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A Constant-Gain Nonlinear Estimator for Linear Switching Systems

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Abstract: In this paper, we develop a nonlinear observer for switching linear systems. The developed state observer offers the properties of being self-tuning without any information about the switching instants and the current mode of the switching system to be observed. Numerical example is provided to highlight the usefulness of the developed results.

Keywords: *Estimation; switching systems; nonlinear observers; LMIs.*

Mathematics Subject Classification (2000): 93B36, 93C05, 93E03; 93E15.

1 Introduction

Estimation of switching systems has rapidly increased in importance with the development of new circuits technologies. Recently, we witnessed an increasing interest in the so-called *switching systems*. We call herein switching systems all dynamical systems described by differential inclusions of the form

$$\dot{x}(t) \in \{f_\sigma(x(t),\, u(t))\}_{\sigma \in \mathcal{A}}\,,$$

where $x(t)$ is the state variable, $u(t)$ is the control input, and $f_\sigma(\,\cdot\,,\,\cdot\,)$ is a collection of continuously differentiable functions parameterized by σ belonging to some given set $\mathcal{A}$. Such systems are composed of both discrete and continuous subsystems. Control, observation, and supervision of this kind of systems appear in many ongoing research projects such as multimedia protocols, electrical circuits, systems subject to failure and so on.

Numerous control procedures are based on the knowledge of all state variables of the considered system. This assumption is not always true since the measurements of the states variables are, in most cases, not possible or simply too expensive. For this reason, observer design has received widespread attention since the introduction of Kalman theory and remains of great importance nowadays.

Estimation of hybrid systems is one of the challenging research problems that necessitate a particular attention. Extension of available results in observation of linear systems to hybrid linear systems is not quite easy due to the variation of nominal models and others technical problems. Switching between different models to compensate or analyze system variations is a well-known technique in modern control theory. It is obvious that if both the switching instants and the switching modes are known, then it is easy to construct a switching gain observer that switches among different gains. We refer the reader to the references [1−5], and [6] for more details.

The question we are addressing in this paper is how one can estimate the unmeasured states of a given switching system if the current mode is unknown? The answer to this question will be detailed in the present work where we assume that there is no switching law that defines the passage of the switching system from a mode to another. The goal of this paper is to develop a new observation technique for switching linear systems. The developed observers are nonlinear and do not necessitate the mode estimation of the system to be observed. We mean by mode estimation, the ability to track a system's discrete dynamics as it moves between different behavioral modes. We show that a constant high-gain observer is sufficient to observe the unmeasured dynamics whatever the changes in the nominal matrices of the considered switching system. The present work eliminates two major frequently-faced problems: detection of the switching instants and identification of the current mode. The whole observer design is efficiently accomplished by using an LMI procedure.

The paper is organized as follows. Section 2 is devoted to the design of the observer for regular switching systems. In Section 3, the results of the previous section are then extended to uncertain switching systems. Section 4 treats a numerical observation example of a switching system. The paper ends with general conclusions and some concluding remarks. Throughout this paper, we note by I and 0 the identity matrix and the null matrix of appropriate dimensions, respectively. $A > 0$ (resp. $A < 0$) denotes that the matrix A is a symmetric and positive-definite (resp. symmetric and negative-definite). We note by A' the matrix transpose of the matrix A. $\|\cdot\|$ stands for the Euclidean norm.

2 Constant-Gain Observer for Switching Systems

Our objective is to conceive an observer for the following switching system

$$\begin{aligned} \frac{dx(t)}{dt} &= A(\sigma(t))x(t) + B(\sigma(t))u(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input, and $y(t) \in R^p$ is the system output. $\sigma(t)$ is a switching signal that maps the index time $[0, +\infty[$ into an index set $\mathcal{S} = \{1, 2, \ldots, s\}$. Each mode $j \in \mathcal{S}$ corresponds to a specific model characterized by $A(j) \in \mathcal{A} = \{A(1), A(2), \ldots, A(s)\}$ and $B(j) \in \mathcal{B} = \{B(1), B(2), \ldots, B(s)\}$. We assume that the switchs in the output matrix C are absent. For the observer design, we suppose that the following assumptions are verified.

Assumption 1 The switch between two different modes is instantaneous and arbitrary.

Assumption 2 There is no information on the current mode of the switching system, and the switching instants are not known.

Assumption 3 For any time t, the control input $u(t)$ is smooth, i.e., it can be written as

$$u(t) = \int_0^t v(\tau)\, d\tau. \tag{3}$$

where $v(t) \in R^m$ is the new control input.

For the class of systems we are considering, different types of observability have been studied in the past and for more details on this subject, we refer the reader to [1] and the references therein. Here, we will assume that the pairs $(A(\sigma(t)), C)$, $\forall\,\sigma(t)$ are observable. This means that the system is observable, in the sense of Kalman, for each mode.

Based on the last assumptions, the switching system is rewritten in the following form:

$$\begin{aligned} \frac{dx(t)}{dt} &= A(\sigma(t))x(t) + B(\sigma(t))u(t), \\ \frac{du(t)}{dt} &= v(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4}$$

For the simplicity of the representation, let

$$\widetilde{A}(\sigma(t)) = \begin{bmatrix} A(\sigma(t)) & B(\sigma(t)) \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$z = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad \widetilde{C} = [\, C \quad 0 \,],$$

then the dynamics (4) is rewritten as:

$$\begin{aligned} \frac{dz(t)}{dt} &= \widetilde{A}(\sigma(t))z + \widetilde{B}v(t), \\ y(t) &= \widetilde{C}z(t). \end{aligned} \tag{5}$$

We propose an observer of the following form:

$$\frac{d\hat{z}(t)}{dt} = \left(\sum_{j=1}^{s} \widetilde{A}(j)\right)\hat{z}(t) + \widetilde{B}v(t) + \left(\sum_{i=1}^{s} P_i\right)^{-1} Y\left(y(t) - \widetilde{C}\hat{z}(t)\right) - \rho(y(t), \hat{z}(t)), \tag{6}$$

where $P_1, P_2, \dots, P_s$ are $(m+n)\times(n+m)$ symmetric and positive definite matrices, Y is a constant matrix of appropriate dimensions, and $\rho(y(t),\, \hat{z}(t))$ is a nonlinear additive term that depends on the output $y(t)$, and the observer state vector $\hat{z}(t)$. The dynamics of the proposed observer is the sum of the dynamics of classical Luemberger observers written for each mode plus a nonlinear additive term $\rho(\cdot,\cdot)$ that attenuates the effects of the difference between the observer and the system outputs. The design of $(P_i)_{1\le i\le s}$, and $\rho(\cdot,\cdot)$ will be given latter. Let $e(t)=\hat{z}(t)-z(t)$ be the observation error, and let

$$\frac{de(t)}{dt}=\left(\sum_{j=1}^{s}\widetilde{A}(j)\right)\hat{z}(t)-\widetilde{A}(\sigma(t))z(t)-\rho(y(t),\hat{z}(t))-\left(\sum_{i=1}^{s}P_i\right)^{-1}Y\widetilde{C}e(t),$$

be the dynamics of the observer error, then we can write

$$\frac{de(t)}{dt}=\left(\widetilde{A}(\sigma(t))-\left(\sum_{i=1}^{s}P_i\right)^{-1}Y\widetilde{C}\right)e(t)+\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t)-\rho(y(t),\hat{z}(t)). \tag{7}$$

The time derivative of the Lyapunov function $V(e(t))=e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)e(t)$ along the trajectory of (7) is

$$\begin{aligned}\frac{dV(e(t))}{dt}&=\frac{de^{\mathrm{T}}(t)}{dt}\left(\sum_{i=1}^{s}P_i\right)e(t)+e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\frac{de(t)}{dt}\\
&=e^{\mathrm{T}}(t)\left(\widetilde{A}^{\mathrm{T}}(\sigma(t))\left(\sum_{i=1}^{s}P_i\right)+\left(\sum_{i=1}^{s}P_i\right)\widetilde{A}(\sigma(t))-\widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}}-Y\widetilde{C}\right)e(t)\\
&\quad-2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\rho(y(t),\hat{z}(t))+e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t)\\
&\quad+\hat{z}^{\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\mathrm{T}}(j)\left(\sum_{i=1}^{s}P_i\right)e(t).\end{aligned}$$

We have for arbitrary vectors w_1 and w_2 and a given positive definite matrix Z of appropriate dimensions [7]

$$2w_1^{\mathrm{T}}w_2\le w_1^{\mathrm{T}}Z^{-1}w_1+w_2^{\mathrm{T}}Zw_2.$$

If we take

$$w_1=\left(\sum_{i=1}^{s}P_i\right)e(t),\quad w_2=\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t),\quad Z=\mu_\sigma I,$$

then

$$\begin{aligned}e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t)+\hat{z}^{\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\mathrm{T}}(j)\left(\sum_{i=1}^{s}P_i\right)e(t)\\
=2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t)\\
\le\mu_\sigma^{-1}e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s}P_i\right)\left(\sum_{i=1}^{s}P_i\right)e(t)+\mu_\sigma\hat{z}^{\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\mathrm{T}}(j)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t).\end{aligned}$$

If the matrices $(P_i)_{1\le i\le s}$ are selected so as to

$$\begin{aligned}\widetilde{A}^{\mathrm{T}}(\sigma(t))\Big(\sum_{i=1}^{s}P_i\Big)+\Big(\sum_{i=1}^{s}P_i\Big)\widetilde{A}(\sigma(t))-\widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}}-Y\widetilde{C}\\ +\mu_\sigma^{-1}\Big(\sum_{i=1}^{s}P_i\Big)\Big(\sum_{i=1}^{s}P_i\Big)=-Q(\sigma)<0,\end{aligned}\tag{8}$$

then we obtain

$$\frac{dV(e(t))}{dt}=-e^{\mathrm{T}}(t)Q(\sigma)e(t)+\mu\hat{z}^{\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\mathrm{T}}(j)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{z}(t)-\rho(y(t),\hat{z}(t)).$$

If we choose $\mu_{\max}=\max\limits_{\sigma}\mu_\sigma$ and

$$\rho(y(t),\hat{z}(t))=\begin{cases}\mu_{\max}\varpi\hat{z}^{\mathrm{T}}(t)\hat{z}(t)\dfrac{\Big(\sum\limits_{i=1}^{s}P_i\Big)^{-1}\widetilde{C}^{\mathrm{T}}\widetilde{C}e(t)}{2\|\widetilde{C}e(t)\|^2} & \text{if}\quad \|\widetilde{C}e(t)\|\neq 0,\\ 0 & \text{if}\quad \|\widetilde{C}e(t)\|=0,\end{cases}\tag{9}$$

where

$$\varpi=\sup_{\sigma(t)}\left\|\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\mathrm{T}}(j)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\right\|,\tag{10}$$

then

$$\frac{dV(e(t))}{dt}\le -e^{\mathrm{T}}(t)Q(\sigma)e(t),$$

which implies that the observer error decays exponentially to the origin.

Remark 1 The formulae of $\rho(\cdot,\cdot)$ given by equation (9) is just a conceptual one. When the observation error is close to zero, it is recommended to modify the nonlinear term $\rho(\cdot,\cdot)$ as follows:

$$\rho(y(t),\hat{z}(t))=\begin{cases}\mu_{\max}\varpi\hat{z}^{\mathrm{T}}(t)\hat{z}(t)\dfrac{\Big(\sum\limits_{i=1}^{s}P_i\Big)^{-1}\widetilde{C}^{\mathrm{T}}\widetilde{C}e(t)}{2\|\widetilde{C}e(t)\|^2} & \text{if}\quad \|\widetilde{C}e(t)\|>\bar{\epsilon},\\ 0 & \text{if}\quad \|\widetilde{C}e(t)\|\le\bar{\epsilon},\end{cases}$$

where $\bar{\epsilon}>0$ is some prescribed small parameter. We summarize the result in the following statement.

Theorem 1 *System*

$$\frac{\hat{z}(t)}{dt}=\left(\sum_{j=1}^{s}\widetilde{A}(j)\right)\hat{z}(t)+\widetilde{B}v(t)+\left(\sum_{i=1}^{s}P_i\right)^{-1}Y\left(y(t)-\widetilde{C}\hat{z}(t)\right)-\rho(y(t),\hat{z}(t)),\tag{11}$$

is an asymptotic observer for system (5) if there exist a set of positive constants $\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_s\}$*, and a set of symmetric and positive definite matrices* $\mathcal{P} = \{P_1, P_2, \dots, P_s\}$ *such that the following coupled LMIs are feasible*

$$\begin{bmatrix} \mathcal{J}(P_1, \dots, P_s, Y, j) & \left(\sum_{i=1}^{s} P_i\right) \\ \left(\sum_{i=1}^{s} P_i\right) & -\mu_j I \end{bmatrix} < 0, \quad 1 \le j \le s,$$

where

$$\mathcal{J}(P_1, \dots, P_s, Y, j) = \widetilde{A}^{\mathrm{T}}(j)\left(\sum_{i=1}^{s} P_i\right) + \left(\sum_{i=1}^{s} P_i\right)\widetilde{A}(j) - \widetilde{C}^{\mathrm{T}} Y^{\mathrm{T}} - Y\widetilde{C}.$$

Proof The LMIs conditions (12) and (8) are equivalent by the Schur complement lemma.

3 Extension to Uncertain Switching Systems

Consider the uncertain switching system

$$\begin{aligned} \frac{dx(t)}{dt} &= (A(\sigma(t)) + \Delta A(\sigma(t)))x(t) + (B(\sigma(t)) + \Delta B(\sigma(t)))u(t), \\ \frac{du(t)}{dt} &= v(t), \\ y(t) &= Cx(t), \end{aligned} \tag{13}$$

which satisfies the assumptions of system (4). The aim of this section is to design a robust nonlinear observer that can estimates the states of (13) without a priori knowledge of the current mode and or the switching instants. The uncertain terms $\Delta A(\sigma(t))$ and $\Delta B(\sigma(t))$ are written respectively as $E_A^{\mathrm{T}} F_A(\sigma(t)) D_A$ and $E_B^{\mathrm{T}} F_B(\sigma(t)) D_B$. he matrices E_A, E_B, D_A, and D_B are constant known matrices and $F_A(\sigma(t))$, $F_B(\sigma(t))$ are unknown matrices satisfying the inequalities $F_A^{\mathrm{T}}(\sigma(t))F_A(\sigma(t)) < I$, $F_B^{\mathrm{T}}(\sigma(t))F_B(\sigma(t)) < I$, respectively. In matrix notation system (13) is rewritten as

$$\begin{aligned} \frac{d\xi(t)}{dt} &= \left(\widetilde{A}(\sigma(t)) + \Delta\widetilde{A}(\sigma(t))\right)\xi(t) + \widetilde{B}v(t), \\ y &= \widetilde{C}\xi(t), \end{aligned} \tag{14}$$

where

$$\Delta\widetilde{A}(\sigma(t)) = \begin{bmatrix} \Delta A(\sigma(t)) & \Delta B(\sigma(t)) \\ 0_{m\times n} & 0_{m\times m} \end{bmatrix}, \qquad \xi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{15}$$

and $\widetilde{A}(\sigma(t))$, $\widetilde{B}$, $\widetilde{C}$ are defined as in equation (5). The uncertain term $\Delta\widetilde{A}(\sigma(t))$ can be rewritten as $\widetilde{E}_A^{\mathrm{T}} \widetilde{F}_A(\sigma(t)) \widetilde{D}_A$ where

$$\widetilde{E}_A = \begin{bmatrix} E_A & 0 \\ E_B & 0 \end{bmatrix}, \quad \widetilde{F}_A(\sigma(t)) = \begin{bmatrix} F_A(\sigma(t)) & 0 \\ 0 & F_B(\sigma(t)) \end{bmatrix}, \quad \widetilde{D}_A = \begin{bmatrix} D_A & 0 \\ 0 & D_B \end{bmatrix}.$$

The observer design is given in the following statement:

Theorem 2 *Consider system (13). If there exist a set of $(n+m)\times(n+m)$ symmetric and positive-definite matrices $(P_i)_{1\le i\le s} > 0$, a matrix Y of appropriate dimensions, and positive constants $(\mu_i)_{1\le i\le s}$, $(\epsilon_A(i))_{1\le i\le s}$, $(\epsilon_B(i))_{1\le i\le s}$ such that the following coupled LMIs hold*

$$\begin{bmatrix} \mathcal{K}(P_1, P_2, \dots, P_s, Y) & \widetilde{E}_A\left(\sum\limits_{i=1}^{s} P_i\right) & \left(\sum\limits_{i=1}^{s} P_i\right) & \left(\sum\limits_{i=1}^{s} P_i\right) \\ \left(\sum\limits_{i=1}^{s} P_i\right)\widetilde{E}_A{}^{\mathrm{T}} & -\epsilon_A(j)I & 0 & 0 \\ \left(\sum\limits_{i=1}^{s} P_i\right) & 0 & -\epsilon_B(j)I & 0 \\ \left(\sum\limits_{i=1}^{s} P_i\right) & 0 & 0 & -\mu_j I \end{bmatrix} < 0, \quad 1\le j\le s, \tag{16}$$

where

$$\mathcal{K}(P_1, P_2, \dots, P_s, Y) = \widetilde{A}^{\mathrm{T}}(j)\left(\sum_{i=1}^{s} P_i\right) + \left(\sum_{i=1}^{s} P_i\right)\widetilde{A}(j) - \widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}} - Y\widetilde{C} + \epsilon_A(j)\widetilde{D}_A^{\mathrm{T}}\widetilde{D}_A.$$

Then system

$$\frac{\hat{\xi}(t)}{dt} = \left(\sum_{j=1}^{s}\widetilde{A}(j)\right)\hat{\xi}(t) + \widetilde{B}v(t) + \left(\sum_{i=1}^{s} P_i\right)^{-1} Y\left(y(t) - \widetilde{C}\hat{\xi}(t)\right) - \rho(y(t),\hat{\xi}(t)),$$

is an asymptotic observer for the uncertain switching system (13) where $\rho(y(t),\hat{\xi}(t))$ is defined as

$$\rho(y(t),\hat{z}(t)) = \begin{cases} \left(\varpi\mu_{\max} + \epsilon_{\max}\|\widetilde{E}_A\|^2\|\widetilde{D}_A\|^2\right)\hat{\xi}^{\mathrm{T}}(t)\hat{\xi}(t)\dfrac{\left(\sum\limits_{i=1}^{s} P_i\right)^{-1}\widetilde{C}^{\mathrm{T}}\widetilde{C}e(t)}{\|\widetilde{C}e(t)\|^2} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad \|\widetilde{C}e(t)\| \ne 0, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{if} \quad \|\widetilde{C}e(t)\| = 0, \end{cases}$$

where $\epsilon_{\max} = \max\limits_{\sigma}(\epsilon_A(\sigma))$ and ϖ is defined as in Theorem 1.

Proof Let $e(t) = \hat{\xi}(t) - \xi(t)$ be the observation error. Then its dynamics is given by

$$\begin{aligned} \frac{de(t)}{dt} &= \left(\sum_{j=1}^{s}\widetilde{A}(j)\right)\hat{\xi}(t) - \widetilde{A}(\sigma)\xi(t) - \Delta A(\sigma(t))\xi(t) - \left(\sum_{i=1}^{s} P_i\right)^{-1} Y\widetilde{C}e - \rho(y,\hat{\xi}(t)) \\ &= \left(\widetilde{A}(\sigma(t)) - \left(\sum_{i=1}^{s} P_i\right)^{-1} Y\widetilde{C} + \Delta\widetilde{A}(\sigma(t))\right)e(t) \\ &\quad + \sum_{\substack{j\in\mathcal{S}\\ j\ne\sigma}}\widetilde{A}(j)\hat{\xi}(t) - \Delta\widetilde{A}(\sigma(t))\hat{\xi}(t) - \rho(y,\hat{\xi}(t)). \end{aligned} \tag{17}$$

Choosing the Lyapunov function as $V(e(t)) = e^{\mathrm{T}}(t)\Big(\sum_{i=1}^{s} P_i\Big)e(t)$, we obtain

$$\begin{aligned}\frac{dV(e(t))}{dt} &= e^{\mathrm{T}}(t)\left(\widetilde{A}^{\mathrm{T}}(\sigma(t)) - \widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}}\left(\sum_{i=1}^{s} P_i\right)^{-1} + \Delta\widetilde{A}^{\mathrm{T}}(\sigma(t))\right)\left(\sum_{i=1}^{s} P_i\right)e(t)\\
&+ e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\left(\widetilde{A}(\sigma(t)) - \left(\sum_{i=1}^{s} P_i\right)^{-1} Y\widetilde{C} + \Delta\widetilde{A}(\sigma(t))\right)e(t)\\
&+ 2e(t)\left(\sum_{i=1}^{s} P_i\right)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{\xi}(t) + 2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\Delta\widetilde{A}(\sigma(t))\hat{\xi}(t)\\
&- 2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\rho(y,\,\hat{\xi}(t)).\end{aligned}$$

We have for any $\mu_\sigma > 0$

$$\begin{aligned}2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\,\hat{\xi}(t) &\le \mu_\sigma^{-1}e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\left(\sum_{i=1}^{s} P_i\right)e(t)\\
&+ \mu_\sigma\hat{\xi}^{\,\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\,\mathrm{T}}(j)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{\xi}(t),\end{aligned}$$

furthermore,

$$\begin{aligned}&\Delta\widetilde{A}^{\,\mathrm{T}}(\sigma(t))\left(\sum_{i=1}^{s} P_i\right) + \left(\sum_{i=1}^{s} P_i\right)\Delta\widetilde{A}(\sigma(t))\\
&= \widetilde{D}_A^{\mathrm{T}}\widetilde{F}_A^{\mathrm{T}}(\sigma(t))\widetilde{E}_A\left(\sum_{i=1}^{s} P_i\right) + \left(\sum_{i=1}^{s} P_i\right)\widetilde{E}_A^{\mathrm{T}}\widetilde{F}_A\widetilde{D}_A\\
&\le \epsilon_A(\sigma)\widetilde{D}_A^{\mathrm{T}}\widetilde{D}_A + \epsilon_A^{-1}(\sigma)\left(\sum_{i=1}^{s} P_i\right)\widetilde{E}_A^{\mathrm{T}}\widetilde{E}_A\left(\sum_{i=1}^{s} P_i\right).\end{aligned}$$

In addition, we have

$$\begin{aligned}2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\Delta\widetilde{A}(\sigma(t))\hat{\xi}(t) &\le \epsilon_B^{-1}(\sigma)e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\left(\sum_{i=1}^{s} P_i\right)e(t)\\
&+ \epsilon_B(\sigma)\hat{\xi}^{\,\mathrm{T}}(t)\Delta\widetilde{A}^{\,\mathrm{T}}(\sigma(t))\Delta\widetilde{A}(\sigma(t))\hat{\xi}(t).\end{aligned}$$

Using the definition of $\rho(\cdot,\cdot)$, we obtain

$$\begin{aligned}&\epsilon_B(\sigma)\hat{\xi}^{\,\mathrm{T}}(t)\Delta\widetilde{A}^{\,\mathrm{T}}(\sigma(t))\Delta\widetilde{A}(\sigma(t))\hat{\xi}(t) + \mu_\sigma\hat{\xi}^{\,\mathrm{T}}(t)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}^{\,\mathrm{T}}(j)\sum_{\substack{j\in\mathcal{S}\\ j\neq\sigma}}\widetilde{A}(j)\hat{\xi}(t)\\
&\qquad - 2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\rho(\cdot,\,\cdot)\\
&\le \left(\varpi\mu_{\max} + \epsilon_{\max}\|\widetilde{E}_A\|^2\|\widetilde{D}_A\|^2\right)\hat{\xi}^{\,\mathrm{T}}(t)\hat{\xi}(t) - 2e^{\mathrm{T}}(t)\left(\sum_{i=1}^{s} P_i\right)\rho(\cdot,\cdot) \le 0.\end{aligned}$$

This implies that

$$
\begin{aligned}
\frac{dV(e(t))}{dt} \le e^{\mathrm{T}}(t)\Bigg(&\widetilde{A}^{\mathrm{T}}(\sigma(t))\Big(\sum_{i=1}^{s} P_i\Big) + \Big(\sum_{i=1}^{s} P_i\Big)\widetilde{A}(\sigma(t)) - \widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}} - Y\widetilde{C} \\
&+ \epsilon_A(\sigma)\widetilde{D}_A'\widetilde{D}_A + \epsilon_B^{-1}(\sigma)\Big(\sum_{i=1}^{s} P_i\Big)\Big(\sum_{i=1}^{s} P_i\Big) + \mu_\sigma^{-1}\Big(\sum_{i=1}^{s} P_i\Big)\Big(\sum_{i=1}^{s} P_i\Big)\Bigg)e(t).
\end{aligned} \tag{18}
$$

If for each mode $1 \le j \le s$ the matrices

$$
\begin{aligned}
&\widetilde{A}^{\mathrm{T}}(j)\Big(\sum_{i=1}^{s} P_i\Big) + \Big(\sum_{i=1}^{s} P_i\Big)\widetilde{A}(j) - \widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}} - Y\widetilde{C} + \epsilon_A(j)\widetilde{D}_A^{\mathrm{T}}\widetilde{D}_A \\
&\quad + \epsilon_A^{-1}(j)\Big(\sum_{i=1}^{s} P_i\Big)\widetilde{E}_A^{\mathrm{T}}\widetilde{E}_A\Big(\sum_{i=1}^{s} P_i\Big) + \epsilon_B^{-1}\Big(\sum_{i=1}^{s} P_i\Big)\Big(\sum_{i=1}^{s} P_i\Big) \\
&\quad + \mu_j^{-1}\Big(\sum_{i=1}^{s} P_i\Big)\Big(\sum_{i=1}^{s} P_i\Big) < 0,
\end{aligned} \tag{19}
$$

then $dV(e(t))/dt$ becomes always negative and the observer error decays exponentially to the origin. The last inequality is equivalent by the Schur complement to (12). This ends the proof.

An observer for uncertain single-mode systems can be deduced from result of Theorem 2. It is sufficient to replace $\Big(\sum\limits_{i=1}^{s} P_i\Big)$ by a one positive definite matrix X in the LMIs of Theorem 2 to deliver a sufficient conditions for the existence of the observer gain. We summarize the result in the following corollary.

Corollary 1 Consider the uncertain system

$$
\begin{aligned}
\frac{dx(t)}{dt} &= (A + \Delta A)x(t) + (B + \Delta B)u(t), \\
\frac{du(t)}{dt} &= v(t), \\
y(t) &= Cx(t),
\end{aligned} \tag{20}
$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y \in R^p$. The uncertain parts of $\Delta A = E_A^{\mathrm{T}}F_A(x(t))D_A$ and $\Delta B(\sigma(t)) = E_B^{\mathrm{T}}F_B(x(t))D_B$ are supposed to satisfy the inequalities $F_A^{\mathrm{T}}(\sigma(t)) \times F_A(\sigma(t)) < I$, $F_B^{\mathrm{T}}(\sigma(t))F_B(\sigma(t)) < I$, respectively. If there exist a matrix $X > 0$, a matrix Y of appropriate dimensions, and positive constants ϵ_A, and ϵ_B such that the following LMI is feasible

$$
\begin{bmatrix} \mathcal{H}(X, Y) & \widetilde{E}_A X & X \\ X\widetilde{E}_A^{\mathrm{T}} & -\epsilon_A I & 0 \\ X & 0 & -\epsilon_B I \end{bmatrix} < 0, \tag{21}
$$

where

$$\mathcal{H}(X,Y) = \widetilde{A}^{\mathrm{T}}X + X\widetilde{A} - \widetilde{C}^{\mathrm{T}}Y^{\mathrm{T}} - Y\widetilde{C} + \epsilon_A \widetilde{D}_A^{\mathrm{T}}\widetilde{D}_A,$$

then system

$$\frac{\hat{\xi}(t}{dt}) = A\hat{\xi}(t) + \widetilde{B}v(t) + X^{-1}Y\left(y(t) - \widetilde{C}\hat{\xi}(t)\right) - \varphi(y(t),\hat{\xi}(t)), \tag{22}$$

is an asymptotic observer for the uncertain switching system (22) where $\varphi(y(t),\hat{\xi}(t))$ is defined as

$$\varphi(y(t),\hat{z}(t)) = \begin{cases} \epsilon_B \|\widetilde{E}_A\|^2 \|\widetilde{D}_A\|^2 \hat{\xi}^{\mathrm{T}}(t)\hat{\xi}(t)\dfrac{X^{-1}\widetilde{C}^{\mathrm{T}}\widetilde{C}e(t)}{\|\widetilde{C}e(t)\|^2} & \text{if} \quad \|\widetilde{C}e(t)\| \neq 0, \\ 0 & \text{if} \quad \|\widetilde{C}e(t)\| = 0, \end{cases}$$

where $\widetilde{E}_A$ and $\widetilde{D}_A$ are defined as in Section 3.

4 Illustrative Example

4.1 Observation of a switching system without uncertainties

Consider the following switching system described by:

$$A(1) = \begin{bmatrix} 0.1 & -0.5 \\ 0 & -1 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -1 & -1 \\ 0.9 & -1 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$B(2) = \begin{bmatrix} 0.1 \\ -1 \end{bmatrix}, \quad C = [\,1 \quad 0\,].$$

Applying the result of Theorem 1 with $\epsilon_A(j) = \epsilon_B(j) = 1\ \forall j,\ \varpi = 1.7818,$ we obtain $\mu = 10^3$ and

$$P_1 = \begin{bmatrix} 27.8287 & -0.1325 & 5.6453 \\ -0.1325 & 53.9905 & 2.6926 \\ 5.6453 & 2.6926 & 39.0959 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 571.9220 & -99.1576 & -201.2708 \\ -99.1576 & 140.3128 & 48.5049 \\ -201.2708 & 48.5049 & 176.2018 \end{bmatrix},$$
$$Y = \begin{bmatrix} 389.7613 \\ -242.2155 \\ 260.9596 \end{bmatrix},$$

and the observer gain is

$$(P_1 + P_2)^{-1}Y = \begin{bmatrix} 1.3246 \\ -1.2869 \\ 2.7216 \end{bmatrix}.$$

The nonlinear term in the observer dynamics can be computed in terms of the solutions P_1, P_2, μ, and ϖ.

4.2 Observation of a switching system with uncertainties

Taking the same switching model with the following additional data:

$$E_A = \begin{bmatrix} 0.2 & 0.5 \\ 0.4 & 0.4 \end{bmatrix}, \qquad D_A = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0 \end{bmatrix}, \qquad E_B = [\,0.3 \quad 0.6\,], \qquad D_B = 0.2.$$

By the application of result of Theorem 2, with $\mu = 10$, and $\epsilon_A(j) = \epsilon_B(j) = 1\ \forall\, j$, we have

$$P_1 = \begin{bmatrix} 0.1337 & -0.0303 & -0.0486 \\ -0.0303 & 0.0486 & 0.0143 \\ -0.0486 & 0.0143 & 0.0605 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.2767 & -0.0625 & -0.1029 \\ -0.0625 & 0.1010 & 0.0303 \\ -0.1029 & 0.0303 & 0.1248 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0.9465 \\ -0.1669 \\ 0.1946 \end{bmatrix}, \qquad (P_1 + P_2)^{-1} Y = \begin{bmatrix} 3.8608 \\ 0.0266 \\ 4.1977 \end{bmatrix}.$$

5 Conclusion

A new observer design methodology is proposed to estimate the unmeasured states of switching systems and uncertain switching systems. We showed that a constant-gain observer is sufficient to observe the system states whatever the switch in the nominal matrices, and the existence of the observer gain is related to the feasibility of a set of coupled LMIs. The proposed observer design is an alternative to the technique of *switching observers* that necessitates both the construction of several observers and estimation of the current modes of the switching system being observed.

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