetilde{w}^{\mathrm T}(t)\mathcal{M}\widetilde{w}(t)$ to and from (3.23), we get

$$\begin{aligned}
\widetilde{\Delta}V(x(t),\imath) \le -z^{\mathrm T}(t)z(t) + \widetilde{w}^{\mathrm T}(t)\mathcal{M}\widetilde{w}(t) + z^{\mathrm T}(t)z(t) + \begin{bmatrix} x(t)\\ \widetilde{w}(t)\end{bmatrix}^{\mathrm T}\\
\times\begin{bmatrix}
\begin{array}{c}[A(\mu,\imath)+B_2(\mu,\imath)K(\mu,\imath)]^{\mathrm T}Q(\imath)\\ +Q(\imath)[A(\imath)+B_2(\mu,\imath)K(\mu,\imath)]\\ +\sum\limits_{k=1}^{s}\lambda_{\imath k}Q(k)+G(\imath)\\ +\frac{1}{(1-\beta)}\,Q(\imath)A_d(\mu,\imath)G^{-1}(\imath)A_d^{\mathrm T}(\mu,\imath)Q(\imath)\end{array} & (*)^{\mathrm T}\\
\widetilde{B}_1^{\mathrm T}(\mu,\imath)Q(\imath) & -\mathcal{M}
\end{bmatrix}\begin{bmatrix} x(t)\\ \widetilde{w}(t)\end{bmatrix},
\end{aligned} \tag{3.24}$$

where $\mathcal{M} = \mathrm{diag}\{I, I, \gamma I, I, I\}$.

Now let us consider the following terms

$$\begin{aligned}
\widetilde{w}^{\mathrm T}(t)\mathcal{M}\widetilde{w}(t) &= \begin{bmatrix} F(x(t),\imath,t)H_1(\mu,\imath)x(t)\\ F(x(t),\imath,t)H_2(\mu,\imath)K(\mu,\imath)x(t)\\ w(t)\\ F(x(t),\imath,t)H_3(\mu,\imath)x(t)\\ F(x(t),\imath,t)H_4(\mu,\imath)K(\mu,\imath)x(t)\end{bmatrix}^{\mathrm T}\mathcal{M}\begin{bmatrix} F(x(t),\imath,t)H_1(\mu,\imath)x(t)\\ F(x(t),\imath,t)H_2(\mu,\imath)K(\mu,\imath)x(t)\\ w(t)\\ F(x(t),\imath,t)H_3(\mu,\imath)x(t)\\ F(x(t),\imath,t)H_4(\mu,\imath)K(\mu,\imath)x(t)\end{bmatrix}\\
&\le \rho^2(\imath)x^{\mathrm T}(t)\{H_1^{\mathrm T}(\mu,\imath)H_1(\mu,\imath) + K^{\mathrm T}(\mu,\imath)H_2^{\mathrm T}(\mu,\imath)H_2(\mu,\imath)K(\mu,\imath)\\
&\quad + H_3^{\mathrm T}(\mu,\imath)H_3(\mu,\imath) + K^{\mathrm T}(\mu,\imath)H_4^{\mathrm T}(\mu,\imath)H_4(\mu,\imath)K(\mu,\imath)\}x(t) + \gamma w^{\mathrm T}(t)w(t)
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
z^{\mathrm T}(t)z(t) &= x^{\mathrm T}(t)[C_1(\mu,\imath) + E_3(\mu,\imath)F(x(t),\imath,t)H_3(\mu,\imath) + D_{12}(\mu,\imath)K(\mu,\imath)\\
&\quad + E_4(\mu,\imath)F(x(t),\imath,t)H_4(\mu,\imath)K(\mu,\imath)^{\mathrm T}[C_1(\mu,\imath) + E_3(\mu,\imath)F(x(t),\imath,t)H_3(\mu,\imath)\\
&\quad + D_{12}(\mu,\imath)K(\mu,\imath) + E_4(\mu,\imath)F(x(t),\imath,t)H_4(\mu,\imath)K(\mu,\imath)]x(t)\\
&= \begin{bmatrix} x(t)\\ \widetilde{w}(t)\end{bmatrix}^{\mathrm T}\begin{bmatrix} \begin{array}{c}[C_1(\mu,\imath)+D_{12}(\mu,\imath)K(\mu,\imath)]^{\mathrm T}\times\\ [C_1(\mu,\imath)+D_{12}(\mu,\imath)K(\mu,\imath)]\end{array} & (*)^{\mathrm T}\\ \widetilde{E}^{\mathrm T}(\mu,\imath)[C_1(\mu,\imath)+D_{12}(\mu,\imath)K(\mu,\imath)] & \widetilde{E}^{\mathrm T}(\mu,\imath)\widetilde{E}(\mu,\imath)\end{bmatrix}\begin{bmatrix} x(t)\\ \widetilde{w}(t)\end{bmatrix},
\end{aligned} \tag{3.26}$$

where

$$\widetilde{E}(\mu,\imath) = [\,0 \quad 0 \quad 0 \quad E_3(\mu,\imath) \quad E_4(\mu,\imath)\,].$$

Substituting (3.25) and (3.26) into (3.24), we have

$$\widetilde{\Delta}V(x(t),\imath) \le -z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t) + \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}^{\mathrm{T}} \Phi(\mu,\imath) \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}, \tag{3.27}$$

where

$$\Phi(\mu,\imath) = \begin{bmatrix} \begin{array}{c} [A(\mu,\imath) + B_2(\mu,\imath)K(\mu,\imath)]^{\mathrm{T}}Q(\imath) \\ + Q(\imath)[A(\mu,\imath) + B_2(\mu,\imath)K(\mu,\imath)] \\ + [C_1(\mu,\imath) + D_{12}(\mu,\imath)K(\mu,\imath)]^{\mathrm{T}} \\ \times [C_1(\mu,\imath) + D_{12}(\mu,\imath)K(\mu,\imath)] \\ + [\widetilde{C}(\mu,\imath) + \widetilde{D}(\mu,\imath)K(\mu,\imath)]^{\mathrm{T}} \\ \times [\widetilde{C}(\mu,\imath) + \widetilde{D}(\mu,\imath)K(\mu,\imath)] \\ + \sum_{k=1}^{s} \lambda_{\imath k}Q(k) + G(\imath) \\ + \frac{1}{(1-\beta)}Q(\imath)A_d(\mu,\imath)G^{-1}(\imath)A_d^{\mathrm{T}}(\mu,\imath)Q(\imath) \end{array} & (*)^{\mathrm{T}} \\ \begin{array}{c} \widetilde{B}_1^{\mathrm{T}}(\mu,\imath)Q(\imath) + \\ \widetilde{E}^{\mathrm{T}}(\mu,\imath)[C_1(\mu,\imath) + D_{12}(\mu,\imath)K(\mu,\imath)] \end{array} & -\mathcal{M} + \widetilde{E}^{\mathrm{T}}(\mu,\imath)\widetilde{E}(\mu,\imath) \end{bmatrix} \tag{3.28}$$

with

$$\begin{aligned} \widetilde{C}(\mu,\imath) &= [\,\rho(\imath)H_1^{\mathrm{T}}(\mu,\imath) \quad \rho(\imath)H_3^{\mathrm{T}}(\mu,\imath) \quad 0 \quad 0\,]^{\mathrm{T}}, \\ \widetilde{D}(\mu,\imath) &= [\,0 \quad 0 \quad \rho(\imath)H_2^{\mathrm{T}}(\mu,\imath) \quad \rho(\imath)H_4^{\mathrm{T}}(\mu,\imath)\,]^{\mathrm{T}}. \end{aligned}$$

Using the fact

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r} \mu_i\mu_j\mu_m\mu_n M_{ij}^{\mathrm{T}}(\imath)N_{mn}(\imath) \le \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j[M_{ij}^{\mathrm{T}}(\imath)M_{ij}(\imath) + N_{ij}(\imath)N_{ij}^{\mathrm{T}}(\imath)],$$

we can rewrite (3.27) as follows:

$$\begin{aligned} \widetilde{\Delta}V(x(t),\imath) &\le -z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}^{\mathrm{T}} \Phi_{ij}(\imath) \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix} \\ &= -z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t) + \sum_{i=1}^{r}\mu_i^2 \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}^{\mathrm{T}} \Phi_{ii}(\imath) \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix} \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r}\mu_i\mu_j \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}^{\mathrm{T}} (\Phi_{ij}(\imath) + \Phi_{ji}(\imath)) \begin{bmatrix} x(t) \\ \widetilde{w}(t) \end{bmatrix}, \end{aligned} \tag{3.29}$$

where

$$\Phi_{ij}(\imath) = \begin{bmatrix} \begin{matrix} [A_i(\imath) + B_{2_i}(\imath)K_j(\imath)]^{\mathrm{T}} Q(\imath) \\ + Q(\imath)[A_i(\imath) + B_{2_i}(\imath)K_j(\imath)] \\ + [C_{1_i}(\imath) + D_{12_i}(\imath)K_j(\imath)]^{\mathrm{T}} \\ \times [C_{1_i}(\imath) + D_{12_i}(\imath)K_j(\imath)] \\ + [\widetilde{C}_i(\imath) + \widetilde{D}_i(\imath)K_j(\imath)]^{\mathrm{T}} \\ \times [\widetilde{C}_i(\imath) + \widetilde{D}_i(\imath)K_j(\imath) \\ + \sum\limits_{k=1}^{s} \lambda_{\imath k} Q(k) + G(\imath) \\ + \frac{1}{(1-\beta)} Q(\imath) A_{d_i}(\imath) G^{-1}(\imath) A_{d_i}^{\mathrm{T}}(\imath) Q(\imath) \end{matrix} & (*)^{\mathrm{T}} \\ \widetilde{B}_{1_i}^{\mathrm{T}}(\imath) Q(\imath) + \widetilde{E}_i^{\mathrm{T}}(\imath)[C_{1_i}(\imath) + D_{12_i}(\imath)K_j(\imath)] & -\mathcal{M} + \widetilde{E}_i^{\mathrm{T}}(\imath)\widetilde{E}_j(\imath) \end{bmatrix}. \tag{3.30}$$

Using (3.20) and pre and post multiplying (3.30) by

$$\Xi(\imath) = \begin{bmatrix} P(\imath) & 0 \\ 0 & I \end{bmatrix},$$

we obtain

$$\Xi(\imath)\Phi_{ij}(\imath)\Xi(\imath) = \begin{bmatrix} \begin{matrix} P(\imath)A_i^{\mathrm{T}}(\imath) + Y_j^{\mathrm{T}}(\imath)B_{2_i}^{\mathrm{T}}(\imath) \\ +A_i(\imath)P(\imath) + B_{2_i}(\imath)Y_j(\imath) \\ +[C_{1_i}(\imath)P(\imath) + D_{12_i}(\imath)Y_j(\imath)]^{\mathrm{T}} \\ \times[C_{1_i}(\imath)P(\imath) + D_{12_i}(\imath)Y_j(\imath) \\ +[\widetilde{C}_i(\imath)P(\imath) + \widetilde{D}_i(\imath)Y_j(\imath)]^{\mathrm{T}} \\ \times[\widetilde{C}_i(\imath)P(\imath) + \widetilde{D}_i(\imath)Y_j(\imath)] \\ + \sum\limits_{k=1}^{s} \lambda_{\imath k} P(\imath)P^{-1}(k)P(\imath) \\ +P(\imath)G(\imath)P(\imath) + \frac{1}{(1-\beta)} A_{d_i}(\imath)G^{-1}(\imath)A_{d_i}^{\mathrm{T}}(\imath) \end{matrix} & (*)^{\mathrm{T}} \\ \widetilde{B}_{1_i}^{\mathrm{T}}(\imath) + \widetilde{E}_i^{\mathrm{T}}(\imath)C_{1_i}(\imath)P(\imath) + \widetilde{E}_i^{\mathrm{T}}(\imath)D_{12_i}(\imath)Y_j(\imath) & -\mathcal{M} + \widetilde{E}_i^{\mathrm{T}}(\imath)\widetilde{E}_j(\imath) \end{bmatrix}. \tag{3.31}$$

Note that (3.31) is the Schur complement of $\Omega_{ij}(\imath)$ defined in (3.7). Using (3.5), (3.6) and (3.31), we learn that

$$\Phi_{ii}(\imath) < 0, \tag{3.32}$$

$$\Phi_{ij}(\imath) + \Phi_{ji}(\imath) < 0. \tag{3.33}$$

Following from (3.29), (3.32) and (3.33), we know that

$$\widetilde{\Delta} V(x(t), \imath) < -z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t). \tag{3.34}$$

Applying the operator $E\left[\int\limits_0^{T_f} (\cdot)\, dt\right]$ on both sides of (3.34), we obtain

$$E\left[\int\limits_0^{T_f} \widetilde{\Delta} V(x(t), \imath)\, dt\right] < E\left[\int\limits_0^{T_f} (-z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t))\, dt\right]. \tag{3.35}$$

From the Dynkin's formula [29], it follows that

$$E\left[\int_0^{T_f} \widetilde{\Delta} V(x(t), \imath)\, dt\right] = E[V(x(T_f), \imath(T_f))] - E[V(x(0), \imath(0))]. \tag{3.36}$$

Substitute (3.36) into (3.35) yields

$$0 < E\left[\int_0^{T_f} (-z^{\mathrm{T}}(t)z(t) + \gamma w^{\mathrm{T}}(t)w(t))\, dt\right] - E[V(x(T_f), \imath(T_f))] + E[V(x(0), \imath(0))].$$

Using (3.34) and the fact that $V(x(0) = 0, \imath(0)) = 0$ and $V(x(T_f), \imath(T_f)) > 0$, we have

$$E\left[\int_0^{T_f} \Big\{z^{\mathrm{T}}(t)z(t) - \gamma w^{\mathrm{T}}(t)w(t)\Big\} dt\right] < 0. \tag{3.37}$$

Hence, the inequality (2.5) holds. This completes the proof of Theorem 3.1.

In order to demonstrate the effectiveness and advantages of the proposed design methodology, an illustrative example is given in next section.

4 An Illustrative Example

Consider an uncertain nonlinear system which is governed by the following state equation [21]

$$\begin{aligned}
\dot{x}_1(t) &= -0.1c(t)x_1^3(t) - \alpha(\eta(t))x_1(t-\tau(t)) - 0.02x_2(t) - 0.67x_2^3(t) \\
&\quad - 0.1x_2^3(t-\tau(t)) - 0.005x_2(t-\tau(t)) + u(t) + 0.1w_1(t), \\
\dot{x}_2(t) &= x_1(t) + 0.1w_2(t), \\
z(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},
\end{aligned} \tag{4.1}$$

where $x_1(t)$ and $x_2(t)$ are the state vectors, $u(t)$ is the control input, $w_1(t)$ and $w_2(t)$ are the disturbance input, $z(t)$ is the regulated output, $\eta(t)$ is the discrete state of the Markov process, $\tau(t) = 4 + 0.5\cos(0.9t)$ and $c(t)$ is the uncertain term, that is, $c(t) \in [0 \quad 2.25]$. It is assumed that

$$x_1(t) \in [-1.5 \quad 1.5] \quad \text{and} \quad x_2(t) \in [-1.5 \quad 1.5].$$

Using the same procedure as in [14], the nonlinear term can be represented as

$$\begin{aligned}
-0.67x_2^3(t) &= M_1 \cdot 0 \cdot x_2(t) - (1 - M_1) \cdot 1.5075x_2(t), \\
-0.1x_2^3(t-\tau(t)) &= M_1 \cdot 0 \cdot x_2(t-\tau(t)) - (1 - M_1) \cdot 0.225x_2(t-\tau(t)).
\end{aligned}$$

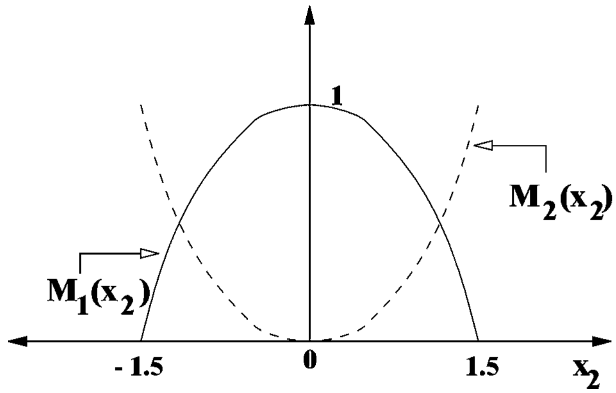

Figure 4.1. Membership functions for two fuzzy set.

Solving the above equations, M_1 is obtained as follows:

$$M_1(x_2(t)) = 1 - \frac{x_2^2(t)}{2.25},$$
$$M_2(x_2(t)) = 1 - M_1(x_2(t)) = \frac{x_2^2(t)}{2.25}.$$

Note that $M_1(x_2(t))$ and $M_1(x_2(t))$ can be interpret as the membership functions of fuzzy set.

Using these two fuzzy set, the uncertain nonlinear Markovian jump system with time-varying delay can be represented by the following TS fuzzy model:

Plant Rule 1: If $x_2(t)$ is $M_1(x_2(t))$ then

$$\dot{x}(t) = [A_1(\imath) + \Delta A_1(\imath)]x(t) + A_{d_1}(\imath)x(t-\tau(t)) + B_1(\imath)w(t) + B_2(\imath)u(t), \quad x(0) = 0,$$
$$z(t) = C_1(\imath)x(t),$$

Plant Rule 2: If $x_2(t)$ is $M_2(x_2(t))$ then

$$\dot{x}(t) = [A_2(\imath) + \Delta A_2(\imath)]x(t) + A_{d_2}(\imath)x(t-\tau(t)) + B_1(\imath)w(t) + B_2(\imath)u(t), \quad x(0) = 0,$$
$$z(t) = C_1(\imath)x(t),$$

where

$$A_1(\imath) = \begin{bmatrix} -0.1125 & -0.02 \\ 1 & 0 \end{bmatrix}, \qquad A_2(\imath) = \begin{bmatrix} -0.1125 & -1.5275 \\ 1 & 0 \end{bmatrix},$$
$$A_{d_1}(\imath) = \begin{bmatrix} -\alpha(\imath) & -0.005 \\ 0 & 0 \end{bmatrix}, \qquad A_{d_2}(\imath) = \begin{bmatrix} -\alpha(\imath) & -0.23 \\ 0 & 0 \end{bmatrix},$$
$$B_1(\imath) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad B_2(\imath) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_1(\imath) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\Delta A_1(\imath) = E_{1_1}(\imath)F(x(t),\imath,t)H_{1_1}(\imath), \qquad \Delta A_2(\imath) = E_{1_2}(\imath)F(x(t),\imath,t)H_{1_2}(\imath),$$

$$x(t) = [x_1^{\mathrm{T}}(t) \quad x_2^{\mathrm{T}}(t)]^{\mathrm{T}} \quad \text{and} \quad w(t) = [w_1^{\mathrm{T}}(t) \quad w_2^{\mathrm{T}}(t)]^{\mathrm{T}}.$$

Assuming $\|F(x(t), \imath, t)\| \leq \rho(\imath) = 1$ and letting

$$E_{1_1}(\imath) = E_{1_2}(\imath) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

we have

$$H_{1_1}(\imath) = H_{1_2}(\imath) = \begin{bmatrix} -1.1250 & 0 \\ 0 & 0 \end{bmatrix}.$$

Assume that the system is a three modes Markov process as shown in Table 4.1.

Table 4.1 Modes of the Markov process.

Mode $\imath$	$\alpha(\imath)$
1	0.0120
2	0.0125
3	0.0130

The transition probability matrix that relates the three modes is given as follows:

$$P_{\imath k} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix}.$$

Using the LMI optimization algorithm and Theorem 3.1 with $\beta = 0.6$, we obtain $\gamma = 0.1680$

$$P(1) = \begin{bmatrix} 2.4912 & -0.2673 \\ -0.2673 & 0.0718 \end{bmatrix}, \qquad W(1) = \begin{bmatrix} 1.1072 & -0.1535 \\ -0.1535 & 16.1836 \end{bmatrix},$$

$$Y_1(1) = [-16.9067 \quad -0.1051], \qquad Y_2(1) = [-17.2552 \quad -0.0235],$$

$$K_1(1) = [-11.5635 \quad -44.5276], \qquad K_2(1) = [-11.5934 \quad -43.5022],$$

$$P(2) = \begin{bmatrix} 2.3815 & -0.2881 \\ -0.2881 & 0.0841 \end{bmatrix}, \qquad W(2) = \begin{bmatrix} 1.1489 & -0.1931 \\ -0.1931 & 16.4120 \end{bmatrix},$$

$$Y_1(2) = [-15.9725 \quad 0.0589], \qquad Y_2(2) = [-16.3401 \quad 0.1485],$$

$$K_1(2) = [-11.3092 \quad -38.0433], \qquad K_2(2) = [-11.3526 \quad -37.1260],$$

$$P(3) = \begin{bmatrix} 2.4793 & -0.2638 \\ -0.2638 & 0.0857 \end{bmatrix}, \qquad W(3) = \begin{bmatrix} 0.9718 & -0.1883 \\ -0.1883 & 15.8428 \end{bmatrix},$$

$$Y_1(3) = [-17.0602 \quad -0.0867], \qquad Y_2(3) = [-17.4006 \quad 0.0530],$$

$$K_1(3) = [-10.3932 \quad -33.0111], \qquad K_2(3) = [-10.3394 \quad -31.2150].$$

The resulting fuzzy controller is

$$u(t) = \sum_{j=1}^{2} \mu_j K_j(\imath) x(t) \tag{4.2}$$

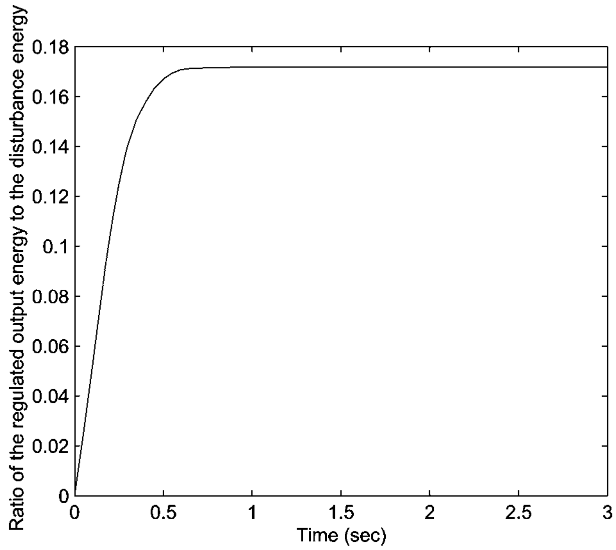

Figure 4.2. The result of the changing between modes during the simulation with the initial mode at Mode 2.

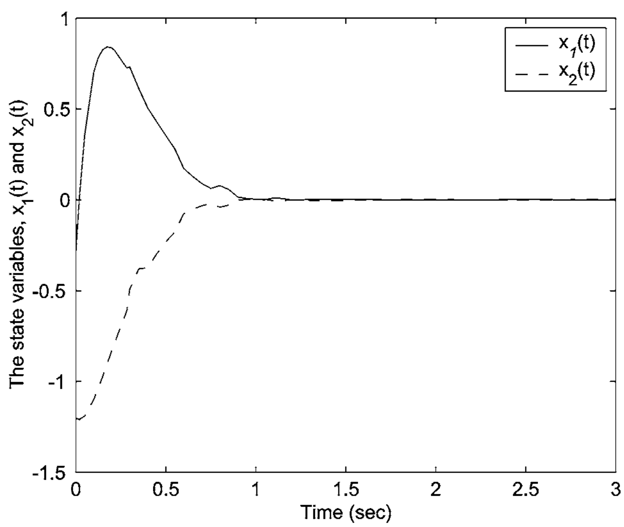

Figure 4.3. The histories of the state variables, $x_1(t)$ and $x_2(t)$.

where

$$\mu_1 = M_1(x_2(t)) \quad \text{and} \quad \mu_2 = M_2(x_2(t)).$$

Remark 4.1 Figure 4.2 shows the changing between modes with the initial mode at Mode 2. The histories of the state variables, $x_1(t)$ and $x_2(t)$ are given in Figure 4.3. The disturbance input signal, $w(t)$, which was used during simulation is given in Figure 4.4. The ratio of the regulated output energy to the disturbance input noise energy obtained by using the $\mathcal{H}_\infty$ fuzzy controller (4.2) is depicted in Figure 4.5. After 3 seconds, the ratio

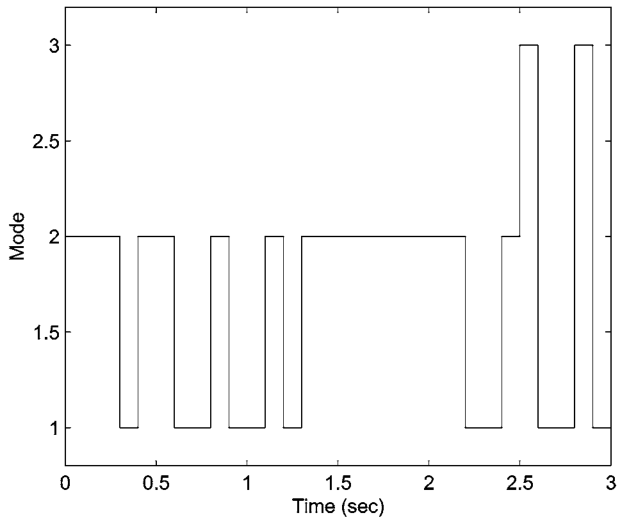

Figure 4.4. The disturbance input noise, $w(t)$.

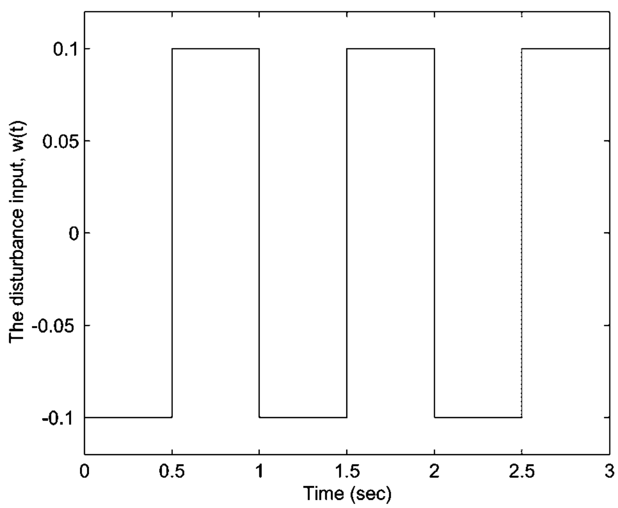

Figure 4.5. The ratio of the regulated output energy to the disturbance noise energy, $\left(\int_0^{T_f} z^{\mathrm{T}}(t)z(t)\,dt \Big/ \int_0^{T_f} w^{\mathrm{T}}(t)w(t)\,dt\right)$.

of the regulated output energy to the disturbance input noise energy tends to a constant value which is about 0.1680. From Figure 4.5, we can conclude that the inequality (2.5) is guaranteed by the fuzzy controller (4.2).

5 Conclusion

In this paper, we have developed a technique for designing a robust $\mathcal{H}_\infty$ fuzzy state-feedback controller for a class of time delay nonlinear Markovian jump systems that guarantees the $\mathcal{L}_2$-gain of the mapping from the exogenous input noise to the regulated output to be less than some prescribed value. In addition, solutions to the problem are given in terms of linear matrix inequalities which make them more useful. Finally, an illustrative example is provided to demonstrate the effectiveness and advantages of the proposed design methodology.

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H_∞ Control for a Class of Nonlinear Stochastic Time-Delay Systems

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Abstract: This paper mainly deals with H_∞ Controller design for a class of nonlinear stochastic time-delay systems with state and control-dependent noise. Some locally (globally) robust H_∞ Controllable conditions are given in terms of matrix inequalities independent of delay length. As applications, some sufficient conditions for the existence of the static state feedback H_∞ control law are presented for linear and special nonlinear stochastic time-delay systems via linear matrix inequalities, respectively.

Keywords: *Stochastic systems; linear matrix inequality; H_∞ control; time-delay systems.*

Mathematics Subject Classification (2000): 93C10, 93D09, 93E15.

1 Introduction

Since the celebrated paper [6] appeared, H_∞ control and filtering problems based on state-space approach, have attracted much more researchers' attention. For example, [1, 11] and [13] treated of the nonlinear uncertain H_∞ control and filtering design, while the H_∞ for linear time-delay systems with norm-bounded uncertainties can be found in [8, 10, 14, 15] and the references therein. The aforementioned works are confined to deterministic systems. Up to date, there are few results on stochastic H_∞ about which the system equation is governed by Itô-type differential equation. Below, we summarize the recent development for stochastic H_∞ briefly.

It is fair to say that [4] is the first paper which systematically dealt with the linear stochastic H_∞ control for state and output feedback control, in which, a very useful stochastic bounded real lemma (SBRL) was also derived, which has been applied to H_∞ filtering design of the stationary continuous time linear stochastic systems [5]. [2] first studied linear stochastic H_2/H_∞ control, in which, necessary and sufficient conditions were given for both finite and infinite horizon H_2/H_∞ via coupled Riccati equations; [16] was on output feedback H_∞ control for linear stochastic systems with norm bounded uncertainty in a state matrix, moreover, an applicable algorithm for designing an H_∞ control law was presented based on linear matrix inequalities (LMIs). In [3], we discussed the general nonlinear stochastic H_∞ control based on dissipative system theory and an associated Hamilton-Jacobi equation, which can be viewed as an extension of the results of [1] in some sense. In conclusion, we can say that stochastic H_∞ has become an attractive topic in recent years.

In spite of deterministic systems or stochastic systems, time-delay phenomena are inevitable arising from many physical problems, which often cause instability of the systems (see [18, 19]). Therefore, the H_∞ control of time-delay systems has received much attention in the past years (e.g. [8, 12]). This paper is on robust H_∞ control for a class of continuous time stochastic time-delay systems with nonlinear perturbation. By imposing a loose limitation on the nonlinear term, a very general theorem is obtained via matrix inequalities, from which, for some special case, we derived many useful sufficient conditions for the existence of a desired H_∞ controller in terms of LMIs. More specifically, as corollary, we also improve the previous conclusions on stochastic stabilization.

The outline of the current paper is organized as follows. In Section 2, we first present a general theorem on local and global H_∞ control by means of matrix inequalities independent of the length of delays, respectively. As corollaries, for linear or nonlinearly perturbed stochastic time-delay systems $(D = 0)$, we are in a position to design an LMI-based state-feedback H_∞ control law, which makes our results more applicable [10].

Section 3 presents two examples to illustrate the effectiveness of our developed theory.

Section 4 concludes this note by some remarks.

For convenience, we adopt the following notations: A' is the transpose of matrix A; $A \geq 0$ $(A > 0)$ is positive semi-definite (positive definite) matrix A; I is identity matrix; $\mathcal{L}^2_{\mathcal{F}}(R_+, R^l)$ is the space of non-anticipative stochastic processes $y(t) \in R^l$ with respect to an increasing σ-algebras $\mathcal{F}_t$ $(t \geq 0)$ satisfying

$$E\int_0^\infty \|y(t)\|^2 dt < \infty.$$

Here $\|\cdot\|$ denotes the standard Euclidean norm of a vector.

2 Main Results

In this section, we investigate the robust H_∞ state feedback control of the following stochastic time-delay system governed by Itô differential equations of the form

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_1u(t) + B_2v(t) + H_0(x(t),\, x(t-\tau),\, u(t)))\, dt \\ &\quad + (Cx(t) + Dx(t-\tau) + D_1u(t) + H_1(x(t), x(t-\tau), u(t)))dw(t), \\ z(t) &= C_2x(t) + D_2u(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \quad \tau > 0. \end{aligned} \tag{1}$$

In the above, $x(t) \in R^n$, $u(t) \in R^m$, $v(t) \in R^r$, and $z(t) \in R^s$ are called the system state, control input, disturbance input, controlled output, respectively. $w(t)$ is the standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ with an increasing filtration $\mathcal{F}_t$ satisfying the usual conditions. Without loss of generality, we can suppose $w(t)$ is one-dimensional, and $C_2' D_2 = 0$. Assume $u(t)$ and $v(t)$ to be adapted and measurable processes with respect to $\mathcal{F}_t$, $H_i(0,\cdot,\cdot) = 0$, $i = 0, 1$, i.e., $x \equiv 0$ is an equilibrium point of (1). A, B, B_1, B_2, C, C_2, D, D_1, and D_2 are constant matrices, $\tau > 0$ is an uncertain time-delay, where we refer the reader to [18] for the notion of $L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n))$. Under very mild conditions on $H_i(\cdot,\cdot,\cdot)$, $i = 0, 1$, (1) exists a unique global solution on $[0, T]$ for any $T > 0$ [18]. It should be pointed out that (1) can represent a class of more general nonlinear stochastic system via Taylor's series expansion at the origin. In what follows, we will show that, for a broader class of nonlinear functions $H_i(\cdot,\cdot,\cdot)$, $i = 0, 1$, LMI-based algorithms for robust H_∞ Control can be given, which is very efficient in practical computation by means of the existing LMI Toolbox [7]. Now, we first introduce the following definitions.

Definition 1 Stochastic time-delay differential system (1) with $v(t) \equiv 0$ is called *locally robustly stabilizable*, if there exists a constant state-feedback control law $u = Kx$, such that the equilibrium point of the closed-loop system

$$\begin{aligned} dx(t) &= ((A + B_1 K)x(t) + Bx(t-\tau) + H_0(x(t),\, x(t-\tau),\, Kx(t)))\,dt \\ &\quad + ((C + D_1 K)x(t) + Dx(t-\tau) + H_1(x(t),\, x(t-\tau),\, Kx(t)))\,dw, \\ x(t) &= \phi(t) \in L^2(\Omega,\, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{2}$$

is asymptotically stable in probability [9] for all $\tau > 0$. It is called globally robustly stabilizable, if the equilibrium point of (2) is asymptotically stable in the large [9] for all $\tau > 0$.

Definition 2 Stochastic time-delay differential system (1) with $\phi(t) \equiv 0$, $u(t) \equiv 0$, is said *to have an H_∞ performance level* $\gamma > 0$, if

$$\|z\|_2 < \gamma \|v\|_2, \quad \forall\, v \neq 0 \in \mathcal{L}^2_{\mathcal{F}}(R_+, R^r) \tag{3}$$

where

$$\|z\|_2^2 = E \int_0^\infty z'(t) z(t)\, dt.$$

Definition 3 Stochastic time-delay differential system (1) is called *locally (globally) robustly H_∞ controllable*, if there exists a constant state-feedback control law $u = Kx$, such that system (1) is locally (globally) stabilizable via state-feedback control law $u(t) = Kx$, and the corresponding closed-loop system has an H_∞ performance level $\gamma > 0$.

For robust stabilization of (1) $(B_2 = 0)$, a very general result is given as follows, which can be proved in the same way as Theorem 1 of [17], but for convenience, we would like to give its detailed proof here.

Lemma 1 *Suppose there exists $\epsilon \geq 0$, such that*

$$\sup_{y \in R^n} \|H_i(x, y, Kx)\| \leq \epsilon \|x\|, \quad i = 0, 1, \tag{4}$$

for all $x \in U$, where U is a neighborhood of the origin, $K \in R^{m\times n}$, $P > 0$ and $Q > 0$ are the solutions of the following matrix inequality

$$Z + Z_1 < 0, \tag{5}$$

then system (1) can be locally robustly stabilized by $u(t) = Kx(t)$. If U is replaced by R^n, then system (1) can be globally robustly stabilized by the same controller. In (5), Z and Z_1 are defined by

$$Z = \begin{bmatrix} \begin{matrix}\{P(A+B_1K)+(A+B_1K)'P+Q \\ +(C+D_1K)'P(C+D_1K)\}\end{matrix} & PB+(C+D_1K)'PD \\ B'P+D'P(C+D_1K) & D'PD-Q \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} (2\epsilon\|C\|+2\epsilon\|D_1\|\,\|K\|+\epsilon\|D\|+2\epsilon+\epsilon^2)\|P\|I & 0 \\ 0 & \epsilon\|D\|\,\|P\|I \end{bmatrix}.$$

Proof We construct the Lyapunov–Krasovskii functional as follows:

$$V(t,x) = x'Px + \int_0^\tau x'(t-s)Qx(t-s)\,ds$$

where $P > 0$ and $Q > 0$ are the solutions of (5). Let $\mathcal{L}_1$ be the infinitesimal generator of the closed-loop system (2) with K a solution to (5), then by Itô's formula, we have

$$\begin{aligned} \mathcal{L}_1V(t,x(t)) = {} & ((C+D_1K)x(t)+Dx(t-\tau)+H_1(x(t),\,x(t-\tau),\,Kx(t)))'P \\ & \times((C+D_1K)x(t)+Dx(t-\tau)+H_1(x(t),x(t-\tau),Kx(t)) \\ & +2[(A+B_1K)x(t)+Bx(t-\tau)+H_0(x(t),x(t-\tau),Kx(t))]'Px(t) \\ & +x'(t)Qx(t)-x'(t-\tau)Qx(t-\tau). \end{aligned} \tag{6}$$

Rearranging (6) yields

$$\begin{aligned} \mathcal{L}_1V(t,x(t)) = {} & x'(t)(P(A+B_1K)+(A+B_1K)'P+Q+(C+D_1K)'P(C+D_1K))x(t) \\ & +2x'(t)(PB+(C+D_1K)'PD)x(t-\tau)+x'(t-\tau)(D'PD-Q)x(t-\tau) \\ & +2H_0'(x(t),x(t-\tau),Kx(t))Px(t)+2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ & +H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)) \\ = {} & \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' Z \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} + 2H_0'(x(t),x(t-\tau),Kx(t))Px(t) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))P(C+D_1K)x(t) \\ & +2H_1'(x(t),x(t-\tau),Kx(t))PDx(t-\tau) \\ & +H_1'(x(t),x(t-\tau),Kx(t))PH_1(x(t),x(t-\tau),Kx(t)). \end{aligned} \tag{7}$$

In addition, by (4), we have

$$\begin{aligned}
2H_0'(x(t), x(t-\tau), Kx(t))Px(t) &+ 2H_1'(x(t), x(t-\tau), Kx(t))P(C + D_1K)x(t) \\
&+ 2H_1'(x(t), x(t-\tau), Kx(t))PDx(t-\tau) \\
&+ H_1'(x(t), x(t-\tau), Kx(t))PH_1(x(t), x(t-\tau), Kx(t)) \\
\le 2\epsilon\|P\|(\|C\| &+ \|D_1\|\,\|K\|)\|x(t)\|^2 + 2\epsilon\|D\|\,\|P\|\,\|x(t)\|\,\|x(t-\tau)\| \\
&+ \epsilon^2\|P\|\,\|x(t)\|^2 + 2\epsilon\|P\|\,\|x(t)\|^2).
\end{aligned} \tag{8}$$

for $(t, x) \in \{t > 0\} \times U$. By inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$, (8) follows

$$\begin{aligned}
2H_0'(x(t), x(t-\tau), Kx(t))Px &+ 2H_1'(x(t), x(t-\tau), Kx(t))P(C + D_1K)x(t) \\
&+ 2H_1'(x(t), x(t-\tau), Kx(t))PDx(t-\tau) \\
&+ H_1'(x(t), x(t-\tau), Kx(t))PH_1(x(t), x(t-\tau), Kx(t)) \\
\le (2\epsilon\|C\| &+ 2\epsilon\|D_1\|\,\|K\| + \epsilon\|D\| + 2\epsilon + \epsilon^2)\|P\|\,\|x(t)\|^2 \\
&+ \epsilon\|D\|\|P\|\,\|x(t-\tau)\|^2 \\
= \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' & Z_1 \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}.
\end{aligned} \tag{9}$$

Substituting (9) into (7), it follows

$$\mathcal{L}_1 V(t, x(t)) \le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' (Z + Z_1) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} < 0$$

due to (5). That is, $\mathcal{L}_1 V(t, x(t)) < 0$ in the domain $\{t > 0\} \times U$ for $x \ne 0$. So the locally robust stabilization is obtained by Corollary 1 of [9] (page 168). By the same discussion, the globally robust stabilization can also be shown by Theorem 4.4 of [9].

Using Lemma 1, a sufficient condition for robust H_∞ control is obtained as follows.

Theorem 1 *Suppose there exists $\epsilon \ge 0$, such that (4) holds for all $x \in U$ with U a neighborhood of the origin, $K \in R^{m\times n}$, $P > 0$ and $Q > 0$ are the solutions to the following matrix inequality*

$$\Sigma = \begin{bmatrix} Z_{11} + C_2'C_2 + K'D_2'D_2K & Z_{12} & PB_2 \\ Z_{12}' & Z_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{10}$$

where

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}' & Z_{22} \end{bmatrix} = Z + Z_1.$$

Then system (1) is locally robustly H_∞ controlled by $u(t) = Kx(t)$. If U is replaced by R^n, then system (1) is globally robustly H_∞ controlled by the same controller.

Proof It is obvious that (5) can be derived from (10), i.e. system (1) is robustly stable. Therefore we only need to prove that the closed-loop system has H_∞ performance level

γ. For any $T > 0$, by (10), it follows

$$
\begin{aligned}
&\|z\|^2_{2,[0,T]} - \gamma^2\|v\|^2_{2,[0,T]} = E\int_0^T [(z'(t)z(t) - \gamma^2 v'(t)v(t))\,dt \\
&= E\int_0^T [(x'(t)C_2'C_2x(t) + x'(t)K'D_2'D_2Kx(t) - \gamma^2 v'(t)v(t))\,dt + d(V(x(t))] - EV(x(T)) \\
&\le E\int_0^T [(x'(t)C_2'C_2x(t) + x'(t)K'D_2'D_2Kx(t) - \gamma^2 v'(t)v(t))\,dt + d(V(x(t))] \qquad (11) \\
&\le E\int_0^T \psi'(t)\Sigma\psi(t) < 0
\end{aligned}
$$

for $\psi \neq 0$, where $\psi = [x'(t)\ \ x'(t-\tau))\ \ v'(t)]'$. Let $T \to \infty$ in (11), (3) is immediately obtained. Theorem 1 is proved.

Generally speaking, Theorem 1 cannot be directly used in practice, because the elements of Z_1 contain the norm of an unknown matrix P. However, from Theorem 1, we can derive some useful results, which can be expressed in terms of LMIs.

Corollary 1 *If the matrix inequality*

$$
\begin{bmatrix} \overline{Z}_{11} + C_2'C_2 + K'D_2'D_2K & \overline{Z}_{12} & PB_2 \\ \overline{Z}_{12}' & \overline{Z}_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{12}
$$

has solutions $P > 0$, $Q > 0$ *and* $K \in R^{m\times n}$, *and*

$$
\lim_{\|x\|\to 0} \sup_{y\in R^n} \|H_i(x,y,Kx)\|/\|x\| = 0, \quad i = 0,1, \tag{13}
$$

where

$$
\begin{bmatrix} \overline{Z}_{11} & \overline{Z}_{12} \\ \overline{Z}_{12}' & \overline{Z}_{22} \end{bmatrix} = Z,
$$

then system (1) can be locally robustly H_∞ *controlled by* $u(t) = Kx(t)$.

Corollary 2 *If* $H_i \equiv 0$, $i = 0,1$, *and the matrix inequality (12) has solutions* $P > 0$, $Q > 0$, *and* $K \in R^{m\times n}$, *then the linear stochastic time-delay system*

$$
\begin{aligned}
dx(t) &= (Ax(t) + Bx(t-\tau) + B_1u(t) + B_2v(t))\,dt \\
&\quad + (Cx(t) + Dx(t-\tau) + D_1u(t))\,dw(t), \\
z(t) &= C_2x(t) + D_2u(t), \\
x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau,0], R^n)), \quad t \in [-\tau, 0],
\end{aligned} \tag{14}
$$

is globally robustly H_∞ *controllable. Especially, if* $D = 0$, *and the following LMI*

$$
\begin{bmatrix}
A\widehat{P} + \widehat{P}A' + B_1Y + Y'B_1' + B\widehat{Q}B' & \widehat{P}C' + Y'D_1' & \widehat{P} & \widehat{P}C_2' & Y'D_2 & B_2 \\
C\widehat{P} + D_1Y & -\widehat{P} & 0 & 0 & 0 & 0 \\
\widehat{P} & 0 & -\widehat{Q} & 0 & 0 & 0 \\
C_2\widehat{P} & 0 & 0 & -I & 0 & 0 \\
D_2Y & 0 & 0 & 0 & -I & 0 \\
B_2' & 0 & 0 & 0 & 0 & -\gamma^2 I
\end{bmatrix} < 0 \tag{15}
$$

admits solutions $\widehat{P} > 0$, $\widehat{Q} > 0$ *and* $Y \in R^{m\times n}$, *then system (14) with* $D = 0$ *is globally robustly* H_∞ *controllable. In this case, the state feedback control law* $u(t) = Kx(t) = Y\widehat{P}^{-1}x(t)$.

Proof If $H_i(\cdot,\cdot,\cdot) \equiv 0$, $i = 0, 1$, we can take $\epsilon = 0$ in (4), then $\mathcal{L}_1 V(t, x(t)) < 0$ for $(t, x) \in \{t > 0\} \times R^n$, except possibly at $x = 0$, and $\Sigma < 0$. Thus, the first part of Corollary 2 is proved.

Furthermore, if $D = 0$, (10) degenerates into

$$\begin{bmatrix} \begin{matrix}\{P(A+B_1K)+(A+B_1K)'P+Q+\\ (C+D_1K)'P(C+D_1K)+C_2'C_2+K'D_2'D_2K\}\end{matrix} & PB & PB_2 \\ B'P & -Q & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{16}$$

Pre- and postmultiply the above matrix inequality by $\mathrm{diag}(P^{-1}, I, I)$, and set $\widehat{P} = P^{-1}$, $Y = KP^{-1} = K\widehat{P}$, $\widehat{Q} = Q^{-1}$. Then by Schur's complement again, (16) is equivalent to (15). Thus the second part of Corollary 2 is also proved.

Corollary 3 *The unforced system*

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_2 v(t))\, dt + (Cx(t) + Dx(t-\tau))\, dw(t), \\ z(t) &= C_2 x(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{17}$$

is robustly stable and has H_∞ *performance level* γ, *if the following LMI*

$$\begin{bmatrix} PA + A'P + C'PC + Q + C_2'C_2 & PB + C'PD & PB_2 \\ B'P + D'PC & D'PD - Q & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{18}$$

has solutions $P > 0$, $Q > 0$.

Corollary 4 *The stochastic linear time-delay controlled system*

$$\begin{aligned} dx(t) &= (Ax(t) + Bx(t-\tau) + B_1 u(t)) + B_2 v(t))\, dt + (Cx(t) + Dx(t-\tau))\, dw(t), \\ z(t) &= C_2 x(t), \\ x(t) &= \phi(t) \in L^2(\Omega, \mathcal{F}_0, C([-\tau, 0], R^n)), \quad t \in [-\tau, 0], \end{aligned} \tag{19}$$

is globally robustly H_∞ *controlled, if the following LMI*

$$\begin{bmatrix} PA + A'P + C'PC + C_2'C_2 + Q & \sqrt{2}PB_1 & PB + C'PD & PB_2 \\ \sqrt{2}B_1'P & -Q & 0 & 0 \\ B'P + D'PC & 0 & D'PD - Q & 0 \\ B_2'P & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{20}$$

admitting solutions $P > 0$ *and* $Q > 0$. *Moreover, the feedback control law* $u(t) = Q^{-1}B_1'Px(t)$.

Proof Applying Theorem 1, this corollary is easily obtained.

Below, for $D=0$, we give another sufficient condition for the local (global) H_∞ control of system (1) in terms of LMIs. Applying the well known inequality

$$X'Y + Y'X \le \varepsilon X'X + \varepsilon^{-1}Y'Y, \quad \forall\, \varepsilon > 0, \tag{21}$$

with $\varepsilon = 1$ for simplicity, we have (if $0 < P \le \dfrac{1}{\alpha} I$ for some $\alpha > 0$)

$$\begin{aligned} &2H_0'(x(t), x(t-\tau), Kx(t))Px(t) + 2H_1'(x(t), x(t-\tau), Kx(t))P(C+D_1K)x(t) \\ &\quad + H_1'(x(t), x(t-\tau), Kx(t))PH_1(x(t), x(t-\tau), Kx(t)) \\ &\quad \le \frac{3\epsilon^2}{\alpha}\|x(t)\|^2 + x'(t)Px(t) + x'(t)(C+D_1K)'P(C+D_1K)x(t). \end{aligned} \tag{22}$$

Substituting (22) into (7), it follows

$$\mathcal{L}_1 V(t, x(t)) \le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' \widehat{Z} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

where

$$\widehat{Z} = \begin{bmatrix} \begin{matrix}\{P(A+B_1K) + (A+B_1K)'P + Q + P + \\ \frac{3\epsilon^2}{\alpha}I + 2(C+D_1K)'P(C+D_1K)\}\end{matrix} & PB \\ B'P & -Q \end{bmatrix}.$$

So if (4) holds for all $x \in U$ ($x \in R^n$), and $\widehat{Z} < 0$, then system (1) can be locally (globally) robustly stabilized by $u(t) = Kx(t)$. Accordingly, (12) is equivalent to

$$\begin{bmatrix} \widehat{Z}_{11} + C_2'C_2 + K'D_2'D_2K & \widehat{Z}_{12} & PB_2 \\ \widehat{Z}_{12}' & \widehat{Z}_{22} & 0 \\ B_2'P & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{23}$$

admitting solutions $0 < P \le \dfrac{1}{\alpha}I$, $Q > 0$ and K, where

$$\begin{bmatrix} \widehat{Z}_{11} & \widehat{Z}_{12} \\ \widehat{Z}_{12}' & \widehat{Z}_{22} \end{bmatrix} = \widehat{Z}.$$

In analogy with the proof of Corollary 2, it is easy to show that (23) is equivalent to that the following LMIs

$$\begin{bmatrix} \begin{matrix} A\widehat{P} + \widehat{P}A' + B_1Y \\ +Y'B_1' + B\widehat{Q}B' + \widehat{P} \end{matrix} & \sqrt{2}(\widehat{P}C' + Y'D_1') & \widehat{P} & \widehat{P} & \widehat{P}C_2' & Y'D_2 & B_2 \\ \sqrt{2}(C\widehat{P} + D_1Y) & -\widehat{P} & 0 & 0 & 0 & 0 & 0 \\ \widehat{P} & 0 & -\widehat{Q} & 0 & 0 & 0 & 0 \\ \widehat{P} & 0 & 0 & -\frac{\alpha}{3\epsilon^2}I & 0 & 0 & 0 \\ C_2\widehat{P} & 0 & 0 & 0 & -I & 0 & 0 \\ D_2Y & 0 & 0 & 0 & 0 & -I & 0 \\ B_2' & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{24}$$

and

$$\widehat{P} \ge \alpha I \tag{25}$$

exist solutions $\widehat{P} > 0$, $\alpha > 0$, $\widehat{Q} > 0$ and $Y = KP^{-1} \in R^{m\times n}$, where $\widehat{P} = P^{-1}$, $Y = KP^{-1} = K\widehat{P}$, and $\widehat{Q} = Q^{-1}$.

Summarize the above discussion, we have the following result.

Theorem 2 *For $D = 0$ in (1), suppose (4) holds for all $x \in U$ $(x \in R^n)$. If LMIs (24) and (25) exist solutions $\widehat{P} > 0$, $\alpha > 0$, $\widehat{Q} > 0$ and $Y \in R^{m\times n}$, simultaneously, then system (1) can be locally (globally) robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t)$.*

Remark 1 All results obtained in this section can be extended without difficulty to systems with multiple delays and independent stochastic perturbations.

Remark 2 Following the same line adopted above, there is no any difficulty to generalize what we have obtained to delay-dependent results with time-varying delay. For instance, if we take $\tau(t)$ to be a time-varying bounded delay satisfying

$$0 < \tau(t) \le h, \dot{\tau}(t) \le d < 1$$

and take the Lyapunov–Krasovskii functional

$$V(x) = x'(t)Px(t) + \int\limits_{t-\tau(t)}^{t} x'(\theta)Rx(\theta)\,d\theta + \int\limits_{-\tau(t)}^{0}\int\limits_{t+\beta}^{t} x'(s)Qx(s)\,ds\,d\beta,$$
$$P > 0, \quad R > 0, \quad Q > 0,$$

correspondingly, then the delay-dependent consequences can be obtained.

In (1), if we take $\tau = 0$, $B = D = B_2 = D_2 = C_2 = 0$, $\phi(0) = x(0)$, then for the system

$$\begin{aligned} dx(t) &= (Ax(t) + B_1u(t) + H_0(x(t), u(t)))\,dt \\ &\quad + (Cx(t) + D_1u(t) + H_1(x(t), u(t)))\,dw(t) \end{aligned} \tag{26}$$

a locally stabilizable condition is concluded by Theorem 2.

Corollary 5 *If for some $\widehat{R} > 0$, $\widehat{Q} > 0$, the following generalized algebraic Riccati equation (GARE)*

$$\widehat{P}A + A'\widehat{P} + C'\widehat{P}C - (\widehat{P}B_1 + C'\widehat{P}D_1)(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C) + \widehat{Q} = 0 \tag{27}$$

has a positive definite solution $\widehat{P} > 0$, and

$$\lim_{\|x\|\to 0} \|H_i(x, \widehat{K}x)\|/\|x\| = 0, \quad i = 0, 1, \tag{28}$$

holds for $\widehat{K} = -(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C)$, then system (27) is locally asymptotically stabilizable. In this case, $u(t) = \widehat{K}x(t) = -(\widehat{R} + D_1'\widehat{P}D_1)^{-1}(B_1'\widehat{P} + D_1'\widehat{P}C)x(t)$ is a stabilizing control law.

It can be seen that Corollary 5 generalizes and improves Proposition 1 of [20].

Remark 3 There is something wrong in Proposition 1 of [20]. By checking its proof therein, we can find that the smallest eigenvalue of $\widehat{Q} \ge 0$ must be larger than zero, i.e., $\widehat{Q} > 0$. In other words, $(\widehat{Q}^{1/2}, A)$ being observable should be replaced by $\widehat{Q} > 0$.

3 Numerical Examples

Now, we present two examples to illustrate the validity of our developed theory in designing the H_∞ controller for nonlinear time-delay system (1).

Example 1 In (1), take $D = 0$, and

$$A = \begin{bmatrix} -4.12 & 1.23 \\ -0.36 & 1.15 \end{bmatrix}, \quad B = \begin{bmatrix} -0.13 & -0.91 \\ 0.22 & -0.76 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.25 \\ 3.48 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2 \\ 0.3 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.02 & -0.09 \\ 0.09 & -0.08 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.16 \\ 0.23 \end{bmatrix}, \quad C_2 = [\,0.1 \quad 0.02\,], \quad D_2 = [\,0.1\,],$$
$$H_0(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} \sin(u(t)x_2(t-\tau))x_1(t) \\ \cos(u(t)x_1(t-\tau))x_2(t) \end{bmatrix},$$
$$H_1(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} e^{-(u(t)+x_1(t-\tau)+x_2(t-\tau))^2}x_2(t) \\ e^{[-u^2(t)x_1^2(t-\tau)]}x_1(t) \end{bmatrix}, \quad \forall\, \tau > 0.$$

Obviously, (4) holds for all $x \in R^n$ with $\epsilon = 1$. Substituting all the above data into (24), and then solving the LMIs (24) and (25) by LMI Toolbox [7], we can obtain solutions, when $\gamma = 1$,

$$\widehat{P} = \begin{bmatrix} 0.3539 & -0.0042 \\ -0.0042 & 0.1263 \end{bmatrix} > 0, \quad \widehat{Q} = \begin{bmatrix} 1.1197 & 0.0008 \\ 0.0008 & 1.0076 \end{bmatrix} > 0,$$
$$Y = [\,-0.2930 \quad -1.3061\,], \quad \alpha = 1.1255 > 0.$$

So by Theorem 2, system (1) can be globally robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t) = -0.8566x_1(t) - 2.4518x_2(t)$.

Example 2 In Example 1, we take

$$H_0(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} (e^{x_1(t)} - 1)\sin u(t) \\ \sin x_2(t)\cos u(t) \end{bmatrix},$$
$$H_1(x(t),\, x(t-\tau),\, u(t)) = \begin{bmatrix} (\cos x_1(t) - 1)e^{-x_2^2(t-\tau)} \\ x_2(t)\sin u(t) \end{bmatrix}, \quad \forall\, \tau > 0.$$

Obviously, we have

$$\|H_0(\cdot,\cdot,\cdot)\| \le \sqrt{(e^{x_1(t)} - 1)^2 + \sin^2 x_2(t)}\,,$$
$$\|H_1(\cdot,\cdot,\cdot)\| \le \sqrt{(\cos x_1(t) - 1)^2 + x_2^2(t)}\,,$$

and

$$\lim_{x_1 \to 0} \frac{(e^{x_1} - 1)}{x_1} = 1, \quad \lim_{x_2 \to 0} \frac{\sin x_2}{x_2} = 1, \quad \lim_{x_1 \to 0} \frac{(\cos x_1 - 1)}{x_1} = 0.$$

So there exists a sufficient small neighborhood U of the origin, such that for all $x \in U$, (4) holds with $\epsilon = 1.05$. Substituting all coefficient matrices of Example 1 into (24) with $\epsilon = 1.05$, when $\gamma = 1$, via solving the LMIs (24) and (25), one has

$$\widehat{P} = \begin{bmatrix} 0.3527 & -0.0060 \\ -0.0060 & 0.1371 \end{bmatrix} > 0, \quad \widehat{Q} = \begin{bmatrix} 1.1064 & -0.0018 \\ -0.0018 & 1.0000 \end{bmatrix} > 0,$$
$$Y = [\,-0.2993 - 0.3101\,], \quad \alpha = 1.0995 > 0.$$

So by Theorem 2, system (1) can be locally robustly H_∞ controlled by $u(t) = Y\widehat{P}^{-1}x(t) = -0.8875x_1(t) - 2.3013x_2(t)$.

4 Conclusions

In the above sections, we have discussed the state feedback H_∞ control for a class of stochastic time-delay systems with nonlinear perturbations. By means of LMIs, some sufficient conditions are given for the existence of an H_∞ control law. Theorem 1 is a very general consequence, from which we derive some useful results for linear time-delay systems, delay-free systems or special nonlinearly perturbed time-delay systems. All consequences except Theorem 1 and Corollary 1 are expressed in terms of LMIs, which makes them more readily applied.

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Robust $\mathcal{H}_\infty$ Filtering for Discrete Stochastic Time-Delay Systems with Nonlinear Disturbances*

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Abstract: This paper deals with the problem of robust $\mathcal{H}_\infty$ filtering for discrete time-delay systems with stochastic perturbation and nonlinear disturbance. It is assumed that the state-dependent noises and the nonlinearities satisfying global Lipschitz conditions enter into both the state and measurement equations, and the system matrices also contain parameter uncertainties residing in a polytope. Attention is focused on the design of robust full-order and reduced-order filters guaranteeing a prescribed noise attenuation level in an $\mathcal{H}_\infty$ sense with respect to all energy-bounded noise inputs for all admissible uncertainties and time delays. Sufficient conditions for the existence of such filters are formulated in terms of a set of linear matrix inequalities, upon which admissible filters can be obtained from the solution of a convex optimization problem. A numerical example is provided to illustrate the applicability of the developed filter design procedure.

Keywords: *Filter design; linear matrix inequality; robust filtering; state-delay systems; stochastic systems; nonlinearity.*

Mathematics Subject Classification (2000): 93E11, 93C10, 93C23.

1 Introduction

During the past decades, stochastic modelling has played an important role in many branches of science such as biology, economics and engineering applications. Therefore, much attention has been drawn to systems with stochastic perturbations from researchers

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working in related areas. By stochastic systems, we generally refer to systems whose parameter uncertainties are modelled as white noise processes. The appearance of these parameter uncertainties are usually due to the random changes of the environment under which the systems are operated, and thus it is a natural way to represent them in the model by stochastic parameters fluctuating around some deterministic nominal values. This kind of systems has been called systems with random parametric excitation [1], stochastic bilinear systems [20, 30] and linear stochastic systems with multiplicative noise [15, 17, 31]. Analysis and synthesis of stochastic systems have been investigated extensively and many fundamental results for deterministic systems have been extended to stochastic cases. To mention a few, the analysis of asymptotic behaviour can be found in [21]; the optimal control problems were reported in [17, 31]; and recently with the development of $\mathcal{H}_\infty$ control theory, the robust control and filtering results have also been extended to stochastic systems through Ricatti-like and linear matrix inequality (LMI) approaches [8, 18].

On the other hand, since time delay exists commonly in dynamic systems and is frequently a source of instability and poor performance, much theoretical work has been produced for time-delay systems. The most powerful approach for solving problems arising in time-delay systems so far has been the so-called Lyapunov-Krasovskii approach, in which the asymptotic stability as well as performances can be established by employing appropriate Lyapunov-Krasovskii functionals. Within this framework, a great number of results have been reported, including stability analysis [26], state-feedback control [5, 23, 28], output-feedback control [9, 10], filter design [12, 13] and model reduction [34], etc.

The simultaneous presence of stochastic uncertainty and time delays results in stochastic time-delay systems (STDS) have attracted much attention in recent years, and some useful research results related to STDS have been reported in the literature. Among these results, the exponential stability and asymptotic stability of stochastic differential delay equations are investigated in [22, 24]; the problems of stabilization and $\mathcal{H}_\infty$ control via a memoryless state-feedback are considered in [32]; and the filtering problems have also been addressed in [2, 19] for different classes of STDS. These useful results have greatly advanced the analysis and synthesis of stochastic systems. However, it is worth noting that most of the aforementioned results are developed for continuous-time systems, while few results are available for discrete time-delay systems with stochastic perturbations which are also important in practical applications.

In this paper, we are interested in the problem of robust $\mathcal{H}_\infty$ filtering for discrete stochastic time-delay systems with parameter uncertainties and nonlinear disturbances. The parameter uncertainty is assumed to be of polytopic-type, and the nonlinearity satisfies global Lipschitz conditions, entering into both state and measurement equations. Attention is focused on the design of robust full-order and reduced-order filters guaranteeing a prescribed noise attenuation level in an $\mathcal{H}_\infty$ sense with respect to all energy-bounded noise inputs for all admissible uncertainties and time delays. Sufficient conditions for the existence of such filters are formulated in terms of a set of linear matrix inequalities, upon which admissible filters can be obtained from the solution of a convex optimization problem. A numerical example is provided to illustrate the applicability of the developed filter design procedure.

Notations The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; R^n denotes the n-dimensional Euclidean space and $R^{m\times n}$ is the set of all real matrices of dimension $m \times n$; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrices; the notation $|\cdot|$ refers to the Euclidean vector norm; $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and diag$\{\dots\}$ stands for a block-diagonal matrix. In addition, $E\{x\}$ and $E\{x|\,y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The space of square summable infinite sequence is denoted by $l_2[0,\infty)$.

2 Problem Formulation

Consider the following discrete stochastic time-delay system with nonlinear disturbance:

$$
\begin{aligned}
\mathcal{S}\colon\quad x_{t+1} &= [Ax_t + A_d x_{t-d} + Ff(x_t, x_{t-d}) + B\omega_t] + [Mx_t + M_d x_{t-d}]v_t,\\
y_t &= [Cx_t + C_d x_{t-d} + Gg(x_t, x_{t-d}) + D\omega_t] + [Nx_t + N_d x_{t-d}]v_t,\\
z_t &= Lx_t,\\
x_t &= \phi_t, \quad t = -d,\, -d+1,\, \dots, 0,
\end{aligned}
\tag{1}
$$

where $x_t \in R^n$ is the state vector; $y_t \in R^m$ is the measured output; $z_t \in R^p$ is the signal to be estimated; $\omega_t \in R^l$ is the disturbance input which belongs to $l_2[0,\infty)$; v_t is a zero mean white noise sequence with covariance I; A, A_d, F, B, M, M_d, C, C_d, G, D, N, N_d, L are system matrices with appropriate dimensions; $d > 0$ is a constant time delay; $\{\phi_t\colon\ t = -d, -d+1, \dots, 0\}$ is a given initial condition sequence; $f(x_t, x_{t-d})$, $g(x_t, x_{t-d})$ are known nonlinear functions. Throughout the paper, we make the following assumptions.

Assumption 1 The nonlinear functions satisfy

(1) $f(0,0) = 0$, $g(0,0) = 0$;

(2) (Lipschitz conditions) there exist known real appropriately dimensioned matrices S_1, S_2, T_1, T_2 such that for all x_1, x_2, y_1, y_2 satisfying

$$
\begin{aligned}
\|f(x_1,x_2) - f(y_1,y_2)\| &\le \|S_1(x_1 - y_1)\| + \|S_2(x_2 - y_2)\|\,,\\
\|g(x_1,x_2) - g(y_1,y_2)\| &\le \|T_1(x_1 - y_1)\| + \|T_2(x_2 - y_2)\|\,.
\end{aligned}
$$

Assumption 2 The system matrices are appropriately dimensioned with partially unknown parameters. We assume that

$$
\Omega \triangleq (A, A_d, F, B, M, M_d, C, C_d, G, D, N, N_d, L) \in \mathcal{R}
$$

where $\mathcal{R}$ is a given convex bounded polyhedral domain described by s vertices

$$
\mathcal{R} \triangleq \left\{\Omega(\lambda)\colon\ \Omega(\lambda) = \sum_{i=1}^{s} \lambda_i \Omega_i;\ \ \sum_{i=1}^{s} \lambda_i = 1,\ \ \lambda_i \ge 0\right\}
$$

and $\Omega_i \triangleq (A_i, A_{di}, F_i, B_i, M_i, M_{di}, C_i, C_{di}, G_i, D_i, N_i, N_{di}, L_i)$ denotes the vertices of the polytope $\mathcal{R}$.

Remark 1 The system under investigation in this paper contains both parameter and nonlinear uncertainties. As can be seen in Assumption 2, the parameter uncertainties are assumed to be of polytopic-type, entering into all the matrices of the system model. The polytopic uncertainty has been widely used in the problems of robust control and filtering for uncertain systems, see, e.g., [3, 7, 14] and the references therein and many practical systems possess parameter uncertainties which can be either exactly modeled or over-bounded by the polytope $\mathcal{R}$. In addition, the nonlinear uncertainty in Assumption 1 has also been widely used in the literature, see, e.g., [16, 29, 33].

Remark 2 Although there is only a single delay taken into consideration in system $\mathcal{S}$, the results developed in this paper can be easily extended to systems with multiple state delays. The reason why we consider single delay systems is to make our derivation more lucid and to avoid complicated notations. It is also worth mentioning that the results obtained in this paper can be readily extended to the case where v_t enters system $\mathcal{S}$ in a summation form, that is, the dynamic and measurement equations in system $\mathcal{S}$ have the following form

$$x_{t+1} = [Ax_t + A_d x_{t-d} + Ff(x_t, x_{t-d}) + B\omega_t] + \sum_{i=1}^{r} [M_i x_t + M_{di} x_{t-d}]\, v_{ti},$$
$$y_t = [Cx_t + C_d x_{t-d} + Gg(x_t, x_{t-d}) + D\omega_t] + \sum_{i=1}^{r} [N_i x_t + N_{di} x_{t-d}]\, v_{ti}.$$

Here we are interested in estimating the signal z_t by a linear dynamic filter of general structure described by

$$\begin{aligned} \mathcal{F}: \quad \hat{x}_{t+1} &= A_F \hat{x}_t + B_F y_t, \\ \hat{z}_t &= C_F \hat{x}_t, \\ \hat{x}_t &= \varphi_t, \quad t = -d, -d+1, \ldots, 0, \end{aligned} \tag{2}$$

where $\hat{x}_t \in R^k$ is the filter state vector and (A_F, B_F, C_F) are appropriately dimensioned filter matrices to be determined. It should be pointed out that here we are interested not only in the full-order filtering problem (when $k = n$), but also in the reduced-order filtering problem (when $1 \leq k < n$). As can be seen in the following, these two filtering problems are solved in a unified framework.

Augmenting the model of $\mathcal{S}$ to include the states of the filter $\mathcal{F}$, we obtain the filtering error system $\mathcal{E}$:

$$\begin{aligned} \mathcal{E}: \quad \xi_{t+1} &= \left[\bar{A}\xi_t + \bar{A}_d K \xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t\right] + \left[\overline{M}\xi_t + \overline{M}_d K \xi_{t-d}\right] v_t, \\ e_t &= \overline{C}\xi_t, \\ \xi_t &= [\,\phi_t^{\mathrm{T}} \quad \varphi_t^{\mathrm{T}}\,]^{\mathrm{T}}, \quad t \in [-d, 0], \end{aligned} \tag{3}$$

where $\xi_t = [\,x_t^{\mathrm{T}} \quad \hat{x}_t^{\mathrm{T}}\,]^{\mathrm{T}}$, $\eta(x_t, x_{t-d}) = [\,f^{\mathrm{T}}(x_t, x_{t-d}) \quad g^{\mathrm{T}}(x_t, x_{t-d})\,]^{\mathrm{T}}$, $e_t = z_t - \hat{z}_t$ and

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} A_d \\ B_F C_d \end{bmatrix}, \quad \overline{F} = \begin{bmatrix} F & 0 \\ 0 & B_F G \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix}, \\ &\overline{M} = \begin{bmatrix} M & 0 \\ B_F N & 0 \end{bmatrix}, \quad \overline{M}_d = \begin{bmatrix} M_d \\ B_F N_d \end{bmatrix}, \quad \overline{C} = [\,L \quad -C_F\,], \quad K = [\,I \quad 0\,]. \end{aligned} \tag{4}$$

We first introduce the following definitions.

Definition 1 The *filtering error system* $\mathcal{E}$ in (3) with $\omega_t = 0$ is said to be *mean-square stable* if for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $E\{|\xi_t|^2\} < \epsilon$, $t > 0$ when $\sup\limits_{-d\le s\le 0} E\{|\xi_s|^2\} < \delta(\epsilon)$. In addition, if $\lim\limits_{t\to\infty} E\{|\xi_t|^2\} = 0$ for any initial conditions, then it is said to be mean-square asymptotically stable.

Definition 2 The *filtering error system* $\mathcal{E}$ in (3) is said to be *mean-square asymptotically stable with an* $\mathcal{H}_\infty$ *disturbance attenuation level* γ if it is mean-square asymptotically stable and under zero-initial conditions $E\{\|e\|_2\} < \gamma\|\omega\|_2$ for all nonzero disturbances $\omega_t \in l_2[0,\infty)$, where

$$E\{\|e\|_2\} \triangleq E\left\{\left(\sum_{t=0}^{\infty} e_t^{\mathrm{T}} e_t\right)^{1/2}\right\}, \quad \|\omega\|_2 \triangleq \left(\sum_{t=0}^{\infty} \omega_t^{\mathrm{T}} \omega_t\right)^{1/2}.$$

Throughout the paper, we make the following assumption.

Assumption 3 System $\mathcal{S}$ in (2) is mean-square asymptotically stable.

Remark 3 Assumption 3 is made based on the fact that there is no control in the system model $\mathcal{S}$ in (1), therefore the original system $\mathcal{S}$ in (1) to be estimated has to be mean-square asymptotically stable, which is a prerequisite for the filtering error system $\mathcal{E}$ in (3) to be mean-square asymptotically stable.

Then the filtering problem to be addressed in this paper is expressed as follows.

Problem RHF (Robust $\mathcal{H}_\infty$ Filtering): Given system $\mathcal{S}$ in (1), develop full-order and reduced-order robust $\mathcal{H}_\infty$ filters of the form $\mathcal{F}$ in (2) such that for all admissible uncertainties, disturbances and time delays the filtering error system $\mathcal{E}$ in (3) is robustly mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ. Filters satisfying this requirement are called robust $\mathcal{H}_\infty$ filters.

Throughout the paper, $(\bar{A}_i, \bar{A}_{di}, \overline{F}_i, \overline{B}_i, \overline{M}_i, \overline{M}_{di}, \overline{C}_i)$ denotes matrices evaluated at each of the vertices of the polytope $\mathcal{R}$. The following lemma will be useful in our derivation.

Lemma 1 *Let Φ_1, Φ_2, Φ_3 and $\Pi > 0$ be given constant matrices with appropriate dimensions. Then, for any scalar $\epsilon > 0$ satisfying $\epsilon I - \Phi_2^{\mathrm{T}}\Pi\Phi_2 > 0$ we have*

$$[\Phi_1 + \Phi_2\Phi_3]^{\mathrm{T}}\Pi[\Phi_1 + \Phi_2\Phi_3] \le \Phi_1^{\mathrm{T}}[\Pi^{-1} - \epsilon^{-1}\Phi_2\Phi_2^{\mathrm{T}}]^{-1}\Phi_1 + \epsilon\Phi_3^{\mathrm{T}}\Phi_3$$

3 Filtering Analysis

This section is concerned with the filtering analysis problem. More specifically, assuming that the matrices (A_F, B_F, C_F) of the filter $\mathcal{F}$ in (2) are already known, we shall study the conditions under which the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ. To ease the exposition of our results, we first consider the stationary case, i.e. $\Omega \in \mathcal{R}$ is fixed. The following theorem shows that the $\mathcal{H}_\infty$ performance of the filtering error system can be guaranteed if there exist some positive definite matrices satisfying certain LMIs. This theorem will play an instrumental role in the filter design problems.

Theorem 1 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ fixed, and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (2) are given. Then the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level bound γ if there exist matrices $P>0$, $Q>0$ and a scalar $\epsilon>0$ satisfying*

$$\begin{bmatrix} -P & 0 & 0 & P\bar{A} & P\bar{A}_d & P\overline{B} & P\overline{F} \\ * & -P & 0 & P\overline{M} & P\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{5}$$

where

$$\Theta_1 \triangleq -P + K^{\mathrm{T}}QK + 2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}}S_1 + T_1^{\mathrm{T}}T_1\right)K,$$
$$\Theta_2 \triangleq -Q + 2\epsilon\left(S_2^{\mathrm{T}}S_2 + T_2^{\mathrm{T}}T_2\right).$$

Proof Let $\mathcal{X}_t \triangleq \{\xi_{t-d}, \xi_{t-d+1}, \ldots, \xi_t\}$, choose a Lyapunov functional candidate for the filtering error system $\mathcal{E}$

$$\begin{gathered} W_t(\mathcal{X}_t) \triangleq W_1 + W_2, \\ W_1 = \xi_t^{\mathrm{T}} P\xi_t, \qquad W_2 = \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} QK\xi_i, \end{gathered} \tag{6}$$

where P, Q are real symmetric positive definite matrices to be determined. Then, along the solution of the filtering error system $\mathcal{E}$ we have

$$\begin{aligned} \mathcal{J} &\triangleq E\{W_{t+1}(\mathcal{X}_{t+1}) \mid \mathcal{X}_t\} - W_t(\mathcal{X}_t) = E\{[W_{t+1}(\mathcal{X}_{t+1}) - W_t(\mathcal{X}_t)] \mid \mathcal{X}_t\} \\ &= E\{\Delta W_1 \mid \mathcal{X}_t\} + E\{\Delta W_2 \mid \mathcal{X}_t\} \end{aligned} \tag{7}$$

where

$$\begin{aligned} E\{\Delta W_1 | \mathcal{X}_t\} &= E\left\{\left(\xi_{t+1}^{\mathrm{T}} P\xi_{t+1} - \xi_t^{\mathrm{T}} P\xi_t\right) \Big| \mathcal{X}_t\right\} \\ &= E\Big\{\Big(\left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t\right]^{\mathrm{T}} P \\ &\quad \times \left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t\right] \\ &\quad + 2\left\{\left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right] v_t\right\}^{\mathrm{T}} P\left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t\right] \\ &\quad + \left\{\left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right] v_t\right\}^{\mathrm{T}} P\left\{\left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right] v_t\right\} - \xi_t^{\mathrm{T}} P\xi_t\Big) \Big| \mathcal{X}_t\Big\}, \end{aligned} \tag{8}$$

$$\begin{aligned} E\{\Delta W_2 | \mathcal{X}_t\} &= E\left\{\left(\sum_{i=t+1-d}^{t} \xi_i^{\mathrm{T}} K^{\mathrm{T}} QK\xi_i - \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} QK\xi_i\right) \Bigg| \mathcal{X}_t\right\} \\ &= E\left\{\left(\xi_t^{\mathrm{T}} K^{\mathrm{T}} QK\xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} QK\xi_{t-d}\right) \mid \mathcal{X}_t\right\}. \end{aligned} \tag{9}$$

Then from (7)–(9), we obtain

$$\begin{aligned}\mathcal{J} = {}& [\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]^{\mathrm{T}} P[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t] \\ & + [\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}]^{\mathrm{T}} P[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}] - \xi_t^{\mathrm{T}} P\xi_t \\ & + \xi_t^{\mathrm{T}} K^{\mathrm{T}} QK\xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} QK\xi_{t-d}. \end{aligned} \tag{10}$$

In addition, using Assumption 1, we have

$$\begin{aligned}\|f(x_t, x_{t-d})\| &\le \|S_1 x_t\| + \|S_2 x_{t-d}\|, \\ \|g(x_t, x_{t-d})\| &\le \|T_1 x_t\| + \|T_2 x_{t-d}\|,\end{aligned}$$

which yields

$$\begin{aligned}\|f(x_t, x_{t-d})\|^2 &\le 2(\|S_1 x_t\|^2 + \|S_2 x_{t-d}\|^2), \\ \|g(x_t, x_{t-d})\|^2 &\le 2(\|T_1 x_t\|^2 + \|T_2 x_{t-d}\|^2).\end{aligned}$$

Then

$$\begin{aligned}\eta^{\mathrm{T}}(x_t, x_{t-d})\eta(x_t, x_{t-d}) &= f^{\mathrm{T}}(x_t, x_{t-d}) f(x_t, x_{t-d}) + g^{\mathrm{T}}(x_t, x_{t-d}) g(x_t, x_{t-d}) \\ &\le 2\left(\|S_1 x_t\|^2 + \|S_2 x_{t-d}\|^2 + \|T_1 x_t\|^2 + \|T_2 x_{t-d}\|^2\right) \\ &= 2\xi_t^{\mathrm{T}} K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K\xi_t + 2\xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) K\xi_{t-d}.\end{aligned} \tag{11}$$

Since (5) implies $\epsilon > 0$ and $\epsilon I - \overline{F}^{\mathrm{T}} P\overline{F} > 0$, by identifying $\Phi_1 = \bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t$, $\Phi_2 = \overline{F}$, $\Phi_3 = \eta(x_t, x_{t-d})$ and $\Pi = P$ in Lemma 1 , we have an upper bound for the first term of $\mathcal{J}$ in (10)

$$\begin{aligned}& [\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]^{\mathrm{T}} P[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t] \\ & \le [\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t]^{\mathrm{T}}\Psi[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t] + \epsilon\eta^{\mathrm{T}}(x_t, x_{t-d})\eta(x_t, x_{t-d}),\end{aligned} \tag{12}$$

where $\Psi = \left[P^{-1} - \epsilon^{-1}\overline{F}\overline{F}^{\mathrm{T}}\right]^{-1}$.

Then from (10)–(12) we have

$$\begin{aligned}\mathcal{J} \le {}& \left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t\right]^{\mathrm{T}}\Psi\left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t\right] \\ & + 2\epsilon\xi_t^{\mathrm{T}} K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K\xi_t + 2\epsilon\xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) K\xi_{t-d} \\ & + \left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right]^{\mathrm{T}} P\left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right] - \xi_t^{\mathrm{T}} P\xi_t \\ & + \xi_t^{\mathrm{T}} K^{\mathrm{T}} QK\xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} QK\xi_{t-d} \\ = {}& \sigma_t^{\mathrm{T}}\Xi\sigma_t,\end{aligned} \tag{13}$$

where

$$\sigma_t = [\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} \quad \omega_t^{\mathrm{T}}\,]^{\mathrm{T}},$$

$$\Xi = \begin{bmatrix} \begin{pmatrix} \bar{A}^{\mathrm{T}}\Psi\bar{A} - P + K^{\mathrm{T}} QK + \overline{M}^{\mathrm{T}} P\overline{M} \\ +2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K \end{pmatrix} & \bar{A}^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}^{\mathrm{T}} P\overline{M}_d & \bar{A}^{\mathrm{T}}\Psi\overline{B} \\ * & \begin{pmatrix} -Q + \bar{A}_d^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}_d^{\mathrm{T}} P\overline{M}_d \\ +2\epsilon\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) \end{pmatrix} & \bar{A}_d^{\mathrm{T}}\Psi\overline{B} \\ * & * & \overline{B}^{\mathrm{T}}\Psi\overline{B} \end{bmatrix}.$$

Therefore, when assuming zero disturbance input $\omega_t = 0$, it follows that

$$\mathcal{J} \leq [\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\,]\,\bar{\Xi}\,[\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\,]^{\mathrm{T}}$$

where

$$\bar{\Xi} = \begin{bmatrix} \begin{pmatrix} \bar{A}^{\mathrm{T}}\Psi\bar{A} - P + K^{\mathrm{T}}QK + \\ 2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}}S_1 + T_1^{\mathrm{T}}T_1\right)K + \overline{M}^{\mathrm{T}}P\overline{M} \end{pmatrix} & \bar{A}^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}^{\mathrm{T}}P\overline{M}_d \\ * & \begin{pmatrix} -Q + \bar{A}_d^{\mathrm{T}}\Psi\bar{A}_d + 2\epsilon\left(S_2^{\mathrm{T}}S_2 + T_2^{\mathrm{T}}T_2\right) \\ +\overline{M}_d^{\mathrm{T}}P\overline{M}_d \end{pmatrix} \end{bmatrix}.$$

By Schur complement [4], LMI (5) implies the negative definiteness of $\bar{\Xi}$, therefore, for $\mathcal{X}_t \neq 0$ we have $\mathcal{J} < 0$, that is,

$$E\left\{W_{t+1}(\mathcal{X}_{t+1})\,|\,\mathcal{X}_t\right\} < W_t(\mathcal{X}_t)$$

which means that there exists $0 < \beta_t < 1$ satisfying

$$E\left\{W_{t+1}(\mathcal{X}_{t+1})|\,\mathcal{X}_t\right\} < \beta_t W_t(\mathcal{X}_t).$$

It is easy to obtain by using this relationship recursively that

$$E\left\{W_t(\mathcal{X}_t)|\,\mathcal{X}_0\right\} < \prod_{i=0}^{t-1}\beta_i W_0(\mathcal{X}_0) \leq \alpha^t W_0(\mathcal{X}_0)$$

where $\alpha = \max_t \beta_t$. Thus $0 < \alpha < 1$ and we have

$$E\left\{\sum_{t=0}^{N}\left[W_t(\mathcal{X}_t)|\,\mathcal{X}_0\right]\right\} < (1 + \alpha + \cdots + \alpha^N)\,W_0(\mathcal{X}_0) = \frac{1-\alpha^{N+1}}{1-\alpha}\,W_0(\mathcal{X}_0).$$

Since $Q > 0$, then

$$\lim_{N\to\infty} E\left\{\sum_{t=0}^{N}\left[x_t^{\mathrm{T}}Px_t\,\middle|\,\mathcal{X}_0\right]\right\} < \frac{1}{1-\alpha}\,W_0(\mathcal{X}_0).$$

Using the Rayleigh quotient inequality, we have

$$\lim_{N\to\infty} E\left\{\sum_{t=0}^{N}\left[x_t^{\mathrm{T}}x_t|\,\mathcal{X}_0\right]\right\} < \frac{1}{(1-\alpha)\lambda_{\min}(P)}\,W_0(\mathcal{X}_0)$$

which means $E\{|x_t|^2\} \to 0$ as $t \to \infty$, then from Definition 1, we know that the filtering error system $\mathcal{E}$ in (3) with $\omega_t = 0$ is mean-square asymptotically stable.

To establish the $\mathcal{H}_\infty$ performance, assume zero initial condition, we have $W_0(\mathcal{X}_0) = 0$. Now consider the following index

$$\mathcal{I} \triangleq E\left\{\sum_{t=0}^{\infty}\left(e_t^{\mathrm{T}}e_t - \gamma^2\omega_t^{\mathrm{T}}\omega_t\right)\right\}. \tag{14}$$

Then, with (13) for all nonzero ω_t we have

$$\mathcal{I} = E\Big\{ \sum_{t=0}^{\infty} \left(e_t^{\mathrm{T}} e_t - \gamma^2 \omega_t^{\mathrm{T}} \omega_t + E\left\{ W_{t+1}(\mathcal{X}_{t+1}) \middle| \mathcal{X}_t \right\} - W_t(\mathcal{X}_t) \right) \Big\} - E\left\{ W_\infty(\mathcal{X}_\infty) \right\}$$
$$\le E\Big\{ \sum_{t=0}^{\infty} \left(e_t^{\mathrm{T}} e_t - \gamma^2 \omega_t^{\mathrm{T}} \omega_t + \mathcal{J} \right) \Big\} = E\Big\{ \sum_{t=0}^{\infty} \sigma_t^{\mathrm{T}} \tilde{\Xi} \sigma_t \Big\}$$

where

$$\tilde{\Xi} = \begin{bmatrix} \begin{pmatrix} \bar{A}^{\mathrm{T}} \Psi \bar{A} - P + K^{\mathrm{T}} Q K \\ + \overline{M}^{\mathrm{T}} P \overline{M} + \overline{C}^{\mathrm{T}} \overline{C} \\ + 2\epsilon K^{\mathrm{T}} \left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1 \right) K \end{pmatrix} & \bar{A}^{\mathrm{T}} \Psi \bar{A}_d + \overline{M}^{\mathrm{T}} P \overline{M}_d & \bar{A}^{\mathrm{T}} \Psi \overline{B} \\ * & \begin{pmatrix} -Q + \bar{A}_d^{\mathrm{T}} \Psi \bar{A}_d + \overline{M}_d^{\mathrm{T}} P \overline{M}_d \\ + 2\epsilon \left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2 \right) \end{pmatrix} & \bar{A}_d^{\mathrm{T}} \Psi \overline{B} \\ * & * & -\gamma^2 I + \overline{B}^{\mathrm{T}} \Psi \overline{B} \end{bmatrix}.$$

Then, by Schur complement, (5) guarantees $\tilde{\Xi} < 0$, which further implies $\mathcal{I} < 0$ and $E\{\|e\|_2\} < \gamma \|\omega\|_2$, then the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ noise attenuation level bound γ, and the proof is completed.

Remark 4 Theorem 1 presents a sufficient condition for the $\mathcal{H}_\infty$ performance of discrete-time stochastic time-delay systems with nonlinear disturbances. It is worth pointing out that the condition presented in Theorem 1 is an LMI condition and therefore can be easily tested by standard numerical software [11]. In the case when we assume $v_t = 0$, that is, no stochastic uncertainty is present in system $\mathcal{S}$, LMI (5) becomes

$$\begin{bmatrix} -P & 0 & P\bar{A} & P\bar{A}_d & P\overline{B} & P\overline{F} \\ * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -\epsilon I \end{bmatrix} < 0. \tag{15}$$

LMI (15) is an $\mathcal{H}_\infty$ performance condition for linear discrete time-delay systems with nonlinear disturbances. In addition, if we further assume $f(x_t, x_{t-d}) = 0$ and $g(x_t, x_{t-d}) = 0$, then LMI (5) becomes

$$\begin{bmatrix} -P & 0 & P\bar{A} & P\bar{A}_d & P\overline{B} \\ * & -I & \overline{C} & 0 & 0 \\ * & * & -P + K^{\mathrm{T}} Q K & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0. \tag{16}$$

LMI (16) is an $\mathcal{H}_\infty$ performance condition for linear discrete time-delay systems.

Then, the following theorem provides a sufficient condition of robust $\mathcal{H}_\infty$ performance for the filtering error system $\mathcal{E}$ in (3).

Theorem 2 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ representing uncertain matrices, and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (2) are given. Then the filtering error system $\mathcal{E}$ in (3) is robustly mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level bound γ if there exist matrices $P_i > 0$, $Q_i > 0$, V and scalars $\epsilon_i > 0$ satisfying*

$$\begin{bmatrix} P_i - V - V^{\mathrm{T}} & 0 & 0 & V^{\mathrm{T}}\bar{A}_i & V^{\mathrm{T}}\bar{A}_{di} & V^{\mathrm{T}}\overline{B}_i & V^{\mathrm{T}}\overline{F}_i \\ * & P_i - V - V^{\mathrm{T}} & 0 & V^{\mathrm{T}}\overline{M}_i & V^{\mathrm{T}}\overline{M}_{di} & 0 & 0 \\ * & * & -I & \overline{C}_i & 0 & 0 & 0 \\ * & * & * & \Pi_1 & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0 \tag{17}$$

$$\forall\, i = 1, \ldots, s,$$

where

$$\Pi_1 = -P_i + K^{\mathrm{T}} Q_i K + 2\epsilon_i K^{\mathrm{T}} \left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1 \right) K,$$
$$\Pi_2 = -Q_i + 2\epsilon_i \left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2 \right).$$

Proof LMIs (17) guarantee that for any fixed $\Omega \in \mathcal{R}$, there exist matrices $P > 0$, $Q > 0$, V and a scalar $\epsilon > 0$ satisfying

$$\begin{bmatrix} P - V - V^{\mathrm{T}} & 0 & 0 & V^{\mathrm{T}}\bar{A} & V^{\mathrm{T}}\bar{A}_d & V^{\mathrm{T}}\overline{B} & V^{\mathrm{T}}\overline{F} \\ * & P - V - V^{\mathrm{T}} & 0 & V^{\mathrm{T}}\overline{M} & V^{\mathrm{T}}\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0. \tag{18}$$

In the following we will show that (18) is equivalent to (5). On one hand, if (5) holds, (18) is readily established by choosing $V = V^{\mathrm{T}} = P$. On the other hand, if (18) holds, we can explore the fact that V is nonsingular. In addition, we have $(P - V)^{\mathrm{T}} P^{-1} (P - V) \geq 0$, which implies that $-V^{\mathrm{T}} P^{-1} V \leq P - V^{\mathrm{T}} - V$. Therefore we can conclude from (18) that

$$\begin{bmatrix} -V^{\mathrm{T}}P^{-1}V & 0 & 0 & V^{\mathrm{T}}\bar{A} & V^{\mathrm{T}}\bar{A}_d & V^{\mathrm{T}}\overline{B} & V^{\mathrm{T}}\overline{F} \\ * & -V^{\mathrm{T}}P^{-1}V & 0 & V^{\mathrm{T}}\overline{M} & V^{\mathrm{T}}\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0. \tag{19}$$

Performing a congruence transformation to (19) by $\mathrm{diag}\left\{ I, V^{-1}P, V^{-1}P, I, I, I, I \right\}$ yields (5), then the proof is completed.

Remark 5 Instead of directly extending Theorem 1 to polytopic uncertain systems based on the notion of quadratic stability, here we incorporate a new result of parameter-dependent stability [6] to reduce the conservatism of filter designs in the quadratic framework. Through the introduction of the slack variable V, the sufficient robust $\mathcal{H}_\infty$ performance condition resulting from Theorem 2 entails different positive definite matrices P_i and Q_i for each vertex of the polytope $\mathcal{R}$, thus enabling us to obtain a parameter-dependent performance criteria. To illustrate the benefit of such performance conditions, let $\bar{\Omega}(\lambda)$ denotes any given point of the polytope $\mathcal{R}$. If we can find feasible solutions in the light of (17), then it is not difficult to show that the Lyapunov matrices defined in (6) for any fixed point $\bar{\Omega}(\lambda)$ can be recovered by

$$P(\lambda) = \sum_{i=1}^{s} \lambda_i P_i, \qquad Q(\lambda) = \sum_{i=1}^{s} \lambda_i Q_i,$$

which implies that there are different Lyapunov functionals for different points in the polytope. Then, the Lyapunov functional defined in (6) for the whole uncertainty domain $\mathcal{R}$ can be expressed as

$$W_t(\mathcal{X}_t, \lambda) = \xi_t^{\mathrm{T}} P(\lambda) \xi_t + \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} Q(\lambda) K \xi_i \tag{20}$$

which is dependent of the parameter λ.

4 Filter Design

In this section we will focus on the design of full-order and reduced-order $\mathcal{H}_\infty$ filters of the form $\mathcal{F}$ based on Theorem 2. That is, to determine the filter matrices (A_F, B_F, C_F) which will guarantee the filtering error system $\mathcal{E}$ to be mean-square asymptotically stable with an $\mathcal{H}_\infty$ performance. The following theorem provides sufficient conditions for the existence of such $\mathcal{H}_\infty$ filters for system $\mathcal{S}$.

Theorem 3 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ representing uncertain matrices. Then an admissible robust $\mathcal{H}_\infty$ filter of the form $\mathcal{F}$ in (2) exists if there exist matrices X, Y, Z, $\bar{A}_F$, $\overline{B}_F$, $\overline{C}_F$, P_{1i}, P_{2i}, P_{3i}, Q_i and scalar $\epsilon_i > 0$ for $i = 1, \dots, s$ satisfying*

$$\begin{bmatrix} \Upsilon_2 & 0 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} & \Upsilon_1 \\ * & \Upsilon_2 & 0 & \Upsilon_5 & \Upsilon_9 & 0 & 0 \\ * & * & -I & \Upsilon_6 & 0 & 0 & 0 \\ * & * & * & \Upsilon_7 & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix} > 0, \tag{22}$$

where

$$\Upsilon_1 = \begin{bmatrix} XF_i & E^{\mathrm{T}} \overline{B}_F G_i \\ Y^{\mathrm{T}} F_i & \overline{B}_F G_i \end{bmatrix},$$

$$
\begin{aligned}
\Upsilon_2 &= \begin{bmatrix} P_{1i} - X - X^{\mathrm{T}} & P_{2i} - Y - E^{\mathrm{T}} Z \\ * & P_{3i} - Z^{\mathrm{T}} - Z \end{bmatrix}, \\
\Upsilon_4 &= \begin{bmatrix} X^{\mathrm{T}} A_i + E^{\mathrm{T}} \overline{B}_F C_i & E^{\mathrm{T}} \bar{A}_F \\ Y^{\mathrm{T}} A_i + \overline{B}_F C_i & \bar{A}_F \end{bmatrix}, \\
\Upsilon_5 &= \begin{bmatrix} X^{\mathrm{T}} M_i + E^{\mathrm{T}} \overline{B}_F N_i & 0 \\ Y^{\mathrm{T}} M_i + \overline{B}_F N_i & 0 \end{bmatrix}, \\
\Upsilon_6 &= \begin{bmatrix} L_i & -\overline{C}_F \end{bmatrix}, \\
\Upsilon_7 &= \begin{bmatrix} -P_{1i} + Q_i + 2\epsilon_i \left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1 \right) & -P_{2i} \\ -P_{2i}^{\mathrm{T}} & -P_{3i} \end{bmatrix}, \\
\Upsilon_8 &= \begin{bmatrix} X^{\mathrm{T}} A_{di} + E^{\mathrm{T}} \overline{B}_F C_{di} \\ Y^{\mathrm{T}} A_{di} + \overline{B}_F C_{di} \end{bmatrix}, \\
\Upsilon_9 &= \begin{bmatrix} X^{\mathrm{T}} M_{di} + E^{\mathrm{T}} \overline{B}_F N_{di} \\ Y^{\mathrm{T}} M_{di} + \overline{B}_F N_{di} \end{bmatrix}, \\
\Upsilon_{10} &= \begin{bmatrix} X^{\mathrm{T}} B_i + E^{\mathrm{T}} \overline{B}_F D_i \\ Y^{\mathrm{T}} B_i + \overline{B}_F D_i \end{bmatrix}, \\
E &= \begin{bmatrix} I_{k \times k} & 0_{k \times (n-k)} \end{bmatrix}.
\end{aligned}
$$

Moreover, if the above condition has a set of feasible solution $(X, Y, Z, \bar{A}_F, \overline{B}_F, \overline{C}_F, P_{1i}, P_{2i}, P_{3i}, Q_i, \epsilon_i)$*, the matrices for an admissible robust* $\mathcal{H}_\infty$ *filter in the form of* $\mathcal{F}$ *in (2) can be calculated by the following steps:*

(1) *find square and nonsingular matrices* $S \in R^{k \times k}$ *and* $T \in R^{k \times k}$ *satisfying* $Z = S^{\mathrm{T}} T^{-1} S$*;*
(2) *calculate the matrices for desired filter matrices by*

$$
\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} S^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F & \overline{B}_F \\ \overline{C}_F & 0 \end{bmatrix} \begin{bmatrix} S^{-1} T & 0 \\ 0 & I \end{bmatrix}. \tag{23}
$$

Proof Since LMIs (21) and (22) implies $P_{3i} - Z - Z^{\mathrm{T}} < 0$ and $P_{3i} > 0$, we can infer that $Z + Z^{\mathrm{T}} > 0$, therefore Z is nonsingular. Then we can always find square and nonsingular $k \times k$ matrices S and T satisfying $Z = S^{\mathrm{T}} T^{-1} S$. Therefore, the matrices (A_F, B_F, C_F) are uniquely defined in (23). Now introduce the following matrix variables:

$$
J = \begin{bmatrix} I & 0 \\ 0 & T^{-1} S \end{bmatrix}, \qquad V = \begin{bmatrix} X & Y S^{-1} T \\ S E & T \end{bmatrix}, \qquad P_i = J^{-T} \begin{bmatrix} P_{1i} & P_{2i} \\ P_{2i}^{\mathrm{T}} & P_{3i} \end{bmatrix} J^{-1}. \tag{24}
$$

Then, it is easy to see that the matrix J defined above is nonsingular and we have $P_i > 0$. In the following we will prove that the filter $\mathcal{F}$ in (2) with state-space realization (A_F, B_F, C_F) defined in (23) is an admissible robust $\mathcal{H}_\infty$ filter such that the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance.

Now, by some algebraic matrix manipulations, it can be established that (21) is equivalent to

$$\begin{bmatrix} \begin{matrix} J^{\mathrm{T}}(P_i - \\ V - V^{\mathrm{T}})J \end{matrix} & 0 & 0 & J^{\mathrm{T}}V^{\mathrm{T}}\bar{A}_iJ & J^{\mathrm{T}}V^{\mathrm{T}}\bar{A}_{di} & J^{\mathrm{T}}V^{\mathrm{T}}\overline{B}_i & J^{\mathrm{T}}V^{\mathrm{T}}\overline{F}_i \\ * & \begin{matrix} J^{\mathrm{T}}(P_i - \\ V - V^{\mathrm{T}})J \end{matrix} & 0 & J^{\mathrm{T}}V^{\mathrm{T}}\overline{M}_iJ & J^{\mathrm{T}}V^{\mathrm{T}}\overline{M}_{di} & 0 & 0 \\ * & * & -I & \overline{C}_iJ & 0 & 0 & 0 \\ * & * & * & J^{\mathrm{T}}\Pi_1J & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0. \tag{25}$$

The equivalence between (21) and (25) can be verified in a reverse order by the following steps. First, by substituting (A_F, B_F, C_F) defined in (23) into (4), the matrices $\big(\bar{A}, \bar{A}_d, \overline{F}, \overline{B}, \overline{M}, \overline{M}_d, \overline{C}\big)$ of the filtering error system $\mathcal{E}$ in (3) can be obtained as

$$\bar{A} = \begin{bmatrix} A & 0 \\ S^{-T}\overline{B}_FC & S^{-T}\bar{A}_FS^{-1}T \end{bmatrix}, \qquad \bar{A}_d = \begin{bmatrix} A_d \\ S^{-T}\overline{B}_FC_d \end{bmatrix},$$

$$\overline{F} = \begin{bmatrix} F & 0 \\ 0 & S^{-T}\overline{B}_FG \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B \\ S^{-T}\overline{B}_FD \end{bmatrix}, \qquad \overline{M} = \begin{bmatrix} M & 0 \\ S^{-T}\overline{B}_FN & 0 \end{bmatrix}, \tag{26}$$

$$\overline{M}_d = \begin{bmatrix} M_d \\ S^{-T}\overline{B}_FN_d \end{bmatrix}, \qquad \overline{C} = [\,L \quad -\overline{C}_FS^{-1}T\,].$$

Then by substituting the matrices J, P_i, V defined in (24) and the matrices $(\bar{A}, \bar{A}_d, \overline{F}, \overline{B}, \overline{M}, \overline{M}_d, \overline{C})$ given by (26) into (25), and by considering the relationship $Z = S^{\mathrm{T}}T^{-1}S$, we obtain inequality (21) after some straightforward matrix manipulations.

Now, performing a congruence transformation to (25) by $\mathrm{diag}\{J^{-1}, J^{-1}, I, J^{-1}, I, I, I\}$ yields (17). Therefore, we conclude from Theorem 2 that the filter $\mathcal{F}$ in (2) with state-space realization (A_F, B_F, C_F) defined in (24) is an admissible robust $\mathcal{H}_\infty$ filter such that the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance, and the proof is completed.

Remark 6 To obtain certain LMI conditions for the existence of desired filters, usually linearization procedures have to be adopted. Since the standard linearization methods adopted in [25, 27] assume the off-diagonal entry of certain matrix (the matrix to be partitioned, in this paper it is V in Theorem 2) to be square and nonsingular, they can only be used to deal with the full-order filtering problem. To keep the reduced-order filter design tractable, here we have sought a different linearization procedure, which solves both the full-order and reduced-order filtering synthesis problems in a unified framework. It is worth noting that the matrix E defined in Theorem 3 plays an instrumental role. For the full-order filtering, the matrix E becomes an identity matrix of dimension n, and for the reduced-order case, we have imposed certain structural restriction on the $(2,1)$ block entry of the matrix V, which introduces some overdesign into the filter design.

Remark 7 Theorem 3 casts the robust $\mathcal{H}_\infty$ filtering problem into an LMI feasibility test, and any feasible solution to the conditions presented in Theorem 3 will yield a suitable filter, which can be obtained by following the two steps presented in Theorem 3. Another formulation of suitable filters upon these feasible solution can be given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} Z^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F & \overline{B}_F \\ \overline{C}_F & 0 \end{bmatrix}. \tag{27}$$

To prove (27), let us denote the filter z transfer function from $y(t)$ to $\hat{z}(t)$ by $T_{\hat{z}y}(z) = C_F(zI - A_F)^{-1}B_F$. By substituting the filter matrices with (23) and by considering the relationship $Z = S^{\mathrm{T}}T^{-1}S$, we have

$$\begin{aligned} T_{\hat{z}y}(s) &= \overline{C}_F S^{-1}T(zI - S^{-T}\bar{A}_F S^{-1}T)^{-1}S^{-T}\overline{B}_F \\ &= \overline{C}_F(zI - Z^{-1}\bar{A}_F)^{-1}Z^{-1}\overline{B}_F. \end{aligned}$$

Therefore, an admissible filter can also be given by (27).

Remark 8 Note that (21) and (22) are LMIs not only over the matrix variables, but also over the scalar γ^2. This implies that the scalar γ^2 can be included as an optimization variable to obtain the minimum noise attenuation level bound. Then the minimum (in terms of the feasibility of (21) and (22)) guaranteed cost of robust $\mathcal{H}_\infty$ filters can be readily found by solving the following convex optimization problems

Problem RHFD (Robust $\mathcal{H}_\infty$ filter design): Minimize γ subject to (21) and (22) over $(X, Y, Z, \bar{A}_F, \overline{B}_F, \overline{C}_F, P_{1i}, P_{2i}, P_{3i}, Q_i, \epsilon_i)$.

Remark 9 Theorem 3 presents a sufficient condition for the existence of robust $\mathcal{H}_\infty$ filters for discrete-time stochastic time-delay systems with nonlinear disturbance. In the case when we assume $v_t = 0$, that is, no stochastic uncertainty is present in system $\mathcal{S}$, LMI (21) becomes

$$\begin{bmatrix} \Upsilon_2 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} & \Upsilon_1 \\ * & -I & \Upsilon_6 & 0 & 0 & 0 \\ * & * & \Upsilon_7 & 0 & 0 & 0 \\ * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0.$$

In addition, if we further assume $f(x_t, x_{t-d}) = 0$ and $g(x_t, x_{t-d}) = 0$, then LMI (21) becomes

$$\begin{bmatrix} \Upsilon_2 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} \\ * & -I & \Upsilon_6 & 0 & 0 \\ * & * & \begin{bmatrix} -P_{1i} + Q_i & -P_{2i} \\ -P_{2i}^{\mathrm{T}} & -P_{3i} \end{bmatrix} & 0 & 0 \\ * & * & * & -Q_i & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0.$$

5 Illustrative Example

In this section, we will provide an example to illustrate the applicability of the above filter design method. Consider the following system:

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 + 0.01\alpha \\ 0 & 0.8187 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} \omega_t \\ &\quad + \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} x_t v_t, \\ y_t &= [\,0.2 \quad 0.1 \quad 0.1 + 0.01\alpha\,]\, x_t + [\,0.1 \quad 0.1 + 0.01\alpha \quad 0\,]\, x_{t-d} \\ &\quad + 0.2 \sin([\,0 \quad 0 \quad 0.2\,]\, x_t + [\,0 \quad 0.1 \quad 0\,]\, x_{t-d}) + 0.1\omega_t, \\ z_t &= [\,0 \quad 0.1 \quad 0.2\,]\, x_t, \end{aligned} \tag{28}$$

where α is an unknown parameter satisfying $-1 \leq \alpha \leq 1$. It is easy to see that system (28) has the structure of system $\mathcal{S}$ in (1) with the following parameters:

$$
\begin{aligned}
A &= \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 + 0.01\alpha \\ 0 & 0.8187 & 0 \end{bmatrix}, \\
M &= \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \\
A_d &= 0_{3\times 3}, \quad F = 0_{3\times 1}, \quad M_d = 0_{3\times 3}, \\
C &= [\,0.2 \quad 0.1 \quad 0.1 + 0.01\alpha\,], \\
C_d &= [\,0.1 \quad 0.1 + 0.01\alpha \quad 0\,], \\
G &= 0.2, \quad D = 0.1, \quad N = 0_{1\times 3}, \quad N_d = 0_{1\times 3}, \\
L &= [\,0 \quad 0.1 \quad 0.2\,], \\
f(x_t, x_{t-d}) &= 0 \\
g(x_t, x_{t-d}) &= 0.2 \sin\left([\,0 \quad 0 \quad 0.2\,]\, x_t + [\,0 \quad 0.1 \quad 0\,]\, x_{t-d}\right).
\end{aligned}
$$

In addition, the nonlinear functions $f(x_t, x_{t-d})$ and $g(x_t, x_{t-d})$ satisfy Assumption 1 with

$$S_1 = S_2 = 0_{1\times 3}, \quad T_1 = [\,0 \quad 0 \quad 0.2\,], \quad T_2 = [\,0 \quad 0.1 \quad 0\,].$$

By solving Problem RHFD, the obtained minimum feasible γ^* and the associated matrices for different cases are as follows:

Third-order Filtering: ($\gamma^* = 0.0200$)

$$
\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{ccc:c} 0.1864 & 1.3287 & 0.1981 & -3.5599 \\ -0.0268 & 0.9945 & 0.0077 & -0.1232 \\ -0.0132 & 0.2543 & 0.0391 & -0.1472 \\ \hdashline 0.0001 & -0.0999 & -0.2048 & 0 \end{array}\right]. \tag{29}
$$

Second-order Filtering: ($\gamma^* = 0.0226$)

$$
\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc:c} 0.9669 & 0.1947 & -0.0624 \\ 0.0002 & 0.9353 & -0.0084 \\ \hdashline -0.0011 & -0.1400 & 0 \end{array}\right]. \tag{30}
$$

First-order Filtering: ($\gamma^* = 0.0228$)

$$
\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{c:c} 0.9589 & -0.1801 \\ \hdashline -0.0031 & 0 \end{array}\right]. \tag{40}
$$

6 Concluding Remarks

The problem of robust $\mathcal{H}_\infty$ filtering for a class of stochastic nonlinear time-delay systems in discrete time has been investigated in this paper. Sufficient conditions are obtained in terms of linear matrix inequality for the existence of desired filters which guarantee the filtering error system to be mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level. A parametrization of the filter matrices can be readily obtained if these conditions have feasible solutions. A numerical example is provided to show the applicability of the developed filter design methods.

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Robust Adaptive Control for a Class of Nonlinear Stochastic Time-delay Systems

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Abstract: The adaptive control problem of a class of stochastic time-delay systems is investigated. Firstly we consider a simple class of stochastic systems with time-varying delays and design the corresponding adaptive controller based on the solution of linear matrix inequalities (LMIs), which can render the closed-loop asymptotically stable in probability. Then we apply the adaptive idea to the interconnected system case. Under the condition that interconnections satisfy the matching condition, we propose a class of decentralized feedback controllers and the corresponding closed-loop systems are also asymptotically stable in probability. Numerical examples on controlling the two classes of stochastic systems are given to show the validity of obtained theoretical results.

Keywords: *Stochastic systems; time-delay systems; interconnected systems; adaptive control.*

Mathematics Subject Classification (2000): 93E15, 93C23, 93D09.

1 Introduction

Time-delay is often encountered in various engineering systems, such as electrical networks, turbojet engines, microwave oscillators, nuclear reactors, rolling mills, chemical processes, manual control, long transmission lines in pneumatic, and hydraulic systems, etc. Its existence is often a source of instability and poor performance. Therefore, the problem of stability analysis and robust control for dynamic time-delay systems has attracted considerable attention of a number of researchers over the past years, see for example, $[1-4]$ and the references therein.

In this paper we will focus on controlling stochastic time-delay systems. In the existing literature, some work has been done on stability analysis and control for stochastic time-delay systems. The robust stability problem of linear stochastic time-delay systems was studied in [5], while robust stability analysis for stochastic delay interval systems is considered in [6]. In [7], the problem of control for uncertain stochastic time-delay systems was considered, and the results were given in the form of LMIs. Filtering problem for uncertain stochastic systems was considered in [8 − 10]. In the meantime, the problem of control for interconnected stochastic time-delay systems was tackled in [11].

Unlike the existing results in literature, in this paper, we investigate the adaptive control problem of stochastic time-delay systems, whose bounds of uncertainties in matching parts are not required to be known. Firstly we consider a simple class of stochastic systems with time-varying delays. Corresponding adaptive controller is designed based on the solution of LMI. Then we apply the adaptive idea to the interconnected system case. Under the condition that interconnections satisfy the matching condition, we propose a class of decentralized feedback controllers, which can render the closed-loop systems asymptotically stable.

2 Problem Formulation

Consider the following time delay system

$$\begin{gathered} dx = (Ax + f(x, x(t - d(t)) + Bu)\, dt + g(x, x(t - h(t)))\, dw, \\ x(t) = \varphi(t), \qquad t \in [-d, 0]. \end{gathered} \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are the state and control input respectively, $d(t)$ and $h(t)$ are time-varying delay parameters, A and B are known constant matrices with appropriate dimensions. w is a zero-mean Wiener process. $f(\cdot)$ and $g(\cdot)$ are uncertain nonlinear function vectors.

For system (1), we introduce the following standard assumptions.

Assumption 2.1 The time-varying time delays $d(t)$ satisfies

$$\dot{d}(t) \le \tau < 1, \qquad \dot{h}(t) \le k < 1. \tag{2}$$

Assumption 2.2 The nonlinear function $f(\cdot)$ can be decomposed into the matched form and the unmatched form

$$f(x, x(t - d(t))) = B\xi(x, x(t - d(t))) + \zeta(x, x(t - d(t))), \tag{3}$$

where $\xi(x, x(t - d(t)))$ and $\zeta(x, x(t - d(t)))$ satisfy

$$\|\xi(x, x(t - d(t)))\| \le \beta_1 \|x\| + \beta_2 \|x(t - d(t))\|, \tag{4}$$

$$\|\zeta(x, x(t - d(t)))\| \le \gamma_1 \|x\| + \gamma_2 \|x(t - d(t))\|, \tag{5}$$

where γ_1 and γ_2 are known positive scalars, β_1 and β_2 are unknown positive scalars.

Assumption 2.3 There exist matrix Y, positive matrix X and positive scalars ε_1 and ε_2 such that the following LMI holds

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} + \varepsilon_1\gamma_1^2 I + \frac{\varepsilon_2}{1-\tau}\gamma_2^2 I & X & X \\ X & -\varepsilon_1 I & 0 \\ X & 0 & -\varepsilon_2 I \end{bmatrix} < 0. \tag{6}$$

Assumption 2.4 The nonlinear function g satisfies

$$\begin{aligned} g^{\mathrm{T}}Pg \le {} & \alpha_2 \left\|B^{\mathrm{T}}Px\right\| \|x\,(t-h\,(t))\| + \alpha_3 \left\|B^{\mathrm{T}}Px\,(t-h\,(t))\right\| \|x\| \\ & + \alpha_1 \left\|B^{\mathrm{T}}Px\right\| \|x\| + \alpha_4 \left\|B^{\mathrm{T}}Px\,(t-h\,(t))\right\| \|x\,(t-h\,(t))\|, \end{aligned} \tag{7}$$

where matrix $P = X^{-1}$, X satisfies LMI (6), α_i $(i = 1,2,3,4)$ are unknown positive scalars.

Remark 1 Assumption 2.1 is often needed on investigating time-delay systems by employing Lyapunov-Krasovskii method. Different from the existing literatures on control of stochastic time-delay systems, we divide the uncertainties into matched and unmatched parts and the bounds of matched parts are not needed to be known in Assumption 2.2. Assumption 2.3 is to guarantee that the system is asymptotically stable without the matching parts and the stochastic parts. In practical systems we may also not know the function g exactly, so Assumption 2.4 is imposed.

Before giving the problem statement in this paper, we first introduce the following definition of stability in probability.

Consider the nonlinear stochastic system

$$dx = f(x, x(t-d))dt + g(x, x(t-d))dw, \tag{8}$$

where $x \in R^n$ is the state, w is an r-dimensional standard Wiener process, and functions f and g are locally Lipschitz and satisfy $f(0,0) = 0$ and $g(0,0) = 0$.

Definition 2.1 [7] The equilibrium $x = 0$ of the system (8) is said to be globally asymptotically stable in probability for given $x(t)$ if for any $s \ge 0$ and $\varepsilon > 0$

$$\lim_{x\to 0} P\left\{\sup_{s<t} |x_t^{s,x}| > \varepsilon\right\} = 0, \qquad P\left\{\lim_{t\to+\infty} |x_t^{s,x}| = 0\right\} = 1,$$

where $x_t^{s,x}$ denotes the solution at time t of a stochastic differential equation starting from the state x at time s for $s \le t$.

Lemma 2.1 [12] *Consider system (8) and suppose there exists a positive definite, radially unbounded, twice continuously differentiable function $V(x)$ such that the following inequality holds*

$$LV(x) = \frac{\partial V(x)}{\partial x}\, f(x) + \frac{1}{2}\,\mathrm{tr}\left[g(x)^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g(x)\right] < 0,$$

then system (8) is globally asymptotically stable in probability.

In this paper, we will firstly consider designing controller to render system (1) globally asymptotically stable in probability under the above assumptions, then further apply the design idea to interconnected stochastic system case and design the corresponding controller.

3 Robust Controller Design

In this section we will investigate designing adaptive state feedback controller to stabilize uncertain stochastic system (1).

Theorem 3.1 *For system (1), the following adaptive state feedback controller*

$$u = Kx - \frac{1}{2}\theta(t)B^{\mathrm{T}}Px, \tag{9}$$

where $K = YX^{-1}$, and matrices P, Y and X satisfy (6), $\theta(t)$ is adaptive parameter whose adaptive law is

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2, \tag{10}$$

where a is an arbitrary positive scalar, can render the closed-loop system robustly stable in probability.

Proof Substituting (9) into (1), we can obtain

$$dx = \left(Ax + f(x, x(t-d(t)) + BK - \frac{1}{2}B\theta(t)B^{\mathrm{T}}Px)\right)dt + g(x, x(t-h(t)))dw. \tag{11}$$

Choose the following Lyapunov–Krasovskii function

$$\begin{aligned} V = x^{\mathrm{T}}Px + \frac{1}{2}a^{-1}\widetilde{\theta}^{\mathrm{T}}\widetilde{\theta} + \left(\delta_6 + \varepsilon_2^{-1}\right)\int\limits_{t-d(t)}^{t}\|x(\xi)\|^2 d\xi \\ + \left(\frac{\delta_2}{1-k} + \frac{\delta_4}{1-k}\right)\int\limits_{t-h(t)}^{t}\|x(\xi)\|^2 d\xi \\ + \left(\frac{\alpha_3^2}{4\delta_3(1-k)} + \frac{\alpha_4^2}{4\delta_4(1-k)}\right)\int\limits_{t-h(t)}^{t}\|B^{\mathrm{T}}Px(\xi)\|^2 d\xi, \end{aligned} \tag{12}$$

where δ_i, $(i = 1, 2, \ldots, 6)$ are positive scalars, $\tilde{\theta} = \theta - \hat{\theta}(t)$, $\hat{\theta}$ is a positive scalar defined in (18).

Taking the time derivative of above Lyapunov function, one can get

$$\begin{aligned} LV \le{}& 2x^{\mathrm{T}}P\left(Ax + f\left(x, x\left(t-d\right)\right) + BK\right) - x^{\mathrm{T}}PB\theta\left(t\right)B^{\mathrm{T}}Px \\ &+ g\left(x, x\left(t-h\right)\right)^{\mathrm{T}}Pg\left(x, x\left(t-h\right)\right) + a^{-1}\tilde{\theta}\dot{\tilde{\theta}} \\ &+ \left(\delta_6 + \varepsilon_2^{-1}\right)\left[\|x\|^2 - (1-\tau)\left\|x\left(t-d\left(t\right)\right)\right\|^2\right] \\ &+ \frac{1}{1-k}\left(\delta_2 + \delta_4\right)\left(\|x\|^2 - (1-k)\left\|x(t-h(t))\right\|^2\right) \\ &+ \frac{1}{(1-k)}\left(\frac{\alpha_3^2}{4\delta_3} + \frac{\alpha_4^2}{4\delta_4}\right)\left(\left\|B^{\mathrm{T}}Px\left(t\right)\right\|^2 - (1-k)\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2\right). \end{aligned} \tag{13}$$

From Assumption 2.4, we obtain that

$$\begin{aligned} g^{\mathrm{T}}Pg \le{}& \alpha_2\left\|B^{\mathrm{T}}Px\right\|\left\|x\left(t-h\left(t\right)\right)\right\| + \alpha_3\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|\|x\| \\ &+ \alpha_1\left\|B^{\mathrm{T}}Px\right\|\|x\| + \alpha_4\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|\left\|x\left(t-h\left(t\right)\right)\right\| \\ \le{}& \frac{\alpha_1^2}{4\delta_1}\left\|B^{\mathrm{T}}Px\right\|^2 + \delta_1\|x\|^2 + \frac{\alpha_2^2}{4\delta_2}\left\|B^{\mathrm{T}}Px\right\|^2 + \delta_2\left\|x\left(t-h\left(t\right)\right)\right\|^2 \\ &+ \frac{\alpha_3^2}{4\delta_3}\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2 + \delta_3\|x\|^2 \\ &+ \frac{\alpha_4^2}{4\delta_4}\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2 + \delta_4\left\|x\left(t-h\left(t\right)\right)\right\|^2. \end{aligned} \tag{14}$$

We know

$$\begin{aligned}
&2x^{\mathrm{T}}P(A+f\,(x,\,x(t-d))+BK)\\
&= x^{\mathrm{T}}\left(PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P\right)x+2x^{\mathrm{T}}PB\xi\left(x,x\left(t-d\left(t\right)\right)\right)\\
&\quad +2x^{\mathrm{T}}P\zeta\left(x,x\left(t-d\right)t\right))\\
&\le x^{\mathrm{T}}\left(PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P\right)x+\frac{\beta_1^2}{\delta_5}\,x^{\mathrm{T}}PBB^{\mathrm{T}}Px+\delta_5\left\|x\right\|^2 \qquad (15)\\
&+\frac{\beta_2^2}{\left(1-\tau\right)\delta_6}\,x^{\mathrm{T}}PBB^{\mathrm{T}}Px+\left(1-\tau\right)\delta_6\left\|x\left(t-d\left(t\right)\right)\right\|^2+\varepsilon_1\gamma_1^2x^{\mathrm{T}}PPx\\
&+\varepsilon_1^{-1}\left\|x\right\|^2+\frac{\varepsilon_2}{\left(1-\tau\right)}\gamma_2^2x^{\mathrm{T}}PPx+\left(1-\tau\right)\varepsilon_2^{-1}\left\|x\left(t-d\left(t\right)\right)\right\|^2.
\end{aligned}$$

Substituting (14), (15) into (13), we can further obtain that

$$LV \le -x^{\mathrm{T}}\Phi x+\left(\widehat{\theta}-\theta\right)\left\|B^{\mathrm{T}}Px\right\|^2+a^{-1}\tilde{\theta}\dot{\tilde{\theta}}, \qquad (16)$$

where

$$\begin{aligned}
-\Phi &= PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P+\varepsilon_1\gamma_1^2PP+\varepsilon_1^{-1}I+\frac{\varepsilon_2}{\left(1-\tau\right)}\,\gamma_2^2PP\\
&\quad +\varepsilon_2^{-1}I+\delta_1+\frac{1}{1-k}\delta_2+\delta_3+\frac{1}{1-k}\,\delta_4+\delta_5+\delta_6,
\end{aligned} \qquad (17)$$

$$\widehat{\theta}=\frac{\beta_1^2}{\delta_5}+\frac{\beta_2^2}{\delta_6\left(1-\tau\right)}+\frac{\alpha_1^2}{4\delta_1}+\frac{\alpha_2^2}{4\delta_2}+\frac{\alpha_3^2}{4\delta_3\left(1-k\right)}+\frac{\alpha_4^2}{4\delta_4\left(1-k\right)}. \qquad (18)$$

As we know if LMI (6) holds, the following inequality stands

$$AX+XA^{\mathrm{T}}+BY+Y^{\mathrm{T}}B^{\mathrm{T}}+\varepsilon_1\gamma_1^2I+\frac{\varepsilon_2}{1-\tau}\,\gamma_2^2I+\varepsilon_1^{-1}X^{\mathrm{T}}X+\varepsilon_2^{-1}X^{\mathrm{T}}X<0. \qquad (19)$$

Further, the following inequality holds (by multiply P on both sides of (19) with $P=X^{-1}$)

$$PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P+\left(\varepsilon_1\gamma_1^2+\frac{\varepsilon_2}{1-\tau}\gamma_2^2\right)PP+\varepsilon_1^{-1}I+\varepsilon_2^{-1}I<0. \qquad (20)$$

Therefore, from (17) and (20) we know there always exist sufficiently small positive scalars δ_i $(\,i=1,2,\dots,6)$ such that

$$\Phi>0. \qquad (21)$$

Substituting (10) into (16), we can obtain

$$LV \le -x^{\mathrm{T}}\Phi x \qquad (22)$$

which implies that the closed-loop system is robustly stable in probability.

Corollary 3.1 *If Assumptions 2.1, 2.4 and Assumption 2.2 with $\zeta(\cdot) = 0$ are satisfied, and the pair $(A,\, B)$ is completely controllable, the following controller*

$$u = -\frac{1}{2}\,\theta(t)B^{\mathrm{T}}Px \tag{23}$$

with adaptive law

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2, \tag{24}$$

where a is a positive scalar, will render the closed-loop system (1) robustly stable in probability.

Proof If $(A,\, B)$ are completely controllable, for a given positive matrix Ω there always exist positive scalar μ such that the following Riccati equality

$$PA + A^{\mathrm{T}}P - \mu PBB^{\mathrm{T}}P = -\Omega \tag{25}$$

has positive matrix solution P. From the above proof, we can design the following controller

$$u = -\frac{1}{2}\,\mu B^{\mathrm{T}}Px - \frac{1}{2}\,\Theta(t)B^{\mathrm{T}}Px \tag{26}$$

with adaptive law

$$\frac{d\Theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2. \tag{27}$$

Further we let $\theta(t) = \Theta(t) + \mu$, where μ is a positive scalar. Thus the controller (26), (27) will give us the desired result.

Corollary 3.2 *If $B = I$ (I is an identity matrix) and Assumption 2.1 holds, the following controller*

$$u_i = -\frac{1}{2}\,\Theta(t)x$$

with adaptive law

$$\frac{d\Theta(t)}{dt} = a\|x\|^2$$

will render the closed-loop system (1) robustly stable in probability.

Proof If $B = I$, it is easy to see $(A,\, B)$ are completely controllable and Assumption 2.4 is satisfied. Therefore, we can design the required adaptive controller to achieve our goal.

Remark 3.1 In the designed controller, we adopt the adaptive law (10). In fact, we can also use the σ-modification adaptive law, that is (10) can be changed into

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2 - \sigma\theta(t), \tag{28}$$

where σ is an adjustable parameter. Compared with the adaptive law (10), the modified adaptive control law (28) can improve the robust performance for the closed-loop systems. Similar to the proof of above, we can also obtain the closed-loop system (1) and (28) is uniformly ultimately bounded stable, and the bounds of the steady-state can be adjusted to be sufficiently small by selecting small parameter σ [4].

4 Control of Interconnected Time Delay Systems

In this section, we investigate a class of interconnected stochastic time-delay systems. A controller is designed to stabilize the underlying system. Different from the literature, instead of using bounds of uncertainties to design the controller, we assume all the bounds unknown. Therefore, the proposed adaptive decentralized feedback controller can be applied to stabilization of a large class of interconnected time-delay systems.

Consider the following interconnected systems whose i-th subsystem is described by

$$\begin{aligned} dx_i &= (A_i x_i + B_i u_i)\,dt + f_i(x_i, x_1, x_2, \dots, x_n, x_1(t-d_{i1}(t)), \dots, x_n(t-d_{in}(t))))\,dt \\ &\quad + g_i(x_i, x_1, x_2, \dots, x_n, x_1(t-h_{i1}(t), \dots, x_n(t-h_{in}(t))))\,dw, \\ &\qquad i = 1, 2, \dots, N. \end{aligned} \tag{29}$$

We impose the following assumptions on system (29).

Assumption 4.1 For $i,\, j = 1, 2, ..., N$, the time-varying time delays satisfy

$$\dot{d}_{ij}(t) \le \tau_j < 1, \qquad \dot{h}_{ij}(t) \le k_j < 1. \tag{30}$$

Assumption 4.2 For $i,\, j = 1, 2, ..., N$ and given $Q_i > 0$, there exist matrix $P_i > 0$ and scalar $\sigma_i > 0$ such that the following equality holds

$$P_i A_i + A_i P_i - \sigma_i P_i B_i B_i^{\mathrm{T}} P_i = -Q_i. \tag{31}$$

Assumption 4.3 For $i = 1, 2, ..., N$, the nonlinear functions $f_i(\cdot)$ satisfy matching condition

$$f_i(\cdot) = B_i \xi_i(\cdot), \tag{32}$$

where $\xi_i(\cdot)$ satisfies

$$\|\xi_i(\cdot)\| \le \sum_{j=1}^{N} \left(\rho_{ij}\,\|x_j\| + \varphi_{ij}\,\|x_j\,(t - d_{ij}\,(t))\|\right). \tag{33}$$

Here ρ_{ij} and φ_{ij} are unknown positive scalars, $i,\, j = 1, 2, ..., N$.

Assumption 4.4 The following inequalities hold

$$\begin{aligned} g_i(\cdot)^{\mathrm{T}} P_i g_i(\cdot) &\le \sum_{j=1}^{N} \left\|B_i^{\mathrm{T}} P_i x_i\right\| \left(\phi_{ij}\,\|x_j\| + \overline{\phi}_{ij}\,\|x_j\,(t - h_{ij}\,(t))\|\right) \\ &\quad + \sum_{j=1}^{N} \left\|B_i^{\mathrm{T}} P_i x_i\,(t - h_{ij})\right\| \left(\psi_{ij}\,\|x_j\| + \overline{\psi}_{ij}\,\|x_j\,(t - h_{ij}\,(t))\|\right), \end{aligned} \tag{34}$$

where ϕ_{ij}, $\overline{\phi}_{ij}$, ψ_{ij} and $\overline{\psi}_{ij}$ are positive scalars, $i, j = 1, 2, ..., N$.

Now we are ready to present our main result in this paper.

Theorem 4.1 *For interconnected stochastic systems (29) under Assumptions 4.1–4.4, the following decentralized feedback controller, for $i = 1, 2, ..., N$,*

$$u_i = -\frac{1}{2}\,\Theta_i\,(t)\,B_i^{\mathrm{T}} P_i x_i \tag{35}$$

with adaptive law

$$\frac{d\Theta_i\,(t)}{dt} = a_i \left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 \tag{36}$$

will render the closed-loop system robustly stable in probability, where a_i is a positive scalar.

Proof Choose the following Lyapunov function

$$\begin{aligned} V = {} & \sum_{i=1}^{N} V_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-\tau_j}\,\delta_{2j} \int_{t-d_{ij}}^{t} \|x_j(\zeta)\|^2 d\zeta + \sum_{i=1}^{N} \frac{1}{2}\,a_i^{-1}\overline{\Theta}_i(t)^2 \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{4j}+\delta_{6j}) \int_{t-h_{ij}}^{t} \|x_j(\zeta)\|^2 d\zeta \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2) \int_{t-h_{ij}}^{t} \|B_i^{\mathrm{T}} P_i x_i(\xi)\|^2 d\xi, \end{aligned} \tag{37}$$

where δ_{sj} $(s \in [1, 6],\ j \in [1, N])$ are positive scalars and

$$\begin{aligned} V_i &= x_i^{\mathrm{T}} P_i x_i, \\ \overline{\Theta}_i\,(t) &= \widehat{\Theta}_i - \Theta_i\,(t)\,, \end{aligned} \tag{38}$$

$\widehat{\Theta}_i$ is defined in (44) (below).

Taking the derivative of V with respect to time t, along the closed-loop system, we obtain

$$\begin{aligned} LV = {} & \sum_{i=1}^{N} LV_i + \sum_{i=1}^{N} a_i \overline{\Theta}_i\,(t)\,\dot{\overline{\Theta}}_i\,(t) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-\tau_j}\delta_{2j} \left(\|x_j\,(t)\|^2 - (1-\tau_j)\,\|x_j\,(t-d_{ij}\,(t))\|^2 \right) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{4j}+\delta_{6j}) \left(\|x_j\,(t)\|^2 - (1-\tau_j)\,\|x_j\,(t-h_{ij}\,(t))\|^2 \right) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j} \left(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2 \right) \\ & \times \left(\left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 - (1-k_j)\,\left\| B_i^{\mathrm{T}} P_i x_i\,(t-h_{ij}\,(t)) \right\|^2 \right). \end{aligned} \tag{39}$$

We know

$$LV_i = 2x_i^{\mathrm{T}} P_i \left(A_i x_i + B_i u_i + f_i\right) + g_i^{\mathrm{T}} P_i g_i \tag{40}$$

and

$$\begin{aligned} 2x_i^{\mathrm{T}} P_i f_i &= 2x_i^{\mathrm{T}} P_i B_i \xi_i\left(\cdot\right) \\ &\le \sum_{j=1}^{N} \left[2\left\|x_i^{\mathrm{T}} P_i B_i\right\| \rho_{ij} \left\|x_j\right\| + 2\left\|x_i^{\mathrm{T}} P_i B_i\right\| \varphi_{ij} \left\|x_j\left(t - d_{ij}\left(t\right)\right)\right\|\right] \\ &\le \sum_{j=1}^{N} \left[\delta_{1j}^{-1} \rho_{ij}^2 \left\|x_i^{\mathrm{T}} P_i B_i\right\|^2 + \delta_{1j} \left\|x_j\right\|^2\right] \\ &\quad + \sum_{j=1}^{N} \left[\delta_{2j}^{-1} \varphi_{2j}^2 \left\|x_i^{\mathrm{T}} P_i B_i\right\|^2 + \delta_{2j} \left\|x_j\left(t - d_{ij}\left(t\right)\right)\right\|^2\right]. \end{aligned} \tag{41}$$

From Assumption 4.3 one can get

$$\begin{aligned} g_i^{\mathrm{T}} P_i g_i &\le \sum_{j=1}^{N} \|B_i^{\mathrm{T}} P_i x_i\| (\phi_{ij}\|x_j\| + \overline{\phi}_{ij}\|x_j(t - h_{ij}(t))\|) \\ &\quad + \sum_{j=1}^{N} \|B_i^{\mathrm{T}} P_i x_i(t - h_{ij}(t))\| (\psi_{ij}\|x_j\| + \overline{\psi}_{ij}\|x_j(t - h_{ij}(t))\|) \\ &\le \sum_{j=1}^{N} \left[\delta_{3j}^{-1} \phi_{ij}^2 \left\|B_i^{\mathrm{T}} P_i x_i\right\|^2 + \delta_{3j} \left\|x_j\right\|^2\right] \\ &\quad + \sum_{j=1}^{N} \left[\delta_{4j}^{-1} \overline{\phi}_{ij}^2 \left\|B_i^{\mathrm{T}} P_i x_i\right\|^2 + \delta_{4j} \left\|x_j\left(t - h_{ij}\left(t\right)\right)\right\|^2\right] \\ &\quad + \sum_{j=1}^{N} \left[\delta_{5j}^{-1} \psi_{ij}^2 \left\|B_i^{\mathrm{T}} P_i x_i\left(t - h_{ij}\left(t\right)\right)\right\|^2 + \delta_{5j} \left\|x_j\right\|^2\right] \\ &\quad + \sum_{j=1}^{N} \left[\delta_{6j}^{-1} \overline{\psi}_{ij}^2 \left\|B_i^{\mathrm{T}} P_i x_i\left(t - h_{ij}\left(t\right)\right)\right\|^2 + \delta_{6j} \left\|x_j\left(t - h_{ij}\left(t\right)\right)\right\|^2\right]. \end{aligned} \tag{42}$$

Substituting (40)–(42) into (39), we obtain

$$\begin{aligned} LV &\le \sum_{i=1}^{N} \left[x_i^{\mathrm{T}} \left(P_i A_i + A_i^{\mathrm{T}} P_i - \sigma_i P_i B_i B_i^{\mathrm{T}} P_i\right) x_i + \sigma_i \left\|B_i^{\mathrm{T}} P_i x_i\right\|^2\right] \\ &\quad + \sum_{i=1}^{N} \left(a_i^{-1} \overline{\Theta}_i\left(t\right) \dot{\overline{\Theta}}_i\left(t\right) - \Theta_i\left(t\right) \left\|x_i^{\mathrm{T}} P_i B_i\right\|^2\right) \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg(\delta_{1j}^{-1} \rho_{ij}^2 + \delta_{2j}^{-1} \varphi_{ij}^2 + \delta_{3j}^{-1} \phi_{ij}^2 + \delta_{4j}^{-1} \overline{\phi}_{ij}^2 \\ &\quad + \frac{1}{1 - k_j} \left(\delta_{5j}^{-1} \psi_{ij}^2 + \delta_{6j}^{-1} \overline{\psi}_{ij}^2\right)\Bigg) \left\|B_i^{\mathrm{T}} P_i x_i\right\|^2 \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\delta_{1j} + \frac{1}{1 - \tau_j} \delta_{2j} + \delta_{3j} + \frac{1}{1 - k_j} \left(\delta_{4j} + \delta_{6j}\right) + \delta_{5j}\right] \left\|x_j\right\|^2. \end{aligned} \tag{43}$$

Let

$$\widehat{\Theta}_i = \sum_{j=1}^{N} \left(\delta_{1j}^{-1}\rho_{ij}^2 + \delta_{2j}^{-1}\rho_{ij}^2 + \delta_{3j}^{-1}\phi_{ij}^2 + \delta_{4j}^{-1}\overline{\phi}_{ij}^2 + \frac{1}{1-k_j}\left(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2\right)\right) + \sigma_i,$$

$$\lambda_i = N\left[\delta_{1i} + \frac{1}{1-\tau_i}\delta_{2i} + \delta_{3i} + \frac{1}{1-k_i}\left(\delta_{4i}+\delta_{6i}\right) + \delta_{5i}\right]. \tag{44}$$

Further, we obtain

$$LV \le -\sum_{i=1}^{N}\left[x_i^{\mathrm{T}}(Q_i - \lambda_i I)x_i + (\widehat{\Theta}_i - \Theta_i(t))\|B_i^{\mathrm{T}}P_i x_i\|^2 + \sum_{i=1}^{N} a_i^{-1}\overline{\Theta}_i(t)\dot{\overline{\Theta}}_i(t)\right]. \tag{45}$$

Substituting (36) into (45), we obtain that

$$LV = -\sum_{i=1}^{N} x_i^{\mathrm{T}}(Q_i - \lambda_i I)x_i. \tag{46}$$

From (46), by selecting sufficiently small parameters δ_{li} ($l \in [1,6]$) we know parameters λ_i can be small enough to ensure

$$Q_i - \lambda_i I > 0.$$

It is readily to see that the closed-loop interconnected time-delay systems are robustly asymptotically stable in probability.

5 Numerical Examples

In this section, simulation examples on time-delay stochastic systems and interconnected stochastic systems are given to demonstrate the validness and feasibility of the obtained theoretic results in previous sections.

Example 1 Consider the following stochastic time-delay system

$$\begin{aligned} dx = & \left\{ \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} x_1\left(t - 0.5\left(1+\sin t\right)\right)\sin t \\ \delta_1 x_2\left(t\right)\cos t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right\} dt \\ & + \begin{bmatrix} \delta_2\left(|x_2|\,|x_1|\right)^{1/2} \\ \delta_3 x_2\left(t - 0.3\left(1+\sin\left(t\right)\right)\right)\cos t \end{bmatrix} dw, \end{aligned} \tag{47}$$

where δ_1, δ_2 and δ_3 are arbitrary scalars. We know the above system satisfying Assumptions 2.1 and 2.2, and when $X = I$, $Y = 0$, $\varepsilon_1 = \varepsilon_2 = 1$, Assumption 2.3 is also satisfied. Further we can verify that Assumption 2.4 also holds.

Therefore, based on Theorem 2.1 we can obtain the following controller

$$u = -\frac{1}{2}\,\theta(t)B^{\mathrm{T}}Px$$

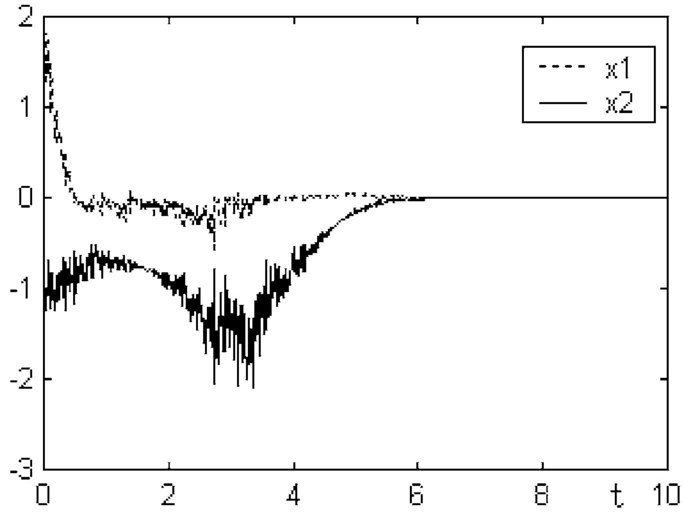

Figure 5.1. The states response curves with $\delta_i = 1$.

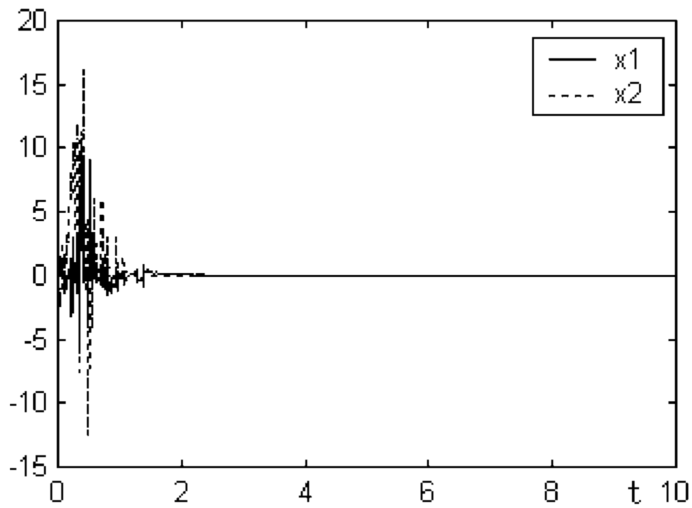

Figure 5.2. The states response curves with $\delta_i = 5$.

with adaptive law

$$\frac{d\theta(t)}{dt} = \|x\|^2.$$

The initial values are chosen as

$$x_1(0) = 2, \qquad x_2(0) = -1, \qquad \theta(0) = 2$$

and the sample time is $0.01s$. The simulation results are shown in Figure 5.1 and Figure 5.2. In Figure 5.1, it shows the response curves with above adaptive controller when $\delta_1 = \delta_2 = \delta_3 = 1$. With the same controller, the response curves are shown in Figure 5.2 when $\delta_i = 5$. From the figures, we can see that the designed controller can render the closed-loop system stable.

Example 2 Consider the following stochastic interconnected time-delay system

$$dx_1 = \left(\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) dt + \begin{bmatrix} \delta_3 \left(|x_{11} x_{21}|\right)^{1/2} \\ \delta_4 x_{12} \left(t - 0.3(1+\sin t)\right) \cos t \end{bmatrix} dw$$
$$+ \begin{bmatrix} 0 \\ \delta_1 x_{21} \left(t - 0.6\left(1+\sin t\right)\right) + \delta_2 x_{11} \left(t - 0.5\left(1+\cos(t)\right)\right) \end{bmatrix} dt,$$
$$dx_2 = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \delta_5 \left(|x_{21}\left(t - 0.6\left(1+\sin(t)\right)\right) x_{12}|\right)^{1/2} \\ 0 \end{bmatrix} \right\} dt$$
$$+ \begin{bmatrix} \delta_6 x_{21} \\ \delta_7 \left(|x_{12}\left(t - 0.3\left(1+\cos(t)\right)\right) x_{21}|\right)^{1/2} \end{bmatrix} dw.$$

We can verify that Assumptions 4.1 − 4.4 hold with $P_i = I$. Therefore the following decentralized feedback controllers can be constructed.

$$u_i = -\frac{1}{2} \widehat{\Theta}_i(t) B_i^{\mathrm{T}} P_i x_i \tag{48}$$

with adaptive law

$$\frac{d\widehat{\Theta}_i(t)}{dt} = \|B_i^{\mathrm{T}} P_i x_i\|^2. \tag{49}$$

The initial values are chosen as

$$x_{11}(0) = 2, \quad x_{12}(0) = 1, \quad x_{21}(0) = -1, \quad x_{22}(0) = -2, \quad \Theta_i(0) = 2.$$

When the parameters $\delta_i = 1$, the states response curves are shown in Figure 5.3, while Figure 5.4 depicts the curves when $\delta_i = 5$. From the two figures, the proposed decentralized feedback controllers guarantee the closed-loop system stable.

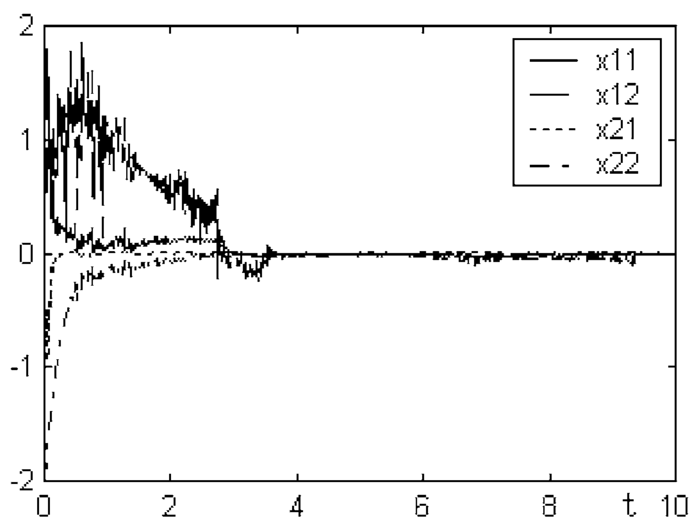

Figure 5.3. The states response curves of interconnected systems with $\delta_i = 1$.

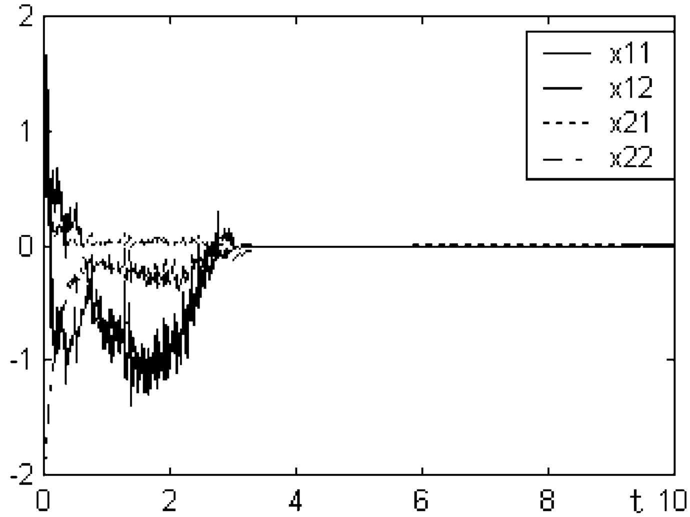

Figure 5.4. The states response curves of interconnected systems with $\delta_i = 5$.

6 Conclusion

In this paper, the robust control problem for uncertain stochastic time-delay systems is investigated. First we considered a simple class of systems and designed the corresponding adaptive feedback controller. Based on L-K method, we proved that the resulting closed-loop system is asymptotically stable. Next, we studied the problem of adaptive control of a class of time-delay interconnected stochastic systems. Sufficient conditions to construct a desired controller are derived. Simulations on controlling the uncertain systems are conducted and the results showed the potential of the proposed techniques.

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Robust Fuzzy Linear Control of a Class of Stochastic Nonlinear Time-Delay Systems

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Abstract: This paper presents the fuzzy linear control design method for a class of stochastic nonlinear time-delay systems with state feedback. First, the Takagi and Sugeno fuzzy linear model is employed to approximate a nonlinear system. Next, based on the fuzzy linear model, a fuzzy linear controller is developed to stabilize the nonlinear system. The control law is obtained to ensure stochastical exponential stability in the mean-square, independent of the time-delay. The sufficient conditions for the existence of such a control are proposed in terms of a certain linear matrix inequality. Finally, a simulation example is given to illustrate the applicability of the proposed design method.

Keywords: *Fuzzy linear control; linear matrix inequality; time-delay systems; stochastic systems; exponential stability.*

Mathematics Subject Classification (2000): 93C42, 93E15, 34K50.

1 Introduction

Most of the systems, which are encountered in control engineering, contain various nonlinearities and are affected by random disturbance signals. Nonlinear systems with time-delay constitute basic mathematical models of real phenomena, for instance in biology, mechanics and economics, see e.g. [8, 18]. Control of time-delay systems has been a subject of great practical importance, which has attracted a great deal of interest for several decades. On the other hand, it turns out that the delayed state is very often the cause for instability and poor performance of systems. Moreover, considerable attention has been given to both the problems of robust stabilization and robust control for linear systems with unavoidable time-varying parameter uncertainties in modelling of dynamical systems and certain types of time-delays [14].

Since the introduction of fuzzy set theory by Zadeh in [30], many people have devoted a great deal of time and effort to both theoretical research and implementation technique for fuzzy logic controllers [15, 22]. With the development of fuzzy systems, it is known that the qualitative knowledge of a system can also be represented in nonlinear functional form. On the basis of this idea, some fuzzy models based control system design methods have appeared in the fuzzy control field [3, 22, 23]. These methods are conceptually simple and straightforward. Fuzzy controllers are usually characterized using Mamdani and T-S type. In general, Mamdani type fuzzy controllers are designed empirically. However, T-S controllers can be designed using the information of several local linearized models of a given system via the so-called parallel-distributed compensation scheme. Various stability conditions of fuzzy systems have been obtained by employing Lyapunov stability theory [4, 9, 10], passivity theory [20], and other methods [5, 12, 22]. Problem of control design based on the state feedback for T-S fuzzy systems using LMI approach has been studied in [28] and the delay-independent stability of T-S fuzzy model for a class of nonlinear time-delay systems was investigated in [7]. Extension of the T-S fuzzy model approach to the stability analysis and control design for both continuous and discrete-time nonlinear systems with time-varying delay has been considered in [2] and also Lee, *et al.* [11] presented design of an output feedback robust H_∞ controller based on T-S fuzzy model for uncertain fuzzy dynamic systems with time-varying delayed state.

Recently, several criteria of input-to-bounded state (IBS) stabilization and bounded-input-bounded-output (BIBO) stabilization in mean-square for nonlinear and quasi-linear stochastic control systems with time-varying uncertainties has been investigated in [6], also, another stability concepts in the mean-square sense such as mean-square stability (MSS) and the internal mean-square stability (IMSS) have been studied in [13]. The stabilization of stochastic systems with multiplicative noise has been studied since the late sixties, particularly in the context of linear quadratic optimal control, see e.g., [17, 24]. Also, a stochastic fuzzy control has been proposed by applying the stochastic control theory, instead of using a traditional fuzzy reasoning in [25] and a class of fuzzy stochastic control systems with random delays investigated in [19].

The main contribution of this paper is to investigate the fuzzy linear control problem for a class of stochastic nonlinear time-delay systems. The attention was focused on the design of state feedback controller which ensures stochastical exponential stability in the mean-square, independent of the time-delay. Finally, the simulation results show that fuzzy linear state feedback controller can achieve the robust stability in the mean-square independent of the time-delay.

Notation The following notations will be used throughout the paper. R^m denotes the m-dimensional Euclidean space and $R^{n\times m}$ denotes the set of all real $n\times m$ matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. $C([-h,0];\, R^n)$ denote the family of continuous functions φ from $[-h,0]$ to R^n with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax|\colon\, |x| = 1\} = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$, where $\lambda_{\max}(A)$ means the largest eigenvalue of A. $L_2[0,\infty]$ is the space of the square integrable vector. Moreover, let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space and $L^P_{F_0}([-h,0];\, R^n)$ denote the family of all F_0-measurable $C([-h,0];\, R^n)$–valued random variables $\zeta = \{\zeta(\theta)\colon\, -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} E|\zeta(\theta)|^P < \infty$ where $E(\cdot)$

stands for the mathematical expectation operator with respect to the given probability measure P.

2 Preliminaries and Problem Formulation

Consider a class of nonlinear continuous-time state delayed stochastic systems described by

$$dx(t) = [A(x(t))x(t) + A_d(x(t))x(t-h) + B(x(t))u(t)]\,dt + E_1\,dw(t), \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-h, 0], \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^{\mathrm{T}} \in R^n$ is the state vector, $u(t) = [u_1(t), u_2(t), \dots, u_m(t)]^{\mathrm{T}} \in R^m$ is the control input, h is the unknown state delay, $\varphi(t)$ is the continuous vector valued initial function and $w(t) = [w_1(t), w_2(t), \dots, w_n(t)]^{\mathrm{T}} \in R^n$ is a scalar Brownian motion defined on the probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$.

A fuzzy dynamic model has been proposed by Takagi and Sugeno [21] to represent local linear input-output relations of nonlinear systems. This fuzzy linear model is described by fuzzy If-Then rules and will be employed here to deal with the control design problem of the nonlinear system (1)–(2). The i-th rule of this fuzzy model for the nonlinear system (1)–(2) is of the following form [9, 21, 23]:

Plant Rule i:

$$\begin{aligned} &\textit{If } z_1(t) \textit{ is } F_{i1} \textit{ and } \dots \textit{ and } z_g(t) \textit{ is } F_{ig}, \\ &\textit{then} \quad dx(t) = [A_i x(t) + A_{id} x(t-h) + B_i u(t)]\,dt + E_1\,dw(t) \end{aligned} \tag{3}$$

for $i = 1, 2, \dots, L$, where F_{ij} is the fuzzy set, $A_i \in R^{n\times n}$, $A_{id} \in R^{n\times n}$, $B_i \in R^{n\times m}$, L is the number of If-Then rules, and $z_1(t), z_2(t), \dots, z_g(t)$ are the premise variables.

The overall fuzzy system is inferred as follows [9, 21, 23]:

$$\begin{aligned} dx(t) &= \frac{\left[\sum\limits_{i=1}^{L} \mu_i(z(t))(A_i x(t) + A_{id} x(t-h) + B_i u(t))\right]}{\sum\limits_{i=1}^{L} \mu_i(z(t))}\,dt + E_1\,dw(t) \\ &= \sum_{i=1}^{L} h_i(z(t))(A_i x(t) + A_{id} x(t-h) + B_i u(t))\,dt + E_1\,dw(t) \end{aligned} \tag{4}$$

where

$$z(t) = [z_1(t), z_2(t), \dots, z_g(t)]^{\mathrm{T}}, \tag{5}$$

$$\mu_i(z(t)) = \prod_{j=1}^{g} F_{ij}(z_j(t)), \tag{6}$$

$$h_i(z(t)) = \frac{\mu_i(z(t))}{\sum\limits_{j=1}^{L} \mu_j(z(t))}, \tag{7}$$

and $F_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in F_{ij}.

Remark 1 In order to consider parametric uncertainties in the T-S fuzzy system (3), we formulate the i-th rule of the fuzzy model as

Plant Rule i:

$$\begin{aligned} &\textit{If } z_1(t) \textit{ is } F_{i1} \textit{ and } \ldots \textit{ and } z_g(t) \textit{ is } F_{ig}, \\ &\textit{then} \quad dx(t) = [(A_i + \Delta A_i^p)x(t) + A_{id}x(t-h) + (B_i + \Delta B_i^p)u(t)]\,dt + E_1\,dw(t) \end{aligned}$$

where ΔA_i^p and ΔB_i^p are assumed norm-bounded matrices with appropriate dimensions, which represent parametric uncertainties in the plant model with the following structure

$$[\Delta A_i^p \;\; \Delta B_i^p] = D_i \Gamma_i(t)\,[F_{1i} \;\; F_{2i}],$$

where D_i, F_{1i} and F_{2i} are known real constant matrices of appropriate dimensions, and $\Gamma_i(t)$ is an unknown matrix function and satisfies $\Gamma_i^{\mathrm{T}}(t)\Gamma_i(t) \le I$ [12].

Assumption 1 We assume $\mu_i(z(t)) \ge 0$ for $i = 1, 2, \ldots, L$ and $\sum_{i=1}^{L} \mu_i(z(t)) > 0$ for all t.

Therefore, we get [9, 23]

$$h_i(z(t)) \ge 0 \tag{8}$$

for $i = 1, 2, \ldots, L$ and

$$\sum_{i=1}^{L} h_i(z(t)) = 1. \tag{9}$$

Therefore, from (1) we get [4]

$$\begin{aligned} dx(t) &= [A(x(t))x(t) + A_d(x(t))x(t-h) + B(x(t))u(t)]\,dt + E_1\,dw(t) \\ &= \Bigg[\sum_{i=1}^{L} h_i(z(t))(A_i x(t) + A_{id}x(t-h) + B_i u(t)) \\ &+ \Bigg\{\Bigg(A(x) - \sum_{i=1}^{L} h_i(z(t))A_i\Bigg)x(t) + \Bigg(A_d(x) - \sum_{i=1}^{L} h_i(z(t))A_{id}\Bigg)x(t-h) \\ &+ \Bigg(B(x) - \sum_{i=1}^{L} h_i(z(t))B_i u(t)\Bigg)\Bigg\}\Bigg]\,dt + E_1 dw(t) \end{aligned} \tag{10}$$

where

$$\begin{aligned} &\Bigg\{\Bigg(A(x) - \sum_{i=1}^{L} h_i(z(t))A_i\Bigg)x(t) + \Bigg(A_d(x) - \sum_{i=1}^{L} h_i(z(t))A_{id}\Bigg)x(t-h) \\ &+ \Bigg(B(x) - \sum_{i=1}^{L} h_i(z(t))B_i u(t)\Bigg)\Bigg\} \end{aligned} \tag{11}$$

denotes the approximation error between the nonlinear system (1) and the fuzzy model (4).

Suppose the following fuzzy controller is employed to deal with the above control system design:

Control Rule j:

$$\begin{aligned}&\textit{If } z_1(t) \textit{ is } F_{j1} \textit{ and } \dots \textit{ and } z_g(t) \textit{ is } F_{jg},\\ &\textit{then} \quad u(t) = K_j x(t)\end{aligned} \tag{12}$$

for $j = 1, 2, \dots, L$. Hence, the overall fuzzy controller is given by

$$u(t) = \frac{\sum\limits_{j=1}^{L} \mu_j(z(t))\,(K_j x(t))}{\sum\limits_{j=1}^{L} \mu_j(z(t))} = \sum_{j=1}^{L} h_j(z(t)) K_j x(t) \tag{13}$$

where $h_j(z(t))$ is defined in (8) and (9) and K_j are the control parameters.

Substituting (13) into (10) yields the closed-loop nonlinear control system as follows:

$$\begin{aligned}dx(t) &= [A(x(t))x(t) + A_d(x(t))x(t-h) + B(x(t))u(t)]\,dt + E_1\,dw(t)\\ &= \Bigg[\Bigg\{\sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))(A_i + B_iK_j)x(t) + A_{id}x(t-h)\Bigg\}\\ &\quad + \Delta A + \Delta A_d + \Delta B\Bigg]\,dt + E_1\,dw(t)\end{aligned} \tag{14}$$

where

$$\Delta A = \Bigg(A(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_i\Bigg)x(t), \tag{15}$$

$$\Delta A_d = \Bigg(A_d(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_{id}\Bigg)x(t-h), \tag{16}$$

$$\Delta B = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(B(x(t)) - B_i)K_j x(t). \tag{17}$$

Assumption 2 There exist bounding matrices ΔA_i, ΔA_{id} and ΔB_i such that for all trajectory $x(t)$

$$\|\Delta A\| \le \Bigg\|\sum_{i=1}^{L} h_i(z(t))\Delta A_i x(t)\Bigg\|, \tag{18}$$

$$\|\Delta A_d\| \le \Bigg\|\sum_{i=1}^{L} h_i(z(t))\Delta A_{id} x(t-h)\Bigg\|, \tag{19}$$

$$\|\Delta B\| \le \Bigg\|\sum_{i=1}^{L} h_i(z(t))\sum_{j=1}^{L} h_j(z(t))\Delta B_i\, K_j x(t)\Bigg\| \tag{20}$$

and the bounding matrices ΔA_i, ΔA_{id} and ΔB_i can be described by

$$\begin{bmatrix} \Delta A_i \\ \Delta A_{id} \\ \Delta B_i \end{bmatrix} = \begin{bmatrix} \delta_i A_p \\ \delta_{id} A_{pd} \\ \eta_i B_p \end{bmatrix}, \tag{21}$$

where $\|\delta_i\| \le 1$, $\|\delta_{id}\| \le 1$ and $\|\eta_i\| \le 1$, for $i = 1, 2, \dots, L$ [1].

According to Assumption 2, we get

$$\begin{aligned} (\Delta A)^{\mathrm{T}}(\Delta A) &= \left(\left(A(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_i\right)x(t)\right)^{\mathrm{T}} \\ &\quad \times \left(\left(A(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_i\right)x(t)\right) \\ &\le \left(\sum_{i=1}^{L} h_i(z(t))\, \Delta A_i x(t)\right)^{\mathrm{T}} \left(\sum_{i=1}^{L} h_i(z(t))\, \Delta A_i x(t)\right) \\ &= \left(\sum_{i=1}^{L} h_i(z(t))\, \delta_i A_p x(t)\right)^{\mathrm{T}} \left(\sum_{i=1}^{L} h_i(z(t))\, \delta_i A_p x(t)\right) \le (A_p x(t))^{\mathrm{T}}(A_p x(t)), \end{aligned} \tag{22}$$

$$\begin{aligned} (\Delta A_d)^{\mathrm{T}}(\Delta A_d) &= \left(\left(A_d(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_{id}\right)x(t-h)\right)^{\mathrm{T}} \\ &\quad \times \left(\left(A_d(x(t)) - \sum_{i=1}^{L} h_i(z(t))A_{id}\right)x(t-h)\right) \\ &\le \left(\sum_{i=1}^{L} h_i(z(t))\, \Delta A_{id} x(t-h)\right)^{\mathrm{T}} \left(\sum_{i=1}^{L} h_i(z(t))\, \Delta A_{id} x(t-h)\right) \\ &= \left(\sum_{i=1}^{L} h_i(z(t))\, \delta_{id} A_{pd} x(t-h)\right)^{\mathrm{T}} \left(\sum_{i=1}^{L} h_i(z(t))\, \delta_{id} A_{pd} x(t-h)\right) \\ &\le (A_{pd} x(t-h))^{\mathrm{T}}(A_{pd} x(t-h)) \end{aligned} \tag{23}$$

and

$$\begin{aligned} (\Delta B)^{\mathrm{T}}(\Delta B) &= \left(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(B(x(t)) - B_i)K_j x(t)\right)^{\mathrm{T}} \\ &\quad \times \left(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(B(x(t)) - B_i)K_j x(t)\right) \\ &\le \left(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\, \Delta B_i K_j x(t)\right)^{\mathrm{T}} \left(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\, \Delta B_i K_j x(t)\right) \end{aligned} \tag{24}$$

$$= \Big(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\,\eta_i B_p\,K_j x(t)\Big)^{\mathrm T}\Big(\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\,\eta_i B_p\,K_j x(t)\Big)$$

$$\le \Big(\sum_{j=1}^{L} h_j(z(t)) B_p K_j x(t)\Big)^{\mathrm T}\Big(\sum_{j=1}^{L} h_j(z(t)) B_p K_j x(t)\Big),$$

i.e. the approximation error in the closed-loop nonlinear system is bounded by the specified structured bounding matrices A_p, A_{pd} and B_p.

Next, observe the closed-loop system (14) and let $x(t,\zeta)$ denote the state trajectory from the initial data $x(\theta) = \zeta(\theta)$ on $-h \le \theta \le 0$ in $L^2_{F_0}([-h,0];\, R^{2n})$. Clearly, the system (14) admits a trivial solution $x(t;0) \equiv 0$ corresponding to the initial data $\zeta = 0$. We introduce the following stability and stabilizability concepts.

Definition 1 [27] For the system (14) and every $\zeta \in L^2_{F_0}([-h,0]\,;\, R^{2n})$, the trivial solution is asymptotically stable in the mean square if

$$\lim_{t\to\infty} E\,|x(t;\,\zeta)|^2 = 0, \tag{25}$$

and is exponentially stable in the mean-square if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$E\,|x(t;\zeta)|^2 \le \alpha e^{-\beta t} \sup_{-h\le\theta\le 0} E\,|\zeta(\theta)|^2\,. \tag{26}$$

Definition 2 [27] We say that the system (1)−(2) is exponentially stabilizable in mean-square if, for every $\zeta \in L^2_{F_0}([-h,0];\, R^{2n})$, there exists a fuzzy linear control law (13) such that the resulting closed-loop system is exponentially stable in mean-square.

The objective of this paper is to design a fuzzy linear control for the stochastic nonlinear time-delay system (1)−(2). More specifically, we are interested in seeking the control parameters K_j, for $j = 1, 2, \dots, L$, such that the closed-loop system (14) is exponentially stable in mean-square, independent of the unknown time-delay h.

3 Main Results and Proofs

We first give the following lemma, which will be used in the proof of our main results.

Lemma 1 [31] *For any matrices X and Y with appropriate dimensions and for any constant $\eta > 0$, we have:*

$$X^{\mathrm T}Y + Y^{\mathrm T}X \le \eta X^{\mathrm T}X + \frac{1}{\eta}\,Y^{\mathrm T}Y. \tag{27}$$

3.1 Stochastic stability analysis

In this section, assuming that the fuzzy linear control is known and we will study the conditions under which the closed-loop system is stochastically exponentially stable in the mean-square. The following theorem will play a key role in the stability analysis of closed-loop system and design of the expected fuzzy linear control.

Theorem 1 *Let the control parameters K_j, for $j = 1, 2, \ldots, L$, be given. If the fuzzy controller (13) is employed in the nonlinear system (1)–(2) and there exists positive scalars ε_1, ε_2, ε_3, ε_4 and a positive definite matrix $P = P^{\mathrm{T}}$ such that the following matrix inequalities*

$$
\begin{aligned}
&(A_i + B_i K_j)^{\mathrm{T}} P + P(A_i + B_i K_j) + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) P^2 \\
&+ \varepsilon_1^{-1} A_{id}^{\mathrm{T}} A_{id} + \varepsilon_2^{-1} A_p^{\mathrm{T}} A_p + \varepsilon_3^{-1} A_{pd}^{\mathrm{T}} A_{pd} + \varepsilon_4^{-1} (B_p K_j)^{\mathrm{T}} (B_p K_j) < 0
\end{aligned}
\tag{28}
$$

are satisfied for all i, $j = 1, 2, \ldots, L$, then the closed-loop nonlinear system (14) is exponentially stable in the mean-square and independent of the unknown time-delay h.

Proof Fix $\zeta \in L^2_{F_0}([-h, 0]; R^{2n})$ arbitrarily, and write $x(t, \zeta) = x(t)$. We define the Lyapunov function candidate

$$
\Upsilon(x(t), t) = x^{\mathrm{T}}(t) P x(t) + \int_{t-h}^{t} x^{\mathrm{T}}(s) Q x(s)\, ds \tag{29}
$$

where $P = P^{\mathrm{T}}$ is the positive definite solution to the matrix inequality (28) and $Q = Q^{\mathrm{T}} > 0$ is defined by

$$
Q = \varepsilon_1^{-1} \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big)^{\mathrm{T}} \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big) + \varepsilon_3^{-1} A_{pd}^{\mathrm{T}} A_{pd}. \tag{30}
$$

The stochastic differential of Υ along a given trajectory is obtained as

$$
\begin{aligned}
d\Upsilon(x(t), t) = \Bigg\{ & x^{\mathrm{T}}(t) \Bigg(\Bigg\{ \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) \Bigg\}^{\mathrm{T}} P + Q \Bigg) x(t) \\
& + x^{\mathrm{T}}(t-h) \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big) x(t-h) \\
& + x^{\mathrm{T}}(t) P \Big(\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) \Big) x(t) \\
& + (\Delta A + \Delta A_d + \Delta B)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P (\Delta A + \Delta A_d + \Delta B) \\
& - x^{\mathrm{T}}(t-h) Q x(t-h) \Bigg\} dt + 2 x^{\mathrm{T}}(t) P E_1\, dw(t).
\end{aligned}
\tag{31}
$$

Now, by Lemma 1, it is trivial to show that for any positive scalars of ε_1, ε_2, ε_3, ε_4 the following matrix inequalities hold:

$$
\begin{aligned}
&\Big(\Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big) x(t-h) \Big)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P \Big(\Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big) x(t-h) \Big) \\
&\le \varepsilon_1 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_1^{-1} x^{\mathrm{T}}(t-h) \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big)^{\mathrm{T}} \Big(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Big) x(t-h),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
(\Delta A)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P (\Delta A) &\le \varepsilon_2 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_2^{-1} (\Delta A)^{\mathrm{T}} (\Delta A) \\
&\le \varepsilon_2 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_2^{-1} (A_p x(t))^{\mathrm{T}} (A_p x(t)) \\
&= x^{\mathrm{T}}(t) (\varepsilon_2 P^2 + \varepsilon_2^{-1} A_p^{\mathrm{T}} A_p) x(t),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
(\Delta A_d)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P (\Delta A_d) &\le \varepsilon_3 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_3^{-1} (\Delta A_d)^{\mathrm{T}} (\Delta A_d) \\
&\le \varepsilon_3 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_3^{-1} (A_{pd} x(t-h))^{\mathrm{T}} (A_{pd} x(t-h)) \\
&= \varepsilon_3 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_3^{-1} x(t-h)^{\mathrm{T}} A_{pd}^{\mathrm{T}} A_{pd} x(t-h)
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
&(\Delta B)^{\mathrm{T}} P x(t) + x^{\mathrm{T}}(t) P (\Delta B) \le \varepsilon_4 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_4^{-1} (\Delta B)^{\mathrm{T}} (\Delta B) \\
&\le \varepsilon_4 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_4^{-1} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j x(t) \Bigg)^{\mathrm{T}} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j x(t) \Bigg) \\
&= x^{\mathrm{T}}(t) \Bigg(\varepsilon_4 P^2 + \varepsilon_4^{-1} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j \Bigg)^{\mathrm{T}} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j \Bigg) \Bigg) x(t).
\end{aligned} \tag{35}
$$

Then, noticing the definition (30), substituting (32)–(35) into (31) result in

$$
\begin{aligned}
d\Upsilon(x(t),t) &\le x^{\mathrm{T}}(t) \Bigg\{ \Bigg(\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) \Bigg)^{\mathrm{T}} P \\
&+ P \Bigg(\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (A_i + B_i K_j) \Bigg) + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) P^2 \\
&+ \varepsilon_1^{-1} \Bigg(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Bigg)^{\mathrm{T}} \Bigg(\sum_{i=1}^{L} h_i(z(t)) A_{id} \Bigg) + \varepsilon_2^{-1} A_p^{\mathrm{T}} A_p + \varepsilon_3^{-1} A_{pd}^{\mathrm{T}} A_{pd} \\
&+ \varepsilon_4^{-1} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j \Bigg)^{\mathrm{T}} \Bigg(\sum_{j=1}^{L} h_j(z(t)) B_p K_j \Bigg) \Bigg\} x(t) dt + 2x^{\mathrm{T}}(t) P E_1 \, dw(t) \\
&\le \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \{ x^{\mathrm{T}}(t) [(A_i + B_i K_j)^{\mathrm{T}} P + P (A_i + B_i K_j) \\
&+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) P^2 + \varepsilon_1^{-1} A_{id}^{\mathrm{T}} A_{id} + \varepsilon_2^{-1} A_p^{\mathrm{T}} A_p + \varepsilon_3^{-1} A_{pd}^{\mathrm{T}} A_{pd} \\
&+ \varepsilon_4^{-1} (B_p K_j)^{\mathrm{T}} (B_p K_j)] x(t) \} \, dt + 2x^{\mathrm{T}}(t) P E_1 \, dw(t) \\
&\le - \sum_{i=1}^{L} \sum_{j=1}^{L} \lambda_{\min}(-\Pi_{ij}) x^{\mathrm{T}}(t) x(t) \, dt + 2x^{\mathrm{T}}(t) P E_1 \, dw(t),
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
\Pi_{ij} &= (A_i + B_i K_j)^{\mathrm{T}} P + P (A_i + B_i K_j) + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) P^2 \\
&+ \varepsilon_1^{-1} A_{id}^{\mathrm{T}} A_{id} + \varepsilon_2^{-1} A_p^{\mathrm{T}} A_p + \varepsilon_3^{-1} A_{pd}^{\mathrm{T}} A_{pd} + \varepsilon_4^{-1} (B_p K_j)^{\mathrm{T}} (B_p K_j).
\end{aligned} \tag{37}
$$

Then, according to the inequality (28), we find

$$\Pi_{ij} < 0, \quad \text{for} \quad i, j = 1, 2, \ldots, L. \tag{38}$$

Consequently, the inequalities (36) and (38) mean that the nonlinear stochastic time-delay closed-loop system (14) is asymptotically stable (in the mean-square) by the fuzzy control law (13).

The expected exponential stability (in the mean-square) of the closed-loop system (14) can be proved by making some standard manipulation on (36), see [16]. Let β_{ij} be the unique root of the equation

$$\lambda_{\min}(-\Pi_{ij}) - \beta_{ij}\lambda_{\max}(P) - \beta_{ij}h\lambda_{\max}(Q)e^{\beta_{ij}h} = 0, \tag{39}$$

where Π_{ij} and Q are defined, respectively, in (37) and (30) and P is the positive definite solution to (28) and h is the unknown time-delay. Then, by [26], we have

$$\begin{aligned} E|x(t)|^2 \le \lambda_{\min}^{-1}(P)\big([\lambda_{\max}(P) + h\lambda_{\max}(Q)] \\ +\ \beta_{ij}\lambda_{\max}(q)h^2 e^{\beta_{ij}h}\big) \sup_{-h\le\theta\le 0} E|\zeta(\theta)|^2 e^{-\beta_{ij}t}. \end{aligned} \tag{40}$$

Notice that, according to (40), the definition of exponential stable in Definition 1 is satisfied and this complete the proof of Theorem 1.

The result of Theorem 1 may be conservative due to the use of inequalities (32)–(35). However, such conservativeness can be significantly reduced by appropriate choices of the parameters ε_1 ε_2, ε_3, ε_4 in a matrix norm sense.

Remark 2 The result of Theorem 1 can be easily extended to the multiple state time-delay case. Consider the following nonlinear continuous-time multidelay stochastic system

$$\begin{gathered} dx(t) = \Big[A(x(t))x(t) + \sum_{i=1}^{r} A_d(x(t))x(t-h_i) + B(x(t))u(t)\Big]dt + \sum_{i=1}^{r} E_i\, dw_i(t), \\ x(t) = \varphi(t), \quad t \in [-h, 0], \quad 0 < h = \max_i(h_i), \end{gathered} \tag{41}$$

where $(w_1, w_2, \ldots, w_m)$ is an m-dimensional Brownian motion, instead of a scalar one in system (1)–(2). Also, instead of (29), we define the Lyapunov function

$$\Upsilon(x(t), t) = x^{\mathrm{T}}(t)Px(t) + \sum_{i=1}^{r} \int_{t-h_i}^{t} x^{\mathrm{T}}(s)Q_i\, x(s)\, ds. \tag{42}$$

Remark 3 We can conclude the following matrix inequality, similar to matrix inequality (28) in Theorem 1, for the T-S fuzzy systems with norm-bounded and structured parametric uncertainties introduced in Remark 1 as

$$\begin{gathered} (A_i + B_iK_j)^{\mathrm{T}}P + P(A_i + B_iK_j) + P\big((\eta_1 + \eta_2)D_iD_i^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)I\big)P \\ + \varepsilon_1^{-1}A_{id}^{\mathrm{T}}A_{id} + \varepsilon_2^{-1}A_p^{\mathrm{T}}A_p + \varepsilon_3^{-1}A_{pd}^{\mathrm{T}}A_{pd} \\ + \eta_1^{-1}F_{1i}^{\mathrm{T}}F_{1i} + \varepsilon_4^{-1}(B_pK_j)^{\mathrm{T}}(B_pK_j) + \eta_2^{-1}(F_{2i}K_j)^{\mathrm{T}}F_{2i}K_j < 0, \end{gathered}$$

where according to Lemma 1 the following matrix inequalities are satisfied for $\forall\, \eta_1, \eta_2 > 0$

$$(\Delta A_i^p)^{\mathrm{T}} P + P\Delta A_i^p \leq \eta_1 P D_i D_i^{\mathrm{T}} P + \eta_1^{-1} F_{1i}^{\mathrm{T}} F_{1i},$$

$$(\Delta B_i^p K_j)^{\mathrm{T}} P + P\Delta B_i^p K_j \leq \eta_2 P D_i D_i^{\mathrm{T}} P + \eta_2^{-1} (F_{2i} K_j)^{\mathrm{T}} F_{2i} K_j.$$

3.2 Fuzzy control design

This subsection is devoted to the design of control parameters K_j, for $j = 1, 2, \dots, L$, by using the result in Theorem 1. We will show that the design of control parameters problem can be solved via the resolution of matrix inequalities. Our approach follows the one developed by Gahinet for the deterministic case [6]. The key tool, which makes this possible, is the stochastic version of the Bounded Real Lemma. From deterministic H_∞ control theory we will need the following lemma, so-called, *Projection Lemma*.

Lemma 2 [29] *Given a symmetric matrix $H \in R^{m\times m}$ and two matrices $N \in R^{l\times m}$ and $M \in R^{n\times m}$, consider the problem of finding some matrix X such that*

$$H + N^{\mathrm{T}} X^{\mathrm{T}} M + M^{\mathrm{T}} X N < 0. \tag{43}$$

Then, (43) is solvable for X if and only if

$$N^{T\perp} H\, N^{T\perp T} < 0, \quad M^{T\perp} H\, M^{T\perp T} < 0. \tag{44}$$

Here, if $\Sigma \in R^{n\times m}$ and $\operatorname{rank}\Sigma = r$, the orthogonal complement $\Sigma^\perp$ is defined as a possibly nonunique $(n-r)\times n$ matrix with rank $n-r$, such that $\Sigma^\perp \Sigma = 0$.

By using the *Schur complement formula*, inequality (28) is equivalent to

$$\begin{bmatrix} (A_i + B_i K_j)^{\mathrm{T}} P + P(A_i + B_i K_j) + \Psi_i^{\mathrm{T}} \Psi_i & (B_p K_j)^{\mathrm{T}} & P \\ B_p K_j & -\varepsilon_4 I & 0 \\ P & 0 & -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^{-1} I \end{bmatrix} < 0, \tag{45}$$

where

$$\Psi_i = \begin{bmatrix} \varepsilon_1^{-1/2} A_{id} \\ \varepsilon_2^{-1/2} A_p \\ \varepsilon_3^{-1/2} A_{pd} \end{bmatrix}. \tag{46}$$

The inequality (45) has the form

$$\Gamma_i + N_i^{\mathrm{T}} \Omega M + M^{\mathrm{T}} \Omega^{\mathrm{T}} N_i < 0, \tag{47}$$

where

$$\begin{gathered} \Omega = K_j, \quad M = [\,I \;\; 0 \;\; 0\,], \quad N_i^{\mathrm{T}} = \begin{bmatrix} PB_i \\ B_p \\ 0 \end{bmatrix} = \begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} B_i \\ B_p \\ 0 \end{bmatrix}, \\ \Gamma_i = \begin{bmatrix} A_i^{\mathrm{T}} P + P A_i + \Psi_i^{\mathrm{T}} \Psi_i & 0 & P \\ 0 & -\varepsilon_4 I & 0 \\ P & 0 & -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^{-1} I \end{bmatrix}. \end{gathered} \tag{48}$$

Then, we have the following result.

Theorem 2 *The closed-loop fuzzy system (14) is exponentially stable in the mean-square and independent of the unknown time-delay h, if the following conditions are satisfied, for* $i = 1, 2, \dots, L$,

$$\begin{aligned} N_i^{T\perp}\Gamma_i N_i^{T\perp T} &< 0, \\ M^{T\perp}\Gamma_i M^{T\perp T} &< 0, \\ P = P^{\mathrm{T}} &> 0, \end{aligned} \tag{49}$$

where M, N_i *and* Γ_i *are defined in (48).*

Proof The proof follows directly from Theorem 1 and Projection lemma.

Let $[\,V_{1i}\ \ V_2\,] = [\,B_i\ \ B_p\,]^{T\perp}$ and, by some calculation, we have

$$N_i^{T\perp} = \begin{bmatrix} V_{1i} & V_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{50}$$

and

$$M^{T\perp} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{51}$$

Then, it follows from (49) that we have:

$$M^{T\perp}\Gamma_i M^{T\perp T} = \begin{bmatrix} -\varepsilon_4 I & 0 \\ 0 & -(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)^{-1}\, I \end{bmatrix} < 0. \tag{52}$$

This further implies that $M^{T\perp}\Gamma_i M^{T\perp T} < 0$ is satisfied for $i = 1, 2, \dots, L$ and

$$N_i^{T\perp}\Gamma_i N_i^{T\perp T} = \begin{bmatrix} W & [\,V_{1i} \quad V_2\,]\begin{bmatrix} I \\ 0 \end{bmatrix} \\ [\,I \quad 0\,]\begin{bmatrix} V_{1i}^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix} & -(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)^{-1} I \end{bmatrix} < 0, \tag{53}$$

where

$$W = [\,V_{1i} \quad V_2\,]\begin{bmatrix} P^{-1}(A_i^{\mathrm{T}}P + PA_i + \Psi_i^{\mathrm{T}}\Psi_i)P^{-1} & 0 \\ 0 & -\varepsilon_4 I \end{bmatrix}\begin{bmatrix} V_{1i}^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix}.$$

Using the Schur complement formula, it is easy to see that (53) is equivalent to

$$A_i^{\mathrm{T}}P + PA_i + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)P^2 + \Psi_i^{\mathrm{T}}\Psi_i < 0. \tag{54}$$

If the LMI in (54) have a positive-definite solution for P, then the closed-loop system (14) is exponentially stable in the mean-square and independent of the unknown time-delay h. Moreover, in this case, a set of particular solutions of control parameters K_j, for $j = 1, 2, \dots, L$, corresponding to a feasible solution P can be obtained by using the result of matrix inequality (54). Then, we obtain the following result.

Theorem 3 *If there exist positive scalars ε_1, ε_2, ε_3, ε_4 such that the linear matrix inequality (54) has positive definite solution P, then, the fuzzy control with parameters $\Omega = K_j$ for $j = 1, 2, \dots, L$ can be easily obtained by solving (47) and will be such that the closed-loop system (14) is exponentially stable in the mean-square and independent of the unknown time-delay h.*

Remark 4 In the case when $E_1 = 0$, that is, the stochastic system (1)−(2) is specialized to a deterministic system. Therefore, Theorems 1, 2 and 3 can be viewed as extensions of existing results from deterministic systems to stochastic systems.

4 Simulation Results

In this section, to illustrate the effectiveness of the proposed method, we will design a fuzzy linear controller for the following stochastic nonlinear time-delay system

$$dx(t) = [-0.06\,x(t)^3 + x(t-h) + u(t)]\,dt + dw(t) \tag{55}$$

$$x(t) = 1, \qquad t \in [-h,\, 0]. \tag{56}$$

Consider $h = 1$ second as the time-delay parameter. To use the fuzzy linear controller design, we consider a fuzzy model, which represents the dynamics of the nonlinear plant. Therefore, we represent the system (55)−(56) by the following T-S fuzzy model

Plant Rule 1:

$$\begin{aligned} &\textit{If } x(t) \textit{ is } F_{11}, \\ &\textit{then} \quad dx(t) = [-3\,x(t) + 0.5\,x(t-h) + 2\,u(t)]\,dt + dw(t). \end{aligned}$$

Plant Rule 2:

$$\begin{aligned} &\textit{If } x(t) \textit{ is } F_{21}, \\ &\textit{then} \quad dx(t) = [-2\,x(t) + 0.1\,x(t-h) + u(t)]\,dt + dw(t). \end{aligned}$$

where the membership functions of F_{11} and F_{21} are given as follows:

$$F_{11} = 1 - \frac{1}{1 + e^{-x^2}}, \quad F_{21} = 1 - F_{11} = \frac{1}{1 + e^{-x^2}},$$

and the bounding matrices are chosen as $A_p = 0.5$, $A_{pd} = 0.5$ and $B_p = 1$.

Substituting the above parameters into Theorem 3, using the LMI toolbox in MATLAB the solutions of (47), i.e., state feedback gains, can be obtained as $K_1 = 0.1$ and $K_2 = 0.1709$ and the positive scalars ε_1, ε_2, ε_3, ε_4 found as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.1$.

Robust stability of the state of system (55) in the presence of disturbance, i.e. Brownian motions has been depicted in Figure 4.1 and it is seen that due to Brownian motion as

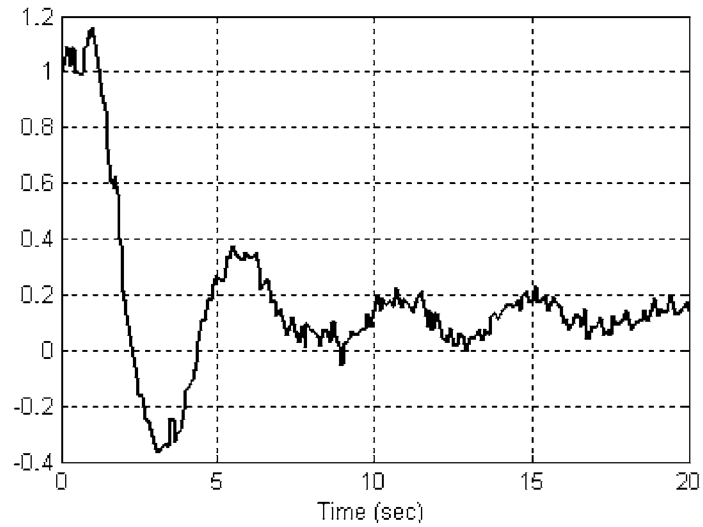

Figure 4.1. Time behavior of the state of system.

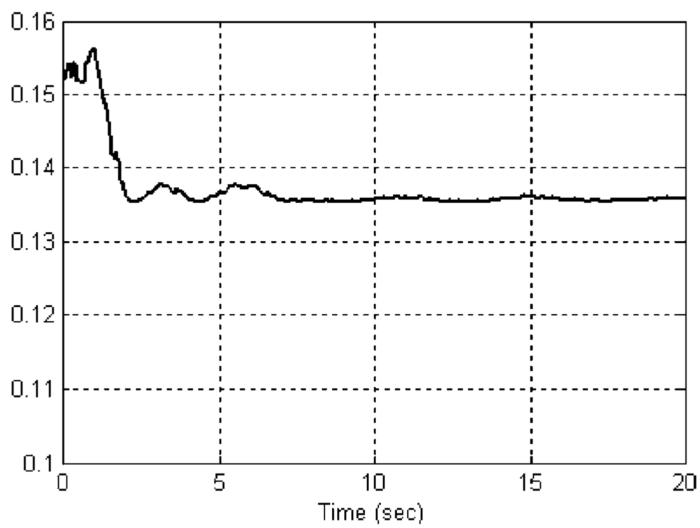

Figure 4.2. Control input.

the external disturbance, state still is bounded. The overall fuzzy controller is shown in Figure 4.2.

5 Conclusions

In this paper, the fuzzy linear control design method for a class of stochastic nonlinear time-delay systems with state feedback was developed. First, the Takagi and Sugeno fuzzy linear model was employed to approximate a nonlinear system. Next, based on the fuzzy linear model, a fuzzy linear controller was developed to stabilize the non-linear system. The control law has been obtained to ensure stochastical exponential stability in the mean-square, independent of the time-delay and the sufficient conditions for the existence of such a control were proposed in terms of certain linear matrix inequality. A simulation example was given to illustrate the applicability of the proposed design method.

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Robust $\mathcal{H}_\infty$ Analysis and Synthesis for Jumping Time-Delay Systems using Transformation Methods

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Abstract: A new transformation method is developed for the $\mathcal{H}_\infty$ analysis and synthesis of a class of uncertain time-delay systems with Markovian jump parameters. In these systems, the jumping parameters are modelled as a continuous-time, discrete-state Markov process and the parametric uncertainties are assumed to be real, time-varying and norm-bounded. The time-delay factor is constant. Complete results for delay dependent stochastic stability and stabilization criteria are developed for all admissible uncertainties. Then a dynamic output feedback controller is designed such that the closed-loop stochastic stability and a prescribed $\mathcal{H}_\infty$-performance are guaranteed. All the developed results are cast in the format of linear matrix inequalities

Keywords: *Time-delay systems; Markovian jump parameters; $\mathcal{H}_\infty$ analysis; $\mathcal{H}_\infty$ synthesis; uncertain parameters.*

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1 Introduction

It becomes increasingly apparent that delays occur in industrial and engineering systems due to various reasons including finite capabilities of information processing among different parts of the system, inherent phenomena like mass transport flow and recycling and/or by product of computational delays [12]. Considerable discussions on delays and their stabilization/destabilization effects in control systems have commanded the interests of numerous investigators in recent years, see [1, 6, 13] and their references. In the course of control design, it turns out that the design goals have to incorporate the impact of parameter shifting, component and interconnection failures which are frequently occurring in practical situations. It is thus appropriate to investigate control processes with the aid of stochastic models. One direction of investigation has been through piecewise deterministic systems or Markovian jump dynamical systems [2] in which the underlying dynamics are governed by different forms depending on the value of an associated finite-state Markov process thus offer a base model of combined continuous and discrete states. Research into this class of systems and their applications span several decades [5, 15]. When the plant modelling uncertainty or external disturbance uncertainty is of major concern in control systems, robust control theory provides tractable design tools using the time domain and the frequency domain. For Markov jumping linear continuous-time systems, the issue of robust stability and $\mathcal{H}_\infty$-control has been investigated in [4, 17] and their references. The class of time- delay systems with jump parameters have been recently considered in [1, 13] and for a modest coverage on the subject, see [2, 14].

The purpose of this paper is to extend the results of [1, 2, 13] further by developing new transformation methods that will help much in the study of stochastic stability and stabilization of a class of uncertain systems with Markovian jump parameters and distributed delays. In these systems, the jumping parameters are treated as continuous-time, discrete-state Markov process and the parametric uncertainties are assumed to be real, time-varying and norm-bounded. The time-delay factor is treated as a constant within a prespecified range. Complete results of delay-dependent stochastic stability criteria are developed for both the nominal and uncertain jumping distributed delay systems with $\mathcal{H}_\infty$ performance measure. Then we move to consider the $\mathcal{H}_\infty$ stabilization problem with instantaneous and delayed state feedback. Finally, we investigate the design of an $\mathcal{H}_\infty$ dynamic output feedback controller that ensures the close-loop stochastic stability. We establish that the $\mathcal{H}_\infty$ stability analysis and synthesis problems for the distributed-delay Markovian jump systems with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled linear matrix inequalities. Several examples are presented to illustrate the theoretical analysis.

Notations and Facts: In the sequel, the Euclidean norm is used for vectors. We use W^t, W^{-1}, $\lambda(W)$ and $\|W\|$ to denote, respectively, the transpose of, the inverse of, the eigenvalues of and the induced norm of any square matrix W. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite matrix W with $\lambda_m(W)$ and $\lambda_M(W)$ being the minimum and maximum eigenvalues of W and I to denote the $n \times n$ identity matrix. The Lebesgue space $\mathcal{L}_2[0, T]$ consists of square-integrable functions on the interval $[0, T]$ equipped with the norm $\|\cdot\|_2$. $\mathbb{E}[\cdot]$ stands for mathematical expectation. Let $\mathcal{S} = \{1, 2, ..., s\}$ be a finite set, $\mathcal{C}[-\tau_j, 0]$ be the space of continuous functions on the interval $[-\tau_j, 0]$ and define $\bar{\mathcal{C}} \triangleq \bigcup_{j \in \mathcal{S}} \mathcal{C}[-\tau_j, 0] \times \{j\}$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

Fact 1: For any real vectors β, ρ and any matrix $Q^t = Q > 0$ with appropriate dimensions, it follows that

$$-2\rho^t \beta \leq \rho^t Q \rho + \beta^t Q^{-1} \beta.$$

Fact 2: For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions and $\Sigma_3^t \Sigma_3 \leq I$, it follows that

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^t \Sigma_3^t \Sigma_1^t \leq \alpha^{-1} \Sigma_1 \Sigma_1^t + \alpha \Sigma_2^t \Sigma_2, \quad \forall\, \alpha > 0.$$

Fact 3: Let Σ_1, Σ_2, Σ_3 and $0 < R = R^t$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \le I$. Then for any $\rho > 0$ satisfying $\rho\Sigma_2^t\Sigma_2 < R$, the following matrix inequality holds:

$$(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t) \le \rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3(R - \rho\Sigma_2^t\Sigma_2)^{-1}\Sigma_3^t.$$

Fact 4 (Schur Complement): Given constant matrices Ω_1, Ω_2, Ω_3, where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \qquad \text{or} \qquad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix}.$$

2 Problem Statement

2.1 System description

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathcal{F}$. Let the random form process $\{\eta_t, t \in [0, \mathcal{T}]\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $\mathcal{S} = \{1, 2, ..., s\}$ with generator $\Im = (\alpha_{ij})$ and transition probability from mode i at time t to mode j at time $t + \delta$, $i, j \in \mathcal{S}$:

$$\mathbf{p}_{ij} = Pr(\eta_{t+\delta} = j \mid \eta_t = i) = \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if} \quad i \ne j, \\ 1 + \alpha_{ij}\delta + o(\delta), & \text{if} \quad i = j \end{cases}, \tag{2.1}$$

with transition probability rates $\alpha_{ij} \ge 0$ for $i, j \in \mathcal{S}$, $i \ne j$ and

$$\alpha_{ii} = -\sum_{m=1,\, m\ne i}^{s} \alpha_{im}, \tag{2.2}$$

where $\delta > 0$ and $\lim_{\delta\downarrow 0} o(\delta)/\delta = 0$. The set $\mathcal{S}$ comprises the various operational modes of the system under study. We consider a class of stochastic uncertain time-delay systems with Markovian jump parameters described over the space $(\Omega, \mathcal{F}, \mathbf{P})$ by:

$$\begin{aligned} (\Sigma_J): \quad \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t, \eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t, \eta_t)]x(t - \tau) + \Gamma(\eta_t)w(t), \\ &= A_{\Delta o}(t, \eta_t)x(t) + A_{\Delta d}(t, \eta_t)x(t - \tau) + \Gamma(\eta_t)w(t) \;\; t \ge 0, \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i, \qquad (2.3) \\ z(t) &= G(\eta_t)x(t) + \Phi(\eta_t)w(t), \qquad (2.4) \end{aligned}$$

where $x(t) \in \Re^n$ is the state vector; $w(t) \in \Re^q$ is the disturbance input which belongs to $\mathcal{L}_2[0, \mathcal{T}]$; $y(t) \in \Re^p$ is the measured output; $z(t) \in \Re^r$ is the controlled output which belongs to $\mathcal{L}_2\big[(\Omega, \mathcal{F}, \mathbf{P}), [0, \mathcal{T}]\big]$ and $\tau \in [0, \tau^*]$ is a constant delay factor. For each possible value $\eta_t = i$, $i \in \mathcal{S}$, we will denote the system matrices of (Σ_J) associated with mode i by

$$\begin{aligned} &A_o(\eta_t) \triangleq A_o(i), \qquad \Gamma(\eta_t) \triangleq \Gamma(i), \qquad G(\eta_t) \triangleq G(i), \\ &A_d(\eta_t) \triangleq A_d(i), \qquad \Phi(\eta_t) \triangleq \Phi(i), \end{aligned} \tag{2.5}$$

where $A_o(i)$, $A_d(i)$, $G(i)$, $\Gamma(i)$ and $\Phi(i)$ are known real constant matrices of appropriate dimensions which describe the nominal system of (Σ_J). The matrices $\Delta A_o(t,\eta_t)$ and $\Delta A_d(t,\eta_t)$ are real, time-varying matrix functions representing the norm-bounded parameter uncertainties. For $\eta_t = i$, the admissible uncertainties are assumed to be modeled in the form:

$$[\Delta A_o(t,i) \quad \Delta A_d(t,i)] = M_a(i)\Delta(t,i)[N_a(i) \quad N_d(i)], \quad \|\Delta(t,i)\|_2 \le 1, \tag{2.6}$$

where $M_a(i) \in \Re^{n\times\alpha}$, $N_a(i) \in \Re^{\beta\times n}$ and $N_d(i) \in \Re^{\beta\times n}$ are known real constant matrices, with $\Delta(t,i) \in \Re^{\alpha\times\beta}$ being unknown, time-varying matrix function whose elements are Lebesgue measurable for any $i \in \mathcal{S}$.

Our purpose in this paper is to develop criteria for $\mathcal{H}_\infty$ analysis and synthesis for system (2.3) – (2.4). Initially, we focus on stochastic stability and $\mathcal{L}_2$-gain criterion and examine their robustness using the performance measure

$$\mathcal{J}(x) \stackrel{\triangle}{=} \mathbb{E}\left\{ \int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t)]\, dt \right\}, \tag{2.7}$$

where $\gamma > 0$ is a desired level of disturbance attenuation.

2.2 Model transformation

For each possible value $\eta_t = i$, $i \in \mathcal{S}$, we introduce the following state transformation

$$\sigma(t) = x(t) + \int_{t-\tau}^{t} A_{\Delta d}(t,i)x(s)\, ds \tag{2.8}$$

into (2.3) to yield

$$\dot{\sigma}(t) = [A_{\Delta o}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t). \tag{2.9}$$

Given a sufficiently small scalar ε, we define the augmented state-vector

$$\zeta(t) = \begin{bmatrix} \sigma(t) \\ \varepsilon x(t) \end{bmatrix} \in \Re^{2n}. \tag{2.10}$$

By combining (2.3) and (2.8) – (2.10) and taking the limit $\varepsilon \to 0$, we obtain the transformed system

$$(\Sigma_T): \quad \dot{\zeta}(t) = \Lambda_\Delta(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon(i)\zeta(s)\, ds + \bar{\Gamma}(i)w(t),$$

$$\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, \tag{2.11}$$

$$z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t), \tag{2.12}$$

where

$$\bar{\Gamma}(i) = \begin{bmatrix} \Gamma(i) \\ 0 \end{bmatrix}, \quad \bar{G}(i) = [0 \quad G(i)], \quad A_{od}(i) = A_o(i) + A_d(i),$$
$$\Lambda_\Delta(i) = \begin{bmatrix} 0 & A_{\Delta o}(t,i) + A_{\Delta d}(t,i) \\ -I & I \end{bmatrix}, \qquad \Upsilon(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{\Delta d}(t,i) \end{bmatrix}. \tag{2.13}$$

For convenience, we introduce the matrices for $i \in \mathcal{S}$

$$\Lambda_o(i) = \begin{bmatrix} 0 & A_{od}(i) \\ -I & I \end{bmatrix}, \qquad \bar{M}(i) = \begin{bmatrix} M_a \\ 0 \end{bmatrix}, \qquad \mathbb{P}(i) = \begin{bmatrix} P_\sigma(i) & 0 \\ P_d(i) & P_x(i) \end{bmatrix},$$
$$N_{ad}(i) = N_a(i) + N_d(i), \qquad \bar{N}_{ad}(i) = [0 \quad N_{ad}(i)], \qquad \bar{P}(i) = U\mathbb{P}(i), \tag{2.14}$$
$$U = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Remark 2.1 Some discussions on the model transformation are in order. On one hand, the σ-variable recovers the delay-dependent dynamics of system (Σ_J). On the other hand, the use of small scalar ε is meant to capture the slow-modes of the system. It is readily seen for absolutely continuous initial functions that systems (Σ_J) and (Σ_T) are equivalent. For single-mode systems $s = 1$, a different approach was developed in [6] based on description-type transformation. In the sequel, it will be shown that our transformation is more flexible.

For system (2.11)–(2.14), we provide the following definition.

Definition 2.1 System (Σ_T) is said to be *delay dependent robustly stochastically stable (DDRSS) with disturbance attenuation* $\gamma > 0$ if for zero initial vector function $\phi \equiv 0$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathcal{S}$

$$\|z(t)\|_{E_2} := \mathbb{E}\left[\int_0^\infty z^t(t)z(t)\,dt\right]^{1/2} < \gamma\|w(t)\|_2$$

for all $0 \neq w(t) \in \mathcal{L}_2[0,\infty)$ and for all admissible uncertainties satisfying (2.6).

3 $\mathcal{L}_2$-Gain Analysis

The theorem and corollaries established in the sequel show that the stability behavior of system Σ_T (or equivalently Σ_J) is related to the existence of a positive definite solution of a family of linear matrix inequalities (LMIs) thereby providing a clear key to designing the feedback controller.

Theorem 3.1 *System Σ_T is DDRSS with disturbance attenuation $\gamma > 0$ if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $0 < P_\sigma(i)$, $P_d(i)$, $P_x(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_2(i) & \Pi_{21}(i) & \Pi_{22}(i) & \Pi_{23}(i) & \Pi_{24}(i) \\ \Pi_{21}^t(i) & -\varepsilon_1(i)I & 0 & 0 & 0 \\ \Pi_{22}^t(i) & 0 & -\tau\varepsilon_2(i)I & 0 & 0 \\ \Pi_{23}^t(i) & 0 & 0 & -\tau Q_x(i) + \tau\varepsilon_2(i)N_d(i)N_d^t(i) & 0 \\ \Pi_{24}^t(i) & 0 & 0 & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) \end{bmatrix} < 0,$$
$$\begin{bmatrix} -Q_x(i) & N_d(i) \\ N_d^t(i) & -\varepsilon_2(i)I \end{bmatrix} < 0, \qquad \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{3.1}$$

where

$$\Pi_2(i) = \begin{bmatrix} -P_d(i) - P_d^t(i) + \sum\limits_{m=1}^{s} \alpha_{im} P_\sigma(m) & -P_x(i) + P_d^t(i) + P_\sigma^t(i) A_{od}(i) \\ -P_x^t(i) + P_d(i) + A_{od}^t(i) P_\sigma(i) & \begin{matrix} P_x(i) + P_x^t(i) + \tau Q_x(i) \\ +G^t(i)G(i) + \rho(i)\tau^2 \sum\limits_{m=1}^{s} \alpha_{im} Q_x(m) \\ +\varepsilon_1(i) \bar{N}_{ad}^t(i) \bar{N}_{ad}(i) \end{matrix} \end{bmatrix}, \quad (3.2)$$

$$\Pi_{21}(i) = \begin{bmatrix} P_\sigma^t(i) E_1 M_a(i) \\ 0 \end{bmatrix}, \qquad \Pi_{22}(i) = \begin{bmatrix} \tau P_\sigma^t(i) E_1 M_a(i) & \tau P_d^t(i) E_1 M_a(i) \\ 0 & \tau P_x^t(i) E_1 M_a(i) \end{bmatrix}, \quad (3.3)$$

$$\Pi_{23}(i) = \begin{bmatrix} \tau P_d^t(i) \\ \tau P_x^t(i) \end{bmatrix}, \qquad \Pi_{24}(i) = \begin{bmatrix} P_\sigma^t(i)\Gamma(i) \\ G^t(i)\Phi(i) \end{bmatrix}. \quad (3.4)$$

Proof Let $\mathbf{x}_s(t) \stackrel{\triangle}{=} x(s+t)$, $t - \tau \le s \le t$ and define the process $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ over the state space $\bar{\mathcal{C}}$. It should be observed that $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ is strong Markovian [9] so is the process $\{(\zeta(t), \eta_t),\ t \ge 0\}$. Now for $\eta_t = i \in \mathcal{S}$, and given $Q(i) = Q^t(i) > 0$, let the Lyapunov functional $V(\cdot)\colon \Re^n \times \Re_+ \times \mathcal{S} \to \Re_+$ of the transformed system be selected as

$$V(t,\zeta,i) = \zeta^t(t)\bar{P}(i)\zeta(t) + \int\limits_{t-\tau}^{t} \int\limits_{\theta}^{t} \zeta^t(s) E_2 Q_x(i) E_2^t \zeta(s)\, ds d\theta. \quad (3.5)$$

The weak infinitesimal operator $\Im_1^\zeta[\cdot]$ of the process $\{\zeta(t), i, t \ge 0\}$ for system (2.11)–(2.14) at the point $\{t, x, i\}$ is given by [5, 9]:

$$\Im_1^\zeta[V] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \zeta}\,\dot{\zeta}(t)\Big|_{\eta_t = i} + \sum_{m=1}^{s} \alpha_{im} V(t,\zeta,i,m). \quad (3.6)$$

Using (2.9)–(2.14) we get:

$$\begin{aligned} \frac{\partial V}{\partial \zeta}\,\dot{\zeta}(t) &= 2\zeta^t(t) U \mathbb{P}^t(i) \dot{\zeta}(t) = 2\sigma^t(t) P_\sigma^t(i) \dot{\sigma}(t) = 2\zeta^t(t)\mathbb{P}^t(i) \begin{bmatrix} \dot{\sigma}(t) \\ 0 \end{bmatrix} \\ &= 2\zeta^t(t)\mathbb{P}^t(i) \begin{bmatrix} A_{\Delta o}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t) \\ -\sigma(t) + x(t) + \int\limits_{t-\tau}^{t} A_{\Delta d}(t,i) x(s)\, ds \end{bmatrix} \\ &= 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_\Delta(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) \\ &\quad + 2\int\limits_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\, d\theta. \end{aligned} \quad (3.7)$$

Hence, it follows from (3.6) – (3.7) that

$$\begin{aligned}\Im_1^{\zeta}[V] &= \zeta^t(t)\Big[\Lambda_\Delta^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_\Delta(i) + \sum_{m=1}^{s}\alpha_{im}\mathbb{P}(m)\Big]\zeta(t) \\ &+ 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta + \int_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta \\ &- \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta + \sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,dsd\theta.\end{aligned} \tag{3.8}$$

Since for some $\rho(i) > 0$, $i \in \mathcal{S}$

$$\sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,dsd\theta \le \tau^2\rho(i)\zeta^t(t)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t\zeta(t) \tag{3.9}$$

and by **Fact 1**, we have

$$\begin{aligned}2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta &= 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)x(\theta)\,d\theta \\ &\le \tau\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}^t(i)\zeta(\theta) + \int_{t-\tau}^{t}x^t(s)Q_x(i)x(s)\,ds \\ &= \tau\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}(i)\zeta(t) + \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta.\end{aligned} \tag{3.10}$$

Now, it follows from (3.8) – (3.10) that

$$\begin{aligned}\Im_1^{\zeta}[V] &\le \zeta^t(t)\Big[\Lambda_\Delta^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_\Delta(i) + \sum_{m=1}^{s}\alpha_{im}\mathbb{P}(m) \\ &+ \rho(i)\tau^2E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}(i)\Big]\zeta(t) \\ &+ \tau E_2Q_x(i)E_2^t + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t).\end{aligned} \tag{3.11}$$

Application of **Facts 2 – 3** to (3.11) yields:

$$\begin{aligned}\Im_1^{\zeta}[V] &\le \zeta^t(t)\Big[\Lambda_o^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_o(i) + \sum_{m=1}^{s}\alpha_{im}\bar P(m) \\ &+ \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar N_{ad}^t(i)\bar N_{ad}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar M(i)\bar M^t(i)\mathbb{P}(i) \\ &+ \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) \\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\Big]\zeta(t) \\ &+ 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) = \zeta^t(t)\Pi_1\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t)\end{aligned} \tag{3.12}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. By taking $w(t) \equiv 0$, the robust stability of system (2.10) readily follows from (3.12) when $\Pi_1 < 0$. Thus we conclude that $\Im_1^\zeta[V] < 0$ for all $\zeta \neq 0$ and $\Im_1^\zeta[V] \le 0$ for all ζ. By Dynkin's formula [9], one has $\mathbb{E}\Big[\int_0^\infty \Im_1^\zeta[V]\,dt\Big] = \mathbb{E}[V(t,x,i)|_{t=\infty}] - V(t,\zeta,i)|_{t=0} \ge 0$. With some manipulations using (2.10) and (3.12), we obtain:

$$
\begin{aligned}
\mathcal{J}(x) &= \mathbb{E}\Big\{\int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_1^\zeta[V] - \Im_1^\zeta[V]]dt\Big\}\\
&\le \mathbb{E}\Big\{\int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_1^\zeta[V]]dt\Big\}\\
&\le \mathbb{E}\Big\{\int_0^\infty \zeta^t(t)\Big[\Lambda_o^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_o(i) + \sum_{m=1}^s \alpha_{im}\bar{P}(m)\\
&\quad + \tau E_2 Q_x(i) E_2^t + \varepsilon_1(i)\bar{N}_{ad}^t(i)\bar{N}_{ad}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i)\\
&\quad + \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i)\\
&\quad + \tau^2\rho(i)E_2\sum_{m=1}^s \alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i)\\
&\quad + [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)]\Big]\zeta(t)\Big\}.
\end{aligned} \tag{3.13}
$$

By using (3.1)−(3.4) and **Fact 4**, it follows from inequality (3.13) that $\mathcal{J}(x) < 0$ and hence system (2.11)−(2.12) is *DDRSS with disturbance attenuation* $\gamma > 0$.

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 3.1 *Consider the nominal jump system*

$$
\begin{aligned}
(\Sigma_{Tn}): \quad \dot{\zeta}(t) &= \Lambda_o(i)\zeta(t) + \int_{t-\tau}^t \Upsilon(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),\\
\zeta(t) &= \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, && (3.14)\\
z(t) &= \bar{G}(i)\zeta(t) + \Phi(i)w(t). && (3.15)
\end{aligned}
$$

System Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ if given matrix sequence $Q(i) = Q^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $P(i) = P^t(i) > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs

$$
\begin{bmatrix} \Pi_{20}(i) & \Pi_{23}(i) & \Pi_{24}(i)\\ \Pi_{23}^t(i) & -\tau Q_x(i) & 0\\ \Pi_{24}^t(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) \end{bmatrix} < 0, \quad \begin{bmatrix} -\gamma^2 I & \Phi^t(i)\\ \Phi(i) & -I \end{bmatrix} < 0, \quad i \in \mathcal{S}, \tag{3.16}
$$

where

$$
\Pi_{20}(i) = \begin{bmatrix} -P_d(i) - P_d^t(i) + \sum_{m=1}^s \alpha_{im}P_\sigma(m) & -P_x(i) + P_d^t(i) + P_\sigma^t(i)A_{od}(i)\\ -P_x^t(i) + P_d(i) + A_{od}^t(i)P_\sigma(i) & \begin{matrix} P_x(i) + P_x^t(i) + \tau Q_x(i)\\ +G^t(i)G(i) + \rho(i)\tau^2 \sum_{m=1}^s \alpha_{im}Q_x(m) \end{matrix} \end{bmatrix}.
$$

Remark 3.1 In the foregoing analysis, τ is assumed to be known and constant. If it turns out to be known, the largest value can be computed by solving a generalized eigenvalue problem of the form:

$$\begin{array}{ll} \text{Maximize} & \tau \\ \text{subject to} & \begin{array}{c} P_\sigma(i) > 0,\ \ P_d(i),\ \ P_x(i), \\ \varepsilon_1(i) > 0,\ \ \varepsilon_2(i) > 0,\ \ \rho(i) > 0,\ \ \gamma > 0 \quad i \in \mathcal{S}. \end{array} \end{array}$$

This problem can be readily solved using the LMI toolbox.

3.1 Example 1

In order to illustrate Theorem 3.1, we consider a pilot-scale multi-reach water quality system [11] which can fall into the type (2.3)−(2.6). Let the Markov process governing the mode switching has generator

$$\Im = \begin{bmatrix} -4 & 3 & 1 \\ 2 & -6 & 4 \\ 4 & 4 & -8 \end{bmatrix}.$$

For the three operating conditions (modes), the associated date are:

Mode 1:

$$A_o(1) = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \qquad A_d(1) = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad \Gamma(1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$G(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad \Phi(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad M_a(1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},$$

$$N_a(1) = [0.2 \quad 0.4], \qquad N_d(1) = [0.1 \quad 0.3].$$

Mode 2:

$$A_o(2) = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \qquad A_d(2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Gamma(2) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G(2) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad \Phi(2) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad M_a(2) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$N_a(2) = [0.2 \quad 0.2], \qquad N_d(2) = [0.1 \quad 0.2].$$

Mode 3:

$$A_o(3) = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_d(3) = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}, \qquad \Gamma(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G(3) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad \Phi(3) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad M_a(3) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$N_a(3) = [0.3 \quad 0.3], \qquad N_d(3) = [0.2 \quad 0.1].$$

Invoking the software environment [7], we solve inequalities (3.1) subject to (3.2) – (3.4) for $i = 1, 2, 3$. The feasible solutions obtained for

$$\begin{aligned} &\varepsilon_1(1) = 0.7825, &&\varepsilon_2(1) = 1.5634, &&\rho(1) = 3.2312,\\ &\varepsilon_1(2) = 1.2671, &&\varepsilon_2(2) = 3.3451, &&\rho(2) = 2.7645,\\ &\varepsilon_1(3) = 4.2355, &&\varepsilon_2(3) = 0.6673, &&\rho(3) = 4.4436 \end{aligned}$$

show water quality system is DDRSS with a disturbance attenuation level of $\gamma = 1.25$ for any constant time delay $\tau \le 0.6715$.

4 Robust $\mathcal{H}_\infty$ Stabilization

In this section, we consider the control uncertain jumping system with $\eta_t = i \in \mathcal{S}$:

$$\begin{aligned} (\Sigma_{JC}):\quad &\dot{x}(t) = A_{\Delta o}(t,i)x(t) + A_{\Delta d}(t,i)x(t-\tau) + B_{\Delta o}(t,i)u(t) + \Gamma(i)w(t), \quad t \ge 0,\\ &x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i, \qquad (4.1)\\ &z(t) = G(i)x(t) + \Phi(i)w(t), \qquad (4.2) \end{aligned}$$

where $u(t) \in \Re^r$ is the control input and

$$B_{\Delta o}(t,i) = B_o(t,i) + M_a(i)\Delta(t,i)N_b(i) \tag{4.3}$$

with $N_b(i) \in \Re^{\beta\times r}$. We will examine two distinct case of state feedback stabilization: instantaneous feedback and delayed feedback.

4.1 Instantaneous state feedback

In this case we use the control law for $\eta_t = i \in \mathcal{S}$

$$u(t) = K(i)x(t), \qquad i \in \mathcal{S} \tag{4.4}$$

such that the use of (2.8) and (4.4) into (4.1) yields for $\eta_t = i$:

$$\begin{aligned} \dot{\sigma}(t) &= [A_{\Delta k}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t),\\ A_{\Delta k}(t,i) &= A_{\Delta o}(t,i) + B_{\Delta o}(t,i)K(i). \end{aligned} \tag{4.5}$$

In this case the transformed system becomes

$$\begin{aligned} (\Sigma_{TK}):\quad &\dot{\zeta}(t) = \Lambda_{\Delta k}(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),\\ &\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, \qquad (4.6)\\ &z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t), \qquad (4.7) \end{aligned}$$

where

$$\Lambda_{\Delta k}(i) = \begin{bmatrix} 0 & A_{\Delta k}(t,i) + A_{\Delta d}(t,i) \\ -I & I \end{bmatrix}. \tag{4.8}$$

Taking into consideration the standard result

$$\mathbb{P}^{-1}(i) = \begin{bmatrix} X_\sigma(i) & 0 \\ X_d(i) & X_x(i) \end{bmatrix},$$
$$X_\sigma(i) = P_\sigma^{-1}(i), \qquad X_x(i) = P_x^{-1}(i), \qquad X_d(i) = -X_x P_d(i) X_\sigma \tag{4.9}$$

we define the following matrices for $i \in \mathcal{S}$:

$$\Lambda_{ok}(i) = \begin{bmatrix} 0 & A_{od}(i) + B_o(i)K(i) \\ -I & I \end{bmatrix}, \qquad \bar{B}_o(i) = \begin{bmatrix} B_o(i) \\ 0 \end{bmatrix}, \qquad Z(i) = \begin{bmatrix} 0 \\ X_\sigma(i) \end{bmatrix},$$
$$\bar{A}_{od}^t(i) = [A_{od}^t(i) \quad I], \qquad N_{kd}(i) = N_{ad}(i) + N_b(i)K(i), \qquad \bar{N}_{kd}(i) = [0 \quad N_{kd}(i)],$$
$$Y(i) = [X_d(i) \quad X_x(i)], \qquad H(i) = [H_2(i) \quad H_1(i)], \qquad N_{dk}(i) = N_d(i) + N_b(i)K_d(i),$$
$$\Omega(\tau, i) = G^t(i)G(i) + \tau E_2 Q_x(i) E_2^t + \rho(i)\tau^2 E_2 \sum_{m=1}^{s} \alpha_{im} Q_x(m) E_2^t + \varepsilon_1(i) N_{ad}^t(i) N_{ad}(i). \tag{4.10}$$

The following theorem establish the main result:

Theorem 4.1 *System Σ_{TK} is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.3) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $H(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_3(i) & \bar{M}(i) & \tau E_1 M_a(i) & \tau E_2 A_d(i) & \bar{\Gamma}(i) + Y^t(i)G(i)\Phi(i) & \mathcal{R}(i) \\ \bar{M}^t(i) & -\varepsilon_1(i)I & 0 & 0 & 0 & 0 \\ M_a^t(i)E_1^t & 0 & -\tau\varepsilon_2(i)I & 0 & 0 & 0 \\ \tau A_d^t(i)E_2^t & 0 & 0 & -\tau Q_x(i) + \tau\varepsilon_2(i)N_d(i)N_d^t(i) & 0 & 0 \\ \bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & 0 & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) & 0 \\ \mathcal{R}^t(i) & 0 & 0 & 0 & 0 & -\mathcal{Y}(i) \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q_x(i) & N_d(i) \\ N_d^t(i) & -\varepsilon_2(i)I \end{bmatrix} < 0, \qquad \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{4.11}$$

where

$$\begin{aligned} \Pi_3(i) &= Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1(i)Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)H(i) \\ &\quad + H^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega(\tau,i)Y(i) + \alpha_{ii}E_1 Z^t(i)E_2 \\ &\quad + \varepsilon_1(i)Y^t(i)N_{ad}^t(i)N_b(i)E_1^t L(i) + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_b(i)E_1^t L(i) \\ &\quad + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_{ad}(i)Y(i) \\ \mathcal{Y}(i) &= \mathrm{diag}\big[E_1 Z^t(1)E_2 \dots E_1 Z^t(i-1)E_2 \quad E_1 Z^t(i+1)E_2 \dots E_1 Z^t(s)E_2\big], \\ \mathcal{R}(i) &= \big[\sqrt{\alpha_{i1}}E_1 Z^t(1)E_2 \dots \sqrt{\alpha_{is}}E_1 Z^t(s)E_2\big], \end{aligned} \tag{4.12}$$

and the state-feedback gain is given by $K(i) = H_1(i)[Y(i)E_1]^{-1}$.

Proof Again, let $\mathbf{x}_s(t) \stackrel{\triangle}{=} x(s+t)$, $t-\tau \le s \le t$ and define the process $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ over the state space $\bar{\mathcal{C}}$. It should be observed that $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ is strong

Markovian [9] so is the process $\{(\zeta(t), \eta_t),\ t \geq 0\}$. Now for $\eta_t = i \in \mathcal{S}$, and given $Q(i) = Q^t(i) > 0$, let the Lyapunov functional $V(\cdot)\colon \Re^n \times \Re_+ \times \mathcal{S} \to \Re_+$ as given by (3.5) and hence the weak infinitesimal operator $\Im_2^\zeta[\cdot]$ of the process $\{\zeta(t), \eta_t,\ t \geq 0\}$ for system (4.6)–(4.9) at the point $\{t, x, \eta_t\}$ is given by (3.6). It is easy to see that:

$$\frac{\partial V}{\partial \zeta}\,\dot\zeta(t) = 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_{\Delta k}(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) + 2\int\limits_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta. \tag{4.13}$$

Hence, it follows from (3.6) and (4.13) that

$$\begin{aligned}\Im_2^\zeta[V] &= \zeta^t(t)\Big[\Lambda_{\Delta k}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{\Delta k}(i) + \sum_{m=1}^{s}\alpha_{im}\bar P(m)\Big]\zeta(t) \qquad (4.14)\\ &+ 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) + 2\int\limits_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta + \int\limits_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta\\ &- \int\limits_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta + \sum_{m=1}^{s}\alpha_{im}\int\limits_{t-\tau}^{t}\int\limits_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2\zeta(s)\,dsd\theta.\end{aligned}$$

By making use of (3.9)–(3.10) into (4.14) and applying Facts 2–3, we get

$$\begin{aligned}\Im_2^\zeta[V] \leq \zeta^t(t)\Big[&\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar P(m) + \varepsilon_1(i)\bar N_{kd}^t(i)\bar N_{kd}(i)\\ &+ \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) + \tau E_2Q_x(i)E_2^t\\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\Big]\zeta(t)\\ &+ \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar M(i)\bar M^t(i)\mathbb{P}(i) + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t)\end{aligned} \tag{4.15}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. By similarity to Theorem 3.1 the robust stability of system Σ_{TK} is guaranteed readily follows from (3.12) and Definition 2.1. Thus we conclude that $\Im_2^\zeta[V] < 0$ for all $\zeta \neq 0$ and $\Im_2^\zeta[V] \leq 0$ for all ζ. Also, by Dynkin's formula [9], one has $\mathbb{E}[\int_0^\infty \Im_2^\zeta[V]dt] = \mathbb{E}[V(t,x,i)|_{t=\infty}] - V(t,\zeta,i)|_{t=0} \geq 0$. With some manipulations using (4.7) and (4.15), it is readily seen that:

$$\begin{aligned}\mathcal{J}(x) &\leq \mathbb{E}\Big\{\int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_2^\zeta[V]]dt\Big\} \qquad (4.16)\\ &\leq \mathbb{E}\Big\{\int_0^\infty \zeta^t(t)\Big[\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar P(m)\\ &+ \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar N_{kd}^t(i)\bar N_{kd}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar M(i)\bar M^t(i)\mathbb{P}(i)\\ &+ \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) + \bar G^t(i)\bar G(i)\\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\\ &+ [\bar P^t(i)\bar\Gamma(i) + \bar G^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar\Gamma^t(i)\mathbb{P}(i) + \Phi^t(i)\bar G(i)]\Big]\zeta(t)\Big\}.\end{aligned}$$

In line of Theorem 3.1, it follows from inequality (4.16) that $\mathcal{J}(x) < 0$ is guaranteed if the following inequality

$$\begin{aligned}\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m) + \tau E_2 Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i)\\ + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)E_1\bar{M}(i)\bar{M}^t(i)E_1^t\mathbb{P}(i)\\ + \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i)\\ + \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i)\\ + [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)] < 0\end{aligned} \tag{4.17}$$

holds. Premultiplying (4.17) by $\mathbb{P}^{-t}(i)$, postmultiplying by $\mathbb{P}^{-1}(i)$, using (4.9) – (4.10) and manipulating with the help of Fact 3, we obtain the LMI (4.11). It follows that system (4.6) – (4.7) is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.4).

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 4.1 *The nominal jump system Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ under the control law (4.4) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $H(i)$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_{30}(i) & \tau E_2A_d(i) & \bar{\Gamma}(i) + Y^t(i)G(i)\Phi(i) & \mathcal{R}(i)\\ \tau A_d^t(i)E_2^t & -\tau Q_x(i) & 0 & 0\\ \bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) & 0\\ \mathcal{R}^t(i) & 0 & 0 & -\mathcal{Y}(i)\end{bmatrix} < 0, \tag{4.18}$$
$$\begin{bmatrix} -\gamma^2 I & \Phi^t(i)\\ \Phi(i) & -I\end{bmatrix} < 0, \qquad i \in \mathcal{S},$$

where

$$\begin{aligned}\Pi_{30}(i) &= Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)H(i)\\ &\quad + H^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega_o(\tau,i)Y(i) + \alpha_{ii}E_1Z^t(i)E_2,\\ \Omega_o(\tau,i) &= G^t(i)G(i) + \tau E_2Q_x(i)E_2^t + \rho(i)\tau^2E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t,\end{aligned}$$

and the state-feedback gain is given by $K(i) = H_1(i)[Y(i)E_1]^{-1}$.

4.2 Delayed state feedback

In this case we use the control law for $\eta_t = i \in \mathcal{S}$ as

$$u(t) = K_d(i)x(t-\tau), \qquad i \in \mathcal{S}, \tag{4.19}$$

along with the following state transformation

$$\sigma(t) = x(t) + \int_{t-\tau}^{t} [A_{\Delta d}(t,i) + B_{\Delta o}(t,i)K_d(i)]x(s)\,ds \tag{4.20}$$

such that the use of (4.19) − (4.20) into (4.1) with (2.13) − (2.14) yields for $\eta_t = i \in \mathcal{S}$:

$$\begin{aligned} \dot{\sigma}(t) &= [A_{\Delta o}(t,i) + A_{\Delta kd}(t,i)]x(t) + \Gamma(i)w(t), \\ A_{\Delta kd}(t,i) &= A_{\Delta d}(t,i) + B_{\Delta o}(t,i)K_d(i). \end{aligned} \tag{4.21}$$

Simple algebra yields the transformed system:

$$(\Sigma_{TD}): \quad \dot{\zeta}(t) = \Lambda_{\Delta d}(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon_k(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),$$

$$\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \tag{4.22}$$

$$z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t), \tag{4.23}$$

where

$$\Lambda_{\Delta d}(i) = \begin{bmatrix} 0 & A_{\Delta o}(t,i) + A_{\Delta kd}(t,i) \\ -I & I \end{bmatrix}, \qquad \Upsilon_k(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{\Delta kd}(t,i) \end{bmatrix}. \tag{4.24}$$

Define

$$\begin{aligned} A_{okd}(i) = A_{od}(i) + B_o(i)K_d(i), &\quad A_{kd}(i) = A_d(i) + B_o(i)K_d(i), \\ L(i) &= [L_2(i) \quad L_1(i)], \\ \Lambda_{od}(i) = \begin{bmatrix} 0 & A_{okd}(i) \\ -I & I \end{bmatrix}, &\quad N_{dr}(i) = N_d(i) + N_b(i)L(i)NR(i). \end{aligned} \tag{4.25}$$

Taking into account the matrices of (4.9) − (4.10), we establish the following theorem:

Theorem 4.2 *System Σ_{TD} is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.19) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $L(i)$, $R(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_4(i) & \bar{M}(i) & \tau E_1 M_a(i) & \begin{matrix} \tau E_2 A_d(i) \\ + \tau E_2 B_o(i) \\ \times L(i) E_2 R(i) \end{matrix} & \begin{matrix} \bar{\Gamma}(i) + Y^t(i) \\ \times G(i)\Phi(i) \end{matrix} & \mathcal{R}(i) \\ \bar{M}^t(i) & -\varepsilon_1(i) I & 0 & 0 & 0 & 0 \\ M_a^t(i) E_1^t & 0 & -\tau\varepsilon_2(i) I & 0 & 0 & 0 \\ \begin{matrix} \tau A_d^t(i) E_2^t \\ + \tau R^t(i) E_2^t L^t(i) B_o^t(i) E_2^t \end{matrix} & 0 & 0 & \begin{matrix} -\tau Q_x(i) + \tau\varepsilon_2(i) \\ \times N_{dr}(i) N_{dr}^t(i) \end{matrix} & 0 & 0 \\ \bar{\Gamma}^t(i) + \Phi^t(i) G^t(i) Y(i) & 0 & 0 & 0 & \begin{matrix} -\gamma^2 I \\ + \Phi^t(i)\Phi(i) \end{matrix} & 0 \\ \mathcal{R}^t(i) & 0 & 0 & 0 & 0 & -\mathcal{Y}(i) \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q_x(i) & N_{dr}(i) \\ N_{dr}^t(i) & -\varepsilon_2(i) I \end{bmatrix} < 0, \qquad \begin{bmatrix} -Y(i)E_1 & I \\ I & -R(i) \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{4.26}$$

where

$$\begin{aligned}\Pi_4(i) = {} & Y^t(i)\bar{A}^t_{od}(i) + \bar{A}_{od}(i)Y(i) - E_1 Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)L(i) \\ & + L^t(i)\bar{B}^t_o(i) + Y^t(i)\Omega(\tau,i)Y(i) + \alpha_{ii}E_1 Z^t(i)E_2 \\ & + \varepsilon_1(i)Y^t(i)N^t_{ad}(i)N_b(i)E_1^t L(i) + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_b(i)E_1^t L(i) \\ & + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_{ad}(i)Y(i)\end{aligned} \tag{4.27}$$

and the delayed-feedback gain is given by $K_d(i) = L(i)E_1R(i)$.

Proof By similarity to Theorem 3.1 and letting the Lyapunov functional $V(\cdot)$ be given by (3.5), the weak infinitesimal operator $\Im_3^\zeta[\cdot]$ of the process $\{\zeta(t),\eta_t,\, t\ge 0\}$ for system (4.22)–(4.23) at the point $\{t,x,\eta_t\}$ is given by (3.6). Hence, it is easy to see that:

$$\begin{aligned}\frac{\partial V}{\partial \zeta}\,\dot\zeta(t) = {} & 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_{\Delta d}(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) \\ & + 2\int_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon_k(i)\zeta(\theta)\,d\theta.\end{aligned} \tag{4.28}$$

Hence, it follows from (3.6) and (4.27) that

$$\begin{aligned}\Im_3^\zeta[V] = {} & \zeta^t(t)\Big[\Lambda^t_{\Delta d}(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{\Delta d}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m)\Big]\zeta(t) \\ & + 2\zeta^t(t)\mathbb{P}^t(i)\bar\Gamma(i)w(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon_k(i)\zeta(\theta)\,d\theta \\ & + \int_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta - \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta \\ & + \sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,ds d\theta.\end{aligned} \tag{4.29}$$

Following parallel developments to Theorem 4.1, we applying Facts 2 – 3, use (3.9), (4.7), (4.10) and (4.24) – (4.25) and manipulate, we get

$$\begin{aligned}\mathcal{J}(x) \le \mathbb{E}\Big\{ & \int_0^\infty \zeta^t(t)\Big[\Lambda^t_{od}(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{od}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m) \\ & + \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}^t_{kd}(i)\bar{N}_{kd}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) \\ & + \tau\mathbb{P}^t(i)E_2A_{kd}(i)[Q_x(i) - \varepsilon_2(i)N_{dk}(i)N^t_{dk}(i)]^{-1}A^t_{kd}(i)E_2^t\mathbb{P}(i) \\ & + \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i) \\ & + [\bar{P}^t(i)\bar\Gamma(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar\Gamma^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)]\Big]\zeta(t)\Big\}\end{aligned} \tag{4.30}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. It follows from inequality (4.30) that $\mathcal{J}(x) < 0$ is guaranteed if the following inequality

$$\begin{aligned}
&\Lambda_{od}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{od}(i) + \sum_{m=1}^{s} \alpha_{im}\bar{P}(m) + \tau E_2 Q_x(i) E_2^t + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i) \\
&\qquad + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) \\
&\quad + \tau\mathbb{P}^t(i)E_2 A_{kd}(i)[Q_x(i) - \varepsilon_2(i)N_{dk}(i)N_{dk}^t(i)]^{-1}A_{kd}^t(i)E_2^t\mathbb{P}(i) \\
&+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1 M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i) \\
&\quad + [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)] < 0
\end{aligned} \tag{4.31}$$

holds. Premultiplying (4.17) by $\mathbb{P}^{-t}(i)$, postmultiplying by $\mathbb{P}^{-1}(i)$, using (4.27) and manipulating with the help of Fact 3, we obtain the LMI (4.26). It follows that system (4.22)−(4.23) is DDRSS with disturbance attenuation $\gamma > 0$ under the state-delayed control law (4.19).

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 4.2 *The nominal jump system Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ under the control law (4.19) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $L(i)$, $R(i)$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix}
\Pi_{40}(i) & \tau E_2[A_d(i) + B_o(i)L(i)E_2R(i)] & \bar{\Gamma}(i) + Y^t(i)G(i)\Phi(i) & \mathcal{R}(i) \\
\tau A_d^t(i)E_2^t & -\tau Q_x(i) & 0 & 0 \\
\bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) & 0 \\
\mathcal{R}^t(i) & 0 & 0 & -\mathcal{Y}(i)
\end{bmatrix} < 0,$$

$$\begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} -Y(i)E_1 & I \\ I & -R(i) \end{bmatrix} \geq 0, \qquad i \in \mathcal{S}, \tag{4.32}$$

where

$$\begin{aligned}
\Pi_{40}(i) = {} & Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1 Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)L(i) \\
& + L^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega_o(\tau, i)Y(i) + \alpha_{ii}E_1 Z^t(i)E_2
\end{aligned} \tag{4.33}$$

and the delayed-feedback gain is given by $K_d(i) = L(i)NR(i)$.

4.2 Example 2

We use the data of Example 1 in addition to

$$B_o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_o(2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B_o(3) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$N_b(1) = [0.1 \quad 0.3], \qquad N_b(2) = [0.2 \quad 0.2], \qquad N_b(3) = [0.3 \quad 0.1]$$

and the level of disturbance attenuation $\gamma = 1.35$. For the data under consideration and in view of Theorem 4.1, the feasible solutions of LMIs (4.11) using the software LMILab [7] yields the gain matrices

$$K(1) = \begin{bmatrix} 0.8532 & 0.9260 \\ -1.4317 & -1.2628 \end{bmatrix}, \qquad K(2) = \begin{bmatrix} 0.9145 & -0.6128 \\ 0.5844 & 1.9912 \end{bmatrix},$$
$$K(3) = \begin{bmatrix} 1.1425 & 0.6603 \\ -0.3123 & 0.4912 \end{bmatrix}$$

for

$$\begin{aligned} &\varepsilon_1(1) = 1.3345, \qquad \varepsilon_2(1) = 0.9144, \qquad \rho(1) = 2.4367, \\ &\varepsilon_1(2) = 2.3567, \qquad \varepsilon_2(2) = 2.5433, \qquad \rho(2) = 1.5321, \\ &\varepsilon_1(3) = 5.2355, \qquad \varepsilon_2(3) = 0.6673, \qquad \rho(3) = 2.3226, \end{aligned}$$

and $\tau \le 0.4772$.

On the other hand, considering Theorem 4.2 we solve the LMIs (4.26) to get the gain matrices

$$K_d(1) = \begin{bmatrix} 0.0454 & -0.9231 \\ 0.0422 & 0.9123 \end{bmatrix}, \qquad K_d(2) = \begin{bmatrix} -0.1636 & 0.2628 \\ -0.5628 & 1.2182 \end{bmatrix},$$
$$K_d(3) = \begin{bmatrix} 0.3144 & 1.1268 \\ -0.7435 & -0.8655 \end{bmatrix}$$

for

$$\begin{aligned} &\varepsilon_1(1) = 3.4225, \qquad \varepsilon_2(1) = 0.7428, \qquad \rho(1) = 1.3452, \\ &\varepsilon_1(2) = 1.7111, \qquad \varepsilon_2(2) = 1.6655, \qquad \rho(2) = 3.0987, \\ &\varepsilon_1(3) = 4.0205, \qquad \varepsilon_2(3) = 0.0876, \qquad \rho(3) = 4.2247 \end{aligned}$$

and $\tau \le 0.4653$.

5 $\mathcal{H}_\infty$-Output Feedback Controller

In this section, we consider the design of an $\mathcal{H}_\infty$-output feedback controller for the jumping system for $\eta = i \in \mathcal{S}$

$$\begin{aligned} &\dot{x}(t) = A_{\Delta o}(t,i)x(t) + A_{\Delta d}(t,i)x(t-\tau) + B_{\Delta o}(t,i)u(t) + \Gamma(i)w(t), \\ &x(t) = \phi(t), \quad t \in [-\tau, 0], \quad t \ge 0, && (5.1) \\ &y(t) = C_o(i)x(t) + D_o(i)w(t), && (5.2) \\ &z(t) = G(i)x(t) + \Phi(i)w(t), && (5.3) \end{aligned}$$

where $y(t) \in \Re^p$ is the measured output and the matrices $C_o(i)$, $D_o(i)$ are constant with appropriate dimensions. Note that system (5.1)–(5.3) is more general (2.3)–(2.4) for control design purposes. A dynamic output feedback controller for $i \in \mathcal{S}$, has the form:

$$\begin{aligned} \dot{x}_C(t) &= A_C(i)x_C(t) + B_C(i)[y(t) - C_o(i)x_C(t)], \\ u(t) &= C_C(i)x_C(t), \qquad (5.4) \end{aligned}$$

where $x_C(t) \in \Re^n$ is the state of the controller and the matrices $A_C(i) \in \Re^{n\times n}$, $B_C(i) \in \Re^{n\times p}$, $C_C(i) \in \Re^{m\times n}$ are controller matrices to be determined. Combining (5.1) – (5.4) for $i \in \mathcal{S}$, we obtain the closed-loop system

$$\begin{aligned}
&\dot{\xi}(t) = A_{JC\Delta}(t,i)\xi(t) + A_{JCd\Delta}(t,i)\xi(t-\tau(t)) + \Gamma_{JC\Delta}(t,i)w(t), \quad t \geq 0,\\
&\xi(t) = \phi_{JC}(t), \quad t \in [-\tau^*, 0],\\
&z(t) = \bar{G}(i)\xi(t) + \Phi(i)w(t),
\end{aligned} \tag{5.5}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ x_C(t) \end{bmatrix} \in \Re^{2n},$$

$$A_{JCd\Delta}(t,i) = \bar{A}_d(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCd}(i), \tag{5.6}$$

$$A_{JC\Delta}(t,i) = \begin{bmatrix} A_{\Delta o}(i) & B_{\Delta o}(i)C_C(i) \\ B_C(i)C_o(i) & A_C(i) - B_C(i)C_o(i) \end{bmatrix} = A_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCa}(i),$$

$$\Gamma_{JC\Delta}(t,i) = \begin{bmatrix} \Gamma(i) \\ B_C D_o(i) \end{bmatrix} = \Gamma_{JCo}(i) + \bar{M}_a(i)\Delta_a\bar{N}_d(i)$$

and

$$\begin{aligned}
&A_{JCo}(i) = \begin{bmatrix} A_o(i) & B_o(i)C_C(i) \\ B_C(i)C_o(i) & A_C(i) - B_C(i)C_o(i) \end{bmatrix},\\
&\bar{M}_{JC}(i) = \begin{bmatrix} 0 \\ \bar{M}_a \end{bmatrix}, \quad \bar{N}_{JCd} = [0 \quad \hat{N}_d],\\
&\bar{M}_a(i) = \begin{bmatrix} M_a(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{N}_a(i) = \begin{bmatrix} N_a(i) & 0 \\ 0 & 0 \end{bmatrix},\\
&\hat{N}_d(i) = \begin{bmatrix} N_d(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_d(i) = \begin{bmatrix} A_d(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_{JCo}(i) = \begin{bmatrix} \Gamma(i) \\ B_C D_o(i) \end{bmatrix}.
\end{aligned} \tag{5.7}$$

Now for each possible value $\eta_t = i$, $i \in \mathcal{S}$, we introduce the following state transformation

$$\mu(t) = \xi(t) + \int_{t-\tau}^{t} A_{JCd\Delta}(t,i)\xi(s)\,ds \tag{5.8}$$

into (5.5) to yield

$$\dot{\mu}(t) = [A_{JC\Delta}(t,i) + A_{JCd\Delta}(t,i)]\xi(t) + \bar{\Gamma}_{JCo}(i)w(t). \tag{5.9}$$

Define the augmented state-vector

$$\omega(t) = \begin{bmatrix} \mu(t) \\ \xi(t) \end{bmatrix} \in \Re^{4n}. \tag{5.10}$$

By combining (5.1) and (5.8) – (5.10), we obtain the transformed system

$$\dot{\omega}(t) = \Lambda_{JC\Delta}(i)\omega(t) + \int_{t-\tau}^{t} \Upsilon_{JC\Delta}(i)\omega(s)\,ds + \Gamma_{JCo}(i)w(t),$$

$$\omega(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \tag{5.11}$$

$$z(t) = \hat{G}(i)\omega(t) + \Phi(i)w(t), \tag{5.12}$$

where

$$\Lambda_{JC\Delta}(i) = \begin{bmatrix} 0 & A_{JC\Delta}(t,i) + A_{JCd\Delta}(t,i) \\ -I & I \end{bmatrix} = \Lambda_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCe}(i),$$

$$\Upsilon_{JC\Delta}(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{JCd\Delta}(t,i) \end{bmatrix} = \Upsilon_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCd}, \tag{5.13}$$

$$\Lambda_{JCo}(i) = \begin{bmatrix} 0 & A_{JCo}(i) + \bar{A}_d(i) \\ -I & I \end{bmatrix}, \qquad \Upsilon_{JCo}(i) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_d(i) \end{bmatrix},$$

$$\bar{\Gamma}_{JCo}(i) = \begin{bmatrix} \Gamma_{JCo}(i) \\ 0 \end{bmatrix}, \qquad \bar{N}_{JCe} = [\hat{N}_d(i) + \hat{N}_a(i) \quad 0], \qquad \hat{G}(i) = [0 \quad \bar{G}(i)].$$

Given matrices

$$0 < \mathcal{P}_\mu(i) \in \Re^{2n}, \qquad \mathcal{P}_d(i) \in \Re^{2n}, \qquad \mathcal{P}_\xi(i) \in \Re^{2n}, \qquad i \in \mathcal{S},$$
$$\mathcal{P}(i) = \begin{bmatrix} \mathcal{P}_\mu(i) & 0 \\ \mathcal{P}_d(i) & \mathcal{P}_\xi(i) \end{bmatrix} \in \Re^{4n} \tag{5.14}$$

such that for $i \in \mathcal{S}$

$$\mathcal{P}^{-1}(i) = \begin{bmatrix} \mathcal{X}_\mu(i) & 0 \\ \mathcal{X}_d(i) & \mathcal{X}_\xi(i) \end{bmatrix}, \qquad \mathcal{X}_\mu(i) = \begin{bmatrix} \mathcal{X}_{\mu1}(i) & 0 \\ 0 & \mathcal{X}_{\mu2} \end{bmatrix},$$
$$\mathcal{X}_\xi(i) = \begin{bmatrix} \mathcal{X}_{\xi1}(i) & 0 \\ 0 & \mathcal{X}_{\xi2} \end{bmatrix}, \qquad \mathcal{X}_d(i) = \begin{bmatrix} \mathcal{X}_{d1}(i) & 0 \\ 0 & \mathcal{X}_{d2} \end{bmatrix}, \tag{5.15}$$
$$\mathcal{X}_\mu(i) = \mathcal{P}_\mu^{-1}(i), \qquad \mathcal{X}_d(i) = -\mathcal{X}_\mu(i)\mathcal{P}_d(i)\mathcal{X}_\xi(i), \qquad \mathcal{X}_\xi(i) = \mathcal{P}_\xi^{-1}(i)$$

and define the matrices:

$$\Sigma(i) = [\mathcal{X}_\mu(i) \quad \mathcal{X}_\xi(i)], \qquad \bar{A}^t{}_{JCod}(i) = [A^t_{JCo}(i) + \bar{A}^t_d(i) \quad I], \qquad \Xi(i) = \begin{bmatrix} 0 \\ \mathcal{X}_\mu(i) \end{bmatrix},$$
$$\Theta(\tau,i) = \tau E_2 I\!R(i) E_2^t + \varepsilon_1(i)\bar{N}^t_{JCd}(i)\bar{N}_{JCd}(i) \tag{5.16}$$
$$+ \bar{G}^t(i)\bar{G}(i) + \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im} I\!R(m)E_2^t.$$

It follows from Theorem 3.1 that given matrix sequence $0 < I\!R(i) = I\!R^t(i)$, $i \in \mathcal{S}$ the transformed system (5.11)–(5.12) is DDRSS with disturbance attenuation $\gamma > 0$ if the algebraic inequality:

$$\Sigma^t(i)\Lambda^t_{JCod}(i) + \Lambda_{JCod}(i)\Sigma(i) - E_1\Xi^t(i) - \Xi(i)E_1^t$$
$$+ E_1\Xi^t(i)E_2\Bigg(\sum_{m=1}^{s}\alpha_{im}[E_2^t\Xi(m)]^{-1}\Bigg)E_2^t\Xi(i)E_1^t + \varepsilon_1^{-1}(i)\bar{M}_a(i)E_1\bar{M}_a^t(i)E_1^t$$
$$+ \tau\varepsilon_2^{-1}(i)\bar{M}_{JC}(i)\bar{M}^t_{JC}(i) + \Sigma^t(i)\Theta(\tau,i)\Sigma(i)(i)$$
$$+ \tau E_2\bar{A}_d(i)[I\!R(i) - \varepsilon_2(i)\bar{N}_{JCd}(i)\bar{N}^t_{JCd}(i)]^{-1}\bar{A}^t_d(i)E_2^t \tag{5.17}$$
$$+ [\bar{\Gamma}_{JCo}(i) + \mathcal{X}^t(i)\hat{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t_{JCo}(i) + \Phi^t(i)\hat{G}(i)\mathcal{X}]$$
$$\triangleq I\!M(\tau,i) = \begin{bmatrix} I\!M_\mu(\tau,i) & I\!M_c(\tau,i) \\ I\!M_c^t(\tau,i) & I\!M_\xi(\tau,i) \end{bmatrix} < 0$$

is satisfied for some positive scalars $\varepsilon_1(i)$, $\varepsilon_2(i)$, $\rho(i)$, $i \in \mathcal{S}$, where

$$\begin{aligned}
I\!M_\mu(\tau,i) &= \begin{bmatrix} I\!M_{\mu1}(\tau,i) & I\!M_{\mu3}(\tau,i) \\ I\!M^t_{\mu3}(\tau,i) & I\!M_{\mu2}(\tau,i) \end{bmatrix}, \qquad (5.18)\\
I\!M_c(\tau,i) &= \begin{bmatrix} I\!M_{c1}(\tau,i) & I\!M_{c3}(\tau,i) \\ I\!M_{c4}(\tau,i) & I\!M_{c2}(\tau,i) \end{bmatrix},\\
I\!M_\xi(\tau,i) &= \begin{bmatrix} I\!M_{\xi1}(\tau,i) & 0 \\ 0 & I\!M_{\xi2}(\tau,i) \end{bmatrix},\\
\Omega_\mu(\tau,i) &= \tau I\!R(i) + \epsilon_1(i)[N_a(i)+N_d(i)][N^t_a(i)+N^t_d(i)] + G^t(i)G(i)\\
&\quad + \tau^2\rho(i)\sum_m \alpha_{im} I\!R(m),\\
I\!M_{\mu1}(\tau,i) &= [A_o(i)+A_d(i)]\mathcal{X}_{d1}(i) + \mathcal{X}^t_{d1}(i)[A^t_o(i)+A^t_d(i)]\\
&\quad + \mathcal{X}^t_{\mu1}(i)\sum_m \mathcal{X}^{-1}_{\mu1}(m)\mathcal{X}_{\mu1}(i) + \epsilon^{-1}M_a(i)M^t_a(i) + \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Gamma^t(i),\\
I\!M_{\mu3}(\tau,i) &= B_o(i)C_C(i)\mathcal{X}_{d2}(i) + \mathcal{X}^t_{d1}(i)C^t_o(i)B^t_C(i)\\
&\quad + \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[D^t_o(i)B^t_C(i) + \Phi^t(i)G(i)\mathcal{X}_{d2}(i)],\\
I\!M_{\mu2}(\tau,i) &= [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) + \mathcal{X}^t_{d2}[A^t_C(i) - C^t_o(i)B^t_C(i)]\\
&\quad + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\sum_m \alpha_{im}\mathcal{X}^{-1}_{\mu2}(m)\mathcal{X}_{\mu2}(i),\\
I\!M_{c1}(\tau,i) &= -\mathcal{X}^t_{\mu1}(i) + \mathcal{X}^t_{d1}(i) + [A_o(i)+A_d(o)]\mathcal{X}^t_{\mu1}(i),\\
I\!M_{c2}(\tau,i) &= -\mathcal{X}^t_{\mu2}(i) + \mathcal{X}^t_{d2}(i) + [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{\mu2}(i)\\
&\quad + [B_C(i)D_o(i) + \mathcal{X}^t_{d2}(i)G^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}_{\mu2}(i)\\
&\quad + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\sum_m \alpha_{im}\mathcal{X}^{-1}_{\mu1}(m)\mathcal{X}_{\xi2}(i),\\
I\!M_{c4}(\tau,i) &= B_C(i)C_o(i)\mathcal{X}^t_{\mu1}(i),\\
I\!M_{c3}(\tau,i) &= \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}^t_{\mu2}(i) + B_o(i)C_C(i)\mathcal{X}^t_{\mu2}(i),\\
I\!M_{\xi1}(\tau,i) &= \mathcal{X}_{\mu1}(i) + \mathcal{X}^t_{\mu1}(i) + \tau\epsilon_2^{-1}M_a(i)M^t_a(i) + \tau A_d(i)[I\!R(i) - \epsilon_2 N_{od}N^t_{od}]^{-1}A^t_d(i),\\
I\!M_{\xi2}(\tau,i) &= \mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i)\\
&\quad + \mathcal{X}^t_{\mu2}(i)G^t(i)\Phi(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}_{\mu2}(i).
\end{aligned}$$

Our objective is to develop conditions that can be used for computing the gains of the dynamic output feedback controller. The following theorem summarizes the main solvability conditions for controller (5.4) guaranteeing that the closed-loop system (5.11) – (5.12) is delay-dependent robustly stochastically stable with disturbance attenuation γ.

Theorem 5.1 *Consider the closed-loop system (5.11) – (5.12) with matrices described in (5.6) – (5.7) and (5.13) – (5.16). Given scalars $\gamma > 0$, $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i)$, $i \in \mathcal{S}$, there exists a dynamic output feedback controller of the type (5.4) such that the closed-loop system (5.11) – (5.12) is DDRSS with a disturbance attenuation γ if there exist matrices $\mathcal{X}_{\mu1}(i)$, $\mathcal{X}_{\mu2}(i)$, $\mathcal{X}_{\xi1}(i)$, $\mathcal{X}_{\xi2}(i)$, $\mathcal{X}_{d1}(i)$, $\mathcal{X}_{d2}(i)$, $i \in \mathcal{S}$ satisfying the following system of simultaneous matrix inequalities and equations*

$$\begin{bmatrix} \mathcal{X}_{\xi 1}(i) + \mathcal{X}_{\xi 1}^t(i) & \tau M_a(i) & \tau A_d(i) \\ \tau M_a^t(i) & -\epsilon_2 I & 0 \\ \tau A_d^t(i) & 0 & -[I\!R - \epsilon_2 N_{od} N_{od}^t] \end{bmatrix} < 0, \tag{5.19}$$

$$\begin{bmatrix} \begin{array}{c}[A_o(i) + A_d(i)]\mathcal{X}_{d1}(i) \\ + \mathcal{X}_{d1}^t(i)[A_o(i) + A_d(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 1}^t(i)\end{array} & M_a(i) & \Gamma(i) & \mathcal{W}_1(i) \\ M_a^t(i) & -\epsilon_1 I & 0 & \\ \Gamma^t(i) & 0 & -[\gamma^2 I - \Phi^t(i)\Phi(i)] & 0 \\ \mathcal{W}_1^t(i) & 0 & 0 & -\mathcal{V}_1(i) \end{bmatrix} < 0, \tag{5.20}$$

$$\begin{bmatrix} \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\mu 2}^t(i) + \mathcal{X}_{\mu 2}^t(i)\Omega_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) & \mathcal{X}_{\mu 2}^t(i) G^t(i)\Phi(i) \\ \Phi^t(i) G(i)\mathcal{X}_{\mu 2}(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] \end{bmatrix} < 0, \tag{5.21}$$

$$\begin{bmatrix} \begin{array}{c}[A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) \\ + \mathcal{X}_{d2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 2}^t(i)\end{array} & \mathcal{X}_{\mu 2}^t(i) & \mathcal{W}_2(i) \\ \mathcal{X}_{\mu 2}(i) & -\Omega_\mu(\tau, i) & 0 \\ \mathcal{W}_2^t(i) & 0 & -\mathcal{V}_2(i) \end{bmatrix} < 0, \tag{5.22}$$

$$\mathcal{X}_{d1}(i) - \mathcal{X}_{\mu 1}(i) + \mathcal{X}_{\xi 1}^t(i)[A_o(i) + A_d(i)]^t = 0, \tag{5.23}$$

$$\mathcal{X}_{d2}(i) - \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\xi 2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \mathcal{X}_{\xi 2}(i)\Omega_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) = 0. \tag{5.24}$$

Then the associated controller matrices are given by:

$$\begin{aligned} A_C(i) &= A_o(i), \\ B_C(i) &= -\mathcal{X}_{d2}^t(i) G^t(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]\Phi(i) D_o^\dagger(i), \\ C_C(i) &= B_o^\dagger(i)\Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]\Phi(i) G(i), \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \mathcal{V}_1(i) &= \mathrm{diag}\big[\mathcal{X}_{\mu 1}^t(1) \dots \mathcal{X}_{\mu 1}^t(i-1)\ \mathcal{X}_{\mu 1}^t(i+1) \dots \mathcal{X}_{\mu 1}^t(s)\big], \\ \mathcal{V}_2(i) &= \mathrm{diag}\big[\mathcal{X}_{\mu 2}^t(1) \dots \mathcal{X}_{\mu 2}^t(i-1)\ \mathcal{X}_{\mu 2}^t(i+1) \dots \mathcal{X}_{\mu 2}^t(s)\big], \\ \mathcal{W}_1(i) &= \big[\sqrt{\alpha_{i1}}\mathcal{X}_{\mu 1}^t(1) \dots \sqrt{\alpha_{is}}\mathcal{X}_{\mu 1}^t(s)\big], \\ \mathcal{W}_2(i) &= \big[\sqrt{\alpha_{i1}}\mathcal{X}_{\mu 1}^t(1) \dots \sqrt{\alpha_{is}}\mathcal{X}_{\mu 1}^t(s)\big] \end{aligned}$$

and $B_o^\dagger(i)$ and $D_o^\dagger(i)$ are the pseudo-inverse of $D_o(i)$ and $B_o(i)$, respectively.

Proof We start from matrix inequality (5.17) and using (5.18) with standard algebraic manipulations, it follows that the choice of the controller matrices (5.25) subject to inequalities (5.19)–(5.24) ensures that $I\!M(\tau, i) < 0$, $i \in \mathcal{S}$ and hence guarantees that system (5.11)–(5.12) is DDRSS with a disturbance attenuation γ and the proof is completed.

In the absence of uncertainties, the closed-loop system (5.11)–(5.12) reduces to

$$\begin{aligned} \dot{\omega}(t) &= \Lambda_{JCo}(i)\omega(t) + \int_{t-\tau}^{t} \Upsilon_{JCo}(i)\omega(s)\, ds + \Gamma_{JCo}(i) w(t), \\ \omega(t) &= \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \end{aligned} \tag{5.26}$$

$$z(t) = \hat{G}(i)\omega(t) + \Phi(i) w(t) \tag{5.27}$$

and for which the following corollary holds:

Corollary 5.1 *Consider the closed-loop system (5.26) – (5.27) with matrices described in (5.6) – (5.7) and (5.13) – (5.16). Given scalars $\rho(i) > 0$, $i \in calS$, $\gamma > 0$, there exists a dynamic output feedback controller of the type (5.4) such that the closed-loop system (5.26) – (5.27) is DDRSS with a disturbance attenuation γ if there exist matrices $\mathcal{X}_{\mu 1}(i)$, $\mathcal{X}_{\mu 2}(i)$, $\mathcal{X}_{\xi 1}(i)$, $\mathcal{X}_{\xi 2}(i)$, $\mathcal{X}_{d1}(i)$, $\mathcal{X}_{d2}(i)$, $i \in \mathcal{S}$ satisfying the following system of simultaneous matrix inequalities and equations*

$$\begin{bmatrix} \mathcal{X}_{\xi 1}(i) + \mathcal{X}_{\xi 1}^t(i)(i) & \tau A_d(i) \\ \tau A_d^t(i) & -I\!R \end{bmatrix} < 0, \tag{5.28}$$

$$\begin{bmatrix} \begin{matrix} [A_o(i) + A_d(i)]\mathcal{X}_{d1}(i) \\ + \mathcal{X}_{d1}^t(i)[A_o(i) + A_d(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 1}^t(i) \end{matrix} & \Gamma(i) & \mathcal{W}_1(i) \\ \Gamma^t(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] & 0 \\ \mathcal{W}_1^t(i) & 0 & -\mathcal{V}_1(i) \end{bmatrix} < 0, \tag{5.29}$$

$$\begin{bmatrix} \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\mu 2}^t(i) + \mathcal{X}_{\mu 2}^t(i)\bar{\Omega}_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) & \mathcal{X}_{\mu 2}^t(i)G^t(i)\Phi(i) \\ \Phi^t(i)G(i)\mathcal{X}_{\mu 2}(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] \end{bmatrix} < 0, \tag{5.30}$$

$$\begin{bmatrix} \begin{matrix} [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) \\ + \mathcal{X}_{d2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 2}^t(i) \end{matrix} & \mathcal{X}_{\mu 2}^t(i) & \mathcal{W}_2(i) \\ \mathcal{X}_{\mu 2}(i) & -\bar{\Omega}_\mu(\tau, i) & 0 \\ \mathcal{W}_2^t(i) & 0 & -\mathcal{V}_2(i) \end{bmatrix} < 0, \tag{5.31}$$

$$\mathcal{X}_{d1}(i) - \mathcal{X}_{\mu 1}(i) + \mathcal{X}_{\xi 1}^t(i)[A_o(i) + A_d(i)]^t = 0, \tag{5.32}$$

$$\mathcal{X}_{d2}(i) - \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\xi 2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \mathcal{X}_{\xi 2}(i)\bar{\Omega}_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) = 0, \tag{5.33}$$

where

$$\bar{\Omega}_\mu(\tau, i) = \tau I\!R(i) + G^t(i)G(i) + \tau^2\rho(i)\sum_m \alpha_{im} I\!R(m) \tag{5.34}$$

and the associated controller matrices are given by (5.25).

5.1 Example 3

We consider the multi-reach water quality system with the data given in Examples 1 and 2 in addition to the following

$$C_o(1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad C_o(2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad C_o(3) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$D_o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_o(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_o(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

With the aid of the LMILab [7], the feasible solutions of LMIs (5.19) – (5.24) yields the controller matrices:

$$A_C(1) = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \quad A_C(2) = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_C(3) = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix},$$
$$B_C(1) = \begin{bmatrix} 0.7854 & -1.3246 \\ 0.2234 & -2.0045 \end{bmatrix}, \qquad B_C(2) = \begin{bmatrix} -1.1157 & 0.8006 \\ 0.7256 & -1.7654 \end{bmatrix},$$

$$B_C(3) = \begin{bmatrix} 0.3423 & -1.0206 \\ -0.5494 & 3.1145 \end{bmatrix},$$

$$C_C(1) = \begin{bmatrix} 0.2238 & 0.0912 \\ 0.5412 & 0.7644 \end{bmatrix}, \qquad C_C(2) = \begin{bmatrix} 0.3458 & 0.9442 \\ -0.1244 & -0.4564 \end{bmatrix},$$

$$C_C(3) = \begin{bmatrix} -0.8121 & 0.8724 \\ 0.8126 & -0.6944 \end{bmatrix}$$

for $\tau \le 0.6545$.

6 Conclusion

This paper has introduced a new transformation method for the $\mathcal{H}_\infty$ analysis and synthesis of a class of uncertain time-delay systems with Markovian jump parameters. It has been established that the new method exhibits the delay-dependence properties of the uncertain jumping system and therefore provides a tractable methodology for stability analysis, stabilization and output feedback control. All the developed results have been cast into the format of linear matrix inequalities and several examples have been worked out to illustrate the theory.

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Stabilization of a Class of Stochastic Nonlinear Time-Delay Systems*

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Abstract: In this paper, the stabilization problem is considered for a class of nonlinear continuous stochastic systems with state delays. The purpose of this problem is to design a state feedback controller such that the closed-loop system is exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic time-delay systems to be stable, and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness of the proposed design method.

Keywords: *Nonlinear systems; stochastic systems; time-delay; Lyapunov stability; algebraic matrix inequalities.*

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1 Introduction

Nonlinear stochastic control has long been an important research field that has attracted many researchers, and enormous results have been published in the literature. In particular, the fundamental nonlinear stochastic stabilization issue has received considerable research interests, and has found successful applications in control and communication problems, such as attitude control of satellites and missile control, macroeconomic system control, chemical process control, etc., see [8] for a survey.

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Recently, there have appeared many methods to tackle different kinds of nonlinear stochastic systems. For example, in [2], a minimax dynamic game approach has been developed for the controller design problem of the nonlinear stochastic systems that employ risk-sensitive performance criteria. The stabilization problem has been investigated in [3, 4] for nonlinear stochastic systems, and a stochastic counterpart of the input-to-state stabilization results has been provided. In [7], under an infinite-horizon risk-sensitive cost criterion, the problem of output feedback control design has been studied for a class of strict feedback stochastic nonlinear systems. In [16], the decentralized global stabilization problem has been dealt with by using a Lyapunov-based recursive design method. On the other hand, the dual nonlinear stochastic filtering problem has also been an active area for three decades [8], and a number of nonlinear filtering approaches have been proposed in the literature, such as extended Kalman filters, bound-optimal filters [13], exponentially bounded filters [14, 20], etc.

It is now a recognized fact that the time delay is frequently a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, and so on. Recently, increasing attention has been focused on robust and/or H_∞ control problems for linear systems with certain types of time-delays, see [1] for a survey. Within the stochastic framework, the stability analysis problem for linear time-delay systems has been studied by many authors. For example, in [11], the stability analysis problem for linear stochastic delay interval systems with Markovian switching has been considered. In [17], an LMI approach has been developed to cope with the robust H_∞ control problem for linear uncertain stochastic systems with state delay. As for nonlinear stochastic time-delay systems, the related results have been scattered, and most of the results have been concerned with the stability analysis issue, see e.g.[5, 9]. So far, the stabilization problem for general nonlinear time-delay systems has not been fully investigated and remains important.

In this paper, we will consider the stabilization problem for a class of nonlinear continuous stochastic systems with state delays. Such a class of systems have been intensively investigated in [18 − 20] for the nonlinear filtering problems. An effective algebraic matrix inequality approach is proposed to design the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness and effectiveness of the proposed design method.

Notation The notations in this paper are quite standard. R^n and $R^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax|\colon |x| = 1\} = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0, \infty]$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-\tau,0];\,R^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau,0];\,R^n)$-valued random variables $\xi=\{\xi(\theta)\colon\ -\tau\leq\theta\leq 0\}$ such that $\sup\limits_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^p<\infty$ where $E\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2 Problem Formulation and Assumptions

Consider the following nonlinear continuous-time state delayed stochastic system in a fixed complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$:

$$dx(t)=[f(x(t),u(t))+g(x(t-\tau))]\,dt+Dx(t)\,dw(t), \tag{1}$$

$$x(t)=\varphi(t),\quad t\in[-\tau,0], \tag{2}$$

where $x(t)\in R^n$ is the state, $u(t)\in R^m$ is the deterministic input, $y(t)\in R^p$ is the measurement output, and $f(\cdot,\cdot)\in R^n$ and $g(\cdot)\in R^n$ are nonlinear vector functions. $\tau>0$ denotes the state delay and $\varphi(t)$ is a continuous vector valued initial function. Here, $w(t)=[w_1(t)\,w_2(t)\ \ldots\ w_m(t)]^{\mathrm{T}}\in R^m$ is an m-dimensional Brownian motion. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(t)$. D is a known constant matrices with appropriate dimensions.

Assumption 1 The nonlinear vector functions $f(\cdot,\cdot)$ and $g(\cdot)$ are assumed to satisfy $f(0,0)=0$, $g(0)=0$ and

$$\left|f(x(t),u(t))-[\,A\quad B\,]\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\right|\leq a_{11}\left|\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\right|+a_{12}, \tag{3}$$

$$|g(x(t-\tau))-A_dx(t-\tau)|\leq a_{21}|x(t-\tau)|+a_{22}, \tag{4}$$

where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $A_d\in R^{n\times n}$ are known constant matrices, and $a_{11}>0$, $a_{12}\geq 0$, $a_{21}>0$ and $a_{22}\geq 0$ are known scalars.

Remark 1 The system (1)–(2) can be used to represent many important physical nonlinear systems subject to inherent state delays and stochastic exogenous noises with known statistics. Similar to [18–20], the nonlinear descriptions (3)–(4) quantify the maximum possible derivations from a linear model with (A,B,A_d) as its system parameter matrices, and are more general than those of [13], [14].

When a state feedback control law

$$u(t)=Kx(t) \tag{5}$$

is applied to the system (1)–(2), the closed-loop system is governed by

$$dx(t)=[f(x(t),\,Kx(t))+g(x(t-\tau))]\,dt+Dx(t)\,dw(t). \tag{6}$$

For notation convenience, we give the following definitions:

$$A_c=A+BK, \tag{7}$$

$$p(t)=f(x(t),Kx(t))-A_cx(t), \tag{8}$$

$$q(t)=g(x(t-\tau))-A_dx(t-\tau), \tag{9}$$

and then obtain from (6) that

$$dx(t) = [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)]\,dt + Dx(t)\,dw(t). \tag{10}$$

Now, let $x(t;\xi)$ denote the state trajectory from the initial data $x(\theta) = \xi(\theta)$ on $-\tau \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-\tau,0];\, R^n)$. It is clear from Assumption 1 that the system (10) admits a trivial solution $x(t;0) \equiv 0$ corresponding to the initial data $\xi = 0$.

Furthermore, we introduce the following concepts for stability and boundedness in the mean square.

Definition 1 Consider the system (10). For every $\xi \in L^2_{\mathcal{F}_0}([-\tau,0];\, R^n)$,

(1) the trivial solution is exponentially stable in the mean square if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$E|x(t;\xi)|^2 \le \alpha x^{-\beta t} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2; \tag{11}$$

(2) the trivial solution is exponentially ultimately bounded in the mean square if there exist constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that

$$E|x(t;\xi)|^2 \le \alpha x^{-\beta t} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 + \gamma. \tag{12}$$

The objective of this paper is to design a controller for the nonlinear time-delay system (1) − (2), such that the closed-loop systems is exponentially stable (or exponentially ultimately bounded) in the mean square. More specifically, we are interested in designing a controller parameter K such that:

(1) in the case of $a_{12} = 0$ and $a_{22} = 0$ (i.e., there are no bounded nonlinearities and uncertain disturbances), the solution of the system (10) is guaranteed to be exponentially stable;
(2) in the case of $a_{12} \ne 0$ or $a_{22} \ne 0$ (i.e., there are bounded nonlinearities or uncertain disturbances), the solution of the system (10) is guaranteed to be exponentially ultimately bounded in the mean square.

3 Main Results and Proofs

In this section, the controller analysis problem will be considered firstly. Given a controller structure, we shall establish the conditions under which the system dynamics is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square. Then, we shall take the controller design problem into account, whose purpose is to derive the explicit expression for the expected controller gain in terms of the positive definite solution to an algebraic matrix inequality.

The following theorem will play an essential role in the design of the expected controllers. It reveals that the exponential stability (or exponential ultimate boundedness) of the controlled nonlinear time-delay stochastic system (10) can be guaranteed if a positive definite solution to a modified algebraic Riccati-like matrix inequality (quadratic matrix inequality) is known to exist.

Theorem 1 *Let the controller parameter K be given. If there exist positive scalars ε_1, ε_2, ε_3, ε_4 such that the following matrix inequality*

$$A_c^{\mathrm{T}}P + PA_c + D^{\mathrm{T}}PD + (\varepsilon_1+\varepsilon_2)P^2 + 4\varepsilon_2^{-1}a_{11}^2(I + K^{\mathrm{T}}K) + Q < 0 \tag{13}$$

where

$$Q = \varepsilon_1^{-1}A_d^{\mathrm{T}}A_d + 4\varepsilon_2^{-1}a_{21}^2 I \tag{14}$$

has a solution $P > 0$, then in the mean square, the system (10) is

(i) *exponentially stable in the case of $a_{12} = 0$ and $a_{22} = 0$;*
(ii) *exponentially ultimately bounded in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$.*

Proof Fix $\xi \in L^2_{\mathcal{F}_0}([-\tau,0]; R^n)$ arbitrarily and write $x(t;\xi) = x(t)$. For $(x(t),t) \in R^n \times R_+$, we define the Lyapunov function candidate

$$V(x(t),t) = x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s)Qx(s)\,ds, \tag{15}$$

where P is the positive definite solution to the matrix inequality (13) and $Q > 0$ is defined in (14).

By Itô's formula (see, e.g., [10]), the stochastic derivative of V along a given trajectory is obtained as

$$\begin{aligned}
dV(x(t),t) &= \big\{x^{\mathrm{T}}(t)P[A_cx(t) + A_dx(t-\tau) + p(t) + q(t)] \\
&\quad + [A_cx(t) + A_dx(t-\tau) + p(t) + q(t)]^{\mathrm{T}}Px(t) \\
&\quad + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau) \\
&\quad + x^{\mathrm{T}}(t)D^{\mathrm{T}}PDx(t)\big\}dt + 2x^{\mathrm{T}}(t)PDx(t)\,dw(t) \\
&= \big\{x^{\mathrm{T}}(t)[A_c^{\mathrm{T}}P + PA_c + D^{\mathrm{T}}PD + Q]x(t) \\
&\quad + x^{\mathrm{T}}(t)PA_dx(t-\tau) + x^{\mathrm{T}}(t-\tau)A_d^{\mathrm{T}}Px(t) \\
&\quad + x^{\mathrm{T}}(t)P[p(t)+q(t)] + [p(t)+q(t)]^{\mathrm{T}}Px(t) \\
&\quad - x^{\mathrm{T}}(t-\tau)Qx(t-\tau)\big\}dt + 2x^{\mathrm{T}}(t)PDx(t)\,dw(t).
\end{aligned} \tag{16}$$

Let ε_1 and ε_2 be two positive scalars. Then the matrix inequality

$$\big[\varepsilon_1^{1/2}x^{\mathrm{T}}(t)P - \varepsilon_1^{-1/2}x^{\mathrm{T}}(t-\tau)A_d^{\mathrm{T}}\big]\big[\varepsilon_1^{1/2}x^{\mathrm{T}}(t)P - \varepsilon_1^{-1/2}x^{\mathrm{T}}(t-\tau)A_d^{\mathrm{T}}\big]^{\mathrm{T}} \geq 0$$

yields

$$\begin{aligned}
&x^{\mathrm{T}}(t)PA_dx(t-\tau) + x^{\mathrm{T}}(t-\tau)A_d^{\mathrm{T}}Px(t) \\
&\qquad \leq \varepsilon_1 x^{\mathrm{T}}(t)P^2x(t) + \varepsilon_1^{-1}x^{\mathrm{T}}(t-\tau)A_d^{\mathrm{T}}A_dx(t-\tau).
\end{aligned} \tag{17}$$

In the sequel, we will use several times the following simple inequality

$$(u+v)^{\mathrm{T}}(u+v) \leq 2u^{\mathrm{T}}u + 2v^{\mathrm{T}}v,$$

where $u \in R^n$ and $v \in R^n$.

Noticing the Assumption 1 and the definitions (7)–(9), we have

$$\begin{aligned} p^{\mathrm{T}}(t)p(t) &= |f(x(t),Kx(t)) - A_c x(t)|^2 \le \left\{ a_{11} \left| \begin{bmatrix} x(t) \\ Kx(t) \end{bmatrix} \right| + a_{12} \right\}^2 \\ &\le 2a_{11}^2 \left| \begin{bmatrix} x(t) \\ Kx(t) \end{bmatrix} \right|^2 + 2a_{12}^2 \le 2a_{11}^2 x^{\mathrm{T}}(t)\big(I + K^{\mathrm{T}}K\big)x(t) + 2a_{12}^2, \end{aligned} \tag{18}$$

$$\begin{aligned} q^{\mathrm{T}}(t)q(t) &= |g(x(t-\tau)) - A_d x(t-\tau)|^2 \le \big\{a_{21}|x(t-\tau)| + a_{22}\big\}^2 \\ &\le 2a_{21}^2 x^{\mathrm{T}}(t-\tau)x(t-\tau) + 2a_{22}^2. \end{aligned} \tag{19}$$

Then, it follows from (18), (19) and

$$\Psi_1 = \varepsilon_2^{1/2} x^{\mathrm{T}}(t)P - \varepsilon_2^{-1/2}[p(t)+q(t)]^{\mathrm{T}}, \quad \Psi_1\Psi_1^{\mathrm{T}} \ge 0$$

that

$$\begin{aligned} & x^{\mathrm{T}}(t)P[p(t)+q(t)] + [p(t)+q(t)]^{\mathrm{T}}Px(t) \\ &\quad \le \varepsilon_2 x^{\mathrm{T}}(t)P^2x(t) + \varepsilon_2^{-1}[p(t)+q(t)]^{\mathrm{T}}[p(t)+q(t)] \\ &\quad \le \varepsilon_2 x^{\mathrm{T}}(t)P^2x(t) + 2\varepsilon_2^{-1}[p^{\mathrm{T}}(t)p(t) + q^{\mathrm{T}}(t)q(t)] \\ &\quad = x^{\mathrm{T}}(t)[\varepsilon_2 P^2 + 4\varepsilon_2^{-1}a_{11}^2(I + K^{\mathrm{T}}K)]x(t) \\ &\qquad + 4\varepsilon_2^{-1}a_{21}^2 x^{\mathrm{T}}(t-\tau)x(t-\tau) + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2). \end{aligned} \tag{20}$$

For simplicity, we denote

$$\Pi = A_c^{\mathrm{T}}P + PA_c + D^{\mathrm{T}}PD + (\varepsilon_1+\varepsilon_2)P^2 + 4\varepsilon_2^{-1}a_{11}^2(I+K^{\mathrm{T}}K) + \varepsilon_1^{-1}A_d^{\mathrm{T}}A_d + 4\varepsilon_2^{-1}a_{21}^2 I, \tag{21}$$

and then (13) and (14) indicate that $\Pi < 0$.

Substituting (14), (17) and (20) into (16) gives

$$dV(x(t),t) \le \big[x^{\mathrm{T}}(t)\Pi x(t) + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\big]dt + 2x^{\mathrm{T}}(t)PDx(t)dw(t). \tag{22}$$

We are now in a position to show the expected exponential stability (or exponential ultimate boundedness) of the system (10), by using the the technique developed in [10]. Let $\beta > 0$ be the unique root of the equation

$$\lambda_{\min}(-\Pi) - \beta\lambda_{\max}(P) - \beta\tau\lambda_{\max}(Q)x^{\beta\tau} = 0 \tag{23}$$

where Π and Q are defined, respectively, in (21) and (14), P is the positive definite solution to (13), and τ is the time-delay.

We can obtain from (22) that

$$\begin{aligned} d\big[x^{\beta t}V(x(t),t)\big] &= x^{\beta t}\big[\beta V(x(t),t)dt + dV(x(t),t)\big] \\ &\le x^{\beta t}\left(-\big[\lambda_{\min}(-\Pi) - \beta\lambda_{\max}(P)\big]|x(t)|^2 + \beta\lambda_{\max}(Q)\int_{t-\tau}^{t} |x(s)|^2 ds \right)dt \\ &\quad + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)x^{\beta t}dt + 2x^{\beta t}x^{\mathrm{T}}(t)PDx(t)w(t)dt. \end{aligned}$$

Then, integrating both sides from 0 to $T > 0$ and taking the expectation result in

$$x^{\beta T} EV(x(T),T) \le \big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 - \big[\lambda_{\min}(-\Pi) - \beta\lambda_{\max}(P)\big] E\int_0^T x^{\beta t}|x(t)|^2 dt + \beta\lambda_{\max}(Q) E\int_0^T x^{\beta t}\int_{t-\tau}^{t} |x(s)|^2 ds dt + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}(x^{\beta T} - 1).$$

Note that

$$\int_0^T x^{\beta t}\int_{t-\tau}^{t} |x(s)|^2 ds dt \le \int_{-\tau}^{T}\left(\int_{\max(s,0)}^{\min(s+\tau,T)} x^{\beta t} dt\right)|x(s)|^2 ds \le \int_{-\tau}^{T} \tau x^{\beta(s+\tau)}|x(s)|^2 ds \le \tau x^{\beta\tau}\int_0^T x^{\beta t}|x(t)|^2 dt + \tau x^{\beta\tau}\int_{-\tau}^{0}|\xi(\theta)|^2 d\theta.$$

Then, considering the definition of β in (23), we have

$$x^{\beta T} EV(x(T),T) \le \big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 + \beta\lambda_{\max}(Q)\tau^2 x^{\beta\tau} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}(x^{\beta T} - 1),$$

and

$$E|x(T)|^2 \le \lambda_{\min}^{-1}(P)\Big(\big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 + \beta\lambda_{\max}(Q)\tau^2 x^{\beta\tau} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2\Big) x^{-\beta T} + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}\lambda_{\min}^{-1}(P)(x^{\beta T} - 1)x^{-\beta T}.$$

Notice that $(x^{\beta T} - 1)x^{-\beta T} < 1$ and let

$$\alpha = \lambda_{\min}^{-1}(P)\big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)(1 + \beta\tau x^{\beta\tau})\big], \quad \gamma = 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}\lambda_{\min}^{-1}(P).$$

Since $T > 0$ is arbitrary, the definition of exponential ultimate boundedness in (12) is then satisfied if $a_{12} \ne 0$ or $a_{22} \ne 0$. If $a_{12} = a_{12} = 0$, it is obvious that the definition of exponential stability in (11) is met. This completes the proof of Theorem 1.

Next, let us focus on deriving the *explicit* expression of expected controller gains by using an algebraic matrix inequality approach. It is worth mentioning that, in most literature concerning nonlinear stochastic stabilization problems, the solution has not been given as an explicit representation.

Based on Theorem 1, we can see that the controller design problem can be transformed into the following two-step problem: (i) find a necessary and sufficient condition for the existence of the positive definite matrix P such that there exists a controller gain K satisfying (13); and (ii) if the controller gain K exists, give the characterization of the set of expected controller gains in terms of the positive definite matrix P and some other free parameters.

Lemma 1 [6] *Let $X \in R^{m_1 \times n_1}$ and $Y \in R^{m_1 \times p_1}$ $(m_1 \leq p_1)$. There exists a matrix $U \in R^{n_1 \times p_1}$ which simultaneously satisfies $Y = XU$ and $UU^{\mathrm{T}} = I$ if and only if $XX^{\mathrm{T}} = YY^{\mathrm{T}}$.*

For presentation convenience, we define

$$\Gamma(\varepsilon_1, \varepsilon_2, P) = A^{\mathrm{T}}P + PA + D^{\mathrm{T}}PD + (\varepsilon_1 + \varepsilon_2)P^2 + 4\varepsilon_2^{-1}a_{11}^2 I + Q, \tag{24}$$

$$\begin{aligned}\Xi(\varepsilon_1, \varepsilon_2, P) = {} & A^{\mathrm{T}}P + PA + D^{\mathrm{T}}PD + P[(\varepsilon_1 + \varepsilon_2)I - 0.25\,\varepsilon_2 a_{11}^{-2}BB^{\mathrm{T}}]P \\ & + 4\varepsilon_2^{-1}(a_{11}^2 + a_{21}^2)I + \varepsilon_1^{-1}A_d^{\mathrm{T}}A_d,\end{aligned} \tag{25}$$

where Q is defined in (14).

The aforementioned two-step problem is solved in the following theorem.

Theorem 2 *There exist positive scalars ε_1, ε_2 and a positive definite matrix P such that the matrix inequality (13) has a solution K if and only if the following quadratic matrix inequality*

$$\Xi(\varepsilon_1, \varepsilon_2, P) < 0 \tag{26}$$

holds, where $\Xi(\varepsilon_1, \varepsilon_2, P)$ is defined in (25). Furthermore, if (26) is true, all gain matrices K satisfying the matrix inequality (13) can be parameterized by

$$K = (0.5\,a_{11}^{-1}\varepsilon_2^{1/2}\Lambda U - 0.25\,a_{11}^{-2}\varepsilon_2 PB)^{\mathrm{T}} \tag{27}$$

where $\Lambda \in R^{n \times m}$ is any matrix satisfying

$$\Lambda\Lambda^{\mathrm{T}} < -\Xi(\varepsilon_1, \varepsilon_2, P) \tag{28}$$

and $U \in R^{m \times m}$ is arbitrary orthogonal matrix (i.e., $UU^{\mathrm{T}} = I$).

Proof Rewrite the matrix inequality (13) as

$$K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + 4\varepsilon_2^{-1}a_{11}^2 K^{\mathrm{T}}K + \Gamma(\varepsilon_1, \varepsilon_2, P) < 0, \tag{29}$$

where $\Gamma(\varepsilon_1, \varepsilon_2, P)$ is defined in (24).

In terms of the definition of $\Xi(\varepsilon_1, \varepsilon_2, P)$ in (25), we can rearrange (29) as

$$(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)^{\mathrm{T}} < -\Xi(\varepsilon_1, \varepsilon_2, P). \tag{30}$$

Obviously, there exists a controller gain matrix K such that the inequality (13) (or equivalently (30)) holds for some positive scalars ε_1, ε_2 and positive definite matrix P if and only if the right-hand side of (30) is positive definite, i.e., $-\Xi(\varepsilon_1, \varepsilon_2, P) > 0$ or (26) holds. The first part of this theorem is proved.

Assume now that (26) is true. Note that the dimension of the controller gain K is $m \times n$. From (30) and the definition of $\Lambda \in R^{n \times m}$ in (28), we could relate a Λ such that

$$(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)^{\mathrm{T}} = \Lambda\Lambda^{\mathrm{T}}. \tag{31}$$

It then follows from Lemma 1 that (31) holds if and only if

$$2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\,\varepsilon_2^{1/2}a_{11}^{-1}PB = \Lambda U, \tag{32}$$

where $U \in R^{m \times m}$ is an arbitrary orthogonal matrix. Therefore, the expression (27) follows immediately. This completes the proof of the theorem.

Finally, our main results can be summarized in the following corollary.

Corollary 1 *Consider the nonlinear discrete-time state delayed stochastic system (1) – (2) with the state feedback controller* $u(t) = Kx(t)$*. If there exist positive scalars* ε_1*,* ε_2*, and a positive definite matrix* P *such that the matrix inequality (26) holds, then the state feedback controller with its gain given in (27) will be such that the system (10) is exponentially stable in the case of* $a_{12} = 0$ *and* $a_{22} = 0$*; or exponentially ultimately bounded in the case of* $a_{12} \neq 0$ *or* $a_{22} \neq 0$*, both in the mean square.*

Remark 2 Corollary 1 solves the addressed stabilization problem for the class of nonlinear time-delay stochastic systems in this paper. In implementation, we could first solve the quadratic matrix inequality (26), and then obtain the expected control parameters from (27) easily. Firstly, based on the algorithms provided in [15] and references therein, we may select appropriate positive scalar parameters ε_1 and ε_2 so as to reduce the conservatism that may have resulted from the inequalities (17) and (20). Then, (26) will be a standard quadratic matrix inequality (QMI) for P. For details concerning the *general* QMIs and relevant algorithms, we refer the reader to [12]. It can also be noticed that, there exists a lot of design freedom in our proposed procedure, such as the choices of matrices Λ and U, which could be used to achieve other expected performance specifications, e.g., reliability constraints.

4 Numerical Simulation

In this section, for the purpose of illustrating the usefulness and flexibility of the theory developed in this paper, we present a simulation example.

Assume that the nonlinear continuous-time stochastic state delayed system (1) – (2) is given by

$$\begin{aligned} dx_1(t) = {} & [-2x_1(t) - 0.1x_2(t) + 0.2\cos(x_1(t) + x_2(t)) \\ & + 0.1x_1(t-0.1) + 0.16\sin x_2(t) + 2.9u_1(t) + 0.2u_2(t)]\,dt + 0.2x_1\,dw(t), \\ dx_2(t) = {} & [-0.1x_1(t) + x_2(t) + 0.15\sin x_2(t) \\ & + 0.1x_2(t-0.1) + 0.15\cos x_1(t) + 0.1u_1(t) - 2.1u_2(t)]\,dt + 0.2x_2\,dw(t). \end{aligned}$$

Considering the system (1) – (2) with the constraints (3) – (4), we can obtain that

$$A = \begin{bmatrix} -2 & -0.1 \\ -0.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2.9 & 0.2 \\ 0.1 & -2.1 \end{bmatrix}, \quad A_d = 0.1I_2, \quad D = 0.2I_2,$$
$$d = 0.1, \quad a_{11} = 0.25; \quad a_{12} = 0.12; \quad a_{21} = 0; \quad a_{22} = 0.$$

We choose $\varepsilon_1 = 4.8$, $\varepsilon_1 = 8.2$, and solve (26) to obtain

$$P = \begin{bmatrix} 0.1287 & 0.0013 \\ 0.0013 & 0.2003 \end{bmatrix}.$$

Then, setting $\Lambda = 2I_2$ which meets (28) and considering two cases of $U = I_2$ and $U = -I_2$, we have two desired gain matrices as follows:

$$\text{Case 1:} \quad K_1 = \begin{bmatrix} -0.7938 & -0.7764 \\ -0.7580 & 25.2439 \end{bmatrix}, \qquad \text{Case 2:} \quad K_2 = \begin{bmatrix} -23.7023 & -0.7764 \\ -0.7580 & 2.3354 \end{bmatrix}.$$

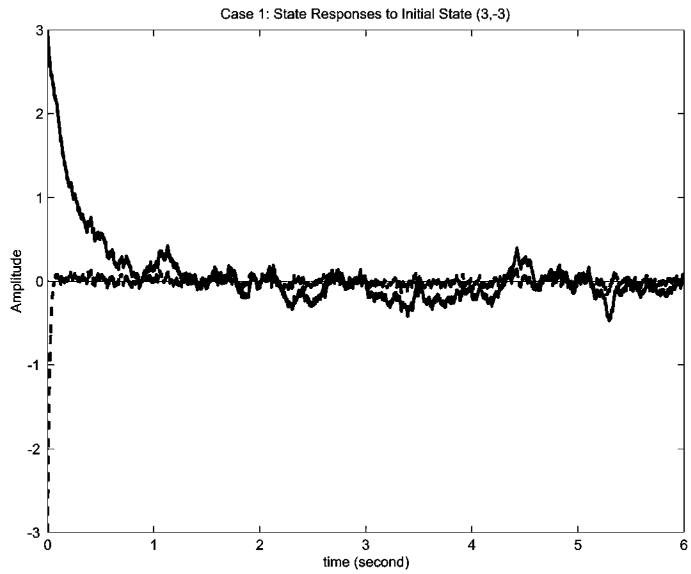

Figure 4.1. x_1 (solid), x_2 (dashed).

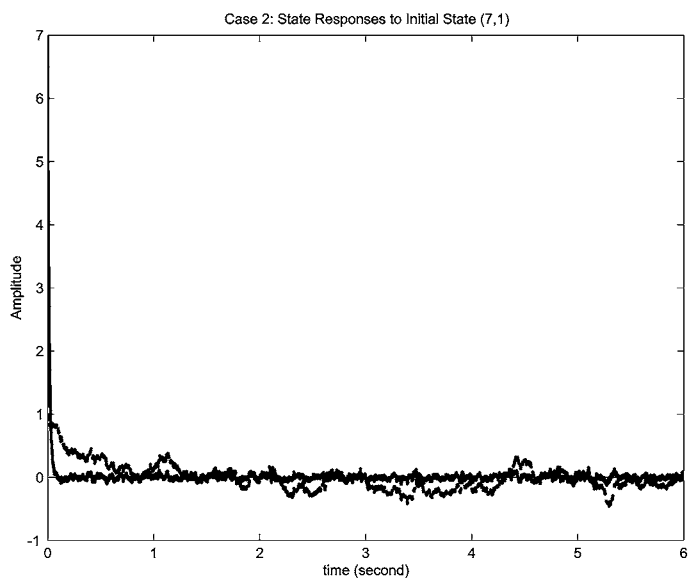

Figure 4.2. x_1 (solid), x_2 (dashed).

The responses of closed-loop system dynamics to initial conditions are shown in Figure 4.1 and Figure 4.2. The simulation results imply that the desired goal is well achieved, i.e., the closed-loop system is exponentially stable in the mean square.

5 Conclusions

In this paper, we have studied the stabilization problem for a class of nonlinear stochastic time-delay systems. The nonlinearities are assumed to have the similar form as those in [18 – 20]. We have developed an effective algebraic matrix inequality approach to designing the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We have investigated the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and have derived the explicit expression of the desired controller gains. A numerical simulation example has been provided to show the usefulness and effectiveness of the proposed design method.

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Robust Observers for a Class of Uncertain Nonlinear Stochastic Systems with State Delays*

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Abstract: This paper investigates the problem of robust observer design for a class of nonlinear stochastic systems with state delays and time-varying norm-bounded parameter uncertainties. The nonlinearities are assumed to satisfy the global Lipschitz conditions and appear in both the state and measured output equations. The purpose is to design a nonlinear observer ensuring mean square asymptotic stability for the error system, irrespective of the uncertainties and the time delays. A sufficient condition for the solvability of this problem is derived in terms of a linear matrix inequality and the explicit formula of a desired robust observer is also given. An example is given to illustrate the proposed approach.

Keywords: *Linear matrix inequality; nonlinear systems; robust observers; stochastic systems; time-delay systems; uncertain systems.*

Mathematics Subject Classification (2000): 93B12, 93E15, 93C23.

1 Introduction

Observer design for linear as well as nonlinear systems has been an active research area in the past years. Various approaches, such as transfer-function, geometric, algebraic, singular value decomposition and so on, have been successfully proposed and many results on the observer design have been reported in the literature. For some representative work on this general topic, to name a few, we refer readers to [6, 7, 9, 10, 12] and the

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references therein. However, one of the limitations of classical observer theory is that it cannot guarantee the observer performance when parameter uncertainty appears in a system model. This has motivated the study of robust observer design problem; see, e.g. [1, 3, 15], and the references cited therein. It is worth noting that in the context of stochastic nonlinear systems, the robust observer design problem has been investigated in [20], in which a method for the design of time-invariant observers with guaranteed exponential convergence has been proposed.

On the other hand, it is well known that time delays are inherent in many physical and engineering systems due to transportation lags, and conduction or computation times [4, 8]. It has been shown that time delay is often a main cause of instability of a dynamic system. A number of estimation and control problems related to time-delay systems have been addressed by many researchers [5, 11, 13, 16 – 18]. Recently, a great deal of interest has been devoted to the observer design for time-delay systems. A general form of linear observers for time-delay systems by using the factorization approach was proposed in [19], where a necessary and sufficient condition for the existence of the state functional observers was presented. For discrete-time delay systems, a memoryless state observer was designed by the state augmentation approach in [13]. However, it should be pointed out that disturbances as well as nonlinearities may be present in time-delay systems. Therefore, the observer design problem for nonlinear time-delay stochastic systems is important in both theory and practice and challenging, thus should be considered. To date, to the authors' best knowledge, little work has been done for such stochastic systems.

In this paper, we are concerned with the problem of robust observer design for a class of nonlinear stochastic systems with state delay and parameter uncertainties. The class of systems under consideration is described by a linear stochastic differential delay equation with the addition of known nonlinearities which depend not only on the state but also on the delayed state and are assumed to satisfy the global Lipschitz conditions. The nonlinearities appear in both the state and measured output equations. The parameter uncertainties are real time-varying norm-bounded and appear in both the state and output matrices of the linear part of the system model. The problem under study is the design of a nonlinear observer that guarantees mean square asymptotic stability of the error dynamics for the whole set of admissible systems. A linear matrix inequality (LMI) approach is proposed to solve this problem and a solution is given in terms of an LMI, which defines a convex set of solutions and can be easily computed by the available LMI algorithms ([2]).

Notation Throughout this paper, for symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); I is the identity matrix with appropriate dimension. The notation M^{T} represents the transpose of the matrix M. While, $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. The notation $\mathcal{E}\{\cdot\}$ stands for the expectation operator; $\|x\|$ stands for the Euclidean norm of the vector x. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation

Consider the following class of nonlinear stochastic systems with state-delay and parameter uncertainties:

$$(\Sigma): \quad dx(t) = [(A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau))]\, dt$$
$$+ [(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)]\, d\omega(t), \tag{1}$$
$$dy(t) = [(C + \Delta C(t))\, x(t) + (C_d + \Delta C_d(t))\, x(t-\tau) + Hh(x(t), x(t-\tau))]\, dt$$
$$+ [(D + \Delta D(t))\, x(t) + (D_d + \Delta D_d(t))\, x(t-\tau)]\, d\omega(t), \tag{2}$$
$$x(t) = \phi(t), \quad \forall\, t \in [-\tau, 0], \tag{3}$$

where $x(t) \in R^n$ is the system state, $y(t) \in R^m$ is the measurement; $\omega(t)$ is a zero-mean real scalar Wiener process on $(\Omega,\, \mathcal{F},\, \mathcal{P})$ relative to an increasing family $(\mathcal{F}_t)_{t>0}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. We assume

$$\mathcal{E}\left\{d\omega(t)\right\} = 0, \qquad \mathcal{E}\left\{d\omega(t)^2\right\} = dt. \tag{4}$$

In system (Σ), $\phi(t)$ is a real-valued continuous initial function on $[-\tau, 0]$, $\tau > 0$ is a known time delay of the system, $g(\cdot,\cdot)\colon R^n \times R^n \to R^{n_g}$ and $h(\cdot,\cdot)\colon R^n \times R^n \to R^{n_h}$ are known nonlinear functions, A, A_d, B, B_d, C, C_d, D, D_d, G and H are known real constant matrices, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$ and $\Delta D_d(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_d(t) \\ \Delta C(t) & \Delta C_d(t) & \Delta D(t) & \Delta D_d(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(t)\, [\, N_1 \quad N_2 \quad N_3 \quad N_4\,], \tag{5}$$

where M_1, M_2, N_1, N_2, N_3 and N_4 are known real constant matrices and $F(\cdot)\colon R \to R^{k\times l}$ is a unknown real-valued time-varying matrix satisfying

$$F(t)^{\mathrm{T}} F(t) \le I, \quad \forall\, t. \tag{6}$$

It is assumed that all the elements of $F(t)$ are Lebesgue measurable. $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$ and $\Delta D_d(t)$ are said to be admissible if both (5) and (6) hold.

Remark 1 The parameter uncertainty structure as in (5) and (6) has been widely used in the problems of robust control and robust filtering of uncertain systems, see, for example, [11, 12, 15] and the references therein and many practical systems possess parameter uncertainties which can be either exactly modeled, or overbounded by (6). Observe that the unknown matrix $F(t)$ in (5) can even be allowed to be state-dependent, i.e. $F(t) = F(t, x(t))$, as long as (6) is satisfied.

Throughout the paper, we make the following assumption on the nonlinear functions in system (Σ).

Assumption 1

(I) $g(0,0) = 0$;

(II) $\|g(x_1, x_2) - g(y_1, y_2)\| \le \|S_{1g}(x_1 - y_1)\| + \|S_{2g}(x_2 - y_2)\|$,
$\|h(x_1, x_2) - h(y_1, y_2)\| \le \|S_{1h}(x_1 - y_1)\| + \|S_{2h}(x_2 - y_2)\|$,
for all x_1, x_2, y_1, $y_2 \in R^n$, where S_{1g}, S_{2g}, S_{1h} and S_{2h} are known real constant matrices.

Before formulating the problem to be addressed in this paper, we first introduce the following concept of stochastic stability.

Definition 1 The *equilibrium* $x = 0$ of the system (1) is said to be *mean square stable* if for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$\mathcal{E}\,|x(t)|^2 < \varepsilon,\ \ t > 0$$

when $\sup\limits_{-\tau \le s \le 0} \mathcal{E}\,|\phi(s)|^2 < \delta(\varepsilon)$. If, in addition,

$$\lim_{t\to\infty} x(t) = 0$$

for any initial conditions, then the equilibrium $x = 0$ of the system (1) is said to be mean square asymptotically stable.

Now, the observer design problem we address in this paper can be formulated as follows: given the uncertain nonlinear stochastic time-delay system (Σ), we are concerned with obtaining an estimate $\hat{x}(t)$ of the state $x(t)$ by using the measurement $y(t)$, such that the error dynamics remain mean square asymptotically stable for all admissible uncertainties satisfying (5) and (6) and the nonlinearities satisfying Assumption 1.

3 Main Results

In this section, an LMI approach is proposed to solve the robust observe design problems formulated in the previous section. Before presenting the main results, we give the following lemmas which will be used in the proof of our main results.

Lemma 1 [14] *Let $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{W}$ and F be real matrices of appropriate dimensions such that $\mathcal{W} > 0$ and $F^{\mathrm{T}}F \le I$. Then we have the following:*

(1) *for scalar $\epsilon > 0$ and vectors $x, y \in R^n$,*

$$2x^{\mathrm{T}}\mathcal{D}F\mathcal{S}y \le \epsilon^{-1}x^{\mathrm{T}}\mathcal{D}\mathcal{D}^{\mathrm{T}}x + \epsilon y^{\mathrm{T}}\mathcal{S}^{\mathrm{T}}\mathcal{S}y;$$

(2) *for any scalar $\epsilon > 0$ such that $\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^{\mathrm{T}} > 0$,*

$$(\mathcal{A} + \mathcal{D}F\mathcal{S})^{\mathrm{T}}\mathcal{W}^{-1}(\mathcal{A} + \mathcal{D}F\mathcal{S}) \le \mathcal{A}^{\mathrm{T}}(\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^{\mathrm{T}})^{-1}\mathcal{A} + \epsilon^{-1}\mathcal{S}^{\mathrm{T}}\mathcal{S}.$$

Theorem 1 *Consider the uncertain nonlinear stochastic time-delay system (1) and (3), that is,*

$$\begin{aligned}(\Sigma_1):\quad dx(t) &= [(A + \Delta A(t))\,x(t) + (A_d + \Delta A_d(t))\,x(t-\tau) + Gg(x(t), x(t-\tau))]\,dt \\ &\quad + [(B + \Delta B(t))\,x(t) + (B_d + \Delta B_d(t))\,x(t-\tau)]\,d\omega(t), \qquad (7)\\ x(t) &= \phi(t), \quad \forall\, t \in [-\tau, 0]. \qquad (8)\end{aligned}$$

Then system (Σ_1) is mean square asymptotically stable if there exist matrices $P > 0$, $Q > 0$ and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$, such that the following LMI holds:

$$\begin{bmatrix} \Omega_1 & PA_d + \epsilon_2 N_1^{\mathrm{T}}N_2 + \epsilon_3 N_3^{\mathrm{T}}N_4 & PG & PM_1 & 0 & B^{\mathrm{T}}P \\ A_d^{\mathrm{T}}P + \epsilon_2 N_2^{\mathrm{T}}N_1 + \epsilon_3 N_4^{\mathrm{T}}N_3 & \Omega_2 & 0 & 0 & 0 & B_d^{\mathrm{T}}P \\ G^{\mathrm{T}}P & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ M_1^{\mathrm{T}}P & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_3 I & M_1^{\mathrm{T}}P \\ PB & PB_d & 0 & 0 & PM_1 & -P \end{bmatrix} < 0 \qquad (9)$$

where

$$\Omega_1 = A^{\mathrm{T}}P + PA + Q + 2\epsilon_1 S_{1g}^{\mathrm{T}} S_{1g} + \epsilon_2 N_1^{\mathrm{T}} N_1 + \epsilon_3 N_3^{\mathrm{T}} N_3 \tag{10}$$

$$\Omega_2 = 2\epsilon_1 S_{2g}^{\mathrm{T}} S_{2g} + \epsilon_2 N_2^{\mathrm{T}} N_2 + \epsilon_3 N_4^{\mathrm{T}} N_4 - Q. \tag{11}$$

Proof Define the following Lyapunov function candidate:

$$V(x_t, t) = x(t)^{\mathrm{T}} P x(t) + \int_{t-\tau}^{t} x(s)^{\mathrm{T}} Q x(s)\, ds \tag{12}$$

where

$$x_t = x(t+\beta), \quad \beta \in [-\tau, 0].$$

By Itô's formula, we obtain the stochastic differential as

$$dV(x_t) = LV(x_t, t)dt + 2x(t)^{\mathrm{T}} P[(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)]\, d\omega(t), \tag{13}$$

where

$$\begin{aligned} LV(x_t, t) = 2x(t)^{\mathrm{T}} P\, &[(A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau))] \\ &+ [(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)]^{\mathrm{T}} \\ &\times P\, [(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)] \\ &+ x(t)^{\mathrm{T}} Q x(t) - x(t-\tau)^{\mathrm{T}} Q x(t-\tau). \end{aligned} \tag{14}$$

From Assumption 1, it follows that

$$\|g(x(t), x(t-\tau))\| \le \|S_{1g} x(t)\| + \|S_{2g} x(t-\tau)\|.$$

Therefore

$$\|g(x(t), x(t-\tau))\|^2 \le 2\, \|S_{1g} x(t)\|^2 + 2\, \|S_{2g} x(t-\tau)\|^2. \tag{15}$$

Considering this and (5) and using Lemma 1, we have that for any scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} &2x(t)^{\mathrm{T}} PGg(x(t), x(t-\tau)) \\ &\le \epsilon_1^{-1} x(t)^{\mathrm{T}} PGG^{\mathrm{T}} P x(t) + \epsilon_1 g(x(t), x(t-\tau))^{\mathrm{T}} g(x(t), x(t-\tau)) \\ &\le \epsilon_1^{-1} x(t)^{\mathrm{T}} PGG^{\mathrm{T}} P x(t) + 2\epsilon_1 \left[x(t)^{\mathrm{T}} S_{1g}^{\mathrm{T}} S_{1g} x(t) + x(t-\tau)^{\mathrm{T}} S_{2g}^{\mathrm{T}} S_{2g} x(t-\tau)\right] \end{aligned} \tag{16}$$

and

$$\begin{aligned} 2x(t)^{\mathrm{T}} P\, [\Delta A(t) x(t) + \Delta A_d(t) x(t-\tau)] &= 2x(t)^{\mathrm{T}} P M_1 F(t)\, [N_1 x(t) + N_2 x(t-\tau)] \\ \le \epsilon_2^{-1} x(t)^{\mathrm{T}} P M_1 M_1^{\mathrm{T}} P x(t) &+ \epsilon_2\, [N_1 x(t) + N_2 x(t-\tau)]^{\mathrm{T}}\, [N_1 x(t) + N_2 x(t-\tau)]. \end{aligned} \tag{17}$$

Furthermore, from (9) it is easy to see that

$$\epsilon_3 I - M_1^{\mathrm{T}} P M_1 > 0$$

which implies

$$P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P > 0.$$

Therefore, by using Lemma 1 again, we have

$$\left[\overline{B} + M_1 F(t) \overline{N}\right]^{\mathrm{T}} P \left[\overline{B} + M_1 F(t) \overline{N}\right] \leq \overline{B}^{\mathrm{T}} P \left(P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P\right)^{-1} P \overline{B} + \epsilon_3 \overline{N}^{\mathrm{T}} \overline{N} \tag{18}$$

where

$$\overline{B} = [\, B \quad B_d \,], \quad \overline{N} = [\, N_3 \quad N_4 \,].$$

Noting

$$\begin{aligned} &[(B + \Delta B(t)) x(t) + (B_d + \Delta B_d(t)) x(t-\tau)]^{\mathrm{T}} P \\ &\times [(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)] \\ = &[\, x(t)^{\mathrm{T}} \quad x(t-\tau)^{\mathrm{T}} \,] \left[\overline{B} + M_1 F(t) \overline{N}\right]^{\mathrm{T}} P \left[\overline{B} + M_1 F(t) \overline{N}\right] \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned}$$

and using (16) – (18) we obtain

$$LV(x_t, t) \leq [\, x(t)^{\mathrm{T}} \quad x(t-\tau)^{\mathrm{T}} \,]\, W \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} W = &\begin{bmatrix} \Omega_1 + \epsilon_1^{-1} P G G^{\mathrm{T}} P + \epsilon_2^{-1} P M_1 M_1^{\mathrm{T}} P & P A_d + \epsilon_2 N_1^{\mathrm{T}} N_2 + \epsilon_3 N_3^{\mathrm{T}} N_4 \\ A_d^{\mathrm{T}} P + \epsilon_2 N_2^{\mathrm{T}} N_1 + \epsilon_3 N_4^{\mathrm{T}} N_3 & \Omega_2 \end{bmatrix} \\ &+ \overline{B}^{\mathrm{T}} P \left(P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P\right)^{-1} P \overline{B}. \end{aligned}$$

On the other hand, pre and post-multiplying (9) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

and using Schur complement, we have $W < 0$, this together with (19) implies

$$LV(x_t, t) < 0$$

for

$$\begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \neq 0,$$

which, by the result in [8], guarantees the mean square asymptotic stability of system (Σ_1).

Now, we are in a position to give a solution to the robust observer design problem formulated in the previous section.

Theorem 2 *Consider the uncertain nonlinear stochastic time-delay system* (Σ) *under Assumption 1. If there exist matrices* $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$ *and* Z *and scalars* $\epsilon_1 > 0$, $\epsilon_2 > 0$ *and* $\epsilon_3 > 0$, *such that the following LMI holds:*

$$\begin{bmatrix} \Xi_1 & \Lambda_1 & \Lambda_2 & 0 & \Lambda_3 \\ \Lambda_1^{\mathrm{T}} & \Xi_2 & 0 & 0 & \Pi_1 \\ \Lambda_2^{\mathrm{T}} & 0 & -\Upsilon_1 & 0 & 0 \\ 0 & 0 & 0 & -\Upsilon_2 & \Pi_2 \\ \Lambda_3^{\mathrm{T}} & \Pi_1^{\mathrm{T}} & 0 & \Pi_2^{\mathrm{T}} & -\Upsilon_3 \end{bmatrix} < 0 \tag{20}$$

where

$$\begin{aligned}
\Xi_1 &= \mathrm{diag}\,(\Xi_{11}, \Xi_{12})\,, \\
\Xi_2 &= \mathrm{diag}\,(\Xi_{21}, \Xi_{22})\,, \\
\Xi_{11} &= A^{\mathrm{T}} P_1 + P_1 A + Q_1 + 2\epsilon_1 S_{1g}^{\mathrm{T}} S_{1g} + \epsilon_2 N_1^{\mathrm{T}} N_1 + \epsilon_3 N_3^{\mathrm{T}} N_3, \\
\Xi_{12} &= A^{\mathrm{T}} P_2 + P_2 A - ZC - C^{\mathrm{T}} Z^{\mathrm{T}} + Q_2 + 2\epsilon_1 S_1^{\mathrm{T}} S_1, \\
\Xi_{21} &= 2\epsilon_1 S_{2g}^{\mathrm{T}} S_{2g} + \epsilon_2 N_2^{\mathrm{T}} N_2 + \epsilon_3 N_4^{\mathrm{T}} N_4 - Q_1, \\
\Xi_{22} &= 2\epsilon_1 S_2^{\mathrm{T}} S_2 - Q_2, \\
\Lambda_1 &= \begin{bmatrix} P_1 A_d + \epsilon_2 N_1^{\mathrm{T}} N_2 + \epsilon_3 N_3^{\mathrm{T}} N_4 & 0 \\ 0 & P_2 A_d - Z C_d \end{bmatrix}, \\
\Lambda_2 &= \begin{bmatrix} P_1 G & 0 & 0 & P_1 M_1 \\ 0 & P_2 G & -ZH & P_2 M_1 - Z M_2 \end{bmatrix}, \\
\Lambda_3 &= \begin{bmatrix} B^{\mathrm{T}} P_1 & B^{\mathrm{T}} P_2 - D^{\mathrm{T}} Z^{\mathrm{T}} \\ 0 & 0 \end{bmatrix}, \\
\Pi_1 &= \begin{bmatrix} B_d^{\mathrm{T}} P_1 & B_d^{\mathrm{T}} P_2 - D_d^{\mathrm{T}} Z^{\mathrm{T}} \\ 0 & 0 \end{bmatrix}, \\
\Pi_2 &= [\, M_1^{\mathrm{T}} P_1 \quad M_1^{\mathrm{T}} P_2 - M_2^{\mathrm{T}} Z^{\mathrm{T}} \,]\,, \\
\Upsilon_1 &= \mathrm{diag}\,(\epsilon_1 I, \epsilon_1 I, \epsilon_1 I, \epsilon_2 I)\,, \\
\Upsilon_2 &= \epsilon_3 I, \\
\Upsilon_3 &= \mathrm{diag}(P_1, P_2).
\end{aligned}$$

Then the robust observer design problem is solvable, where

$$S_1 = \begin{bmatrix} S_{1g} \\ S_{1h} \end{bmatrix}, \qquad S_2 = \begin{bmatrix} S_{2g} \\ S_{2h} \end{bmatrix}. \tag{21}$$

Furthermore, when LMI (20) is satisfied, a suitable nonlinear observer is given as follows:

$$\begin{aligned} d\hat{x}(t) &= [A\hat{x}(t) + A_d \hat{x}(t-\tau) + Gg(\hat{x}(t), \hat{x}(t-\tau))]\, dt \\ &\quad + L\,[dy(t) - (C\hat{x}(t) + C_d \hat{x}(t-\tau) + Hh(\hat{x}(t), \hat{x}(t-\tau)))\, dt]\,, \end{aligned} \tag{22}$$

where $L = P_2^{-1} Z$.

Proof Let

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

then from (1)–(3) and (22), we obtain

$$\begin{aligned} d\tilde{x}(t) = {} & [(A - LC)\tilde{x}(t) + (A_d - LC_d)\tilde{x}(t-\tau) + (\Delta A(t) - L\Delta C(t))\,x(t) \\ & + (\Delta A_d(t) - L\Delta C_d(t))\,x(t-\tau) + \bar{G}\xi(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))]\,dt \\ & + [((B - LD) + (\Delta B(t) - L\Delta D(t)))\,x(t) \\ & + ((B_d - LD_d) + (\Delta B_d(t) - L\Delta D_d(t)))\,x(t-\tau)]\,d\omega(t), \end{aligned} \tag{23}$$

where $\bar{G} = [\,G \quad -LH\,]$ and

$$\xi(x(t),\, x(t-\tau),\, \hat{x}(t),\, \hat{x}(t-\tau)) = \begin{bmatrix} g(x(t),\, x(t-\tau)) - g(\hat{x}(t),\, \hat{x}(t-\tau)) \\ h(x(t),\, x(t-\tau)) - h(\hat{x}(t),\, \hat{x}(t-\tau)) \end{bmatrix}.$$

Setting

$$\eta(t)^{\mathrm{T}} = [\,x(t)^{\mathrm{T}} \quad \tilde{x}(t)^{\mathrm{T}}\,]^{\mathrm{T}}$$

and considering (1)–(3) and (18), we have

$$\begin{aligned} d\eta(t) = {} & [(A_c + \Delta A_c(t))\,\eta(t) + (A_{cd} + \Delta A_{cd}(t))\,\eta(t-\tau) \\ & + G_c\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))]dt \\ & + [(B_c + \Delta B_c(t))\,\eta(t) + (B_{cd} + \Delta B_{cd}(t))\,\eta(t-\tau)]\,d\omega(t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} A_c &= \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}, & \Delta A_c(t) &= \begin{bmatrix} \Delta A(t) & 0 \\ \Delta A(t) - L\Delta C(t) & 0 \end{bmatrix}, \\ A_{cd} &= \begin{bmatrix} A_d & 0 \\ 0 & A_d - LC_d \end{bmatrix}, & \Delta A_{cd}(t) &= \begin{bmatrix} \Delta A_d(t) & 0 \\ \Delta A_d(t) - L\Delta C_d(t) & 0 \end{bmatrix}, \\ B_c &= \begin{bmatrix} B & 0 \\ B - LD & 0 \end{bmatrix}, & \Delta B_c(t) &= \begin{bmatrix} \Delta B(t) & 0 \\ \Delta B(t) - L\Delta D(t) & 0 \end{bmatrix}, \\ B_{cd} &= \begin{bmatrix} B_d & 0 \\ B_d - LD_d & 0 \end{bmatrix}, & \Delta B_{cd}(t) &= \begin{bmatrix} \Delta B_d(t) & 0 \\ \Delta B_d(t) - L\Delta D_d(t) & 0 \end{bmatrix}, \end{aligned}$$
$$G_c = \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix}$$

and

$$\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau)) = [\,g(x(t), x(t-\tau))^{\mathrm{T}} \quad \xi(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))^{\mathrm{T}}\,]^{\mathrm{T}}.$$

Using Assumption 1 yields

$$\|\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))\|^2 \leq 2\left\|\tilde{S}_1\eta(t)\right\|^2 + 2\left\|\tilde{S}_2\eta(t-\tau)\right\|^2, \tag{25}$$

where

$$\tilde{S}_1 = \begin{bmatrix} S_{1g} & 0 \\ 0 & S_1 \end{bmatrix}, \qquad \tilde{S}_2 = \begin{bmatrix} S_{2g} & 0 \\ 0 & S_2 \end{bmatrix}. \tag{26}$$

Noting (5), it can be easily seen that

$$[\,\Delta A_c(t) \quad \Delta A_{cd}(t) \quad \Delta B_c(t) \quad \Delta B_{cd}(t)\,] = M_{1c}F(t)\,[\,N_{1c} \quad N_{2c} \quad N_{3c} \quad N_{4c}\,],$$

where

$$M_{1c} = \begin{bmatrix} M_1 \\ M_1 - LM_2 \end{bmatrix}, \qquad N_{1c} = [\,N_1 \quad 0\,], \qquad N_{2c} = [\,N_2 \quad 0\,],$$
$$N_{3c} = [\,N_3 \quad 0\,], \qquad N_{4c} = [\,N_4 \quad 0\,].$$

Define

$$\begin{aligned}
P_c &= \operatorname{diag}(P_1, P_2), \\
Q_c &= \operatorname{diag}(Q_1, Q_2), \\
\Omega_{1c} &= A_c^{\mathrm{T}} P_c + P_c A_c + Q_c + 2\epsilon_1 \tilde{S}_1^{\mathrm{T}} \tilde{S}_1 + \epsilon_2 N_{1c}^{\mathrm{T}} N_{1c} + \epsilon_3 N_{3c}^{\mathrm{T}} N_{3c}, \\
\Omega_{2c} &= 2\epsilon_1 \tilde{S}_2^{\mathrm{T}} \tilde{S}_2 + \epsilon_2 N_{2c}^{\mathrm{T}} N_{2c} + \epsilon_3 N_{4c}^{\mathrm{T}} N_{4c} - Q_c,
\end{aligned}$$

then by some algebraic manipulations and noting (20), it follows that

$$\begin{bmatrix}
\Omega_{1c} & \begin{matrix} P_c A_{cd} + \epsilon_2 N_{1c}^{\mathrm{T}} N_{2c} \\ + \epsilon_3 N_{3c}^{\mathrm{T}} N_{4c} \end{matrix} & P_c G_c & P_c M_{1c} & 0 & B_c^{\mathrm{T}} P_c \\
\begin{matrix} A_{cd}^{\mathrm{T}} P_c + \epsilon_2 N_{2c}^{\mathrm{T}} N_{1c} \\ + \epsilon_3 N_{4c}^{\mathrm{T}} N_{3c} \end{matrix} & \Omega_{2c} & 0 & 0 & 0 & B_{cd}^{\mathrm{T}} P_c \\
G_c^{\mathrm{T}} P_c & 0 & -\epsilon_1 I & 0 & 0 & 0 \\
M_{1c}^{\mathrm{T}} P_c & 0 & 0 & -\epsilon_2 I & 0 & 0 \\
0 & 0 & 0 & 0 & -\epsilon_3 I & M_{1c}^{\mathrm{T}} P_c \\
P_c B_c & P_c B_{cd} & 0 & 0 & P_c M_{1c} & -P_c
\end{bmatrix}$$
$$= \begin{bmatrix}
\Xi_1 & \Lambda_1 & \Lambda_2 & 0 & \Lambda_3 \\
\Lambda_1^{\mathrm{T}} & \Xi_2 & 0 & 0 & \Pi_1 \\
\Lambda_2^{\mathrm{T}} & 0 & -\Upsilon_1 & 0 & 0 \\
0 & 0 & 0 & -\Upsilon_2 & \Pi_2 \\
\Lambda_3^{\mathrm{T}} & \Pi_1^{\mathrm{T}} & 0 & \Pi_2^{\mathrm{T}} & -\Upsilon_3
\end{bmatrix} < 0.$$

Finally, using this inequality and Theorem 1, the desired result follows immediately.

Remark 2 Theorem 2 provides an LMI method for designing robust observers for system (Σ). It is worth pointing out that the LMI in (20) can be solved by means of numerically efficient convex programming algorithms, and no tuning of parameters is required [2, though there are several parameters and matrices to be determined.

4 Numerical Example

In this section, we provide an example to demonstrate the effectiveness of the proposed method.

Consider the following class of nonlinear stochastic systems with state-delay and parameter uncertainties:

$$\begin{aligned}
dx_1(t) = {} & [-1.8x_1(t) + (0.2 - 0.4f(t))x_2(t) - (0.1 + 0.2f(t))x_1(t-1.5) + 0.2x_2(t-1.5) \\
& + 0.3\sin(-0.2x_1(t) + 0.1x_2(t) + 0.1x_1(t-0.5) + 0.2x_2(t-1.5))]\,dt \\
& + [(0.1 + 0.2f(t))x_1(t) + (0.3 + 0.2f(t))x_2(t) \\
& + 0.4f(t)x_1(t-1.5) - 0.2x_2(t-1.5)]\,d\omega(t), \\
dx_2(t) = {} & [-0.4x_1(t) - (2.5 + 0.2f(t))x_2(t) - 0.1f(t)x_1(t-1.5) - 0.1x_2(t-1.5) \\
& + 0.2\sin(-0.2x_1(t) + 0.1x_2(t) + 0.1x_1(t-1.5) + 0.2x_2(t-1.5))]\,dt \\
& + [(0.1f(t) - 0.4)x_1(t) + (1 + 0.1f(t))x_2(t) \\
& + (0.6 + 0.2f(t))x_1(t-1.5) + 0.1x_2(t-1.5)]\,d\omega(t), \\
dy(t) = {} & [0.1x_1(t) - (0.4 + 0.2f(t))x_2(t) + (0.4 - 0.1f(t))x_1(t-1.5) + 0.6x_2(t-1.5) \\
& + 0.5\sin(0.2x_1(t) - 0.1x_2(t) + 0.2x_1(t-1.5))]\,dt \\
& + [0.1f(t)x_1(t) + (0.1f(t) - 0.2)x_2(t) \\
& + (0.2f(t) - 0.5)x_1(t-1.5) + 0.2x_2(t-1.5)]\,d\omega(t),
\end{aligned}$$

where $f(t)$ is unknown but satisfies $|f(t)| \le 1$. It is easy to see that the above system has the form (1) and (2) with parameters as follows

$$\begin{aligned}
A &= \begin{bmatrix} -1.8 & 0.2 \\ -0.4 & -2.5 \end{bmatrix}, & A_d &= \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 1 \end{bmatrix}, & B_d &= \begin{bmatrix} 0 & -0.2 \\ 0.6 & 0.1 \end{bmatrix}, \\
C &= \begin{bmatrix} 0.1 & -0.4 \end{bmatrix}, & C_d &= \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \\
D &= \begin{bmatrix} 0 & -0.2 \end{bmatrix}, & D_d &= \begin{bmatrix} -0.5 & 0.2 \end{bmatrix}, \\
G &= \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, & H &= 0.5, \\
M_1 &= \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}, & M_2 &= 0.2, \\
N_1 &= \begin{bmatrix} 0 & -1 \end{bmatrix}, & N_2 &= \begin{bmatrix} -0.5 & 0 \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, & N_4 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
S_{1g} &= \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}, & S_{2g} &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \\
S_{1h} &= \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}, & S_{2h} &= \begin{bmatrix} 0.2 & 0 \end{bmatrix}.
\end{aligned}$$

Now, using the Matlab LMI Control Toolbox, we obtain the solution to the LMI (20) as follows:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 5.0934 & -0.7812 \\ -0.7812 & 4.3022 \end{bmatrix}, & P_2 &= \begin{bmatrix} 2.8203 & -0.5012 \\ -0.5012 & 1.6465 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 10.9532 & -0.5227 \\ -0.5227 & 3.6335 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 4.3914 & -0.7795 \\ -0.7795 & 4.7745 \end{bmatrix},
\end{aligned}$$

$$Z = \begin{bmatrix} 0.1271 \\ -2.1537 \end{bmatrix},$$

$$\epsilon_1 = 4.8400, \qquad \epsilon_2 = 2.6078, \qquad \epsilon_3 = 2.7588.$$

Therefore, by Theorem 2, it follows that the robust observer design problem is solvable, and the desired nonlinear observer can be chosen by

$$\begin{aligned} d\hat{x}(t) = \Bigg(&\begin{bmatrix} -1.8 & 0.2 \\ -0.4 & -2.5 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix} \hat{x}(t-1.5) \\ &+ \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \sin([\,-0.2 \quad 0.1\,]\,\hat{x}(t) + [\,0.1 \quad 0.2\,]\,\hat{x}(t-1.5)) \Bigg) dt \\ + \begin{bmatrix} -0.1981 \\ -1.3684 \end{bmatrix} &(dy(t) - ([\,0.1 \quad -0.4\,]\,\hat{x}(t) + [\,0.4 \quad 0.6\,]\,\hat{x}(t-1.5) \\ &+ 0.5 \sin([\,-0.2 \quad 0.1\,]\,\hat{x}(t) + [\,0.1 \quad 0.2\,]\,\hat{x}(t-1.5))) \, dt) \,. \end{aligned}$$

5 Conclusions

In this paper, we have studied the robust observer design problem for a class of nonlinear stochastic systems with state delays and time-varying norm-bounded parameter uncertainties. In terms of an LMI, a nonlinear observer has been developed to guarantee mean square asymptotic stability of the error dynamics for all admissible uncertainties. A numerical example has been provided to show the effectiveness of the proposed methods.

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