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Topological Sequence Entropy and Chaos of Star Maps*

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Abstract: Let $\mathbb{X}_n = \{z \in \mathbb{C} \colon z^n \in [0,1]\}$, $n \in \mathbb{N}$, and let $f \colon \mathbb{X}_n \to \mathbb{X}_n$ be a continuous map such that $f(0) = 0$. In this paper we prove that f is chaotic in the sense of Li–Yorke iff there is a strictly increasing sequence of positive integers A such that the topological sequence entropy of f relative to A is positive.

Keywords: *Star maps; Li–Yorke chaos; topological sequence entropy.*

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1 Introduction

Let (X,d) be a compact metric space and let $C(X)$ denote the set of continuous maps $f \colon X \to X$. For any $f \in C(X)$, the pair (X,f) is called a *discrete (semi)dynamical system*. Given $x \in X$, the sequence $(f^i(x))_{i=0}^{\infty}$ is the *trajectory* of x (also *orbit* of x). Recall that a point $x \in X$ is *periodic* if $f^i(x) = x$ for some $i \in \mathbb{N}$. Denote by $\mathrm{Per}\,(f)$ the set of periodic points of f. The map f is said to be *chaotic in the sense of Li–Yorke* (or simply *chaotic*) if there is an uncountable set $S \subset X \backslash \mathrm{Per}\,(f)$ such that for any $x,y \in S$, $x \neq y$, and any $p \in \mathrm{Per}\,(f)$ it holds

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \tag{1}$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \tag{2}$$

$$\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0. \tag{3}$$

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The set S is called a *scrambled* set of f (see [18] or [21]).

The notion of chaos in the sense of Li–Yorke has been studied in the case of $X = [0,1]$ and $X = S^1$ (see e.g. [10, 14, 16, 17, 22] or [21]). In this setting, topological sequence entropy plays a special role to characterize chaotic maps. Given a strictly increasing sequence of positive integers A, denote by $h_A(f)$ the *topological sequence entropy* of f with respect to A (see the definition in the next section). Then

Theorem 1 *Let $f \in C([0,1]) \cup C(S^1)$. Then:*

(1) *f is chaotic iff there is a strictly increasing sequence of positive integers A such that $h_A(f) > 0$;*
(2) *for any sequence A there is a chaotic map $f_A \in C(I)$ (resp. $f_A \in C(S^1)$) such that $h_A(f_A) = 0$.*

Statement (1) was proved by Franzová and Smítal [14] for interval maps and by Hric [17] for circle maps. Statement (2) was also proved by Hric [16, 17]).

Theorem 1 (1) is false in general in the case of two dimensional maps (see [13, [20]). So, the following question remains open: is Theorem 1 true for continuous maps defined on finite graphs?

In this paper we give a partial answer to this question. More precisely, we consider the n-star $\mathbb{X}_n = \{z \in \mathbb{C} \colon z^n \in [0,1]\}$, $n \in \mathbb{N}$. Dynamical systems generated by continuous maps on the n-star have been studied in the literature (see [1, 3−6, 8]). Moreover, the construction of chaotic n-star maps holding (2) in Theorem 1 was made in [17]. Let $C_0(\mathbb{X}_n)$ be the set of continuous maps $f \in C(\mathbb{X}_n)$ such that $f(0) = 0$. The aim of this paper is to prove the following result which extends Theorem 1 (1) to the space $C_0(\mathbb{X}_n)$.

Theorem 2 *Let $f \in C_0(\mathbb{X})$. Then f is chaotic iff there is an strictly increasing sequence of positive integers A such that $h_A(f) > 0$.*

This paper is organized as follows. Next section is devoted to introduce basic notation and definitions. In Section 3 we prove useful technical results which are used in the last section, where the main result is proved.

2 Basic Notation

First we introduce the notion of topological sequence entropy for continuous maps defined on compact metric spaces. Let (X, d) be a compact metric space and let $f \in C(X)$. Consider a strictly increasing sequence of positive integers $A = (a_i)_{i=1}^{\infty}$ and let $Y \subseteq X$ and $\varepsilon > 0$. We say that a subset $E \subset Y$ is $(A, \varepsilon, m, Y, f)$-separated if for any $x, y \in E$, $x \neq y$, there is an $i \in \{1, \ldots, m\}$ such that $d(f^{a_i}(x), f^{a_i}(y)) > \varepsilon$. Denote by $s_m(A, \varepsilon, Y, f)$ the cardinality of any maximal $(A, \varepsilon, m, Y, f)$-separated set. Define

$$s(A, \varepsilon, Y, f) = \limsup_{m \to \infty} \frac{1}{m} \log s_m(A, \varepsilon, Y, f). \tag{4}$$

Let

$$h_A(f, Y) = \lim_{\varepsilon \to 0} s(A, \varepsilon, Y, f). \tag{5}$$

The *topological sequence entropy* of f respect to the sequence A is

$$h_A(f) = h_A(f, X). \tag{6}$$

When $A = (i)_{i=0}^{\infty}$, we receive the usual definition of topological entropy (see [2, Chapter 4]).

For $x \in X$, let $\omega(x, f)$ denote the set of limit points of the sequence $(f^i(x))_{i=0}^{\infty}$. $\omega(x, f)$ is called the *omega limit set* of f at x. Let $\omega(f) = \bigcup_{x \in X} \omega(x, f)$ be the *omega limit set* of f.

Now, we introduce some definitions on n-star maps. The point $0 \in \mathbb{X}_n$ is called the *branching point* of $\mathbb{X}_n$. The connected components of $\mathbb{X}_n \backslash \{0\}$ are called *branches* of $\mathbb{X}$, denoted by $B_1, \ldots, B_n$. For $Y \subset \mathbb{X}_n$, $\overline{Y}$ denotes the closure of Y. $|x|$ denotes the module of $x \in \mathbb{X}$. For a fixed $i \in \{1, \ldots, n\}$ and $x, y \in \overline{B}_i$, we write $x < y$ (resp. $x \leq y$) to denote $|x| < |y|$ (resp. $|x| \leq |y|$). For $x, y \in \overline{B}_i$, $x \leq y$, by an *interval* we understand the set $[x, y] = \{z \in \overline{B}_i \colon x \leq z \leq y\}$, $(x, y]$, $[x, y)$ and (x, y) will be understand in the obvious way. Then, for $1 \leq i \leq n$, the closure $\overline{B}_i = [0, z_i]$, with $z_i^n = 1$. Now, define a metric on $\mathbb{X}_n$ as follows. For any $x, y \in \mathbb{X}_n$, let

$$\rho(x, y) = \begin{cases} |x - y| & \text{if } x \text{ and } y \text{ lie in the same branch;} \\ |x| + |y| & \text{if } x \text{ and } y \text{ do not lie in the same branch.} \end{cases}$$

For any $x \in \mathbb{X}_n$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{y \in \mathbb{X}_n \colon \rho(x, y) < \varepsilon\}$.

Finally, we recall the notion of horseshoe (see [19]). Let $k \in \mathbb{N}$. We say that f has a k-*horseshoe* if there is a closed interval J contained in one branch of $\mathbb{X}_n$ and there are k closed subintervals $J_i \subset J$, $1 \leq i \leq k$, with pairwise disjoint interiors such that $J \subseteq f(J_i)$ for $1 \leq i \leq k$.

3 Preliminary Results

This section is devoted to state some results which help us to prove the main theorem. We use basically two ideas in the proof. The first one is based on the following proposition.

Proposition 3 *Let (X, d) be a compact metric space and let $f \in C(X)$. Then, for any $k \in \mathbb{N}$ the following statements hold:*

(1) *f^k is chaotic iff f is chaotic;*
(2) *for any strictly increasing sequence A there is a strictly increasing sequence B such that $h_B(f^k) \geq h_A(f)$.*

Proof (1) is a well-known fact which is due to the uniform continuity of f. (2) was proved in [17].

We begin with the n-star case. For any $x \in \mathbb{X}_n$, let $s(x) = (s_i)_{i=0}^{\infty} \in \{0, 1, \ldots, n\}^{\mathbb{N}}$ be defined by $s_i = j$ iff $f^i(x) \in B_j$ for some $j \in \{1, \ldots, n\}$, and $s_i = 0$ iff $f^i(x) = 0$. We say that $s(x)$ is *eventually constant* if there is $k \in \mathbb{N}$ such that $s_i = s_k$ for all $i \geq k$. We say that $f \in C_0(\mathbb{X}_n)$ has *property* $\mathcal{P}$ if the condition $x \in \operatorname{Per}(f)$ implies that $s(x)$ is a constant sequence. The following remark is immediate but useful in what follows. We will use it without citation.

Remark 1 If for some $k \in \mathbb{N}$ we have $f^k(x) = 0$ then $f^l(x) = 0$ for each $l \geq k$ and, hence, $\omega(x, f) = \{0\}$ and $s(x)$ is eventually constant.

Property $\mathcal{P}$ is the other key which allows us to prove the main result. As we will see later, any map from $C_0(\mathbb{X}_n)$ with zero topological entropy has an iterate which holds property $\mathcal{P}$. This fact jointly with Proposition 3 are the main ideas for proving our result. Notice that maps from $C_0(\mathbb{X}_n)$ having property $\mathcal{P}$ have every periodic orbit contained in one branch, which is useful for proving next three lemmas proved previously in the proof of Lemmas 5 and 6 from [11].

Lemma 4 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$. Let $x \in B_i$, $1 \leq i \leq n$, be such that $f(x) \notin B_i$. Then for any $k \in \mathbb{N}$ such that $f^k(x) \in \overline{B}_i$ it follows that $f^k(x) < x$.*

Lemma 5 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$. Let $x \in \mathbb{X}_n$ be such that $s(x)$ is not eventually constant. Then $\lim_{k\to\infty} f^k(x) = 0$, that is, $\omega(x, f) = \{0\}$.*

For any $Y \subset \mathbb{X}_n$, let $\tau_Y : \mathbb{X}_n \to Y$ be the natural retraction from $\mathbb{X}_n$ to Y. For $f \in C_0(\mathbb{X}_n)$, let $f_Y \in C(Y)$ be defined by $f_Y = \tau_Y \circ f|_Y$, where $f|_Y$ means the restriction of f to Y. For each $i \in \{1, \dots, n\}$ define $f_i = f_{\overline{B}_i} \in C(\overline{B}_i)$. Notice that f_i is conjugated to a map $g \in C([0, 1])$ holding $g(0) = 0$.

Lemma 6 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$. Then for all $j \in \{1, \dots, n\}$, $\omega(f) \cap \overline{B}_j = \omega(f_j)$. In particular, $\omega(f) = \bigcup_{j=1}^n \omega(f_j)$ and hence $\omega(f)$ is compact.*

Let $x, y \in \mathbb{X}_n$. We write $x \prec y$ to mean that either $x < y$ or $x \in B_i$ and $y \in B_j$ for some $i, j \in \{1, \dots, n\}$, $i \neq j$. Given $S, T \subset \mathbb{X}$, we say that $S \prec T$ if $s \prec t$ for all $s \in S$ and $t \in T$.

Lemma 7 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$ and $h(f) = 0$. Let $x \in B_i$ for some $1 \leq i \leq n$ and let $a \in B_i$ be such that $x \prec a$ and $f(1) = 0$. Then $f^i(x) \prec a$ for all $i \in \mathbb{N}$.*

Proof Assume the contrary and let $j \in \mathbb{N}$ be such that $a \prec f^j(x)$. Then $a \leq f^j(x)$, $f^j(1) = f^j(0) = 0$ and $0 < x < a$. Therefore $[0, a] \subseteq f^j([0, x])$ and $[0, a] \subseteq f^j([x, a])$, that is, f^j has a 2-horseshoe. By [19, Theorem A], $h(f^j) > 0$. By [2, Chapter 4], $h(f^j) = h(f)j$. Then $h(f) > 0$, which leads us to a contradiction.

Let $x \in \mathbb{X}_n$ and $0 < \varepsilon < \min\{|x|, 1 - |x|\}$. Denote by $x_{-\varepsilon}$ and x_ε the elements such that $x_{-\varepsilon} < x < x_\varepsilon$ and $|x - x_\varepsilon| = |x - x_{-\varepsilon}| = \varepsilon$.

Lemma 8 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$ and $h(f) = 0$. Let $J \subset \mathbb{X}$ be an open interval such that $J \cap \omega(f) = \varnothing$. Then, for any $y \in J$ there is an interval J_y, $y \in J_y$, containing at most two points of each orbit.*

Proof For $y \in J \subset B_j$, $j \in \{1, \dots, n\}$, we distinguish three cases: $f(y) \notin \overline{B}_j$, $f(y) \in B_j$ and $f(y) = 0$.

First, assume that $f(y) \notin \overline{B}_j$. Let $(a, b) \subset B_j$ be such that $y \in (a, b)$, $f(1) = 0$ and $f(a, b) \cap B_j = \varnothing$. If $f^i(a, b) \prec (a, b)$ for all $i \in \mathbb{N}$, then the proof concludes. So, let $m \in \mathbb{N}$ be the first integer such that $f^m(a, b) \cap (a, b) \neq \varnothing$. Assume that if any positive integer i is big enough, then it is held $f^m(y_{-\varepsilon_i}, y_{\varepsilon_i}) \cap (y_{-\varepsilon_i}, y_{\varepsilon_i}) \neq \varnothing$ with $\varepsilon_i = 1/i$. Hence $\cap_i (f^m(y_{-\varepsilon_i}, y_{\varepsilon_i}) \cap (y_{-\varepsilon_i}, y_{\varepsilon_i})) = \{y\}$. Since f^m is continuous, we would have $f^m(y) = y$, which leads us to a contradiction. So there is $i \in \mathbb{N}$ such that $f^m(y_{-\varepsilon_i}, y_{\varepsilon_i}) \prec (y_{-\varepsilon_i}, y_{\varepsilon_i})$ (cf. Lemma 4) and $(y_{-\varepsilon_i}, y_{\varepsilon_i}) \subset (a, b)$. Now, we distinguish two cases. If $a = 0$ then by Lemma 4 $f^k(y_{-\varepsilon_i}, y_{\varepsilon_i}) \prec (y_{-\varepsilon_i}, y_{\varepsilon_i})$ for all $k \geq m$ and the proof concludes. If $a \neq 0$, then applying Lemmas 4 and 7, $f^k(y_{-\varepsilon_i}, y_{\varepsilon_i}) \prec (y_{-\varepsilon_i}, y_{\varepsilon_i})$ for all $k \geq m$, which finishes the proof.

Now, assume that $f(y) \in B_j$. Let $(a, b) \subset B_j$ be such that $y \in (a, b)$ and $f(a, b) \subset B_j$. Assume that any open subinterval J containing y contains at least three points of some orbit, that is, there is an $x \in \mathbb{X}$ and there are $n_1 < n_2 < n_3$ such that $f^{n_i}(x) \in J$, $1 \leq i \leq 3$. By Lemma 6 and [10, Proposition 11, Chapter 4], there is an interval J_y holding that for any $x \in \mathbb{X}_n$ with $f^{n_i}(x) \in J_y$, $1 \leq i \leq 3$, there is $k \in \mathbb{N}$, $n_1 < k < n_3$, such that $f^k(x) \notin B_j$. Then, $f^{k-1}(x) \in (c, d)$ with $f(c) = 0$ and $f(c, d) \cap B_j = \varnothing$.

By Lemma 7, $(c,d) \prec J_y$. Then, by Lemma 4, for all integer $m > k$ it holds that $f^m(x) \prec J_y$, a contradiction.

Finally, assume that $f(y) = 0$. Since $f(0) = 0$ and f is uniformly continuous, there are real numbers $\varepsilon_n > \dots > \varepsilon_1 > 0$ such that

$$f(B(0,\varepsilon_j)) \subset B(0,\varepsilon_{j+1}) \quad \text{for} \quad j = 1,2,\dots,n-1. \tag{7}$$

Since $f(y) = 0$, there is $\delta > 0$ such that $f(y_{-\delta}, y_\delta) \subset B(0,\varepsilon_1)$. On the other hand, let $K = \max\{|f_j(z)|\colon z \in [0,y]\}$ and let $z_0 \in [0,y]$ be such that $|f(z_0)| = |f_j(z_0)| = K$. Clearly ε_n and δ can be chosen such that

$$(y_{-\delta}, y_\delta) \bigcap [0, f_j(z_0)] = \varnothing \quad \text{(cf. Lemma 7)} \tag{8}$$

and $(y_{-\delta}, y_\delta) \cap B(0,\varepsilon_n) = \varnothing$. Now, let $x \in (y_{-\delta}, y_\delta)$ and notice that $f(x) \in B(0,\varepsilon_1)$. If $f_j^i(x) = f^i(x)$ for all $i \in \mathbb{N}$ then, as $(y_{-\delta}, y_\delta) \cap [0, f_j(z_0)] = \varnothing$, we conclude that $f^i(x) \notin (y_{-\delta}, y_\delta)$ for all $i \in \mathbb{N}$ and we finish. So, let m be the first integer such that $f^m(x) \notin B_j$. If $m > 1$, and $k > m$ holds $f^k(x) \in \overline{B}_j$, then by Lemma 4, $f^k(x) < f(x)$ and hence $f^k(x) \in B(0,\varepsilon_1)$. This, jointly with (8) gives us $\{f^i(x)\colon i \in \mathbb{N}\} \cap (y_{-\delta}, y_\delta) = \varnothing$. To finish the proof, assume $m = 1$ and let k be the smallest integer such that $f^k(x) \in \overline{B}_j$ (if such k does not exist we finish). Let $l < n$ be the number of branches in which the set $\{f^i(x)\colon 1 \le i \le k\}$ lies. Notice that if an element $z \in B(0,\varepsilon_s) \cap \overline{B}_r$, $1 \le r,s \le n$, $f^i(z) \in \overline{B}_r$ for some $i \in \mathbb{N}$ and $f^{i+1}(z) \notin \overline{B}_r$, then by Lemmas 4 and 7, $f^i(z) \in B(0,\varepsilon_s)$. Then, by (7), $f^k(x) \in B(0,\varepsilon_l)$ and by Lemma 4 and (8) we conclude that $\{f^i(x)\colon i \in \mathbb{N}\} \cap (y_{-\delta}, y_\delta) = \varnothing$, which ends the proof.

The argument of the proof of the following result is very similar to the analogous result for interval continuous maps.

Corollary 9 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$ and $h(f) = 0$. For any open set $U \supset \omega(f)$ there is a positive integer $q = q(U)$ such that at most q points of any trajectory lie outside U.*

Proof The set $\mathbb{X}_n \backslash U$ is compact. By Lemma 8, for any $y \in \mathbb{X}_n \backslash U$, there is an open interval J_y (relative to $\mathbb{X}_n \backslash U$) containing at most two points of any orbit. Since $\mathbb{X}_n \backslash U$ is a compact set we can obtain a finite number of such intervals covering $\mathbb{X}_n \backslash U$, which ends the proof.

Proposition 10 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$. Then f is chaotic iff $f_i \in C(\overline{B}_i)$ is chaotic for some $i \in \{1,\dots,n\}$.*

Proof First, assume that f_i is chaotic for some $i \in \{1,2,\dots,n\}$, and let $S \subset \overline{B}_i$ be a scrambled set. Notice that if $x \in S$ then $f_i^j(x) \ne 0$ for all $j \in \mathbb{N}$ (in other case $\omega(x,f) = \{0\}$ and $x \notin S$). Hence the trajectory of any $x \in S$ is contained in B_i which implies that the trajectories of x under f_i and f are the same. Then S is also a scrambled set for f.

Now, assume that f is chaotic and let $S \subset \mathbb{X}_n$ be an uncountable scrambled set of f. Let $x \in S$. By Lemma 5, the sequence $s(x)$ must be eventually constant, because in other case the orbit of x would be attracted by the fixed point 0. Let r be such that $s_j = s_r$ for all $j \ge r$, but $s_{r-1} \ne s_r$. On the other hand, let $y \in S$. Since $\liminf_{j\to\infty} d(f^j(x), f^j(y)) = 0$, we have that the trajectory of y is eventually contained in B_{s_r}. Let $[0,a] = \bigcap_{j\ge 0} (f_{s_r})^j(\overline{B}_{s_r})$. Since f_{s_r} is an interval map, by [9, Lemma 3.5], if $\{(f_{s_r})^j(f^r(x)) \colon j \in \mathbb{N}\} \cap [0,a]$ is empty, then the trajectory of $f^r(x)$ will be attracted by a periodic orbit and then x cannot belong to any scrambled set. So, there is $j_x \geq r$ such that $f^{j_x}(x) \in [0,a]$. Since $f_{s_r}|_{[0,a]}$ is surjective, there is $x_0 \in [0,a]$ such that $f^{j_x}(x_0) = f^{j_x}(x)$. Similarly, there are $y_0 \in [0,a]$ and $j_y \geq r$ such that $f^{j_y}(y_0) = f^{j_y}(y)$. Let $S_0 = \{y_0 \in B_{s_e} \colon y \in S\} \subset [0,a]$. Then it is straightforward to see that S_0 is a scrambled set for f_{s_r} and therefore f_{s_r} is chaotic.

In order to finish the preparatory work to prove our main result, we prove the following lemma, which is an extension of a similar lemma from [14].

Lemma 11 *Assume $f \in C_0(\mathbb{X}_n)$ has property $\mathcal{P}$ and $h(f) = 0$. Suppose f is non-chaotic. Then, for any $\varepsilon > 0$ there are points $x_1, \ldots, x_k \in \omega(f)$, and a set $U \supset \omega(f)$, relatively open in $\mathbb{X}_n$, with the following property: if*

$$f^j(x) \in U \quad \textit{for} \quad 0 \leq j \leq r,$$

then there is some i such that for any j with $0 \leq j \leq r$

$$d(f^j(x), f^j(x_i)) < \varepsilon.$$

Proof Let f be non-chaotic. By Proposition 10, f_i is non-chaotic for $i = 1, \ldots, n$. Then, by [12, Theorem 2.3], for $i = 1, \ldots, n$ it holds that $f_i|_{\omega(f_i)}$ are Lyapunov stable (it has equicontinuous powers), and any point $y \in \omega(f_i)$ is almost periodic (for any neighborhood G of y there is an integer $m > 0$ such that $f^{m \cdot j}(y) \in G$ for any $j \geq 0$). By Lemma 6 it is easy to see that

$$f|_{\omega(f)} \quad \text{is Lyapunov stable} \tag{9}$$

and

$$\text{every point in } \omega(f) \text{ is almost periodic.} \tag{10}$$

Then, using (9) and (10) and following the proof of the lemma from [14] we obtain the result.

4 Proof of Theorem 2

First, consider the case $h(f) = 0$. Following the proof of Theorem 1.5 from [5] we see that f^N, $N = n!(n-1)! \ldots 2!1!$, holds property $\mathcal{P}$. Additionally, by [2, Chapter 4], $h(f^N) = Nh(f) = 0$. So, by Proposition 3 we may assume without loss of generality that f has property $\mathcal{P}$.

First, assume f is non-chaotic and let A be a strictly increasing sequence of positive integers. Then, applying Lemma 11 and Corollary 9 and proceeding as in the first part of the proof of the main result of [14], we obtain that $h_A(f) = 0$ for all A. Now assume that f is chaotic. By Proposition 10, f_i is chaotic for some $i \in \{1, \ldots, n\}$. Following [14], there is a interval $J \subset B_i$, with $f_i^{2^j}(J) = f^{2^j}(J) = J$ for some $j \in \mathbb{N}$ and such that $f_i^k(J) \subset B_i$ for $1 \leq k \leq 2^j$. Additionally, $f_i^{2^j}|_J$ is chaotic and hence $h_A(f_i^{2^j}|_J) > 0$. Then

$$h_{2^j \cdot A}(f) \geq h_{2^j \cdot A}(f, J) = h_{2^j \cdot A}(f_i, J) = h_A(f_i^{2^j}|_J) = h_A(f_i^{2^j}, J) > 0,$$

where $2^j \cdot A = (2^j \cdot a_i)_{i=1}^{\infty}$.

Finally, assume $h(f) > 0$. By [19], there is an $l \in \mathbb{N}$ such that f^l has a k-horseshoe. Since $h(f^l) = lh(f) > 0$, by Proposition 3 we may assume that $l = 1$. So, there is an interval J and k subintervals $J_1, \dots, J_k$ with pairwise disjoint interiors and such that $J \subseteq f(J_i)$. There is an invariant compact subset Y included in at most two branches such that $f|_Y$ is semiconjugate to a shift map defined on $\Sigma = \{(x_j)_{j=1}^{\infty} : \ x_j \in \{0,1\}\}$ (see e.g. [10, Chapter 2]). Then, it is straightforward to check that f is chaotic, and the proof concludes.

Corollary 12 *Let $f \in C_0(\mathbb{X}_n)$ be such that $0 \in Per(f)$. Then f is chaotic iff there is an increasing sequence of positive integers A such that $h_A(f) > 0$.*

Proof Just apply Proposition 3 and Theorem 2.

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