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PERSONAGE IN SCIENCE

Professor V.Lakshmikantham

J.H. Dshalalow[1] and A.A. Martynyuk[2*]

[1] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA*

[2] *S.P.Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Nesterov str. 3, 03057, Kiev-57, Ukraine*

V. LAKSHMIKANTHAM was born on March 16, 1924, in an Indian village where there was no school. Only at the age of 13 he started learning English and then entered a high school, which he finished three years later. Following the school, he completed two years of college. Then Lakshmikantham joined a bank and has worked there for five years. His photographic memory (which he apparently inherited from his mother) and extraordinary intellectual abilities enabled him to pick up and process many things very rapidly, and as the result of this, Lakshmikantham progressed extremely expedient whatever he studied or worked on. After a very strenuous period of his life he successfully completed his Bachelor's and Master's degrees and started his work on his doctoral thesis. Some time later, inspired by a series of papers by Wintner, Hartman, Bellman, Coddington and Levinson, Lakshmikantham picked out differential equations as his primary area of interest.

A crucial moment in this direction was his discovery of a new phenomenon in convergence for differential equations. He established convergence under general Kamke's uniqueness condition utilizing differential (instead of commonly used integral) inequalities with no restrictive monotony assumption used by all of his predecessors. He read one of the Auriel Wintner's articles (in 1946) who first stated this as a conjecture. His direct communication with Professor Wintner regarding this problem encouraged Lakshmikantham and lead him to his first and very important paper [A2] that appeared in the Proceedings of American Mathematical Society, in 1957. In this paper a truly significant progress on convergence criteria has been made. The reader is referred to much more details and discussions on this and other issues in a very informative 2000 article [A1] about Lakshmikantham by R.P. Agarwal and S.G. Leela.

After that Lakshmikantham went back and forth to the US, Canada, and India spending several years at UCLA, University of Wisconsin at Madison, RIAS, Baltimore, and University of Calgary (among other schools). Then he returned to India by joining Marathawada University in the capacity of a department Head of mathematics. In 1966 he finally resided in the US having held appointments at three schools.

*Corresponding author: anmart@stability.kiev.ua

In summary, over the past 41 years, Lakshmikantham held mathematics department Head positions at four universities:

* Marathawada University, Aurangabad, India (1964 – 1965);
* University of Rhode Island, Kingston, Rhode Island, USA (1966 – 1972);
* The University of Texas at Arlington, Texas, USA (1973 – 1986);
* Florida Institute of Technology, Melbourne, Florida, USA (1989 – Present).

The presently acknowledged prominence of the University of Rhode Island, University of Texas at Arlington, and Florida Institute of Technology (FIT) is due to his leadership and his phenomenal academic and scholarly activities. In all named schools, Lakshmikantham (in contrast with the efforts by his predecessors) created doctoral programs and produced a record number of gifted doctoral students. Some of them have become internationally known.

While Lakshmikantham had established himself as a world leading mathematician prior to 1989, a lion portion of his contributions fell into the period of time during which Lakshmikantham served as the Division and Department Head of Mathematical Sciences at FIT. Indeed, in 1991 he established the International Federation of Nonlinear Analysts (IFNA) that has become a central organization of world activities in the area of Nonlinear Analysis and interdisciplinary mathematics. In 1992, under the auspices of IFNA, Lakshmikantham established the Journal of Nonlinear World (later on renamed as Nonlinear Studies) and organized the first World Congress of Nonlinear Analysts held in Tampa, Fl, USA, followed by three more congresses in Athens, Greece (1996), Sicily, Italy (2000), and Orlando, FL, USA (2004). All four congresses have become major events in the world mathematical community where not only mathematics people from all over the world have assembled, but also representatives of science, technology, physics, biology and chemistry. It has been a unique gathering of seemingly diverse sciences separated by many decades and even centuries and brought and resurrected by Lakshmikantham in one huge group.

While at University of Texas at Arlington, Lakshmikantham founded in 1976 what has now become a chief academic journal in Nonlinear Analysis: Nonlinear Analysis, Theory Methods and Applications (copyrighted by Pergamon Press), followed by Nonlinear Analysis, Real World Applications, in 2000 (by Pergamon Press). He also founded and has been a chief editor of Stochastic Analysis and Its Applications (copyrighted by Marcel Dekker), Mathematical Problems in Engineering (copyrighted by Gordon and Breach and now by Hindawi), and Nonlinear Studies (initially copyrighted by Gordon and Breach and presently, by I&S Publishers). Through his academic and scholarly efforts, Nonlinear Analysis has spread throughout the world (including non-mathematical communities) and it has become one of the principal mathematical disciplines. We cannot help noticing that through his tireless efforts the world of mathematics, physics, and engineering has changed as the result of an almost unprecedented influence of just one person (not to mention a huge army of his followers).

Lakshmikantham's mathematics interests extended far beyond differential equations. His almost unique versatility encompassed fairly broad topics shown in his numerous publications. We singled out some major areas of his research activities reflected in his books:

* Differential and Integral Inequalities [1,2,11,24]
* Differential Equations in Abstract Spaces [3,23]
* Boundary-Value Problems [4]
* Stochastic Differential Equations [5]

* Monotone Iterative Technique [7,19]
* Stability Analysis [1,2,11 - 17]
* Systems with Finite and Infinite Delays [8,12,21]
* Nonlinear Problems in Abstract Cones [10,26]
* Dynamic of Systems on Time Scale [22]
* Impulsive Differential Equations [11,12,14,17]
* Uniqueness and Nonuniqueness [18]
* Method of Variation of Parameters [25]
* Method of Quasilinearization [26]
* History of Mathematics [27]
* Stochastic Analysis [28]
* Difference Equations and Numerical Analysis [29]
* Fuzzy Differential Equations [31]
* Computational Error and Complexity [32]

Lakshmikantham authored or co-authored over 400 books, monographs and research articles, all of which are widely referred to. In addition, Lakshmikantham, along with Martynyuk, edited an International series of Scientific Monographs in "Stability and Control: Theory, Methods and Applications", initially by Gordon and Breach and now by Taylor and Francis, London. Under their joint editorship, the series produced 22 volumes in various areas of stability and control theory. Lakshmikantham's mathematical results has now become essential in various applications to physics, engineering and mechanical engineering, and technology.

Besides his phenomenal involvement in mathematics, editorship, teaching, administrative activities, and supervising students, Lakshmikantham is a down-to-earth person. Among his auxiliary interests, he enjoys world history. In 1999 he even published a very interesting and informative book, "The Origin of Human Past, Children of Immortal Bliss." In this book, Lakshmikantham investigated many aspects of the origin of civilizations, both from Western points of view and from various old and ancient Indian sources, among them he discussed the origins of the theory of Aryan invasion and reasons for its dismissal. He also went on to rectify some errors in chronology of the world history and analyzed Vedic literature.

As a consequence of his passion to the world history, Lakshmikantham is also seriously interested in the history of mathematics. Year after the "Origin of Human Past" has appeared, he published the book "The Origin of Mathematics" [27] (jointly with Leela) where the authors outline major events in the history of ancient mathematics and shed new light on its chronology.

Through his enormous contributions to academia and mathematical sciences, Lakshmikantham is highly revered throughout the scientific world. He has become an honorary editor of many internationally renowned journals and associate and advisory editor of a few dozens more. It is without any doubt that Lakshmikantham is among the most influential people in the second half of the twentieth century, and he continues to enrich the mathematics world throughout the twenty first century having a tremendous impact on the lives of each and everyone who has ever come in contact with him.

We would like to finish our article with words of one Indian teacher: "*The one who is not afraid of revising the principles of a doctrine to refine the knowledge is already right. The one who is not afraid of remaining misunderstood stands with us. The one who is not afraid of putting together the beds of large streams is our friend.*" We are certain that Professor Lakshmikantham is "the one who is not afraid" of any endeavor and we wish him to be mightily blessed.

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On the Minimum Free Energy for the Ionosphere

Giovambattista Amendola*

Dipartimento di Matematica Applicata "U.Dini", Facoltà di Ingegneria, via Diotisalvi 2, 56126-Pisa, Italy

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Abstract: Within the linear theory of the electromagnetism for the ionosphere we give a general closed expression for the minimum free energy in terms of Fourier-transformed quantities, when the integrated history of the electric field is chosen to characterize the state of the material. Another equivalent expression is derived and also used to study the particular case of a discrete spectrum model.

Keywords: *Electromagnetism; fading memory; free energy.*

Mathematics Subject Classification (2000): 78A25, 74A15.

1 Introduction

In a previous work [14] we have considered the problem of finding an explicit form for the minimum free energy of a linear conductor, characterized, in particular, by a constitutive equation for the current density expressed by a local functional of the history of the electric field.

This hereditary theory, which well describes the electromagnetic phenomena in the ionosphere [8, 9, 18], has been studied in particular in [10], where some thermodynamic potentials are derived, as well as the maximal free enthalpy and the maximal free energy; these representations depend on the choice of the state variables and several possibilities are considered. In [13] we have also considered a different constitutive equation, between the current density and the electric field, which associates the presence of memory effects with the actual action of the electric field; moreover, in particular, in this work we have derived the thermodynamic restrictions on the assumed constitutive equations.

As we have done in [14], still now we follow Golden's lines of [12], where the analogous problem is studied for a linear viscoelastic material in the scalar case, and the procedure used in [11] for the same problem always in viscoelasticity, see also [7, 15, 19].

*Corresponding author: amendola@dma.unipi.it

In [14], to characterize the state of the material we have chosen the instantaneous values of the electric and magnetic fields together with the history of the electric field. In the present work we assume the state made by the integrated history of the electric field in the place of its history, to give a different formulation of the problem, which requires, in particular, an appropriate new definition of the continuation of histories and processes.

In Section 2, we introduce some fundamental relationships and recall some useful results derived in previous works [13, 16]. Then, in Section 3, we give the definition of states and processes; moreover, we define the equivalence between integrated histories. Another definition of the equivalence of two integrated histories is also given by using the boundedness of the electromagnetic work, in Section 4 and Section 5. After, in Section 6, we derive the expression of the maximum recoverable work we can obtain by starting from a given state. Then, in Section 7, we derive a new form of the minimum free energy, which is applied in the last Section 8 to study the particular case of a discrete spectrum material.

2 Notation and Preliminaries

Let $\mathcal{B}$ be a rigid conducting material, whose electromagnetic behavior is characterized by these linear constitutive equations

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t), \qquad \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t), \tag{2.1}$$

$$\mathbf{J}(\mathbf{x},t) = \int_0^{+\infty} \alpha(s)\mathbf{E}^t(\mathbf{x},s)\,ds, \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{D}$ and $\mathbf{B}$ denote the electric displacement and the magnetic induction and $\mathbf{J}$ is the current density expressed in terms of the history of the electric field, $\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},\, t-s)$ $\forall\, s \in \mathbb{R}^+ = [0,+\infty)$, up to time t; moreover, the position vector is denoted by $\mathbf{x} \in \Omega$, the region occupied by the solid $\mathcal{B}$.

We suppose that the body is a homogeneous and isotropic material; hence, the dielectric constant $\varepsilon > 0$, as well as the magnetic permeability $\mu > 0$ and the memory kernel α are constant in $\overline{\Omega}$. We assume that this relaxation function $\alpha\colon \mathbb{R}^+ \to R$ is such that $\alpha \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$.

The region Ω occupied by the conductor $\mathcal{B}$ is a bounded and simply-connected domain of the three-dimensional Euclidean space, with a smooth boundary $\partial\Omega$ with the unit outward normal $\mathbf{n}$.

If we introduce the integrated history of $\mathbf{E}$, which is the function $\overline{\mathbf{E}}^t(\mathbf{x},\cdot)\colon \mathbb{R}^+ \to R^3$ defined by

$$\overline{\mathbf{E}}^t(\mathbf{x},s) = \int_0^s \mathbf{E}^t(\mathbf{x},\lambda)\,d\lambda = \int_{t-s}^t \mathbf{E}(\mathbf{x},\tau)\,d\tau, \tag{2.3}$$

the constitutive equation (2.2), with an integration by parts and taking into account the function α expressed by means of

$$\alpha(t) = \alpha_0 + \int_0^t \dot{\alpha}(\tau)\,d\tau \quad \forall\, t \in \mathbb{R}^+, \tag{2.4}$$

where α_0 is the initial value at time $t = 0$, with $\lim_{t\to+\infty} \alpha(t) = 0$, can be rewritten in the following equivalent form

$$\mathbf{J}(\mathbf{x}, t) = -\int_0^{+\infty} \alpha'(s)\bar{\mathbf{E}}^t(\mathbf{x}, s)\, ds. \tag{2.5}$$

Let us introduce the formal Fourier transform of any function $f\colon R \to R^n$ denoted by f_F and given by

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(\xi)e^{-i\omega\xi}\, d\xi = f_+(\omega) + f_-(\omega), \tag{2.6}$$

where

$$f_+(\omega) = \int_0^{+\infty} f(\xi)e^{-i\omega\xi} d\xi, \qquad f_-(\omega) = \int_{-\infty}^{0} f(\xi)e^{-i\omega\xi} d\xi. \tag{2.7}$$

Besides the definitions of $f_\pm$, it is useful to consider the half-range Fourier cosine and sine transforms expressed by

$$f_c(\omega) = \int_0^{+\infty} f(\xi)\cos(\omega\xi)\, d\xi, \qquad f_s(\omega) = \int_0^{+\infty} f(\xi)\sin(\omega\xi)\, d\xi; \tag{2.8}$$

the definitions of f_c, f_s as well as f_+ hold also if the function f is defined on $\mathbb{R}^+$, while the definition of f_- stands for f defined on $\mathbb{R}^- = (-\infty, 0]$.

Furthermore, we observe that any function defined on $\mathbb{R}^+$ can be extended on R. If we identify functions on $\mathbb{R}^+$ with functions on R which vanish for any $s \in \mathbb{R}^{--}$, the strictly negative reals, we have

$$f_F(\omega) = f_c(\omega) - if_s(\omega). \tag{2.9}$$

The extension made with an even function, that is such that $f(\xi) = f(-\xi)\ \forall\, \xi < 0$, yields

$$f_F(\omega) = 2f_c(\omega), \tag{2.10}$$

while, when the function becomes an odd function, i.e. $f(\xi) = -f(-\xi)\ \forall\, \xi < 0$, such an extension gives

$$f_F(\omega) = -2if_s(\omega). \tag{2.11}$$

From the definitions (2.7) of $f_\pm$ it follows that we can consider these functions defined for any complex $z \in \mathbf{C}$, the complex plane; thus, $f_\pm(z)$ are analytic for $z \in \mathbf{C}^{(\mp)}$ where

$$\mathbf{C}^{(-)} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{--}\right\}, \quad \mathbf{C}^{(+)} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{++}\right\}. \tag{2.12}$$

This analyticity can be extended by hypothesis on $\mathbf{C}^{\mp}$ [12], where

$$\mathbf{C}^{-} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{-}\right\}, \quad \mathbf{C}^{+} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R_{+}\right\}, \tag{2.13}$$

thus including the real axis, which in (2.12) is excluded.

In the following we shall use the notation $f_{(\pm)}(z)$ to denote that the function has zeros and singularities in $\mathbf{C}^{\pm}$.

The thermodynamic restrictions on the constitutive equations were studied in [13, 16]; they yield the following constraint

$$\alpha_c(\omega) > 0 \quad \forall\, \omega \in R, \tag{2.14}$$

where we have added the hypothesis $\alpha_c(0) > 0$.

We recall some results of [10, 14]. Introducing the electric conductivity

$$\nu(t) = \int_0^t \alpha(\xi)\, d\xi, \tag{2.15}$$

it follows that its asymptotic value is

$$\nu_\infty = \int_0^{+\infty} \alpha(\xi)\, d\xi = \alpha_c(0) > 0. \tag{2.16}$$

Moreover, from the inverse Fourier transform

$$\alpha(t) = \frac{2}{\pi} \int_0^{+\infty} \alpha_c(\omega) \cos(\omega t)\, d\omega \tag{2.17}$$

we get

$$\alpha(0) = \frac{2}{\pi} \int_0^{+\infty} \alpha_c(\omega)\, d\omega > 0, \qquad \alpha'(0) \le 0. \tag{2.18}$$

Finally, we have

$$\alpha'_s(\omega) = -\omega \alpha_c(\omega), \quad \lim_{\omega \to \infty} \omega \alpha'_s(\omega) = -\lim_{\omega \to \infty} \omega^2 \alpha_c(\omega) = \alpha'(0) \le 0 \tag{2.19}$$

if $\alpha'' \in L^2(\mathbb{R}^+)$ and $|\alpha'(0)| < +\infty$.

We assume

$$\alpha'(0) < 0. \tag{2.20}$$

Given a history $\mathbf{E}^t(s) = \mathbf{E}(t-s)$ for any $s \in R^+$, we consider its static continuation of duration $\tau \in R^{++}$ defined as

$$\mathbf{E}^{t(\tau)} = \begin{cases} \mathbf{E}(t), & s \in [0, \tau], \\ \mathbf{E}^t(s-\tau), & s > \tau, \end{cases} \tag{2.21}$$

to which corresponds the following integrated history

$$\overline{\mathbf{E}}^{t+\tau}(s) = \begin{cases} \int_0^s \mathbf{E}(t)\, d\eta = s\mathbf{E}(t), & 0 \le s \le \tau, \\ \tau \mathbf{E}(t) + \int_0^{s-\tau} \mathbf{E}^t(\rho)\, d\rho, & s > \tau. \end{cases} \tag{2.22}$$

This static continuation yields a current density, which, taking into account (2.5) where the integrated history to be considered is expressed by (2.22), assumes the following form

$$\mathbf{J}(t+\tau) = \nu(\tau)\mathbf{E}(t) - \int_0^{+\infty} \alpha'(\tau+\rho)\overline{\mathbf{E}}^t(\rho)\,d\rho. \tag{2.23}$$

3 States, Processes and Equivalent Integrated Histories

The constitutive equations (2.1) – (2.2), assumed to describe the electromagnetic behavior of our conductor $\mathcal{B}$, characterize a simple material [3, 4], whose states and processes can be defined as follows.

We observe that all the functions introduced in the previous section depend on $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$; in the following we shall understand the dependence on $\mathbf{x}$, which, therefore will not be written.

Contrary to the choice made in [14], we now consider the electromagnetic state of $\mathcal{B}$ given by

$$\sigma(t) = (\mathbf{E}(t),\, \mathbf{H}(t),\, \overline{\mathbf{E}}^t), \tag{3.1}$$

where the integrated history of the electric field is chosen to give the memory effects on the instantaneous values of the current density. We denote by Σ the set of the admissible states of $\mathcal{B}$.

The electromagnetic process is defined by the function $P\colon\ [0,d) \to R^3 \times R^3$, supposed piecewise continuous on the time interval $[0,d) \subset \mathbb{R}^+$ and expressed by

$$P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau)) \quad \forall\, \tau \in [0,d), \tag{3.2}$$

where $d \in R_+$ is said the duration of the process and $\dot{\mathbf{E}}_P(\tau)$ and $\dot{\mathbf{H}}_P(\tau)$ are the time derivatives of the electric and magnetic fields $\mathbf{E}_P$ and $\mathbf{H}_P$ at any instant of the time interval. The set of all admissible processes is denoted by Π. Sometimes, we shall consider the restriction of a process P to the interval $[t_1,t_2) \subset [0,d)$, denoted by $P_{[t_1,t_2)}$.

Given any initial state $\sigma^i \in \Sigma$ and a process $P \in \Pi$, the evolution function $\rho\colon\ \Sigma \times \Pi \to \Sigma$ provides the final state $\sigma^f = \rho(\sigma^i, P)$. If $\sigma(0)$ is the initial state and we apply $P_{[0,t)}$, the evolution function yields the state $\sigma(t) = \rho(\sigma(0), P_{[0,t)})\ \forall\, t \in [0,d]$; in particular, the pair (σ, P) is called a cycle if $\sigma(d) = \rho(\sigma(0),\, P) = \sigma(0)$.

The response of the material is the function

$$U(t) = (\mathbf{D}(t),\, \mathbf{B}(t),\, \mathbf{J}(t)). \tag{3.3}$$

In (3.3) the instantaneous values of $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ obviously depend on the pair (σ, P); therefore, the output function $U(t)$ is a function $\widetilde{U}\colon\ \Sigma \times \Pi \to R^3 \times R^3 \times R^3$ such that

$$U = \widetilde{U}(\sigma, P). \tag{3.4}$$

In (3.3) the triplet $(\mathbf{D}(t), \mathbf{B}(t), \mathbf{J}(t))$ is expressed in terms of (2.1) and (2.2), the last of which allows us to introduce the linear functional $\tilde{\mathbf{J}}\colon\ \Gamma \to R^3$ such that

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\int_0^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s)\,ds, \tag{3.5}$$

which is defined in the following function space

$$\Gamma = \left\{ \overline{\mathbf{E}}^t : \ (0, +\infty) \to R^3; \ \left| \int_0^{+\infty} \alpha'(s+\tau) \overline{\mathbf{E}}^t(s) \, ds \right| < +\infty \quad \forall \tau \geq 0 \right\}. \tag{3.6}$$

The process (3.2) is defined in the time interval $[0, d)$, but its application can be done at any initial instant $t \geq 0$; therefore, we distinguish the following two cases.

If the initial state is $\sigma(0) = (\mathbf{E}_*(0), \mathbf{H}_*(0), \overline{\mathbf{E}}^0_*) \in \Sigma$ and the process is applied at time $t = 0$, then the process is $P(t) = (\dot{\mathbf{E}}_P(t), \dot{\mathbf{H}}_P(t)) \in \Pi$, since in this case $\tau \equiv t$ in (3.2). Thus, the process yields a family of states $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t) \ \forall t \in [0, d]$, where

$$\mathbf{E}(t) = \mathbf{E}_*(0) + \int_0^t \dot{\mathbf{E}}_P(\rho) \, d\rho, \qquad \mathbf{H}(t) = \mathbf{H}_*(0) + \int_0^t \dot{\mathbf{H}}_P(\rho) \, d\rho, \tag{3.7}$$

that is $\mathbf{E}_P(\tau) \equiv \mathbf{E}_P(t) = \mathbf{E}(t)$ and similarly $\mathbf{H}_P(\tau) \equiv \mathbf{H}_P(t) = \mathbf{H}(t)$, and

$$\overline{\mathbf{E}}^t(\tau') = \begin{cases} \int_{t-\tau'}^{t} \mathbf{E}(s) \, ds, & 0 \leq \tau' < t, \\ \overline{\mathbf{E}}^0_*(\tau' - t) + \int_0^t \mathbf{E}(s) \, ds, & \tau' \geq t, \end{cases} \tag{3.8}$$

$\mathbf{E}(s)$ being given by $(3.7)_1$.

When a process P is applied at time $t > 0$ and the initial state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$, we relate the process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau)) \ \forall \tau \in [0, d)$, d being its duration, to

$$\mathbf{E}_P : \quad (0, d] \to R^3, \quad \mathbf{E}_P(\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(s') \, ds', \tag{3.9}$$

$$\mathbf{H}_P : \quad (0, d] \to R^3, \quad \mathbf{H}_P(\tau) = \mathbf{H}(t) + \int_0^\tau \dot{\mathbf{H}}_P(s') \, ds', \tag{3.10}$$

for every $\tau \in (0, d]$. Consequently, the process P induces a prolongation of the initial integrated history $\overline{\mathbf{E}}^t$, related to the final value $\mathbf{E}_P(d)$ and the continuation of the initial $\overline{\mathbf{E}}^t$ by means of $\mathbf{E}_P(\tau) = \mathbf{E}(t + \tau)$, with $t + \tau \leq t + d$, as follows

$$\overline{\mathbf{E}}^{t+d}(s) = (\mathbf{E}_P * \overline{\mathbf{E}})^{t+d}(s) = \begin{cases} \overline{\mathbf{E}}^d_P(s) = \int_0^s \overline{\mathbb{E}}^d_P(\rho) \, d\rho, & 0 \leq s < d, \\ \overline{\mathbf{E}}^d_P(d) + \overline{\mathbf{E}}^t(s - d), & s \geq d, \end{cases} \tag{3.11}$$

where the integrated histories

$$\overline{\mathbf{E}}^d_P(d) = \int_0^d \mathbf{E}_P(\eta) \, d\eta, \qquad \overline{\mathbf{E}}^t(s - d) = \int_{t-(s-d)}^{t} \mathbf{E}(\xi) \, d\xi \tag{3.12}$$

are related to the process in $[0,d)$ and the initial integrated history, respectively.

We are able to evaluate the final value of the current density assumed by means of a process P applied to any initial state $\sigma(t)$ during the time interval $[0,d)$, i.e. $\tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}})^{t+d})$, taking account of (3.5) and (3.11).

If the restriction $P_{[0,\tau)}$ is applied to the initial state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$, from (3.5), considering the integral between zero and infinity as the sum of two integrals, the first of which between zero and τ and the second one between τ and infinity, and substituting d with τ in (3.11), we have

$$\begin{aligned}
\tilde{\mathbf{J}}(\overline{\mathbf{E}}^{t+\tau}) &= -\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \overline{\mathbf{E}})(t+\tau-s)\,ds \\
&= -\int_0^{\tau} \alpha'(\eta)\overline{\mathbf{E}}_P^{\tau}(\eta)\,d\eta - \int_\tau^{+\infty} \alpha'(s)[\overline{\mathbf{E}}_P^{\tau}(\tau) + \overline{\mathbf{E}}^t(s-\tau)]ds \\
&= \alpha(\tau)\overline{\mathbf{E}}_P^{\tau}(\tau) - \int_0^{\tau} \alpha'(\eta)\overline{\mathbf{E}}_P^{\tau}(\eta)\,d\eta - \int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s-\tau)\,ds.
\end{aligned} \tag{3.13}$$

Definition 3.1 Two states $\sigma_i \in \Sigma$, $i = 1,2$, are said to be equivalent if

$$\widetilde{U}(\sigma_1, P) = \widetilde{U}_2(\sigma, P) \quad \forall\, P \in \Pi. \tag{3.14}$$

In other words, two states are equivalent if the response of the material is the same whatever may be the applied admissible process. Therefore, it follows that two initial states which produce the same response during the application of any process are indistinguishable.

This definition [15] yields an equivalent relation $\mathcal{R}$ in the space Σ of the states, which induces the introduction of the quotient space $\Sigma_{\mathcal{R}}$, whose elements are the classes of equivalent states denoted by $\sigma_{\mathcal{R}}$.

Definition 3.2 A state of the body is said minimal if it is characterized by the minimum set of data.

We only observe that Definition 3.1 implies the equality of some quantities present in the definition of the states; these conditions are absorbed by the following definition of equivalence between two integrated histories of the electric field.

Definition 3.3 Given two states $(\mathbf{E}_i(t), \mathbf{H}_i(t), \overline{\mathbf{E}}_i^t)$, $i = 1,2$, corresponding to the same value of the magnetic field, $\mathbf{H}_i(t) = \mathbf{H}(t)$, $i = 1,2$, the integrated histories of the electric field $\overline{\mathbf{E}}_i^t$ $(i = 1,2)$ are called equivalent if

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_1)^{t+\tau}) = \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_2)^{t+\tau}) \tag{3.15}$$

for every $\mathbf{E}_P \colon (0,\tau] \to R^3$ and for every $\tau > 0$, whatever may be $\mathbf{H}_P \colon (0,\tau] \to R^3$.

This definition characterizes the integrated histories associated to the same current density; moreover, we note that it has no effects on the magnetic field $\mathbf{H}_P(\tau)$, whose values are independent of $\overline{\mathbf{E}}_i^t$, $i = 1,2$.

The zero integrated history is the particular history such that $\overline{\mathbf{E}}^t(s) = \bar{\mathbf{0}}^\dagger(s) = \mathbf{0}$ $\forall\, s \in \mathbb{R}^+$; its continuation by means of the process $P_{[0,\tau)}$, applied to the initial state $\sigma(t) = (\mathbf{E}(t),\, \mathbf{H}(t),\, \bar{\mathbf{0}}^\dagger)$ is given by

$$(\mathbf{E}_P * \bar{\mathbf{0}}^\dagger)^{t+\tau}(s) = \begin{cases} \overline{\mathbf{E}}_P^\tau(s), & 0 \le s < \tau, \\ \overline{\mathbf{E}}_P^\tau(\tau), & s \ge \tau. \end{cases} \tag{3.16}$$

Taking account of (3.13), it follows that an integrated history $\overline{\mathbf{E}}^t$ is equivalent to the zero integrated history $\bar{\mathbf{0}}^\dagger$ if

$$\int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s-\tau)\, ds = \int_0^{+\infty} \alpha'(\tau+\rho)\overline{\mathbf{E}}^t(\rho)\, d\rho = \mathbf{0}. \tag{3.17}$$

This condition yields a new equivalence relation, which characterizes the same equivalence between two integrated histories of Definition 3.3. In fact, two integrated histories $\overline{\mathbf{E}}_i^t$ $(i = 1,2)$, equivalent in the sense of Definition 3.3, must satisfy (3.15), from which it follows that this equality

$$\int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}_1^t(s-\tau)\, ds = \int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}_2^t(s-\tau)\, ds \tag{3.18}$$

must hold. Introducing the difference $\overline{\mathbf{E}}^t(s-\tau) = \overline{\mathbf{E}}_1^t(s-\tau) - \overline{\mathbf{E}}_2^t(s-\tau)$, we see that (3.18) coincides with (3.17), which assures the equivalence between the integrated history given by this difference and the zero integrated history.

4 Electromagnetic Work

The electromagnetic work done on a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau))$, defined for every $\tau \in [0,d)$, starting from the initial state $\sigma^i(t) = (\mathbf{E}_i(t),\, \mathbf{H}_i(t),\, \overline{\mathbf{E}}_i^t)$, is expressed by [1, 2, 16]

$$\begin{aligned} \widetilde{W}(\mathbf{E}_i(t), \mathbf{H}_i(t), \overline{\mathbf{E}}_i^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \int_t^{t+d} \Big[&\dot{\mathbf{D}}(\mathbf{E}(\xi)) \cdot \mathbf{E}(\xi) + \dot{\mathbf{B}}(\mathbf{H}(\xi)) \cdot \mathbf{H}(\xi) \\ &+ \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_i)^\xi) \cdot \mathbf{E}(\xi) \Big] d\xi \end{aligned} \tag{4.1}$$
$$= \int_0^d \Big[\dot{\mathbf{D}}(\mathbf{E}_P(\tau)) \cdot \mathbf{E}_P(\tau) + \dot{\mathbf{B}}(\mathbf{H}_P(\tau)) \cdot \mathbf{H}_P(\tau) + \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_i)^{t+\tau}) \cdot \mathbf{E}_P(\tau) \Big] d\tau,$$

where the subscript P indicates that the fields are expressed by (3.9) – (3.10) in terms of $\tau \in (0,d]$.

In (4.1) we have considered $\widetilde{W}$ as a function of all the quantities which characterize the states and the processes of $\mathcal{B}$; therefore, the work is a function of the couple $(\sigma(t), P)$. Taking account of (2.1) and (2.5), we can give (4.1) the following form

$$W(\sigma(t), P) = \frac{1}{2}\left[\varepsilon \mathbf{E}^2(t+d) + \mu \mathbf{H}^2(t+d)\right] - \frac{1}{2}\left[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)\right] - \int_t^{t+d}\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \overline{\mathbb{E}}_i)^\xi(s)\, ds \cdot \mathbf{E}(\xi)\, d\xi, \tag{4.2}$$

where $(\mathbf{E}_P * \overline{\mathbb{E}}_i)^\xi(s)$ is expressed by (3.11).

It is interesting to distinguish the part of the work due only to the process. For this purpose, let us examine the particular case, corresponding to the initial state $\sigma(0) = (\mathbf{0}, \mathbf{0}, \bar{\mathbf{0}}^\dagger)$, i.e. when $\mathbf{E}(0) = \mathbf{0}$, $\mathbf{H}(0) = \mathbf{0}$, $\overline{\mathbf{E}}^0(s) = \bar{\mathbf{0}}^\dagger(s) = \mathbf{0}$. Thus, the process P of duration d is applied at time $t = 0$ and yields the ensuing fields (3.7) – (3.8), which now reduce to

$$\mathbf{E}_0(t) = \int_0^t \dot{\mathbf{E}}_P(s)\, ds, \qquad \mathbf{H}_0(t) = \int_0^t \dot{\mathbf{H}}_P(s)\, ds \tag{4.3}$$

and

$$(\mathbf{E}_0 * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \overline{\mathbf{E}}_0^t(s), & 0 \le s < t, \\ \overline{\mathbf{E}}_0^t(t), & s \ge t. \end{cases} \tag{4.4}$$

Definition 4.1 A process $P = (\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P)$ of duration d, applied at time $t = 0$ and related to $\mathbf{E}_0(t)$, $\mathbf{H}_0(t)$ and $(\mathbf{E}_0 * \bar{\mathbf{0}}^\dagger)^t$, given by (4.3) – (4.4), is said to be a finite work process if

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \int_0^d \Big[\dot{\mathbf{D}}(\mathbf{E}_0(t)) \cdot \mathbf{E}_0(t) + \dot{\mathbf{B}}(\mathbf{H}_0(t)) \cdot \mathbf{H}_0(t) + \tilde{\mathbf{J}}((\mathbf{E}_0 * \bar{\mathbf{0}}^\dagger)^t) \cdot \mathbf{E}_0(t)\Big] dt < +\infty. \tag{4.5}$$

Lemma 4.1 *The work considered in Definition 4.1 satisfies the inequality*

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger;\, \dot{\mathbf{E}}_P,\, \dot{\mathbf{H}}_P) > 0. \tag{4.6}$$

Proof The work (4.5), taking into account (4.3) – (4.4), becomes

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \frac{1}{2}\left[\varepsilon \mathbf{E}_0^2(d) + \mu \mathbf{H}_0^2(d)\right] - \int_0^d \left[\int_0^t \alpha'(s)\overline{\mathbf{E}}_0^t(s)\, ds + \int_t^{+\infty} \alpha'(s)\overline{\mathbf{E}}_0^t(t)\, ds\right] \cdot \mathbf{E}_0(t)\, dt, \tag{4.7}$$

which, integrating by parts in the first integral and evaluating the second one, reduces to

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^\dagger; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \frac{1}{2}[\varepsilon \mathbf{E}_0^2(d) + \mu \mathbf{H}_0^2(d)] + \int_0^d \int_0^t \alpha(s)\mathbf{E}_0^t(s)\, ds \cdot \mathbf{E}_0(t)\, dt. \tag{4.8}$$

Application of the Plancherel theorem, on assuming that for any $t > d$ the functions in (4.8) are equal to zero, allows us to transform the integral in (4.8) as follows

$$\int_0^d \int_0^t \alpha(s)\mathbf{E}_0^t(s)\,ds \cdot \mathbf{E}_0(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha_F(\omega)\mathbf{E}_{0_F}(\omega)\cdot\mathbf{E}^*_{0_F}(\omega)\,d\omega, \tag{4.9}$$

where * denotes the complex conjugate. The functions we have transformed are defined on $\mathbb{R}^+$ and equal to zero on R^{--}; therefore we can use (2.9), which expresses the Fourier transforms in terms of the corresponding cosine and sine transforms, which are even and odd functions, respectively. Thus the integral in (4.9) reduces to

$$\int_{-\infty}^{+\infty} \alpha_F(\omega)\mathbf{E}_{0_F}(\omega)\cdot\mathbf{E}^*_{0_F}(\omega)\,d\omega = \int_{-\infty}^{+\infty} \alpha_c(\omega)[\mathbf{E}^2_{0_c}(\omega)+\mathbf{E}^2_{0_s}(\omega)]d\omega > 0, \tag{4.10}$$

by virtue of (2.14), and hence the positiveness of the work follows.

Let the duration of a process be $d < +\infty$, the process can be defined on $\mathbb{R}^+$ by assuming $P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau)) = (\mathbf{0},\mathbf{0})\ \forall\,\tau \ge d$; we assume $\mathbf{E}_P(\tau) = \mathbf{0},\ \mathbf{H}_P(\tau) = \mathbf{0}$ $\forall\,\tau > d$. Then, taking into account that the initial integrated history is zero, (4.5) gives

$$\begin{aligned}
\widetilde{W}(\mathbf{0},\mathbf{0},\mathbf{0}^\dagger;\dot{\mathbf{E}}_P,\,\dot{\mathbf{H}}_P) &= \int_0^d [\varepsilon\dot{\mathbf{E}}_0(t)\cdot\mathbf{E}_0(t) + \mu\dot{\mathbf{H}}_0(t)\cdot\mathbf{H}_0(t)]\,dt \\
&\quad - \int_0^{+\infty}\left[\int_0^\eta \alpha'(s)\overline{\mathbf{E}}^\eta_P(s)\,ds + \int_\eta^{+\infty}\alpha'(s)\overline{\mathbf{E}}^\eta_P(\eta)\,ds\right]\cdot\mathbf{E}_P(\eta)\,d\eta \\
&= \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \int_0^{+\infty}\int_0^\eta \alpha(s)\mathbf{E}^\eta_P(s)\,ds\cdot\mathbf{E}_P(\eta)\,d\eta \\
&= \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\alpha(|\eta-\rho|)\mathbf{E}_P(\rho)\cdot\mathbf{E}_P(\eta)\,d\rho d\eta.
\end{aligned} \tag{4.11}$$

Here the last integral can be transformed by applying the Plancherel theorem and taking into account that the Fourier transform of the even function $\alpha(|\eta-\rho|)$ can be written in terms of the Fourier cosine transform; thus, we get

$$\widetilde{W}(\mathbf{0},\mathbf{0},\mathbf{0}^\dagger;\,\dot{\mathbf{E}}_P,\,\dot{\mathbf{H}}_P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d)+\mu\mathbf{H}_0^2(d)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}^*_{P+}(\omega)\,d\omega. \tag{4.12}$$

With this result we can introduce the function space

$$\widetilde{H}_\alpha(\mathbb{R}^+,R^3) = \left\{\mathbf{g}\colon\ \mathbb{R}^+\to R^3;\quad \int_{-\infty}^{+\infty}\alpha_c(\omega)\mathbf{g}_+(\omega)\cdot\mathbf{g}^*_+(\omega)\,d\omega < +\infty\right\},$$

which characterizes the finite work processes. We can consider as the space of the processes, to which $\mathbf{E}_P$ is related, the Hilbert space $H_\alpha(\mathbb{R}^+, R^3)$, obtained by the complation of $\widetilde{H}_\alpha$ with respect to the norm corresponding to the inner product

$$(\mathbf{g}_1, \mathbf{g}_2)_\alpha = \int\limits_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{g}_{1+}(\omega) \cdot \mathbf{g}_{2+}^*(\omega)\, d\omega.$$

Let us now consider the general case, when the initial state is $\sigma(t) = (\mathbf{E}(t),\ \mathbf{H}(t),\ \overline{\mathbf{E}}^t)$ and the work done on any process P, of duration $d < +\infty$, supposed to be zero for any $\tau \geq d$ and related to $\mathbf{E}_P(\tau) = \mathbf{0}$, $\mathbf{H}_P(\tau) = \mathbf{0}\ \forall\, \tau > d$, is given by (4.1), which yields

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \int\limits_0^d [\varepsilon \dot{\mathbf{E}}_P(\tau) \cdot \mathbf{E}_P(\tau) + \mu \dot{\mathbf{H}}_P(\tau) \cdot \mathbf{H}_P(\tau)]\, d\tau \\ + \int\limits_0^{+\infty} \Bigg[\alpha(\tau)\overline{\mathbf{E}}_P^\tau(\tau) - \int\limits_0^\tau \alpha'(s)\overline{\mathbf{E}}_P^\tau(s)\, ds &- \int\limits_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi\Bigg] \cdot \mathbf{E}_P(\tau)\, d\tau\end{aligned} \tag{4.13}$$

by using (3.13).

Evaluating the first integral and integrating by parts in the second one, (4.13) becomes

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \frac{1}{2}[\varepsilon \mathbf{E}_P^2(d) + \mu \mathbf{H}_P^2(d)] - \frac{1}{2}[\varepsilon \mathbf{E}_P^2(0) + \mu \mathbf{H}_P^2(0)] \\ &+ \int\limits_0^{+\infty} \Bigg[\int\limits_0^\tau \alpha(\tau-\eta)\mathbf{E}_P(\eta)\, d\eta - \int\limits_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi\Bigg] \cdot \mathbf{E}_P(\tau)\, d\tau.\end{aligned} \tag{4.14}$$

Now, we put

$$\mathbf{I}(\tau, \overline{\mathbf{E}}^t) = \int\limits_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi, \qquad \tau \geq 0. \tag{4.15}$$

and note that $\mathbf{I}(\tau, \overline{\mathbf{E}}^t)$ has the regularity induced by (2.23) where the continuation has the duration τ; moreover, we can evaluate its Fourier transform, which is given by

$$\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) = \int\limits_0^{+\infty} e^{-i\omega\tau}\mathbf{I}(\tau, \overline{\mathbf{E}}^t)\, d\tau, \tag{4.16}$$

since $\mathbf{I}(\tau, \overline{\mathbf{E}}^t)$ is defined on $\mathbb{R}^+$.

Thus, (4.14) can be written as

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \frac{1}{2}\left\{\varepsilon \mathbf{E}_P^2(d) + \mu \mathbf{H}_P^2(d) - [\varepsilon \mathbf{E}_P^2(0) + \mu \mathbf{H}_P^2(0)]\right\} \\ &+ \int\limits_0^{+\infty} \Bigg[\frac{1}{2}\int\limits_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}_P(\eta)\, d\eta - \mathbf{I}(\tau, \overline{\mathbf{E}}^t)\Bigg] \cdot \mathbf{E}_P(\tau)\, d\tau,\end{aligned} \tag{4.17}$$

or using Plancherel's theorem, in the following equivalent form

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t),\mathbf{H}(t),\overline{\mathbf{E}}^t;\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P) &= \frac{1}{2}\left\{\varepsilon\mathbf{E}^2(t+d)+\mu\mathbf{H}^2(t+d)-[\varepsilon\mathbf{E}^2(t)+\mu\mathbf{H}^2(t)]\right\}\\ &+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}^*_{P+}(\omega)\,d\omega-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{I}_+(\omega,\bar{\mathbf{E}}^t)\cdot\mathbf{E}^*_{P+}(\omega)\,d\omega.\end{aligned} \tag{4.18}$$

5 Equivalence Between Two Integrated Histories Done by Means of the Work

In Section 3 we have called equivalent two integrated histories $\overline{\mathbf{E}}_1^t$ and $\overline{\mathbf{E}}_2^t$ which yield the same current density when they are subjected to the same process; hence, because of the equality imposed to the values of the electric and magnetic fields at the initial instant of the process, we have the same response of the material. An analogous equivalence relation between two integrated histories may be introduced in terms of the work.

Definition 5.1 Let $(\mathbf{E}_i,\mathbf{H}_i,\overline{\mathbf{E}}_i^t),\ i=1,2,$ be two given states of the electromagnetic material $\mathcal{B}$, the two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are called w-equivalent if the equality

$$\widetilde{W}(\mathbf{E}_1(t),\,\mathbf{H}_1(t),\,\overline{\mathbf{E}}_1^t;\,\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P)=\widetilde{W}(\mathbf{E}_2(t),\,\mathbf{H}_2(t),\,\overline{\mathbf{E}}_2^t;\,\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P) \tag{5.1}$$

holds for every $\dot{\mathbf{E}}_P\colon\ [0,\tau)\to R^3,\ \dot{\mathbf{H}}_P\colon\ [0,\tau)\to R^3$ and for every $\tau>0$.

The equivalence of the two definitions we have given follows easily from this theorem.

Theorem 5.1 *For any electromagnetic material described by the constitutive equations (2.1) and (3.5), two integrated histories of the electric field are equivalent in the sense of Definition 3.3 if and only if they are w-equivalent.*

Proof If two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are equivalent in the sense of Definition 3.3, then, for every $\mathbf{E}_P\colon\ (0,d]\to R^3,\ \mathbf{H}_P\colon\ (0,d]\to R^3$ and for every d, the works done on the process, of duration d and related to $\mathbf{E}_P$ and $\mathbf{H}_P$ by (3.9) – (3.10), applied to the states $(\mathbf{E}_1(t),\,\mathbf{H}_1(t),\,\overline{\mathbf{E}}_1^t)$ and $(\mathbf{E}_2(t),\,\mathbf{H}_2(t),\,\overline{\mathbf{E}}_2^t)$, characterized by $\mathbf{E}_1(t)=\mathbf{E}_2(t)$ and $\mathbf{H}_1(t)=\mathbf{H}_2(t)$, are the same since we have

$$\begin{aligned}&\int_0^d[\dot{\mathbf{D}}(\mathbf{E}_P(\tau))\cdot\mathbf{E}_P(\tau)+\dot{\mathbf{B}}(\mathbf{H}_P(\tau))\cdot\mathbf{H}_P(\tau)+\tilde{\mathbf{J}}((\mathbf{E}_P*\overline{\mathbf{E}}_1)^{t+\tau})\cdot\mathbf{E}_P(\tau)]d\tau\\ &\quad=\int_0^d[\dot{\mathbf{D}}(\mathbf{E}_P(\tau))\cdot\mathbf{E}_P(\tau)+\dot{\mathbf{B}}(\mathbf{H}_P(\tau))\cdot\mathbf{H}_P(\tau)+\tilde{\mathbf{J}}((\mathbf{E}_P*\overline{\mathbf{E}}_2)^{t+\tau})\cdot\mathbf{E}_P(\tau)]d\tau\end{aligned} \tag{5.2}$$

because of (3.9) – (3.10) and $(3.15)_2$.

If we now suppose that two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are w-equivalent, then (5.1) must be satisfied whatever may be the process P and its duration d. Taking into account the expression (4.17) of the work, together with (3.9) – (3.10), which give $\mathbf{E}_P$

and $\mathbf{H}_P$ in function of the initial values $(\mathbf{E}_i(t), \mathbf{H}_i(t))$, $i = 1, 2$, and the same process $(\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P)$, we derive from (5.1) a relation in which the process and its duration are arbitrary. This arbitrariness yields

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \mathbf{H}_1(t) = \mathbf{H}_2(t), \quad \mathbf{I}(\tau, \overline{\mathbf{E}}_1^t) = \mathbf{I}(\tau, \overline{\mathbf{E}}_2^t). \tag{5.3}$$

The last of these equalities is expressed by (4.15) and implies

$$\int_0^{+\infty} \alpha'(\tau + \xi)[\overline{\mathbf{E}}_1^t(\xi) - \overline{\mathbf{E}}_2^t(\xi)]\, d\xi = \mathbf{0}, \tag{5.4}$$

namely, the difference $\overline{\mathbf{E}}^t = \overline{\mathbf{E}}_1^t - \bar{\mathbf{E}}_2^t$ satisfies (3.17), which assures the equivalence of the two integrated histories $\overline{\mathbf{E}}_i^t$ $(i = 1, 2)$.

6 Formulation of the Maximum Recoverable Work

The maximum recoverable work expresses the maximum work we can obtain from the material at the given state σ, that is the amount of energy which is available at σ. It is defined as follows [19].

Definition 6.1 Let σ be a given state of the body $\mathcal{B}$, the maximum recoverable work starting from σ is

$$W_R(\sigma) = \sup\left\{-W(\sigma, P) \colon \ P \in \Pi\right\}, \tag{6.1}$$

where Π denotes the set of finite work processes.

Since the null process belongs to Π and yields a null work, $W_R(\sigma)$ is nonnegative and is bounded from above, i.e. $W_R(\sigma) < +\infty$, as a consequence of the thermodynamics. The work defined by (6.1) has been shown to coincide with the minimum free energy, that is denoted by $\psi_m(\sigma)$; thus, we have [11, 12, 19]

$$\psi_m(\sigma) = W_R(\sigma). \tag{6.2}$$

We want to find an expression for the maximum recoverable work and hence for the minimum free energy $\psi_m(\sigma)$. For this purpose we consider as initial state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$ at a fixed time t and we apply a process $P \in \Pi$, which is related to $\mathbf{E}_P$ and $\mathbf{H}_P$ by means of (3.9)–(3.10). Let d be the finite duration of P, we define P on $\mathbb{R}^+$ with its extension on $[d, +\infty)$, where we assume $P = 0$ together with $\mathbf{E}_P(d) = \mathbf{0}$ and $\mathbf{H}_P(d) = \mathbf{0}$. The work done on such a process is given by (4.17), which reduces to

$$\begin{aligned} W(\sigma, P) = &-\frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha(|\tau - \eta|)\mathbf{E}_P(\eta) \cdot \mathbf{E}_P(\tau)\, d\eta d\tau - \int_0^{+\infty} \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \cdot \mathbf{E}_P(\tau)\, d\tau. \end{aligned} \tag{6.3}$$

To determine the maximum of $-W(\sigma, P)$, expressed by (6.1), we consider the set of processes which are related to

$$\mathbf{E}_P(\tau) = \mathbf{E}^{(m)}(\tau) + \gamma \mathbf{e}(\tau), \quad \tau \in \mathbb{R}^+, \tag{6.4}$$

where $\mathbf{E}^{(m)}(\tau)$ is related to the process which yields the required maximum recoverable work, γ is a real parameter and $\mathbf{e}$ is an arbitrary smooth function with $\mathbf{e}(0) = \mathbf{0}$.

From (6.3) and (6.4), we get

$$\frac{d}{d\gamma}[-W(\sigma,P)]|_{\gamma=0} = -\int_0^{+\infty} \left[\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta - \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \right] \cdot \mathbf{e}(\tau)\, d\tau = 0 \tag{6.5}$$

and hence, being $\mathbf{e}$ arbitrary, we obtain

$$\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta = \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \quad \forall\, \tau \in \mathbb{R}^+. \tag{6.6}$$

This relation is a the Wiener-Hopf equation of the first type, which is not solvable in general. However, by using theorems of factorization and the thermodynamic properties of the kernel α, the solution $\mathbf{E}^{(m)}$ of (6.6) can be determined and gives the maximum recoverable work [5, 6, 20], which, taking account of (6.3) and (6.6), becomes

$$W_R(\sigma) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2} \int_0^{+\infty}\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta) \cdot \mathbf{E}^{(m)}(\tau)\, d\eta\, d\tau. \tag{6.7}$$

Application of the Plancherel theorem allows us to transform (6.7) as follows

$$W_R(\sigma) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega) \cdot (\mathbf{E}_+^{(m)}(\omega))^* d\omega. \tag{6.8}$$

To solve the Wiener–Hopf equation (6.6), we write it as follows

$$\int_{-\infty}^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta = \mathbf{I}(\tau, \overline{\mathbf{E}}^t) + \mathbf{r}(\tau), \quad \forall\, \tau \in R, \tag{6.9}$$

where we have added the function $\mathbf{r}$ defined by

$$\mathbf{r}(\tau) = \int_{-\infty}^{+\infty} \alpha(|\tau-s|)\mathbf{E}^{(m)}(s)\, ds \quad \forall\, \tau \in \mathbb{R}^- \tag{6.10}$$

and supposed equal to zero on R^{++}. Therefore, $supp(\mathbf{E}^{(m)}) \subseteq \mathbb{R}^+$, $supp\,(\mathbf{I}(\cdot, \overline{\mathbf{E}}^t)) \subseteq \mathbb{R}^+$, $supp\,(\mathbf{r}) \subseteq \mathbb{R}^-$ and hence (6.9) reduces to (6.6) for $\tau \in \mathbb{R}^+$ and to (6.10) for $\tau \in \mathbb{R}^-$.

In (6.7) we have the cosine Fourier transform $\alpha_c(\omega)$, which can be factorized as

$$\alpha_c(\omega) = \alpha_{(+)}(\omega)\alpha_{(-)}(\omega). \tag{6.11}$$

By introducing

$$K(\omega) = (1+\omega^2)\alpha_c(\omega), \tag{6.12}$$

which, taking into account the properties (2.18)-(2.19), is a function without zeros for every $\omega \in R$, also at infinity, and can be factorized as

$$K(\omega) = K_{(+)}(\omega)K_{(-)}(\omega), \tag{6.13}$$

we have

$$\alpha_{(+)}(\omega) = \frac{1}{1+i\omega}K_{(+)}(\omega), \qquad \alpha_{(-)}(\omega) = \frac{1}{1-i\omega}K_{(-)}(\omega). \tag{6.14}$$

Therefore, taking the Fourier transform of (6.9),

$$2\alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega) = \mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) + \mathbf{r}_-(\omega), \tag{6.15}$$

where $\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)$ is given by (4.16), we can evaluate the following quantity

$$\alpha_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) = \frac{1}{2}\left[\frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)}{\alpha_{(-)}(\omega)} + \frac{\mathbf{r}_-(\omega)}{\alpha_{(-)}(\omega)}\right], \tag{6.16}$$

which appears in (6.8).

Let us define

$$\mathbf{P}^t(z) = \frac{1}{4\pi i}\int_{-\infty}^{+\infty} \frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)/\alpha_{(-)}(\omega)}{\omega - z}\,d\omega, \qquad \mathbf{P}^t_{(\pm)}(\omega) = \lim_{\beta\to 0^\mp}\mathbf{P}^t(\omega + i\beta); \tag{6.17}$$

hence we see that the function $\mathbf{P}^t_{(\pm)}(z)$ has zeros and singularities in $\mathbf{C}^\pm$, which implies that $\mathbf{P}^t(z) = \mathbf{P}^t_{(\pm)}(z)$ is analytic in $\mathbf{C}^{(\mp)}$ and also in R because of the hypothesis assumed for the Fourier-transformed functions on R [12]. Applying the Plemelj formulae, (6.17) yield

$$\frac{1}{2}\frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)}{\alpha_{(-)}(\omega)} = \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^t_{(+)}(\omega), \tag{6.18}$$

which can be substituted into (6.16) to obtain the following equality

$$\alpha_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) + \mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^t_{(-)}(\omega) + \frac{1}{2}\frac{\mathbf{r}_-(\omega)}{\alpha_{(-)}(\omega)}. \tag{6.19}$$

In this relation the quantity at the left-hand side considered as a function of z, i.e. $\alpha_{(+)}(z)\mathbf{E}_+^{(m)}(z) + \mathbf{P}^t_{(+)}(z)$, is analytic for $z \in \mathbf{C}^-$, whereas from the right-hand side we see that $\mathbf{P}^t_{(-)}(z) + \dfrac{1}{2}\dfrac{\mathbf{r}_-(z)}{\alpha_{(-)}(z)}$ is analytic for $z \in \mathbf{C}^+$. Moreover, denoting by $\mathbf{U}(\omega)$ the function at the left-hand side of (6.19), such a function has an analytic extension on the whole complex plane and vanishes at infinity; therefore, it must be zero so that from (6.19) we get

$$\mathbf{E}_{+}^{(m)}(\omega) = -\frac{\mathbf{P}_{(+)}^{t}(\omega)}{\alpha_{(+)}(\omega)}, \qquad \mathbf{P}_{(-)}^{t}(\omega) = -\frac{1}{2}\frac{\mathbf{r}_{-}(\omega)}{\alpha_{(-)}(\omega)}. \tag{6.20}$$

From (6.20), we can evaluate $\mathbf{E}_{+}^{(m)}(\omega) \cdot \left(\mathbf{E}_{+}^{(m)}(\omega)\right)^{*}$, which allows to obtain the required expression of the minimum free energy

$$\psi_m(\sigma(t)) = \hat{\psi}(\mathbf{E}(t), \mathbf{H}(t), \mathbf{P}_{(+)}^{t}) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\mathbf{P}_{(+)}^{t}(\omega)|^2 d\omega. \tag{6.21}$$

7 Another Formulation for ψ_m

A different but equivalent form of the minimum free energy can be derived by expressing $\mathbf{P}_{(+)}^{t}(\omega)$ in terms of the integrated history $\overline{\mathbf{E}}^{t}$.

For this purpose we shall extend the memory kernel α' on R^{--} with an odd function $\alpha'^{(o)}(s)$, such that $\alpha'^{(o)}(s) = \alpha'(s)\ \forall s \geq 0$ and $\alpha'^{(o)}(s) = -\alpha'(-s)\ \forall s < 0$, whose Fourier transform is therefore given by (2.11), i.e. $\alpha_F'^{(o)}(\omega) = -2i\alpha_s'(\omega)$; moreover, we consider the causal extension to R^{--} for $\overline{\mathbf{E}}^{t}$, that is $\overline{\mathbf{E}}^{t}(s) = \mathbf{0}$ for all $s < 0$.

With these assumptions, (4.15) can be put in this form

$$\mathbf{I}(\tau, \overline{\mathbf{E}}^{t}) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^{t}(\xi)\, d\xi, \quad \tau \geq 0, \tag{7.1}$$

and can be extended to R by means of

$$\mathbf{I}^{(N)}(\tau, \overline{\mathbf{E}}^{t}) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^{t}(\xi)\, d\xi, \quad \tau < 0, \tag{7.2}$$

as follows

$$\mathbf{I}^{(R)}(\tau, \overline{\mathbb{E}}^{t}) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^{t}(\xi)\, d\xi = \begin{cases} \mathbf{I}(\tau, \overline{\mathbf{E}}^{t}) & \forall\, \tau \geq 0, \\ \mathbf{I}^{(N)}(\tau, \overline{\mathbf{E}}^{t}) & \forall\, \tau < 0. \end{cases} \tag{7.3}$$

Let us introduce $\overline{\mathbf{E}}_{N}^{t}(s) = \overline{\mathbf{E}}^{t}(-s)\ \forall s \leq 0$ with its extension $\mathbf{E}_{N}^{t}(s) = \mathbf{0}\ \forall s > 0$; thus, (7.3) becomes

$$\mathbf{I}^{(R)}(\tau, \overline{\mathbf{E}}^{t}) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau - s)\overline{\mathbf{E}}_{N}^{t}(s)\, ds \tag{7.4}$$

and it gives

$$\mathbf{I}_{F}^{(R)}(\omega, \overline{\mathbf{E}}^{t}) = -2i\alpha_s'(\omega)\left(\overline{\mathbf{E}}_{+}^{t}(\omega)\right)^{*}, \tag{7.5}$$

since

$$\overline{\mathbf{E}}^t_{N_F}(\omega) = \overline{\mathbf{E}}^t_{N_-}(\omega) = \left(\overline{\mathbf{E}}^t_+(\omega)\right)^*. \tag{7.6}$$

From (7.3) we get

$$\mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \int_{-\infty}^{+\infty} \mathbf{I}^{(R)}(\tau, \overline{\mathbf{E}}^t) e^{-i\omega\tau} d\tau = \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t) + \mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) \tag{7.7}$$

and hence

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t) + \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^t_{(+)}(\omega) \tag{7.8}$$

on the basis of (6.18).

Use of the Plamelj formulae yields

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \mathbf{P}^{(1)\,t}_{(-)}(\omega) - \mathbf{P}^{(1)\,t}_{(+)}(\omega), \tag{7.9}$$

where $\mathbf{P}^{(1)\,t}_{(\pm)}(\omega)$ are defined as in (6.17) and, when they are considered as functions of $z \in \mathbf{C}$, have zeros and singularities in $\mathbf{C}^{\pm}$.

By substituting (7.9) into (7.8), we arrive at the function

$$\mathbf{V}(\omega) \equiv \mathbf{P}^t_{(+)}(\omega) - \mathbf{P}^{(1)\,t}_{(+)}(\omega) = \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^{(1)\,t}_{(-)}(\omega) + \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t), \tag{7.10}$$

which is analytic in $\mathbf{C}^-$ for its first expression and in $\mathbf{C}^+$ for the second one, moreover, it vanishes at infinity. Thus, we must have $\mathbf{V}(\omega) = \mathbf{0}$ and hence

$$\mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^{(1)\,t}_{(+)}(\omega), \quad \mathbf{P}^t_{(-)}(\omega) = \mathbf{P}^{(1)\,t}_{(-)}(\omega) - \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t). \tag{7.11}$$

From (7.5), taking into account $(2.19)_1$ and (6.11), we get

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = i\omega\alpha_{(+)}(\omega)\big(\overline{\mathbf{E}}^t_+(\omega)\big)^*, \tag{7.12}$$

which can be substituted into the relation (6.17) written for $\mathbf{P}^{(1)\,t}_{(+)}(\omega)$ to obtain from $(7.11)_1$

$$\mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^{(1)\,t}_{(+)}(\omega) = \lim_{z \to \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega'\alpha_{(+)}(\omega')\big(\overline{\mathbf{E}}^t_+(\omega')\big)^*}{\omega' - z} d\omega'. \tag{7.13}$$

The conjugate of (7.13) yields

$$\big(\mathbf{P}^t_{(+)}(\omega)\big)^* = i \lim_{w \to \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\omega'\alpha_{(-)}(\omega')\overline{\mathbf{E}}^t_+(\omega')}{\omega' - w} d\omega' \tag{7.14}$$

where, applying the Plamelj formulae, we have

$$\omega\alpha_{(-)}(\omega)\overline{\mathbf{E}}^t_+(\omega) = \mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega) \tag{7.15}$$

with

$$\mathbf{Q}^t_{(\pm)}(\omega) = \lim_{z\to\omega^\mp} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\omega'\alpha_{(-)}(\omega')\overline{\mathbf{E}}^t_+(\omega')}{\omega' - z}\, d\omega' \tag{7.16}$$

such that $\mathbf{Q}^t_{(\pm)}(z)$ has zeros and singularities for $z \in \mathbf{C}^\pm$.

Finally, (7.14) and (7.16) give

$$\left(\mathbf{P}^t_{(+)}(\omega)\right)^* = i\mathbf{Q}^t_{(-)}(\omega), \tag{7.17}$$

which allows us to transform (6.21) as follows

$$\psi_m(t) = \hat{\psi}(\mathbf{E}(t),\, \mathbf{H}(t),\, \mathbf{Q}^t_{(-)}) = \frac{1}{2}\left[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)\right] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\mathbf{Q}^t_{(-)}(\omega)|^2 d\omega. \tag{7.18}$$

We observe that the current density given by (3.5) may be written in terms of the quantities now deduced. In fact, applying the Plancherel theorem to (3.5) and taking account of (2.11) for $\alpha'_F(\omega)$, since α' is extended to R as an odd function, and $(2.19)_1$, we have

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \omega\alpha_c(\omega)\overline{\mathbf{E}}^t_+(\omega)\, d\omega = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\left[\mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega)\right] d\omega, \tag{7.19}$$

where we have considered (6.11) and (7.15) too.

In $(7.19)_2$ we can consider two integrals, one of which is zero because of the analyticity of the integrand $\alpha_{(+)}(\omega)\mathbf{Q}^t_{(+)}(\omega)$ in $\mathbf{C}^-$; thus, it follows that the other integral must be real and gives the current density, that is

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\mathbf{Q}^t_{(-)}(\omega)\, d\omega. \tag{7.20}$$

8 A Particular Model

The discrete spectrum model is characterized by a particular class of response functions, which are expressed by a linear combination of exponentials. Thus we assume the following kernel

$$\alpha(t) = \begin{cases} \sum_{i=1}^{n} g_i e^{-\alpha_i t}, & t \geq 0, \\ 0 & t < 0, \end{cases} \tag{8.1}$$

with $g_i,\ \alpha_i \in R^{++}$ $(i = 1, 2, \dots, n)$, $n \in \mathbf{N}$ and $\alpha_1 < \alpha_2 < \dots < \alpha_n$.

We first observe that with these assumptions $(2.18)_1$ is satisfied, being

$$\alpha(0) = \sum_{i=1}^{n} g_i > 0.$$

We have

$$\alpha_F(\omega) = \sum_{i=1}^{n} \frac{g_i}{\alpha_i + i\omega} \quad \forall\, \omega \in R, \tag{8.2}$$

whence

$$\alpha_c(\omega) = \sum_{i=1}^{n} \frac{\alpha_i g_i}{\alpha_i^2 + \omega^2}, \qquad \alpha_s(\omega) = \omega \sum_{i=1}^{n} \frac{g_i}{\alpha_i^2 + \omega^2} \quad \forall\, \omega \in R. \tag{8.3}$$

Then, (6.12) yields

$$K(\omega) = (1 + \omega^2) \sum_{i=1}^{n} \frac{\alpha_i g_i}{\alpha_i^2 + \omega^2}, \qquad K_\infty = \lim_{\omega \to \pm\infty} K(\omega) = \sum_{i=1}^{n} \alpha_i g_i > 0. \tag{8.4}$$

The expression $(8.4)_1$ coincides with the one obtained in [14]. Thus, we recall that, putting $z = -\omega^2$, we have $K(\omega) = f(z)$, which is a function with n simple poles at α_i^2 if $\alpha_i^2 \neq 1\ \forall\, i \in \{1, 2, \dots, n\}$, while it has $n-1$ simple poles if one of α_i's, and only one since α_i's are ordered and distinct numbers, is equal to 1.

Let $n \neq 1$.

We first suppose that $\alpha_i^2 \neq 1$ $(i = 1, 2, \dots, n)$.

If $1 < \alpha_1^2$ or $\alpha_n^2 < 1$, $f(z)$ has n simple poles α_i^2 $(i = 1, 2, \dots, n)$ and has n simple zeros denoted by $\gamma_1^2 = 1$ and γ_j^2 $(j = 2, 3, \dots, n)$. It rests to consider the case when there exists p such that $\alpha_p^2 < 1 < \alpha_{p+1}^2$, where p may assume only one of the values: $1, 2, \dots, n-1$; in this case we have the zero γ_{p+1}^2 such that $\gamma_{p+1}^2 \gtreqless 1 = \gamma_1^2$ and therefore it can be equal to 1, which becomes a zero of multiplicity 2; hence $f(z)$ has $n-1$ distinct zeros. In any case the zeros denoted by γ_j^2 $(j = 2, 3, \dots, n)$ are so ordered

$$\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < \dots < \alpha_p^2 < \gamma_{p+1}^2 < \alpha_{p+1}^2 < \dots < \alpha_{n-1}^2 < \gamma_n^2 < \alpha_n^2 \tag{8.5}$$

and we can write (8.4) as follows

$$K(\omega) = K_\infty \prod_{i=1}^{n} \left\{ \frac{\gamma_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\}, \tag{8.6}$$

where $\gamma_1^2 = 1$ and only one of the other zeros, say γ_{p+1}^2, can be equal to 1, which thus becomes a zero with multiplicity 2. Hence, it follows that the factorization (6.13) of $K(\omega)$ expressed by (8.6) yields, as in [14],

$$K_{(-)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \quad K_{(+)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega - i\gamma_i}{\omega - i\alpha_i} \right\}, \quad k_\infty = (K_\infty)^{1/2}. \tag{8.7}$$

Since in (7.18) we have the integral of $|\mathbf{Q}^t_{(-)}(\omega)|^2$, we must consider (7.16), where $\alpha_{(-)}(\omega)$, contrary to what occurs in [14], is now multiplied by ω. Thus, from $(6.14)_2$, taking account of $(8.7)_1$, we obtain the required product

$$\omega\alpha_{(-)}(\omega) = \omega \frac{ik_\infty}{\omega + i} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \tag{8.8}$$

which also vanishes at $\gamma_0 = 0$.

Hence, since $\gamma_1 = 1$, putting $\delta_1 = \gamma_0 = 0$ and $\delta_j = \gamma_j$ $(j = 2, 3, \dots, n)$, we get

$$\omega\alpha_{(-)}(\omega) = ik_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\delta_i}{\omega + i\alpha_i} \right\} = ik_\infty \left(1 + i \sum_{r=1}^{n} \frac{A_r}{\omega + i\alpha_r} \right), \tag{8.9}$$

whose coefficients A_r are given by

$$A_r = (\delta_r - \alpha_r) \prod_{i=1, i\neq r}^{n} \left\{ \frac{\delta_i - \alpha_r}{\alpha_i - \alpha_r} \right\}, \quad r = 1, 2, \dots, n, \tag{8.10}$$

where it may occur that there exists p such that $\delta_{p+1} = \gamma_{p+1} = 1$.

Then, we suppose that $\alpha_p^2 \equiv \gamma_1^2 = 1$, where p may assume only one of the values: $1, 2, \dots, n$.

We can distinguish three cases. If $\alpha_1^2 = 1$ or $\alpha_n^2 = 1$, $f(z) = K(\omega)$ has $n-1$ zeros γ_j^2 $(j = 2, 3, \dots, n)$ and $n-1$ poles α_j^2 $(j = 2, 3, \dots, n)$ or α_i^2 $(i = 1, 2, \dots, n-1)$ in the two cases; however, the presence of the factor $\dfrac{\omega}{\omega + i}$ in (8.8) still now yields the introduction of $\delta_1 = 0$ and hence (8.9) with (8.10), where $\alpha_1 = 1$ or $\alpha_n = 1$, hold again. Also in the third case with $\alpha_p^2 = 1$, $1 < p < n$, we have $n-1$ zeros γ_j^2 $(j = 2, 3, \dots, n)$ and $n-1$ poles α_i^2 $(i = 1, 2, \dots, p-1, p+1, \dots, n)$ ordered as (8.5) shows with this condition $\alpha_{p-1}^2 < \gamma_p^2 < 1 < \gamma_{p+1}^2 < \alpha_{p+1}^2$; moreover, as before we get (8.9)–(8.10), where we have $\alpha_p = 1$.

Let $n = 1$.

This case must be studied separately. Substituting $(8.7)_1$, written with $n = 1$, into $(6.14)_2$, we have the following relation

$$\omega\alpha_{(-)}(\omega) = ik_\infty \frac{\omega}{\omega + i\alpha_1} = ik_\infty \left(1 + i \frac{A_1}{\omega + i\alpha_1} \right), \tag{8.11}$$

where

$$A_1 = -\alpha_1, \quad k_\infty = (\alpha_1 g_1)^{1/2}. \tag{8.12}$$

In the general case, with $n \neq 1$, $\mathbf{Q}^t_{(-)}(\omega)$, taking into account its definition (7.16) and the expression $(8.9)_2$, is given by

$$\mathbf{Q}^t_{(-)}(\omega) = ik_\infty \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\mathbf{E}}^t_+(\omega')}{\omega' - \omega^+}\, d\omega' - \sum_{r=1}^{n} k_\infty A_r \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\mathbf{E}}^t_+(\omega')/(\omega' - \omega^+)}{\omega' - (-i\alpha_r)}\, d\omega'. \tag{8.13}$$

Since $\overline{\mathbf{E}}^t_+$ as function of $z \in \mathbf{C}$ is analytic in $\mathbf{C}^{(-)}$, the first integral over the real axis can be extended to an infinite contour on $\mathbf{C}^{(-)}$ without altering its value, which is zero because of the analyticity of the integrand. It rests to evaluate the other integrals by closing again on $\mathbf{C}^{(-)}$ and taking account of the sense of the integrations. Thus, we get

$$\mathbf{Q}^t_{(-)}(\omega) = -k_\infty \sum_{r=1}^{n} \frac{A_r}{\omega + i\alpha_r} \overline{\mathbf{E}}^t_+(-i\alpha_r), \tag{8.14}$$

and hence

$$\left(\mathbf{Q}^t_{(-)}(\omega)\right)^* = -k_\infty \sum_{r=1}^{n} \frac{A_r}{\omega - i\alpha_r} \left(\overline{\mathbf{E}}^t_+(-i\alpha_r)\right)^*, \tag{8.15}$$

where, on the basis of $(2.7)_1$, we have

$$\overline{\mathbf{E}}^t_+(-i\alpha_r) = \int_0^{+\infty} e^{-\alpha_r s} \overline{\mathbf{E}}^t(s)\, ds = \left(\overline{\mathbf{E}}^t_+(-i\alpha_r)\right)^*. \tag{8.16}$$

We can now evaluate the integral

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathbf{Q}^t_{(-)}(\omega)|^2 d\omega \\ &\quad = k^2_\infty \sum_{r,l=1}^{n} A_r A_l \overline{\mathbf{E}}^t_+(-i\alpha_r) \cdot \overline{\mathbf{E}}^t_+(-i\alpha_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i/(\omega + i\alpha_r)}{\omega - i\alpha_l} d\omega \\ &\quad = K_\infty \sum_{r,l=1}^{n} \frac{A_r A_l}{\alpha_r + \alpha_l} \overline{\mathbf{E}}^t_+(-i\alpha_r) \cdot \overline{\mathbf{E}}^t_+(-i\alpha_l), \end{aligned} \tag{8.17}$$

which, substituted into (7.18), by virtue of $(8.16)_1$, yields

$$\begin{aligned} \psi_m(t) = &\frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] \\ &+ \frac{1}{2} \int_0^{+\infty}\!\! \int_0^{+\infty} 2K_\infty \sum_{r,l=1}^{n} \frac{A_r A_l}{\alpha_r + \alpha_l} e^{-(\alpha_r s_1 + \alpha_l s_2)} \overline{\mathbf{E}}^t(s_1) \cdot \overline{\mathbf{E}}^t(s_2)\, ds_1 ds_2. \end{aligned} \tag{8.18}$$

In the particular case, when $n = 1$, taking account of (8.12), the expression of the minimum free energy assumes the simpler form

$$\psi_m(t) = \frac{1}{2}\left[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)\right] + \frac{1}{2}\alpha_1^2 g_1 \left[\int_0^{+\infty} e^{-\alpha_1 s} \overline{\mathbf{E}}^t(s)\, ds\right]^2. \tag{8.19}$$

We observe that, integrating by parts in this last relation, we obtain the same result derived in [14], where the history of **E** is considered instead of its integrated history.

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A Duality Principle in the Theory of Dynamical Systems

C. Corduneanu[1]* and Yizeng Li[2]

[1] *The University of Texas at Arlington, Box 19408 UTA, Arlington, TX 76019, USA*
[2] *Tarrant County College, Hurst, Ft. Worth, TX 76053, USA*

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Abstract: The aim of this paper is to formulate and illustrate a duality principle for dynamical systems. There is a one-to-one correspondence between causal (nonanticipative) systems, and the anticipative ones. Several cases are dealt with, based on the nature of the functional equations describing the dynamics.

Keywords: *Dynamical systems; causal; antisipative; duality principle.*

Mathematics Subject Classification (2000): 93A99, 93B99, 93C30.

1 Introduction

The dynamical systems we shall consider in this paper will be described by functional equations of various types.

The duality principle we are going to formulate and illustrate establishes a one-to-one correspondence between the class of *causal systems*, and the class of *anticipative systems.* The first class is also known as abstract Volterra systems, while the second class contains the so-called anti-Volterra systems.

The principle of duality states that: *to any causal system, one can associate an anticipative systems, and vice-versa.*

Moreover, the mathematical treatment is basically the same for causal/ anticipative couples which are in correspondence.

The idea of formulating this duality principle came from writing our joint paper [3], in which the mathematical apparatus used in dealing with anticipative systems (the corresponding describing equations are with advanced argument), has revealed a striking

*Corresponding author: concord@uta.edu

resemblance with the one used when investigating causal systems (usually described by functional equations of Volterra type).

As we can expect, the initial state in the causal system becomes the terminal state in the anticipative systems, and vice-versa.

Apparently, the anticipative systems, which sometimes (see, for instance, Dubois [4, 5]) are called *anticipatory* systems, present interest in various applied areas, including some economic problems. Several proceedings volumes have been published, under the editorship of Dubois [4, 5]. They illustrate the significance of various types of anticipative systems, both theoretically and from the point of view of applications.

2 A Class of Discrete Systems

Let us consider a dynamical system with a finite number of states, say $x(t_i)$, $i = 1, 2, \dots, n$, with t_i an increasing sequence of reals. We assume $x(t_i) \in R$, $i = 1, 2, \dots, n$, even though we could deal with more general spaces than R, e.g., a Banach space E. We shall denote, for brievety, $x(t_i) = x_i$, $i = 1, 2, \dots, n$.

Let us further assume that the dynamics of the system is described by n equations of the form

$$\begin{aligned} x_1 &= f_1(x_1), \\ x_2 &= f_2(x_1, x_2), \\ &\dots\dots\dots\dots\dots\dots \\ x_n &= f_n(x_1, x_2, \dots, x_n). \end{aligned} \tag{1}$$

The particular form of the system (1) expresses the fact that we deal with a *causal system.* As each equation shows, the state of the system at the moment t_k, $1 \le k \le n$, depends only on the states at moments preceding or equal to t_k.

Now let us operate a change of variables $t_k = -\tau_{n-k+1}$, $x_k = y_{n-k+1}$, $k = 1, 2, \dots, n$. Then, the system (1) becomes

$$\begin{aligned} y_1 &= f_n(y_n, y_{n-1}, \dots, y_1), \\ y_2 &= f_{n-1}(y_n, y_{n-1}, \dots, y_2), \\ &\dots\dots\dots\dots\dots\dots\dots \\ y_n &= f_1(y_n). \end{aligned} \tag{2}$$

From (2), we see that the system is of *anticipative* type (or, as sometimes called, *anticipatory*).

Since the times t_k, $1 \le k \le n$, form an increasing sequence, there follows that the new times τ_k, $1 \le k \le n$, also form an increasing sequence: $\tau_1 < \tau_2 < \dots < \tau_n$.

It is obvious that the systems (1) and (2) are identical, which tells us that from mathematical point of view, we have to solve the same problem for either of the associated causal and anticipative systems.

There are several questions rising from the above discussion related to the system (1), (2). Namely, the equations describing the dynamics of the system have been chosen in such a way that we deal with a “determined” system. In other words, we assume that the system (1) has a unique solution. This situation can be easily achieved. To take just an elementary example, we will assume that $|\partial f_k / \partial x_k| \le m_k < 1$, $k = 1, 2, \dots, n$. This implies the existence of a unique real solution to the first equation; then substituting in the second equation this value for x_1, we will again determine a unique value for x_2, and so on. At each step we have to apply the Banach contraction mapping principle, in order to determine the unique value of the variable characterizing the state of the system. Many other conditions can be imposed in order to achieve the existence result specified above.

In case the system (1) has several solutions, say $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $(\bar{\bar{x}}_1, \bar{\bar{x}}_2, \dots, \bar{\bar{x}}_n)$, then we can define two anticipative systems the same way we have proceeded in the case of a unique solution. In other words, if there exist two (or several) causal systems described by the equations (1), we can accordingly associate two (or several) anticipative systems with "reverse" dynamics.

Another aspect to be considered corresponds to the situation when the system (1) is not determined, in the sense that some of the variables can be chosen arbitrarily (for instance, the n equations are not independent). What is, in such a case, the adequate manner to attach to (1) an anticipative system? Apparently, this is possible because if we assign values to some of the x_k's, the remaining equations still describe a causal system.

Finally, we would like to formulate an open problem (apparently) related to the topics discussed above. Namely, if the system (1) is replaced by a more general system of equations like $f_k(x_1, x_2, \dots, x_n) = 0,\ k = 1, 2, \dots, n$, *under what conditions can we state that they describe the dynamics of a causal system*?

3 Systems Described by Integral Equations

In this section we shall illustrate the duality principle in the case the dynamics of the system is described by an integral equation of anti-Volterra type. Therefore, we shall start with an anticipative system, and construct the causal system whose dynamics is determined by the same data as those of the given anticipative system.

The dynamical system under consideration in this section is defined by means of a function $x = x(t),\ 0 \le t \le T$, the values of x being taken in a Banach space E. The describing equation for the dynamics is of the form

$$x(t) = f(t) + \int_t^T k(t, s, x(s))\, ds, \tag{3}$$

with $f \in C([0,T], E)$, and $k(t,s,x)$ defined and continuous on $\Delta \times E$, with values in E, where $\Delta = \{(t,s)\colon\ 0 \le t \le s \le T\} \subset R^2$. If we also admit for $k(t,s,x)$ a Lipschitz type condition in $\Delta \times E$,

$$\|k(t,s,x) - k(t,s,y)\| \le L\|x - y\|, \quad L > 0, \tag{4}$$

then we get existence and uniqueness of the solution $x = x(t),\ 0 \le t \le T$, which is in $C([0,T], E)$.

The proof of existence and uniqueness of the solution to (3) can be conducted on the classical pattern, by the method of successive approximations

$$x_{n+1}(t) = f(t) + \int_t^T k(t, s, x_n(s))\, ds, \quad n \ge 0, \tag{5}$$

with $x_0(t) = f(t)$, $t \in [0, T]$. It has been carried out in our paper [3].

The case when we have existence on the whole interval $[0, T]$, but not necessary uniqueness, has been also discussed in [3].

Now let $x = x(t)$ be a solution of (3) defined on $[0, T]$, in either case of uniqueness or nonuniqueness. In order to define the causal system corresponding to the anticipative system described by (3), we shall proceed as follows: we operate the change of variables $t = -\tau$, $s = -u$, and denote $f(-\tau) = \tilde{f}(\tau)$, $x(-\tau) = y(\tau)$, $\tau \in [-T, 0]$. Then (3) becomes

$$y(\tau) = \tilde{f}(\tau) + \int_{-T}^{\tau} k(-\tau, -u, y(u))\, du, \tag{6}$$

with $\tilde{f} \in C([-T, 0], E)$, and k defined on $\Delta \times E$, with values in E, where $\overline{\Delta} = \{(\tau, u)\colon -T \leq u \leq \tau \leq 0\} \subset R^2$. Obviously, $k(-\tau, -u, y)$ satisfies a Lipschitz condition derived from (4).

It is obvious from (6) that the dynamics described by this equation is of causal type. It is possible to “shift” the considerations from the interval $[-T, 0]$, to any interval $[a, b] \subset R$.

We have again illustrated the duality principle, this time for continuous time dynamical systems for which the law of the dynamics is given by means of an integral equation of anti-Volterra type.

We shall see below that other types of dynamical systems can be reduced, in principle, to the case examined in this section.

4 A Case with General Causal Operators

In this section we will consider a Cauchy type problem, for a differential equation involving a linear causal operator, as well as a nonlinear part. More precisely, we shall deal with the equation

$$\dot{x}(t) = (Lx)(t) + (fx)(t), \quad t \in [0, T], \tag{7}$$

with the initial condition

$$x(0) = x^0 \in R^n, \quad n \geq 1. \tag{8}$$

The linear causal operator L is acting continuously on the space $L^2([0, T], R^n)$, while $f\colon L^2([0, T], R^n) \to L^2([0, T], R^n)$ is a continuous causal operator, generally nonlinear. It is understood that any solution we consider is of Carathéodory type, i.e., is in $AC([0, T], R^n)$ and satisfies the differential equation (7) a.e. on $[0, T]$.

For general properties of such equations we send the reader to the book [2] by C. Corduneanu. The formula of variation of parameters is given in the paper [6] by Yizeng Li.

As shown in the above mentioned references, the problem (7), (8) is equivalent to the integral equation of Volterra type

$$x(t) = X(t, 0)x^\circ + \int_0^t X(t, s)(fx)(s)\, ds, \tag{9}$$

where $X(t, s)$, $0 \leq s \leq t \leq T$, is the Cauchy operator attached to the linear operator L in (7). In [2, 3], it is dealt with existence, and some properties are emphasized. See also the paper [7] by Mahdavi, in which L^2 is substituted by any L^p, $1 < p < \infty$.

The integral equation (9) is not exactly of the classical Volterra type, due to the presence of the operator f under the integral. In order to place ourselves in the classical framework, we shall assume that the operator f is a Niemytskii operator, i.e.,

$$(fx)(t) = F(t, x(t)), \quad t \in [0, T]. \tag{10}$$

In order to assure the fact that F is acting on $L^2([0,T];\, R^n)$, we can impose the growth condition

$$\|F(t, x)\| \le c\|x\| + a(t), \tag{11}$$

with $c > 0$ and $a \in L^2([0,T];\, R)$. Of course, we need some measurability conditions on F, and the Carathéodory assumptions are just adequate (i.e., continuity in x for almost all $t \in [0, T]$, and measurability in t for all $x \in R^n$).

With the choice (10) for the operator f in the equation (7), the integral equation (9) becomes

$$x(t) = X(t, 0)x^\circ + \int_0^t X(t, s)F(s, x(s))\, ds. \tag{12}$$

Equation (12) is of classical Volterra type, and we can compare it with the equation (6). Due to the properties of $X(t, s)$, any solution of (12) belongs to the space $AC([0,T],\, R^n)$ of absolutely continuous maps, and satisfies the differential equation (7) almost everywhere on $[0, T]$.

There remains to write the integral equation of anticipative type, which describes the dynamics of the dual system associated to the system described by the equation (9), with $f = F$.

By the same substitution used in the preceding section, namely $t = -\tau$, $s = -u$, $x(-\tau) = y(\tau)$, the equation (12) leads to the integral equation of anticipative type on $[-T, 0]$,

$$y(\tau) = X(-\tau, 0)x^\circ + \int_\tau^0 X(-\tau, -u)F(-u,\, y(u))\, du. \tag{13}$$

Conditions for existence/uniqueness of solution to the equation (13) can be found in the above mentioned references [2, 3].

Our aim was to illustrate once again the validity of the duality principle stated in this paper. In this case, the equations (12) and (13) describe the dynamics of the associated systems (causal and anticipative). At the same time, we have presented an example which relies on the use of general causal operators.

5 Conclusions and Open Problems

The examples discussed above show that whole classes of dynamical system can be used to illustrate the *duality principle* enunciated in this paper. What is really interesting, from mathematical point of view, is the fact that the mathematical apparatus is, basically, the same for the couple of associated systems. Moreover, from any result concerning the causal systems, one can derive a similar result for the associated anticipative systems.

We propose to the reader the following exercise: start with a result on causal systems in the book [2], and describe the corresponding result for the associated anticipative system. A first step would be to find the functional equation describing the dynamics of the anticipative system.

There seems to be a problem in dealing with the duality principle when we have infinite time in our initial system (causal or anticipative). In other words, when the time interval $[0,T]$ is replaced by the semi-axis $R_+ = [0,\infty)$. If for the describing functional equation we look for the so-called *transient* solutions, tending at infinity towards a stationary state, then the duality principle appears to be easy to be formulated. Another venue should be found when, for instance, we deal with an oscillatory solution of the describing functional equation. We mention this situation as an *open* problem.

Another open problem is to check the validity of the duality principle in case of dynamical systems whose dynamics is described by functional equations of the form $x(t) = f(t, x_t)$, with the usual notation $x_t(s) = x(t+s)$, $s \in [-T,0]$. The case of differential equations with delay

$$\dot{x}(t) = f(t, x_t), \quad x(s) = x_0(s), \quad s \in [-T,0],$$

is covered by the above mentioned functional equation $x(t) = f(t, x_t)$, with $x_0(s)$ assigned on $[-T,0]$. One may succeed in this respect, by considering "dual" equations of the form $y(t) = \tilde{f}(t, y^t)$, with obvious meaning for y^t, namely $y^t(s) = y(t+s)$, $s \in [0,T]$.

A comprehensive approach, in order to produce an adequate framework for the statement and illustration of the duality principle, will require a more general concept of dynamical systems, in which the terms *causal* and *anticipative* make sense.

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Output Synchronization of Chaotic Systems: Model-Matching Approach with Application to Secure Communication

D. López-Mancilla[1] and C. Cruz-Hernández[2]*

[1] *Electronics & Telecomunications Department, Scientific Research and Advanced Studies of Ensenada (CICESE), Km. 107 Carretera Tijuana-Ensenada, 22860 Ensenada, B. C., México*
[2] *Telematics Direction, Scientific Research and Advanced Studies of Ensenada (CICESE), P.O.Box 434944, San Diego, CA 92143-4944, USA*

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Abstract: In this paper, a method for synchronizing chaotic systems in continuous-time is presented. The approach, which exploits the model-matching problem from nonlinear control theory, is advantageously applied to achieve complete synchronization and output synchronization of identical and nonidentical chaotic systems, respectively. Some potential applications to secure communication for audio and binary information signals are also given.

Keywords: *Chaos synchronization; model-matching problem; encryption; secure communication.*

Mathematics Subject Classification (2000): 37N35, 65P20, 68P25, 70K99, 94A99.

1 Introduction

Undoubtedly, data security has been an issue of increasing importance in communications as the Internet and personal communication systems are being made accessible world-wide. Recently, increasing efforts have been made to use chaotic systems for enhancing some features of communication systems. In particular, chaotic synchronization to design secure communication systems. Chaos and cryptography have some common features, the most prominent being extremely sensitivity to parameter changes. Chaos has already been used to design cryptography systems [9]. One common feature of most existing chaos-based secure communication schemes is that a chaotic signal is used for

*Corresponding author: ccruz@cicese.mx

transmitting the message. More precisely, by a proper modulation of the chaotic transmitter dynamics, the private message is hidden and sent to the chaotic receiver dynamics. At the receiving side, a synchronous chaotic system is built to synchronize with the transmitter to recover the original message. Different approaches for chaos synchronization have been proposed to synchronize identical systems, see e.g., [3 – 5,7,8,11,13,16,17,19,24–26, 30] and references inside. Although synchronization of identical chaotic systems is useful to transmit private information, some researchers have proposed different methods to synchronize nonidentical chaotic systems, which has been suggested to several potential applications. Adaptive control has been used to synchronize nonidentical chaotic systems in [2, 31]. Feedback linearization and adaptive feedback linearization has been proposed in [28]. A method to get an equivalence between two nonidentical chaotic attractors was presented in [12]. Synchronization of nonidentical chaotic systems is useful in many cases of practical interest, and significantly when it occurs in living systems, like synchronization of the activity of groups of neurons located in different brain areas [20, 27] or, in the synchronization between heart and respiratory rates [15] or, the coupling of biological oscillators [29]. However, the synchronization of nonidentical chaotic systems is a much more difficult problem. The aim of this paper is to illustrate an effective method for synchronizing chaotic systems in continuous-time. This objective is achieved by using results from nonlinear control theory; in particular, we use the model-matching problem [6, 10]. This synchronization method presents the following advantages:

- It is systematic.
- It is useful to synchronize identical and nonidentical chaotic systems.
- It uses unidirectional coupling, that let the coupling signal requires less transmission channels, because of the model/master does not need to know any information from the plant/slave.

Moreover, with this methodology chaotic synchronization has applications on transmission of private information schemes. To this purpose, the attention is at first focused on Rössler–Rössler and Lorenz–Rössler synchronization. Finally, we give an application to private/secure communication for transmission and recovering of audio and binary messages (i.e., analog and digital signals) using different chaotic communication schemes.

The paper is organized as follows: Section 2 states the problem formulation. Briefly, the model-matching problem from nonlinear control theory is reviewed in Section 3. In Section 4, we apply this approach to synchronize identical and nonidentical chaotic systems based on Lorenz and Rössler systems. In Section 5, we propose three private/secure communication schemes based on chaotic synchronization for transmission and recovering of audio and binary messages. Finally, Section 6 summarizes the concluding remarks.

2 Problem Statement

Consider a dynamical system described by state equations of the form

$$P\colon \quad \begin{cases} \dfrac{dx}{dt} = f(x) + g(x)u, \\ y = h(x), \end{cases} \tag{1}$$

where the state $x(t) \in R^n$, the input $u(t) \in R$, and the output $y(t) \in R$, being $f(x)$ and $g(x)$ smooth and analytical functions. In addition, consider another nonlinear system described by

$$M\colon \quad \begin{cases} \frac{dx_M}{dt} = f_M(x_M) + g_M(x_M)u_M, \\ y_M = h_M(x_M), \end{cases} \tag{2}$$

where the state $x_M(t) \in R^{n_M}$, the input $u_M(t) \in R$, and the output $y_M(t) \in R$, being $f_M(x_M)$ and $g_M(x_M)$ smooth and analytical functions too. We assume that x° is an equilibrium point of system (1), i.e., $f(x^\circ) = 0$. Similarly, x_M° is an equilibrium point of system (2). Assume that dynamical systems (1) and (2) under certain conditions have *chaotic* behavior. Then, the chaotic system (1) *synchronizes* with the chaotic system (2), if

$$\lim_{t\to\infty} |y(t) - y_M(t)| = 0, \tag{3}$$

no matter which initial conditions $x(0)$ and $x_M(0)$ have, and for suitable input signals $u(t)$ and $u_M(t)$.

Note that, we are only considering *output synchronization problem* between chaotic systems (1) and (2). Moreover, no matter if the chaotic systems (1) and (2) are identical or nonidentical. In the next section, we will describe how to satisfy the output synchronization condition (3) from the perspective of the model-matching problem.

On the other hand, in the context of secure/private communications based on the chaotic synchronization between systems (1) and (2); in the chaotic transmitter system, the private message is hidden/encrypted and sent to the chaotic receiver system via public channel. Finally, the original message is retrieved/decrypted at the receiver end. For this purpose, we will use the chaotic masking and chaotic switching techniques.

3 Model-Matching Problem

Now, consider the dynamical systems (1) and (2) like a *plant* P and *model* M, respectively. We want to design a feedback control law $u(t)$ for the plant P which, irrespectively of the initial states of P and M, makes the output $y(t)$ asymptotically converges to the output $y_M(t)$ produced by M under an arbitrary input $u_M(t)$. This problem is the so-called asymptotic *model-matching problem* from nonlinear control theory. It is also well-known that different approaches to solve the model-matching problem have been proposed in the literature, see e.g. [6, 10]. In this work, we adopt the following solution: the model-matching problem is reduced into a problem of decoupling the output of a suitable auxiliary system from the input $u_M(t)$ to the model M. To this purpose the *auxiliary system* is defined as follows

$$E\colon \quad \begin{cases} \dfrac{dx_E}{dt} = f_E(x_E) + \hat{g}(x_E)u + \hat{g}_M(x_E)u_M, \\ y_E = h_E(x_E), \end{cases} \tag{4}$$

with state $x_E = (x, x_M)^T \in R^{n+n_M}$, inputs $u(t)$ and $u_M(t)$, and

$$f_E(x_E) = \begin{pmatrix} f(x) \\ f_M(x_M) \end{pmatrix}, \qquad \hat{g}(x_E) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix},$$

$$\hat{g}_M(x_E) = \begin{pmatrix} 0 \\ g_M(x_M) \end{pmatrix}, \qquad h_E(x_E) = h(x) - h_M(x_M).$$

That corresponds to a system having as "output" the difference between the output of P and the output of M. We consider $u_M(t)$ as a "disturbance" acting on the auxiliary system (4), and we want to decouple it from the output $y_E(t)$. We are allowed to use disturbance "measurements" because $u_M(t)$ is the input of M, and thus we may use a control law of the form

$$u = \alpha(x_E) + \gamma(x_E)u_M + \beta(x_E)v, \tag{5}$$

with $v(t)$ an additional input signal to obtain asymptotic stability in the closed-loop auxiliary system, which corresponds to the convergence rate of output synchronization.

The control objective of the model-matching problem is contained in the following definition.

Definition 1 (Model-matching problem) Given the plant P and the model M around their respective equilibrium points x° and x_M°, and a point x_E°. The model-matching problem consists in finding a feedback control law $u(t) \in R$ for auxiliary system E equation (4) such that, the output $y_E(t)$ of system E (feedback by $u(t)$ of the form (5)), $y_E(t) \to 0$ as $t \to \infty$.

In the sequel, the model matching problem will be treated in terms of a relative degree associated with the output $y(t)$ of P and the output $y_M(t)$ of M.

Definition 2 (Relative degree [10]) The single-input single-output nonlinear system (1), is said to have *relative degree* r at a point x° if

(1) $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and for all $k < r-1$;

(2) $L_g L_f^{r-1} h(x^\circ) \neq 0$.

In Definition 2, $L_f h(x) = \dfrac{\partial h(x)}{\partial x} f(x)$ and $L_g L_f^k h(x) = \dfrac{\partial (L_f^k h(x))}{\partial x} g(x)$. A similar definition can be given for the relative degree of model (2), r_M near x_M°. Suppose that the output $y(t)$ of P and the output $y_M(t)$ of M have a finite relative degree r and r_M, respectively. It is well-known that the model matching problem is locally solvable if, and only if [10],

$$r \leq r_M. \tag{6}$$

Now, we show the auxiliary system E equation (4) feedback by (5) in terms of P and M in a different coordinate frame. In this work, we restrict our results on output synchronization to fully linearizable plants P, i.e., for $r = n$. From definition of relative degrees r and r_M; $h(x), \dots, L_f^{n-1}h(x)$, and $h_M(x_M), \dots, L_{f_M}^{n-1}h_M(x_M)$ are sets of independent functions from P and M, and can be chosen as new coordinates $\xi_i(x) = L_f^{i-1}h(x)$ and $\xi_{Mi}(x_M) = L_{f_M}^{i-1}h_M(x_M)$ with $i = 1, \dots, n$, around x° and x_M°, respectively. Let us now consider the auxiliary system E and the new coordinates [10]

$$(\zeta(x_E),\, x_M) = \phi(x_E) = \phi(x, x_M),$$

where $\zeta(x_E) = (\zeta_1(x_E), \dots, \zeta_n(x_E))^{\mathrm{T}}$, and $\zeta_i(x_E) = L_{f_E}^{i-1}h_E(x_E) = \xi_i(x) - \xi_{Mi}(x_M)$, $i = 1, \dots, n$.

Thus, the closed-loop auxiliary system E, using the following feedback control law

$$u = \frac{1}{L_g L_f^{n-1}h(x)}\left(v - L_f^n h(x) + L_{f_M}^n h_M(x_M) + L_{g_M} L_{f_M}^{n-1} h_M(x_M) u_M\right), \tag{7}$$

takes the form

$$
\begin{aligned}
\frac{d\zeta_i}{dt} &= \zeta_{i+1}, \quad i = 1, \dots, n-1, \\
\frac{d\zeta_n}{dt} &= v = -c_0\,\zeta_1 - \;\dots\; - c_{n-1}\,\zeta_n, \\
\frac{dx_M}{dt} &= f_M(x_M) + g_M(x_M)u_M, \\
y_E &= \zeta_1.
\end{aligned} \tag{8}
$$

From (8) we see that the output $y(t)$ of the closed-loop system P differs from the output $y_M(t)$ of the model M by a signal $y_E(t)$ obeying the linear differential equation

$$
y_E^{(n)} + c_{n-1}y_E^{(n-1)} + \ldots + c_1 y_E^{(1)} + c_0 y_E = 0,
$$

where $c_0, \dots, c_{n-1}$ are constant real coefficients, thus allowing us to make the output $y(t)$ converges to $y_M(t)$. We can also identify two subsystems in the closed-loop system (8), namely:

1. The subsystem described by

$$
\frac{dx_M}{dt} = f_M(x_M) + g_M(x_M)u_M,
$$

which represents the dynamics of M, and

2. The subsystem described by

$$
\frac{d\zeta}{dt} = A^*\zeta
$$

with

$$
A^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-1} \end{pmatrix},
$$

which represents the dynamics of $y_E(t)$.

The model M is stable by assumption, and if we choose the control law $u(t)$ so that the eigenvalues of matrix A^* have real part negative, then the closed-loop system will be exponentially stable, and the output synchronization condition (3) holds.

Remark 1 Since $y_E(t) = \zeta_1(t) = \xi_1(x) - \xi_{M_1}(x_M) \to 0$ as $t \to \infty$, notice that $\xi(x)$ and $\xi_M(x_M)$ are diffeomorphisms. Then, if P and M are *identical* chaotic systems, $\xi(x) \to \xi_M(x_M)$ and, if the mappings have the same structure and tends to be equals, then the arguments too, i.e., $x(t) \to x_M(t)$. Moreover, from the control law (7) we can see that, $u(t) \to u_M(t)$, with the purpose to decouple the input $u_M(t)$ from the auxiliary system E. Thus, for identical chaotic systems, *complete synchronization* is achieved, i.e., the condition

$$
\lim_{t\to\infty} \|x(t) - x_M(t)\| = 0,
$$

holds. However, for *nonidentical* chaotic systems only *output synchronization* is guarantied, i.e., the condition (3) holds.

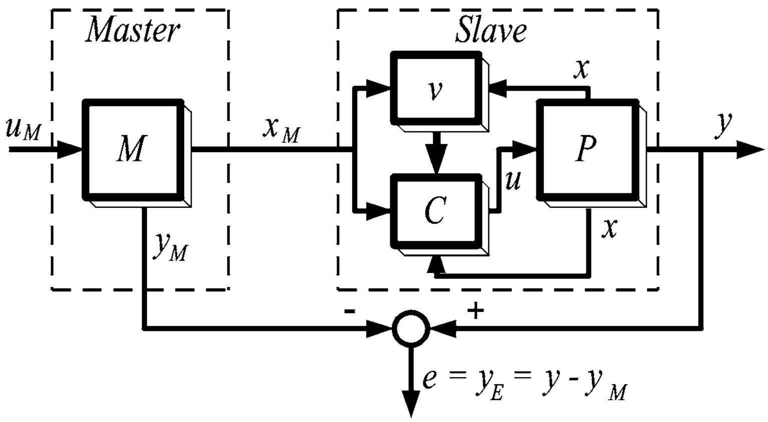

Figure 4.1. Block diagram of chaotic synchronization through model-matching approach.

4 Chaotic Synchronization through Model-Matching Approach

In this section, we use the previous material to show how synchronization of two chaotic systems can be achieved. We consider two cases of study using identical and nonidentical chaotic systems. Figure 4.1 shows the block diagram of chaotic synchronization through model-matching approach. Controller C has like input signals to $x(t)$, $x_M(t)$ and $v(t)$. It has like output signal to $u(t)$ that is the input signal of the plant P. And $e(t) = y_E(t) = y(t) - y_M(t)$ is the *output synchronization error* between the output signals of P and M.

Rössler and Lorenz systems are used to illustrate chaotic synchronization, although the proposed approach can be applied to any chaotic system that holds (6) and for all plant P with a strong relative degree.

4.1 Rössler–Rössler synchronization

Consider the Rössler system given by [21]

$$\begin{aligned} \frac{dx_1}{dt} &= -(x_2 + x_3), \\ \frac{dx_2}{dt} &= x_1 + \widehat{\alpha} x_2, \\ \frac{dx_3}{dt} &= \widehat{\alpha} + x_3(x_1 - \mu). \end{aligned} \tag{9}$$

With the parameter values $\widehat{\alpha} = 0.2$ and $\mu = 7$, the Rössler system (9) exhibits chaotic dynamics. We can write it in the form (1) by means of adding a control law $u(t)$ into some equation, we choose rewrite it as follows

$$P\colon \quad \left\{ \begin{array}{l} \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} -(x_2 + x_3) \\ x_1 + \widehat{\alpha} x_2 \\ \widehat{\alpha} + x_3(x_1 - \mu) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \\ y = x_2. \end{array} \right. \tag{10}$$

The system (10) will be considered *the plant* P. The relative degree of P is $r = 3$. Let us propose a reference model M for P, using another Rössler system writing it in the form (2) and taking the same relative degree $r_M = 3$. Notice that, if both systems have the same relative degree, $r = r_M$, that is, if (6) holds, then, there exists solution to model-matching problem, and so we can achieve synchronization between systems (10) and (11), i.e., the condition (3) is satisfied. So, we have

$$M\colon \quad \begin{cases} \begin{pmatrix} \dfrac{dx_{M_1}}{dt} \\ \dfrac{dx_{M_2}}{dt} \\ \dfrac{dx_{M_3}}{dt} \end{pmatrix} = \begin{pmatrix} -(x_{M_2} + x_{M_3}) \\ x_{M_1} + \widehat{\alpha} x_{M_2} \\ \widehat{\alpha} + x_{M_3}(x_{M_1} - \mu) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_M, \\ y_M = x_{M_2}. \end{cases} \tag{11}$$

We consider the same parameter values in P and M. To solve the model-matching problem, and with this, the original output synchronization problem, we have to take an auxiliary system (4) and thus we reduce the problem described before to disturbance decoupling problem. Then we take $u_M(t)$ like a " disturbance" signal and we seek the control law (7) for system E that is given by

$$\begin{aligned} u = -v + (\widehat{\alpha}^2 - 1)(x_1 - x_{M_1}) + \widehat{\alpha}(\widehat{\alpha}^2 - 2)(x_2 - x_{M_2}) + \widehat{\alpha}(x_3 - x_{M_3}) \\ + x_3(x_1 - \mu) - x_{M_3}(x_{M_1} - \mu) + u_M. \end{aligned} \tag{12}$$

The auxiliary system (4), after a change of coordinates $\zeta_1 = x_2 - x_{M_2}$, $\zeta_2 = x_1 - x_{M_1} + \widehat{\alpha}(x_2 - x_{M_2})$, and $\zeta_3 = \widehat{\alpha}(x_1 - x_{M_1}) + (\widehat{\alpha}^2 - 1)(x_2 - x_{M_2}) - (x_3 - x_{M_3})$, takes the form (8), with $v = -C\zeta$, or,

$$v = -c_0\,\zeta_1 - c_1\,\zeta_2 - c_2\,\zeta_3.$$

Choosing the poles in -3, we have $C = (\,27 \quad 27 \quad 9\,)$. We can consider that the synchronization between both outputs is given too when $u_M(t) = 0$, but in this case we used $u_M(t) = 0.3\,\sin(t)$. And thus we keep the model with chaotic dynamics but in the presence of a disturbance signal. Some numerical simulations were done. The initial conditions $x(0)$ and $x_M(0)$ were $(1,1,1)$ and $(2,-2,2)$, respectively. Figure 4.2 shows the output of the plant, $y(t) = x_2(t)$ following the output of the model $y_M(t) = x_{M_2}(t)$ (top of figure), the error signal $e(t) = y_E(t) = y(t) - y_M(t)$ (middle of figure), and the typical phase plot confirming synchronization between the outputs $y(t)$ and $y_M(t)$ (bottom of figure). The control law $u(t)$ takes action after 20 seconds.

Here, we obtain complete synchronization, i.e., all states of P and M synchronize, because we considered identical chaotic systems.

4.2 Lorenz–Rössler synchronization

Now consider the coupling between two nonidentical chaotic systems as plant and model; for example, a Lorenz system [14] like a model with relative degree $r_M = 3$ (for all x_M

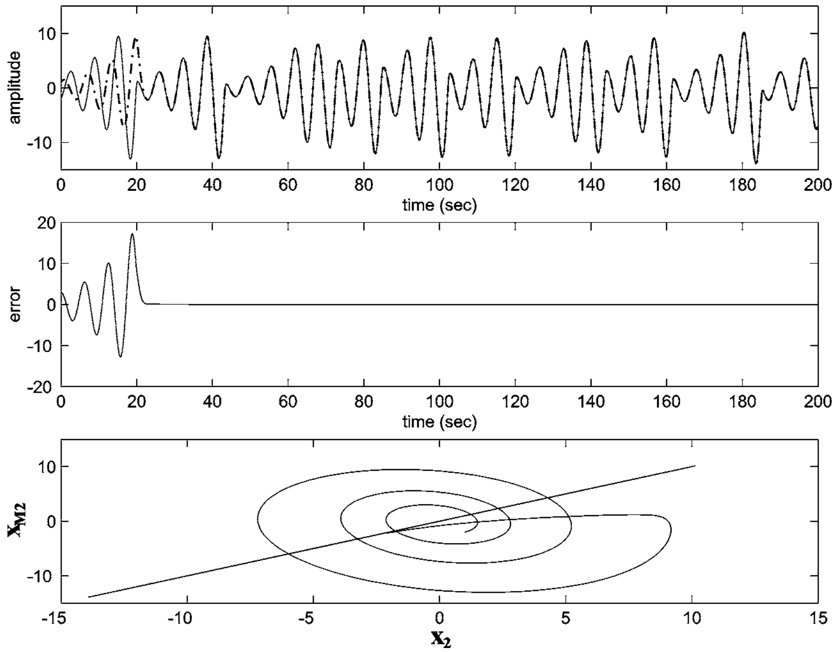

Figure 4.2. Rössler–Rössler synchronization. Solid line $y_M = x_{M_2}$, dashed line $y = x_2$ (top of figure). Error signal $e = y_E = y - y_M$ (middle of figure). Output synchronization between x_{M_2} and x_2 (bottom of figure). Control u takes action when $t = 20$ sec.

such that $x_{M_1} \neq 0$) as follows:

$$M\colon \quad \begin{cases} \begin{pmatrix} \dfrac{dx_{M_1}}{dt} \\ \dfrac{dx_{M_2}}{dt} \\ \dfrac{dx_{M_3}}{dt} \end{pmatrix} = \begin{pmatrix} \sigma(x_{M_2} - x_{M_1}) \\ \hat{r}x_{M_1} - x_{M_2} - x_{M_1}x_{M_3} \\ x_{M_1}x_{M_2} - bx_{M_3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_M, \\ y_M = x_{M_1}. \end{cases} \tag{13}$$

Let us to consider again the same plant described by equation (10). Thus, the control law $u(t)$ for output synchronization between (10) and (14) is given by

$$\begin{aligned} u = -\{v &- [(\widehat{\alpha}^2 - 1)x_1 - \widehat{\alpha}(\widehat{\alpha}^2 - 2)x_2 - -\widehat{\alpha}x_3 - \widehat{\alpha} - x_3(x_1 - \mu)] \\ &+ \sigma[\sigma(\sigma + \hat{r} - x_{M_3})(x_{M_2} - x_{M_1}) - (\sigma + 1)(\hat{r}x_{M_1} - x_{M_2} - x_{M_1}x_{M_3}) \\ &- x_{M_1}(x_{M_1}x_{M_2} - bx_{M_3})] - \sigma x_{M_1}u_M\}. \end{aligned} \tag{14}$$

The results are illustrated by means of some numerical simulations. The initial conditions for plant and model are $x(0) = (3, 1, 1)$ and $x_M(0) = (1, 1.5, 0.1)$, respectively. The parameter values are $\sigma = 10$, $\hat{r} = 28$, $b = 8/3$, $\widehat{\alpha} = 0.2$ and $\mu = 7$.

Figure 4.3 shows how the output of the plant $y(t) = x_2(t)$ follows $y_M(t) = x_{M_1}(t)$ for Lorenz–Rössler output synchronization: a) output of Lorenz/model $y_M(t) = x_{M_1}(t)$,

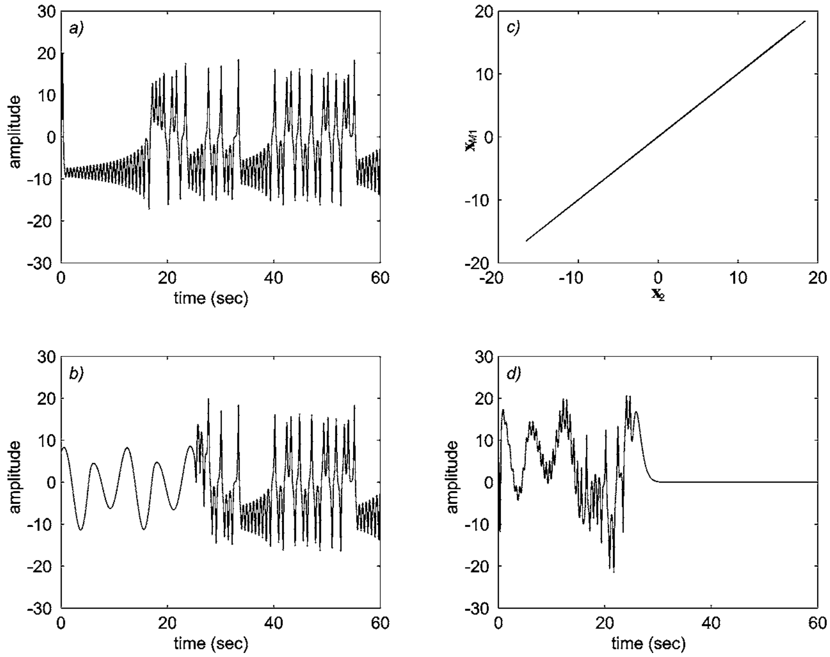

Figure 4.3. Lorenz–Rössler output synchronization: a) $y_M = x_{M_1}$, b) $y = x_2$ following $y_M = x_{M_1}$ after 25 seconds when control law takes action, c) x_2 versus x_{M_1}, and d) error signal $e = y_E = y - y_M$.

b) output of Rössler/plant $y(t) = x_2(t)$ following the output $y_M(t) = x_{M_1}(t)$ of Lorenz/model, c) $x_2(t)$ versus $x_{M_1}(t)$, and d) error signal $e(t) = y_E(t) = y(t) - y_M(t)$.

Remark 2 In this case, unlike the previous one, synchronization between the outputs of both systems was only obtained. No other state of the plant synchronizes with those of the model.

5 Private/Secure Communication Systems

This section does not pretend to propose secure chaos-based communication systems. It tries to illustrate the flexibility of the model-matching approach for chaotic communication. Nevertheless, certain properties of security are found.

5.1 Chaotic communication using two channels

In order to illustrate the proposed approach to transmit private information signals, a chaotic communication scheme using two transmission channels is now designed. It is based on the output synchronization between identical and nonidentical chaotic systems. To this purpose, consider that $u(t)$ equation (5) can be separated in the following form

$$\begin{aligned} u &= \alpha(x, x_M) + \beta(x, x_M)v + \gamma(x, x_M)u_M \\ &= \gamma_2(x)\{[\alpha_1(x_M) + \beta_1(x_M)v_1(x_M) + \gamma_1(x_M)u_M] + [\alpha_2(x) + \beta_2(x)v_2(x)]\} \\ &= \gamma_2(x)[u_1(x_M) + u_2(x)], \end{aligned}$$

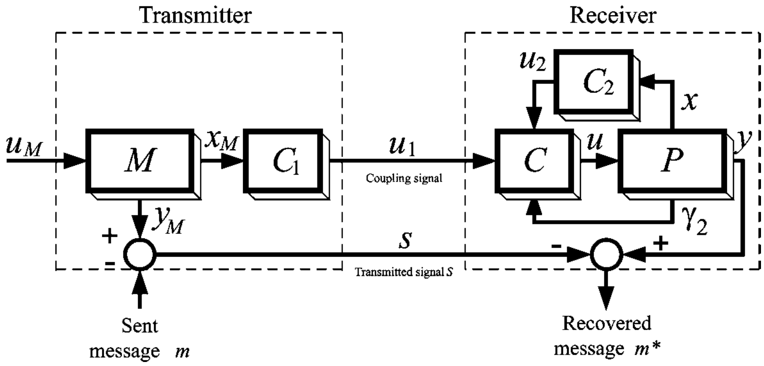

Figure 5.1. Analog communication system using two transmission channels.

with

$$\begin{aligned}
u_1(x_M) &= v_1(x_M) + L_{f_M}^n h_M(x_M) + L_{g_M} L_{f_M}^{n-1} h_M(x_M) u_M, \\
u_2(x) &= v_2(x) - L_f^n h(x), \\
\gamma_2(x) &= \frac{1}{L_g L_f^{n-1} h(x)}, \\
v_1(x_M) &= c_0 \xi_{M_1}(x_M) + \cdots + c_{n-1} \xi_{Mn}(x_M), \\
v_2(x) &= -c_0 \xi_1(x) - \cdots - c_{n-1} \xi_n(x),
\end{aligned}$$

like we can see from (7).

This let us to propose the following coupling scheme shown in Figure 5.1, in which $u_1(x_M, u_M)$ is the output from a new control block C_1, $u_2(x)$ is the output of C_2 and $u(x, x_M, u_M)$ or, simply $u(t)$ is the output of controller C. This scheme has two transmission channels, one channel is used to send $u_1(x_M, u_M)$ for output synchronization only, with no connection to the private message. The other channel is used to transmit the hidden message $m(t)$ through $s(t) = y_M(t) - m(t)$. This message is recovered by comparison between the output $y(t)$ and the signal $s(t)$ at the receiver end, i.e., $m^*(t) = y(t) - s(t)$. Some numerical simulations that illustrate the transmission of private message using this scheme were done.

Figure 5.2 shows an audio signal like the private message (top of figure), the transmitted chaotic signal including the hidden message (middle of figure), and the recovered message using Rössler–Rössler output synchronization (bottom of figure). The transmission of the message is through the output of the model $x_{M_2}(t)$.

A remarkable feature is that, in the proposed scheme, the signal that is sent in order to obtain output synchronization is a nonlinear function of the state $x_M(t)$, but is not the own state. So, with this scheme we obtain high privacy because it is possible to hide a message through the coupling signal $u_s(t) = u_1(t) + m(t)$, and with this to increase the security of cryptography, because now $s(t) = y_M(t)$ does not contain any message. So, a third person cannot recover the hidden message with the reported methods in [22, 23]. Figure 5.3 shows a binary signal obtained from a picture like the private message (top of figure), the transmitted chaotic coupling signal including the hidden binary message (middle of figure), and the recovered message at the receiver end (bottom of figure), using Lorenz–Rössler output synchronization.

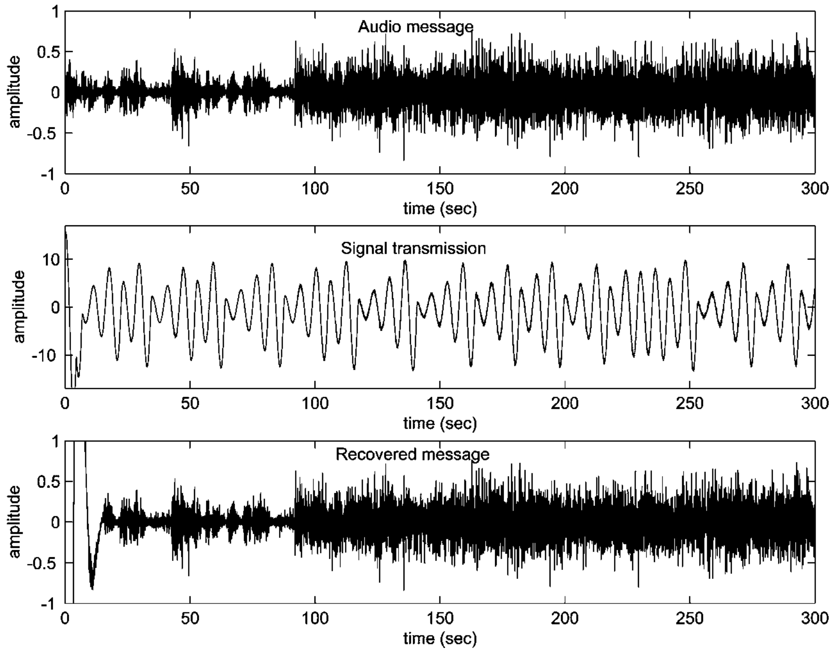

Figure 5.2. Transmission and recovering of an audio message using Rössler–Rössler output synchronization.

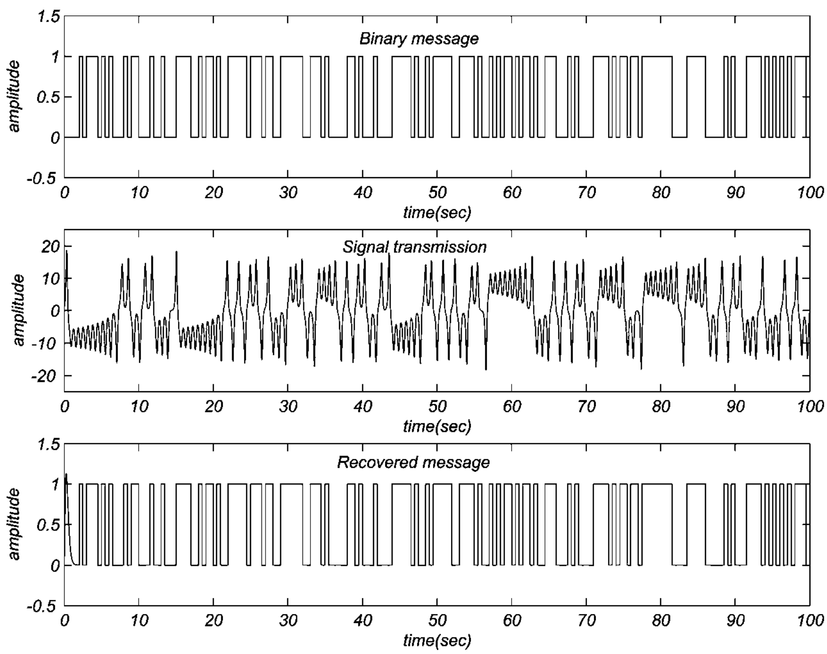

Figure 5.3. Transmission and recovering of a binary message obtained from a digital image using Lorenz–Rössler output synchronization.

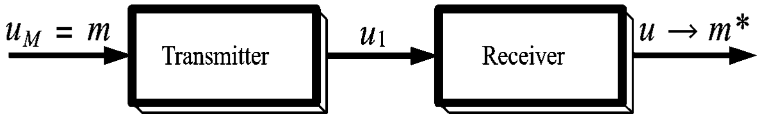

Figure 5.4. Analog communication system using a single transmission channel.

5.2 Chaotic communication using a single channel

Another scheme of transmission that can be used in the case of the synchronization of only identical chaotic systems by model-matching approach is using a single transmission channel to obtain synchronization and to transmit private information signals. This scheme is shown in Figure 5.4. The message $m(t)$ is injected into the transmitter through the input signal $u_M(t)$. The output signal of the transmitter is a nonlinear function $u_1(x_M, u_M)$ whereas it is possible to take like output of receiver to $u(t)$, which, when synchronization is achieved between the outputs of P and M, then $u(t) \to u_M(t) = m(t)$, and thus we obtain the recovered message $m^*(t)$. This scheme is only useful for identical systems because in this case all states of P synchronizes with those of M and $u(t)$ has not to compensate any asynchronous states, so that $u(t) \to u_M(t)$. Figure 5.5 samples numerically the transmission through a single transmission channel using Rössler–Rössler synchronization, in which, control $u(t)$ takes action after 20 seconds and the private message is sent after 40 seconds, when complete synchronization has been achieved.

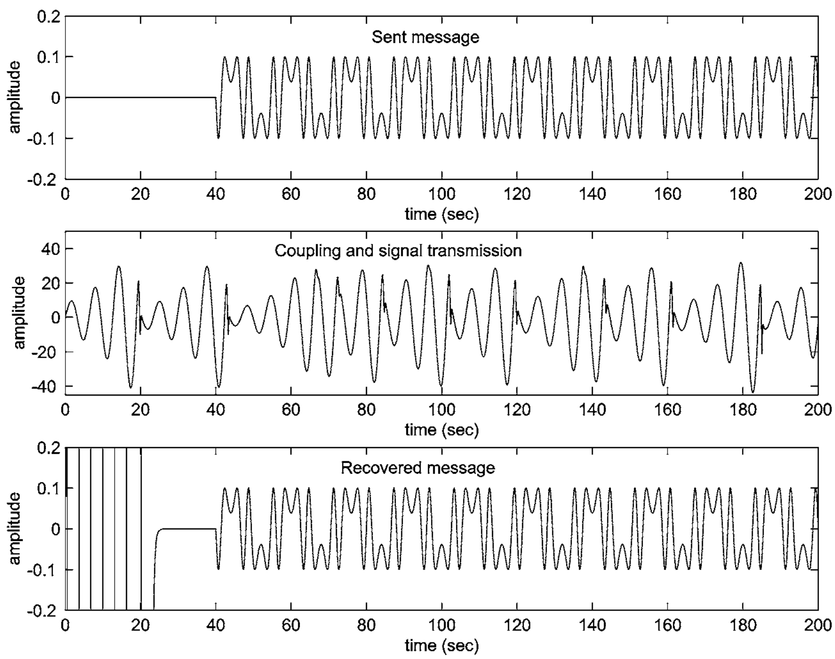

Figure 5.5. Transmission of private information using a single channel.

Since, this scheme does not send any single chaotic signal, but it sends the nonlinear

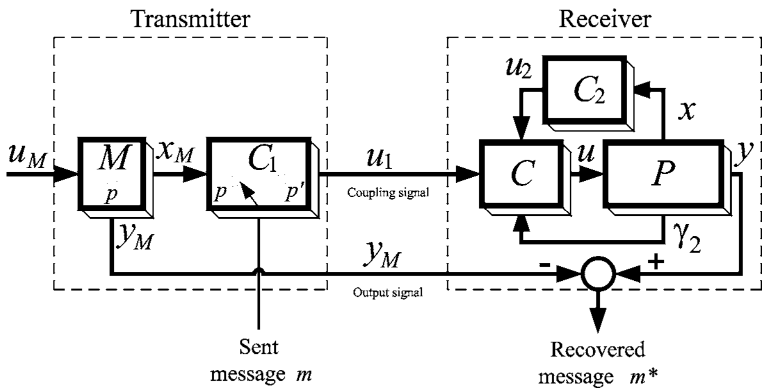

Figure 5.6. Digital communication system of private information by chaotic switching.

function $u_1(t)$, any chaotic attractor can be reconstructed in order to extract the hidden message by means of the reported existing methods in [22, 23].

5.3 Chaotic switching

In the following scheme that is shown in Figure 5.6, we have proposed p like the parameters of P. The same way, p and p' have been proposed like the parameters for controller C_1. During both P and C_1 are on p then there exists synchronization or, at least, output synchronization and during C_1 is on p' there exists an error different from zero. This scheme commonly is known like chaotic switching or chaos shift keying.

Figure 5.7 shows how varying a parameter in the model (transmitter) it is possible to send binary information and to recover it in the plant (receiver) using nonidentical systems: Lorenz–Rössler output synchronization. To make this possible consider that $e_2(t) \to 0$ when $m = 0$ and $e_2(t) \nrightarrow 0$ when $m = 1$, interpreting $e(t) = 0$ like "0" logical and $e(t) \neq 0$ like "1" logical. In this example, the parameter $\hat{r}$ of Lorenz system (13) is switching in C_1 between two values: $p = \hat{r} = 28$ when $m = 0$ and $p' = \hat{r}' = 29$ when $m = 1$ in accordance with $p^* = \hat{r} + m$, with $p^* = (p, p')$. The message is recovered faithfully after a brief iterative signal processing.

Since, this scheme does not switch between two chaotic attractors of identical systems, but it switches a controller parameter, it is a secure cryptography system, where the hidden message through the coupling signal cannot be reconstructed by means of the reported existing methods in literature (see e.g. [18, 23]).

6 Concluding Remarks

In this work we have presented a systematic method to synchronize chaotic systems in continuous-time. In particular, we used the model-matching problem from the nonlinear control theory (see [1] for the discrete-time context). We have obtained *complete synchronization* of Rössler/plant and Rössler/model, and *output synchronization* of Rössler/plant and Lorenz/model. In addition, we have proposed some communication schemes based on complete and output synchronization: using two transmission channels, using a single transmission channel, and using chaotic switching. The advantages over

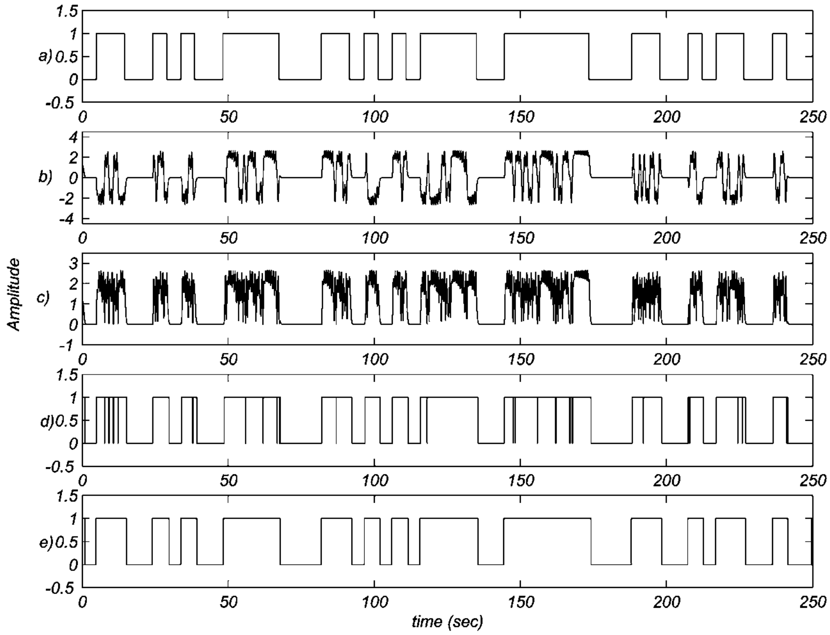

Figure 5.7. Transmission of a binary signal by chaotic switching using Lorenz–Rössler output synchronization: a) original message, b) recovered message at the receiver by output synchronization error detection, c) absolute magnitude of the error signal, d) rounding and iterative signal processing, and e) recovered binary message.

the other cited approaches to synchronize nonidentical chaotic systems are the following: This approach is systematic, it uses unidirectionally coupled systems, gains for controller are small and synchronization is obtained after a short transient behavior. Moreover, this methodology is useful to transmit private information through only one transmission channel (only for identical systems). In addition, this transmission scheme is secure because the coupling signal, including the private message, is a nonlinear function of the state, which is not useful to recover any chaotic attractor and thus it is a difficult if not impossible task that some third person can recover the private message.

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Stability of Dynamical Systems in Metric Space

A.A Martynyuk*

Stability of Processes Department, S.P.Timoshnko Institute of Mechanics,
National Academy of Sciences of Ukraine,
03057, Nesterov str., 3, 03057, Kiev-57, Ukraine

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Abstract: In the paper a new approach is developed for stability analysis of motions of dynamical systems defined on metric space using matrix-valued preserving mappings. These results are applicable to a much larger class of systems then existing results, including dynamical systems that cannot be determined by the usual classical equations and inequalities. We apply our results in the stability analysis of hybrid systems in general and two-component hybrid systems.

Keywords: *Dynamical system; metric space; hybrid system; asymptotic stability; stability matrix-valued preserving mapping.*

Mathematics Subject Classification (2000): 34G20, 35B35, 37K45, 37L15, 93A15, 93D30.

1 Introduction

This paper presents an approach to stability analysis of dynamical systems determined in metric space. The method of analysis of invariant sets of dynamical systems was proposed by Zubov [11] on the basis of generalized direct Liapunov method. In our approach a generalized comparison principle is used together with the idea of multicomponent mapping (cf. matrix-valued Liapunov functions [5, 6]).

In the present paper, we first developed a matrix-valued preserving mapping for stability analysis of general dynamical systems defined on metric space. To accomplish this, we utilize, as in our earlier work (see [7]), stability preserving matrix-valued mappings. We use the above results to establish the principal Lyapunov theorems for dynamical systems on metric space. Finally, we analyze a class of hybrid systems, using some of these results with particular application to two-component hybrid system.

*Corresponding author: anmart@stability.kiev.ua

2 Basic Concepts and Definitions

Let X be a set of elements (no matter of what nature) and a measure $\rho(x,y)$ be defined for $x,\, y \in X$. The Definitions 2.1−2.6 presented here follow in the spirit of the works [1, 2, 9, 11] even if some of the formulations are different.

Definition 2.1 (X, ρ) is a metric space if the following conditions are fulfilled for any $x, y, z \in X$:

(1) $\rho(x,y) \geq 0$,

(2) $\rho(x,y) = 0 \Leftrightarrow x = y$,

(3) $\rho(x,y) = \rho(y,x)$,

(4) $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$,

and, additionally, for any $X_0 \subseteq X$, $\rho(x, X_0) = \inf\limits_{y \in X_0} \rho(x,y)$.

Definition 2.2 A metric space (T, ρ) is called a temporal space if:

(1) T is completely ordered by the ordering "$<$";

(2) T has a minimum element $t_{\min} \in T$, i.e. $t_{\min} < t$ for any $t \in T$, such that $t \neq t_{\min}$;

(3) for any $t_1, t_2, t_3 \in T$ such that $t_1 < t_2 < t_3$ it holds that

$$\rho(t_1, t_3) = \rho(t_1, t_2) + \rho(t_2, t_3);$$

(4) T is unbounded from above; i.e., for any $M > 0$, there exists $t \in T$ such that $\rho(t, t_{\min}) > M$.

Definition 2.3 Let (X, ρ) be a metric space with a subset $A \subseteq X$ and let (T, ρ) be a temporal space with subset $T \subseteq R_+$. A mapping $p(\cdot, a, \tau_0)\colon T_{a,\tau_0} \to X$ is called *a motion* if $p(\tau_0, a, \tau_0) = a$, where $a \in A$, $\tau_0 \in T$ and $T_{a,\tau_0} = [\tau_0, \tau_1) \cap T$ for $\tau_1 > \tau_0$, with τ_1 being a finite value or infinity.

Definition 2.4 Let $T_{a,\tau_0} \times \{a\} \times \{\tau_0\} \to X$ denote the set of mappings of $T_{a,\tau_0} \times \{a\} \times \{\tau_0\}$ into X, $\Lambda = \bigcup_{(a,\tau_0) \in A \times T} (T_{a,\tau_0} \times \{a\} \times \{\tau_0\} \to X)$ and S be a family of motions; i.e.,

$$S \subseteq \{p(\,\cdot\,, a, \tau_0) \in \Lambda\colon\ p(\tau_0, a, \tau_0) = a\}.$$

Then the four-tuple (T, X, A, S) of sets and spaces is called *a dynamical system*.

Note that Definition 2.4 possesses some generality. Specifically,

(i) if X is a normed linear space and every motion $p(\tau, a, \tau_0)$ is assumed to be continuous with respect to τ, a and τ_0, then Definition 2.4 corresponds to the concept of a family of motions in Hahn [3];

(ii) under some additional conditions imposed on $p(\tau, a, \tau_0)$ (see [11], pp.183–184), Definition 2.4 reduces to the concept of a general system introduced by Zubov.

In what follows, we consider dynamical systems satisfying the standard semigroup property

$$p(\tau_2,\, p(\tau_1, a, \tau_0)) = p(\tau_2 + \tau_1,\, a, \tau_0)$$

for all $a \in A$ and any $\tau_1, \tau_2 \in R_+$.

Definition 2.5 A dynamical system (R_+, X, A, S) is called *continuous* if any of its motions $p \in S$ is continuous; i.e., any mapping $p(\,\cdot\,, a, \tau_0)\colon\ T_{a,\tau_0} \to X$ is continuous.

Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces, and let (R_+, X_1, A_1, S_1) be a continuous dynamical system. We assume that the space X_1 is a Descartes product of spaces $X_{11}, X_{12}, \dots, X_{1m}$, on which the multicomponent mapping (see [7])

$$U(t,x)\colon\ T \times X_{11} \times X_{12} \times \ldots \times X_{1m} \to X_2 \tag{1}$$

is acting.

It is assumed that the mapping $U\colon\ R_+ \times X_{11} \times X_{12} \times \ldots \times X_{1m} \to X_2$ has the following properties: for any motion $p(\,\cdot\,, a, t_0) \in S_1$, the function $q(\,\cdot\,, b, t_0) = U(\,\cdot,\ p(\,\cdot, a, t_0),\,)$ with initial value $b = U(t_0, a)$ is another motion for which $T_{a,t_0} = T_{b,t_0}$, $b \in A_2 \subset X_2$.

Let S_2 denote the set of motions q determined by initial values $a \in A_1$ and $t_0 \in R_+$. Then (R_+, X_2, A_2, S_2) is a continuous dynamical system.

The mapping given by (1) induces a mapping of S_1 into S_2, denoted by $\mathfrak{M}$; i.e., $S_2 = \mathfrak{M}(S_1)$. Moreover, we denote by $M_1 \subset A_1$ and $M_2 \subset A_2$ some sets invariant under S_1 and S_2, respectively. The set M_2 is then defined by the formula

$$\begin{gathered} M_2 = U(R_+ \times M_1) = \{x_2 \in X_2\colon\ x_2 = U(t', x_1) \\ \text{for some } x_1 \in M_1 \text{ and } t' \in R_+,\}. \end{gathered} \tag{2}$$

In what follows, we consider continuous dynamical systems (R_+, X_1, A_1, S_1) and (R_+, X_2, A_2, S_2) with invariant sets $M_1 \subset A_1$ and $M_2 \subset A_2$, respectively.

Definition 2.6 Multicomponent mapping (1)

$$U\colon\ R_+ \times X_{11} \times X_{12} \times, \dots, \times X_{1m} \to X_2 \tag{3}$$

preserves some type of stability of a continuous dynamical system if the sets

$$\begin{gathered} S_2 = \mathfrak{M}(S_1) \triangleq \{q(\,\cdot\,, b, t_0)\colon\ q(t, b, t_0) = U(t,\, p(t, a, t_0)), \\ p(\,\cdot\,, a, t_0) \in S_1, \quad \eta \in R^m, \quad b = U(t_0, a), \\ T_{b,t_0} = T_{a,t_0}, \quad a \in A_1, \quad t_0 \in R_+\} \end{gathered} \tag{4}$$

and M_2 (see formula (2)) satisfy the following conditions:

(1) the invariance of (S_1, M_1) is equivalent to the invariance of (S_2, M_2);
(2) some type of stability of (S_1, M_1) is equivalent to the same type of stability of (S_2, M_2).

3 Sufficient Conditions for Stability of Dynamical System

Note that the mapping U induces a mapping $\mathfrak{M}\colon\ S_1 \to S_2$, that preserves some types of stability of (S_1, M_1) and $(S_2,\, U(R_+ \times M_1))$.

Theorem 3.1 *Let a dynamical system* (R_+, X_1, A_1, S_1) *be assigned a comparison system* (R_+, X_2, A_2, S_2) *by means of a multicomponent mapping* $U(t,p)\colon R_+ \times X_1 \to X_2$*. Suppose that there exist closed sets* $M_i \subset A_i$*,* $i = 1,2$*, and following conditions are fulfilled:*

(1) *for* $\mathfrak{M}(S_1)$ *and* S_2*,* $\mathfrak{M}(S_1) = S_2$*;*

(2) *there exist constant* $m \times m$ *matrix* A_i*,* $i = 1,2$*, and comparison functions* $\psi_1, \psi_2 \in K$ *such that*

$$\psi_1^{\mathrm{T}} A_1 \psi_1 \le \rho_2(U(t,p),\, M_2) \le \psi_2^{\mathrm{T}} A_2 \psi_2 \tag{5}$$

for all $p \in X_1$ *and* $t \in R_+$*, where*

$$\psi_1 = (\psi_{11}(\rho_1(p, M_1)),\, \dots,\, \psi_{1m}(\rho_1(p, M_1)))^{\mathrm{T}},$$
$$\psi_2 = (\psi_{21}(\rho_1(p, M_1)),\, \dots,\, \psi_{2m}(\rho_1(p, M_1)))^{\mathrm{T}}.$$

Here, ρ_1 *and* ρ_2 *are metrics defined on* X_1 *and* X_2*, respectively.*

If the matrices A_i*,* $i = 1,2$*, are positive definite, then the following is true:*

(1) *the invariance of* (S_2, M_2) *implies the invariance of* (S_1, M_1)*;*

(2) *the stability, uniform stability, asymptotic stability, or uniform asymptotic stability of* (S_2, M_2) *implies the respective type of stability of* (S_1, M_1)*;*

(3) *if in estimate (5)* $\psi_1^{\mathrm{T}} A_1 \psi_1 = a(\rho_1(p, M_1))^b$*, where* $a > 0$ *and* $b > 0$*, then the exponential stability of* (S_2, M_2) *implies the exponential stability of* (S_1, M_1)*.*

Proof of item (1) Let (S_2, M_2) be an invariant pair. Then, for any $a \in M_1$ and any motion $p(\,\cdot\,; a, t_0) \in S_1$, we find that $q(\,\cdot\,; b, t_0) = U(t,\, p(\,\cdot\,; a, t_0)) \in S_2$, where $b = U(t_0, a)$. This follows from condition (1) in Theorem 3.1 and from the definition of $\mathfrak{M}(S_1)$ by formula (4). Moreover, the invariance of (S_2, M_2) implies that $q(t; b, t_0) = U(t,\, p(t; a, t_0)) \in M_2$ for all $t \in T_{b,t_0} = T_{a,t_0}$. Since M_1 and M_2 are closed and the matrices A_1 and A_2 are positive definite and satisfy (5), we conclude that $p(t; a, t_0) \in M_1$ for all $t \in T_{a,t_0}$. This implies the invariance of (S_1, M_1).

Proof of item (2) Assume that (S_2, M_2) is stable. Then, by the definition of stability, for any $\varepsilon_2 > 0$ and $t_0 \in R_+$, there exists $\delta_2 = \delta_2(t_0, \varepsilon_2) > 0$ such that $\rho_2(q(t; b, t_0),\, M_2) < \varepsilon_2$ for all $q(\,\cdot\,; b, t_0) \in S_2$ and all $t \in T_{b,t_0}$ whenever $\rho_2(b, M_2) < \delta_2(t_0, \varepsilon_2)$. Estimates (5) can be transformed into

$$\lambda_m(A_1)\widetilde{\psi}_1(\rho_1(p, M_1)) \le \rho_2(U(t,p),\, M_2) \le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(p, M_1)). \tag{6}$$

Here $\lambda_m(A_1) > 0$ and $\lambda_M(A_2) > 0$ are the minimum and maximum eigenvalues of the positive definite matrices A_1 and A_2, and $\widetilde{\psi}_1, \widetilde{\psi}_2 \in K$ are such that

$$\psi_1^{\mathrm{T}}(\rho_1(p, M_1))\psi_1(\rho_1(p, M_1)) \ge \widetilde{\psi}_1(\rho_1(p, M_1))$$

and

$$\psi_2^{\mathrm{T}}(\rho_1(p, M_1))\psi_2(\rho_1(p, M_1)) \ge \widetilde{\psi}_2(\rho_1(p, M_1)).$$

Since (S_2, M_2) is stable, for any $\varepsilon > 0$ and any $t_0 \in R_+$, we choose $\varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon)$ and $\delta_1 = \lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(\delta_2)$. Assuming that $\rho_1(a, M_1) < \delta_1$ and taking into account (6), we obtain

$$\begin{aligned}\rho_2(b, M_2) &\le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\delta_1)\\ &= \lambda_M(A_2)\widetilde{\psi}_2(\lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(\delta_2)) = \delta_2.\end{aligned}$$

It follows that, for all motions $q(\,\cdot\,; b, t_0) \in S_2$, the estimate $\rho_2(q(t; b, t_0), M_2) < \varepsilon_2$ holds for all $t \in T_{b,t_0}$. Returning to estimates (6), we find that, for all $p(\,\cdot\,; a, t_0) \in S_1$ and all $t \in T_{a,t_0} = T_{b,t_0}$, where $b = U(t_0, a)$, we have

$$\begin{aligned}\rho_1(p(t; a, t_0), M_1) &\le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(V(p(t; a, t_0)), M_2))\\ &\le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(q(t; b, t_0), M_2)) \le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\lambda_m(A_1)\widetilde{\psi}_1(\varepsilon)) = \varepsilon,\end{aligned}$$

whenever $\rho_1(a, M_1) < \delta_1$. It follows that (S_1, M_1) is stable.

It is well known that a system motion is asymptotically stable if it is stable and attracting. Assume that (S_2, M_2) is attracting. Then, for any $t_0 \in R_+$ there exists $\Delta_2 = \Delta_2(t_0) > 0$ such that, for all $q(\,\cdot\,; b, t_0) \in S_2$, the limit relation

$$\lim_{t\to\infty} \rho_2(q(t; b, t_0), M_2) = 0,$$

holds true whenever $\rho_2(b, M_2) < \Delta_2$. In other words, for any $\varepsilon_2 > 0$, there exists $\tau = \tau(\varepsilon_2, t_0, q) > 0$ with $q = q(\,\cdot\,; b, t_0) \in S_2$ such that $\rho_2(q(t; b, t_0), M_2) < \varepsilon_2$ for all $t \in T_{b,t_0+\tau}$, whenever $\rho_2(b, M_2) < \Delta_2$. According to condition (1) in Theorem 3.1, for any motion $p(\,\cdot\,; a, t_0) \in S_1$, we set $b = U(t_0, a)$. Then $q(\,\cdot\,; b, t_0) = U(p(\,\cdot\,; a, t_0)) \in S_2$. Furthermore, for any $\varepsilon_1 > 0$, we choose $\varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon_1)$ and set $\Delta_1 = \lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(A_2)$. For any motion $p(\,\cdot\,; a, t_0) \in S_1$, we then have

$$\rho_2(b, M_2) \le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\Delta_1) = \Delta_2$$

whenever $\rho_1(a, M_1) < \Delta_1$ and $t \in T_{a,t_0+\tau} = T_{b,t_0+\tau}$. Hence, $\rho_2(q(t; a, t_0), M_2) < \varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon_1)$ for all $t \in T_{a,t_0+\tau}$. Returning to estimate (2), we find that

$$\rho_1(p(t; a, t_0), M_1) \le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(q(t; a, t_0), M_2)) < \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\varepsilon_1),$$

i.e., (S_1, M_1) is an attractive pair. Thus, if (S_2, M_2) is asymptotically stable, then (S_1, M_1) is asymptotically stable as well.

The statements on uniform stability and uniform asymptotic stability are proved following the same scheme, but δ_2 and Δ_2 are chosen to be independent of $t_0 \in R_+$.

Let us prove statement (3) of the theorem. Assume that (S_2, M_2) is exponentially stable. Then there exists $\alpha_2 > 0$ and, for any $\varepsilon_2 > 0$, there exists $\delta_2 = \delta_2(\varepsilon_2) > 0$ such that for any motion $q(\,\cdot\,; b, t_0) \in S_2$ and all $t \in T_{b,t_0}$

$$\rho_2(q(t; b, t_0), M_2) < \varepsilon_2 e^{-\alpha_2(t-t_0)}$$

whenever $\rho_2(b, M_2) < \delta_2$. According to condition (1) in Theorem 3.1, for any motion $p(\,\cdot\,; a, t_0) \in S_1$, there exists a motion $q(\,\cdot\,; b, t_0) = U(p(\,\cdot\,; a, t_0)) \in S_2$, where $b =$

$U(t_0, a)$. Furthermore, for any $\varepsilon_1 > 0$, we choose $\varepsilon_2 = a\varepsilon_1^b$. Let $\alpha_1 = \alpha_2/b$ and $\delta_1 = \lambda_M^{-1}(A_2)\psi_2^{-1}(\delta_2)$. For $p(t; a, t_0) \in M_1$ with $\rho_1(a, M_1) < \delta_1$, in view of (6), we obtain

$$\rho_2(b, M_2) \leq \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\delta_1) = \delta_2.$$

Consequently,

$$\rho_2(q(t; b, t_0), M_2) < \varepsilon_2 e^{-\alpha_2(t-t_0)}$$

for all $t \in T_{b,t_0}$.

According to the hypothesis of Theorem 3.1, we have to set

$$\psi_1^{\mathrm{T}} A_1 \psi_1 = a(\rho_1(p, M_1))^b$$

in (1.6.6). It is easy to see that

$$\rho_1(p(t; a, t_0), M_1) < \left(\frac{\varepsilon_2}{a}\right)^{1/b} e^{-\frac{\alpha_2}{b}(t-t_0)} = \varepsilon_1 e^{-\alpha_1(t-t_0)}$$

for all $t \in T_{a,t_0}$. Thus, (S_1, M_1) is exponentially stable.

4 Stability Analysis of Hybrid System

Many physical and technical problems of real world are modelled by mixed systems of equations and correlations. For example, in motion control theory the feedback consists of several interconnected blocks. These blocks are described by equations of different types. Such systems are called hybrid (see [9]). Under certain assumptions real hybrid system σ can correspond to the dynamical system (T, X, A, S) in metric space.

Assume that (X, ρ) and (X_i, ρ_i), $i = 1, 2, \dots, m$, are metric spaces. Let $X = X_1 \times X_2 \times \ldots \times X_m$ and there exist constants $a_1, a_2 > 0$ such that

$$a_1\rho(x, y) \leq \sum_{i=1}^{m} \rho_i(x_i, y_i) \leq a_2\rho(x, y) \tag{7}$$

for all $x, y \in X$, where $x = (x_1, \dots, x_m)^{\mathrm{T}}$, $y = (y_1, \dots, y_m)^{\mathrm{T}}$, $x_i \in X_i$, $y_i \in X_i$, $i = 1, 2, \dots, m$. Further on we will assume that

$$\rho(x, y) = \sum_{i=1}^{m} \rho_i(x_i, y_i). \tag{8}$$

Definition 4.1 (cf. [9]) Dynamical system (T, X, A, S) is *hybrid*, if its metric space (X, ρ) consists of metric spaces (X_i, ρ_i), $i = 1, 2, \dots, m$, where X_i are nontrivial unsplitted with metrics $\rho_i(x_i, y_i)$, and if there exist at least two metric spaces X_i and X_j, $1 \leq i \neq j \leq m$, which are not isometric.

The proposition below is necessary when the multicomponent mapping is made by matrix-valued functional.

Proposition 4.1 *Let multicomponent mapping $U(t,x)\colon T\times X\to X_2$ be performed by matrix-valued functional $U(t,x)=[v_{ij}(t,x)]$, $i,j=1,2,\dots,m$, for the elements of which:*

(a) *$v_{ii}\in C(R_+\times X,\,R_+)$, $i=1,2,\dots,m$, $v_{ij}\in C(R_+\times X,\,R)$ for all $i\neq j$ and for all $x\in X$ and $t\in R_+$;*

(b) *there exist comparison functions $\varphi_{i1},\varphi_{i2}$ of class K, positive constants $\underline{c}_{ii}>0$, $\overline{c}_{ii}>0$ and arbitrary constants $\underline{c}_{ij}\in R$, $\overline{c}_{ij}\in R$ for $i\neq j$ such that*

$$
\begin{gathered}
\underline{c}_{ii}\varphi_{i1}^2(\rho_i(x_i,M_i))\le v_{ii}(t,x)\le \overline{c}_{ii}\varphi_{i2}^2(\rho_i(x_i,M_i)),\\
\underline{c}_{ij}\varphi_{i1}(\rho_i(x_i,M_i))\varphi_{j1}(\rho_j(x_j,M_j))\le v_{ij}(t,x)\\
\le \overline{c}_{ij}\varphi_{i2}(\rho_i(x_i,M_i))\varphi_{j2}(\rho_j(x_j,M_j))
\end{gathered}
\tag{9}
$$

for all $x_i\in X_i$, $x\in X$ and $t\in R_+$.

Then for the functional

$$
v(t,x,\eta)=\eta^{\mathrm T}U(t,x)\eta,\quad \eta\in R_+^m,\quad \eta_i>0,
$$

the bilateral inequality

$$
\begin{gathered}
u_1^{\mathrm T}(\rho(x,M))H^{\mathrm T}\underline{C}Hu_1(\rho(x,M))\le v(t,x,\eta)\\
\le u_2^{\mathrm T}(\rho(x,M))H^{\mathrm T}\underline{C}Hu_2(\rho(x,M))
\end{gathered}
\tag{10}
$$

holds for all $x\in X$ and $t\in R_+$, where

$$
\begin{gathered}
H=\operatorname{diag}(\eta_1,\eta_2,\dots,\eta_m),\\
\underline{C}=[\underline{c}_{ij}],\quad \overline{C}=[\overline{c}_{ij}],\quad i,j=1,2,\dots,m,\\
u_1(\cdot)=(\varphi_{i1}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m1}(\rho_m(x_m,M_m)))^{\mathrm T},\\
u_2(\cdot)=(\varphi_{i2}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m2}(\rho_m(x_m,M_m)))^{\mathrm T}.
\end{gathered}
$$

Proof Estimate (10) is obtained by direct substitution by estimates (b) of Proposition 4.1 in the expression

$$
v(t,x,\eta)=\sum_{i=1}^m\sum_{j=1}^m v_{ij}(t,x)\eta_i\eta_j.
$$

Theorem 4.1 *Assume that behaviour of the hybrid system Σ is correctly described by the dynamical system (T,X,A,S), where $T=R_+$, $X=X_1\times\ldots\times X_m$ and X_i are subspaces with metrics ρ_i, $i=1,2,\dots,m$. Let $M_i\subset X_i$ and $M=M_1\times M_2\times\ldots\times M_m$ be an invariant set. If*

(1) *there exist functionals $v_{ij}(t,x)$ mentioned in Proposition 4.1;*

(2) *given functionals $v_{ij}(t,x)$ and a vector $\eta\in R_+^m$, $\eta>0$, there exist bounded for all $x\in X$ functions $\Phi_{ij}(x,\eta)$, $i,j=1,2,\dots,m$, and comparison functions φ_{i3} of class K such that*

$$
D^+v(t,x,\eta)|_{(S)}\le u_3^{\mathrm T}\Phi(x,\eta)u_3
$$

on system of motions S for all $x\in X$ and $t\in R_+$, where

$$
u_3(\rho(x,M))=(\varphi_{13}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m3}(\rho_m(x_m,M_m)))^{\mathrm T}.
$$

Then

(a) *If matrices* $B_1 = H^{\mathrm{T}}\underline{C}H,\ B_2 = H^{\mathrm{T}}\overline{C}H$ *are positive definite and constant* $m\times m$ *matrix* $\overline{\Phi} \geq \dfrac{1}{2}\,(\Phi^{\mathrm{T}}(x,\eta) + \Phi(x,\eta))$ *for all* $x \in X$ *is negative semidefinite, then the couple* (S, M) *is uniformly stable.*

(b) *If matrices* B_1 *and* B_2 *are positive definite and matrix* $\overline{\Phi}$ *is negative definite, then the couple* (S, M) *is uniformly asymptotically stable.*

(c) *If matrices* B_1 *and* B_2 *are positive definite, matrix* $\overline{\Phi}$ *is negative semidefinite, the set* M *is bounded and the comparison functions* $\varphi_{i1}, \varphi_{i2} \in KR$ *class* $i = 1, 2, \dots, m$*, then the family of motion* S *is uniformly bounded.*

(d) *If in condition* (c) *the matrix* $\overline{\Phi}$ *is negative definite, then the family of motions* S *is uniform-ultimately bounded and the couple* (S, M) *is uniformly asymptotically stable in the whole.*

(e) *If there exist constants* $a_1,\ a_2,\ b,\ c$ *such that*

$$a_1 r^b \leq u_1^{\mathrm{T}}(\rho(x,M))H^{\mathrm{T}}\underline{C}u_1(\rho(x,M)),$$
$$u_2^{\mathrm{T}}(\rho(x,M))H^{\mathrm{T}}\overline{C}u_2(\rho(x,M)) \leq a_2 r^b,$$
$$\varphi_3^{\mathrm{T}}\overline{\Phi}\varphi_3 \geq c r^b$$

for all $r \in R_+$, *then the couple* (S, M) *is exponentially stable in the whole.*

Proof Let us prove statement (a) of Theorem 4.1. Under condition (1) of Theorem 4.1 the functional $v(t, x, \eta)$ is positive definite and decreascent because matrices B_1 and B_2 are positive definite. Under condition (2) of Theorem 4.1 the functional $D^+v(t, x, \eta)$ on the system of motions S is negative semidefinite due to restrictions on matrix $\overline{\Phi}$. In this case the functional $v(t, x, \eta)$ is nonincreasing for all $t \geq 0$ along the system of motions S. Further, given $\varepsilon > 0$, we compute $\lambda = \inf\limits_{t\geq 0} v(t, x, \alpha)$ for $\rho(x, M) = \varepsilon$. Because of estimate (9) we can find by value λ the value $\delta > 0$ such that for $\rho(x, M) < \delta$ the estimate $v(t, x, \alpha) < \lambda$ holds for all $t \geq 0$. Now we show that the obtained value $\delta > 0$ corresponds to the given $\varepsilon > 0$, i.e. for $\rho(x, M) < \delta$ the inequality

$$\rho(q(t; a, t_0),\ M) < \varepsilon$$

holds for all $t \geq 0$. Assume on the contrary, let there exist a motion $q(t; a, t_0) \in S$ such that for some value $t^* \in R_+$ the inequality $\rho(q(t^*; a, t_0),\ M) = \varepsilon$ takes place. Then we get

$$v(t,\ q(t^*; a, t_0), \alpha) \geq \lambda,$$

but due to condition (a) of Theorem 4.1 the functional $v(t, x, \alpha)$ is nonincreascent along the system of motions S. Therefore

$$v(t,\ q(t; a, t_0), \alpha) \leq v(t, x, \alpha) < \lambda$$

for any $q(t; a, t_0) \in S$.

The contradiction obtained shows that the system of motions S of the hybrid system Σ is uniformly (S, M) stable.

The proof of statements (b) – (e) of Theorem 4.1 is similar to that of statement (a) following the Liapunov (ε, δ)-technique.

5 Stability Analysis of Two-Component Systems

We consider a hybrid two-component system [4]

$$\frac{dx}{dt} = X(t, x(t)) + g_1(t, z, x(t), w(t, z)), \quad x(t_0) = x_0, \tag{11}$$

$$\frac{\partial w}{\partial t} = L(t, x, \partial/\partial z)w + g_2(t, z, x(t), w(t, z)), \tag{12}$$

where

$$w(t_0, z) = w^0(z), \quad M(t, z, \partial/\partial z)w|_{\partial\Omega} = w^1(t, s), \quad s \in \partial\Omega, \quad \Omega \subset R^k,$$
$$X\colon\ T_0 \times U \to R^n, \quad L\colon\ B_1 \to B_2, \quad M\colon\ B_1 \to B_3, \quad w^0 \in B_4,$$

L, M are some differential operators and $B_1, \dots, B_4$ are Banach spaces.

A hybrid system (11) and (12) consists of the independent subsystems

$$\frac{dx}{dt} = X(t, x(t)), \tag{13}$$

$$\frac{\partial w}{\partial t} = L(t, z, \partial/\partial z)w \tag{14}$$

and interconnection functions between them

$$g_1 = g_1(t, z, x, w)\colon\ T_0 \times \Omega \times H \times Q \to R^n,$$
$$g_2 = g_2(t, z, x, w)\colon\ T_0 \times \Omega \times H \times Q \to R^m.$$

Let us introduce the assumptions on subsystems (13), (14) and interconnection functions between them.

Assumption 5.1 There exist functions $v_{ij} \in C(R_+ \times H \times Q, R)$, $i, j = 1, 2$, $v_{ij}(t, x, w)$ is locally Lipschitzian in x and w, functions of comparison $\varphi_i, \psi_i \in K$, $i = 1, 2$, and positive constants $\underline{\alpha}_{ii}, \overline{\alpha}_{ii} > 0$, $i = 1, 2$, and arbitrary constants $\underline{\alpha}_{12}$, $\overline{\alpha}_{12}$ such that

$$\underline{\alpha}_{11}\varphi_1^2(\|x\|) \le v_{11}(t, x, w) \le \overline{\alpha}_{11}\varphi_2^2(\|x\|);$$
$$\underline{\alpha}_{22}\psi_1^2(\|x\|) \le v_{22}(t, x, w) \le \overline{\alpha}_{22}\psi_2^2(\|x\|);$$
$$\underline{\alpha}_{12}\varphi_1(\|x\|)\psi_1(\|x\|) \le v_{12}(t, x, w) \le \overline{\alpha}_{12}\varphi_2(\|x\|)\psi_2(\|x\|)$$

for all $x \in H$, $w \in Q$ and $t \ge 0$.

Lemma 5.1 *If all conditions of Assumption 5.1 are fulfilled and the matrices*

$$A_1 = \begin{pmatrix} \underline{\alpha}_{11} & \underline{\alpha}_{12} \\ \underline{\alpha}_{21} & \underline{\alpha}_{22} \end{pmatrix}, \quad \underline{\alpha}_{12} = \underline{\alpha}_{21},$$
$$A_2 = \begin{pmatrix} \overline{\alpha}_{11} & \overline{\alpha}_{12} \\ \overline{\alpha}_{21} & \overline{\alpha}_{22} \end{pmatrix}, \quad \overline{\alpha}_{12} = \overline{\alpha}_{21},$$

are positive definite, then the function

$$v(t,x,w) = \eta^{\mathrm{T}} U(t,x,w)\eta, \tag{15}$$

where $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$, $\eta_i > 0$, *is positive definite and decreasing.*

Proof We introduce the notations

$$r = (\varphi_1(\|x\|), \psi_1(\|w\|))^{\mathrm{T}}, \quad q = (\varphi_2(\|x\|), \psi_2(\|w\|))^{\mathrm{T}}, \quad B = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}.$$

Under the conditions of Assumption 5.1 for the function (15) the bilateral estimation

$$r^{\mathrm{T}} B^{\mathrm{T}} A_1 B r \le \eta^{\mathrm{T}} U(t,x,w)\eta \le q^{\mathrm{T}} B^{\mathrm{T}} A_2 B q \tag{16}$$

holds.

By virtue of conditions of Lemma 5.1 it follows from the estimation (16) that the function $v(t,x,w)$ is positive definite and decreasing.

Assumption 5.2 There exist:

(1) functions $v_{11}(t,x)$, $v_{22}(t,w)$ and functions $v_{12}(t,x,w) = v_{21}(t,x,w)$;

(2) constants β_{ik}, $i = 1,2$, $k = 1,\dots,8$, and functions $\xi_1 = \xi_1(\|x\|)$ and $\xi_2 = \xi_2(\|w\|)$ of the K-class such that

(a) $D_t^+ v_{11}(t,x) + D_x^+ v_{11}(t,x)|_X \le \beta_{11}\xi_1^2$;

(b) $D_x^+ v_{11}(t,x)|_{g_1} \le \beta_{12}\xi_1^2 + \beta_{13}\xi_1\xi_3$;

(c) $D_t^+ v_{22}(t,w) + D_w^+ v_{22}(t,w)|_L \le \beta_{21}\xi_2^2$;

(d) $D_w^+ v_{22}(t,w)|_{g_2} \le \beta_{22}\xi_2^2 + \beta_{23}\xi_1\xi_2$;

(e) $D_t^+ v_{12(t,x,w)} + D_x^+ v_{12}(t,x,w)|_X \le \beta_{14}\xi_1^2 + \beta_{15}\xi_1\xi_2$;

(f) $D_w^+ v_{12}(t,x,w)|_L \le \beta_{24}\xi_1^2 + \beta_{25}\xi_1\xi_2$;

(g) $D_x^+ v_{12}(t,x,w)|_{g_1} \le \beta_{16}\xi_1^2 + \beta_{17}\xi_1\xi_2 + \beta_{18}\xi_2^2$;

(h) $D_w^+ v_{12}(t,x,w)|_{g_2} \le \beta_{26}\xi_1^2 + \beta_{27}\xi_1\xi_2 + \beta_{28}\xi_2^2$.

Lemma 5.2 *If all conditions of Assumption 5.2 are fulfilled and the matrix*

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad c_{12} = c_{21},$$

with the elements

$$\begin{aligned}
c_{11} &= \eta_1^2(\beta_{11} + \beta_{12}) + 2\eta_1\eta_2(\beta_{14} + \beta_{16} + \beta_{26}),\\
c_{22} &= \eta_2^2(\beta_{21} + \beta_{22}) + 2\eta_1\eta_2(\beta_{18} + \beta_{24} + \beta_{28}),\\
c_{12} &= \frac{1}{2}\,(\eta_1^2\beta_{13} + \eta_2^2\beta_{23}) + \eta_1\eta_2(\beta_{15} + \beta_{25} + \beta_{17} + \beta_{27})
\end{aligned}$$

is negative definite, then the derivative

$$D^+ v(t,x,w) = \eta^{\mathrm{T}} D^+ U(t,x,w)\eta$$

of the function $v(t,x,w)$ is a negative definite function by virtue of the system (11), (12).

Proof By virtue of the estimations (a) – (d) of Assumption 5.2 the estimation

$$D^+ v(t,x,w) \le p^{\mathrm{T}} C p$$

holds, where $p = (\xi_1(\|x\|),\, \xi_2(\|x\|))^{\mathrm{T}}$.

A definite negativeness of the derivative follows from the condition of Lemma 5.2.

Theorem 5.1 *If the two-component system (11), (12) is such that all conditions of Lemmas 5.1 and 5.2 are fulfilled, then the state of equilibrium* $x = 0$, $w = 0$ *of the system is uniform asymptotically stable.*

If in Assumption 5.1 $N_x = R^k$, $N_w = Q$, *functions* φ_i, ψ_i, ξ_i *belong to the KR-class and conditions of Lemmas 5.1, 5.2 are fulfilled, then the state of equilibrium* $x = 0$, $w = 0$ *of the system (11), (12) is uniform asymptotically stable in the whole.*

Proof Under the enumerated conditions the function $v(t, x, w)$ and its full derivative satisfy all conditions of Theorem 4.1. It proves the statement of Theorem 5.1.

Remark 5.1 If in estimations (a) – (d) of Assumption 5.2 we change the sign of the inequality for the opposite one and leave in the inequalities of Assumption 5.1 only estimation from below, then it isn't difficult to define conditions of instability of the state $x = 0$, $w = 0$ of the system (11), (12).

6 Concluding Remarks

Similar to Theorem 3.1 in the paper [7] the theorem was proved for discontinuous dynamical system. The mappings preserving stability in metric space were first considered by Thomas [10] and Hahn [3]. In the papers [8] and the book [9] and other mappings of the type were studied in the stability analysis of large-scale systems.

The application of multicomponent mapping $U(t, p)\colon\ R_+ \times X_1 \to X_2$ adds more flexibility to the approach to stability analysis of dynamical system in metric space, because this mapping admits a wider class of components for its elements $v_{ij}(t, p)$,.

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Bi-Impulsive Control to Build a Satellite Constellation*

A.F.B.A. Prado†

Instituto Nacional de Pesquisas Espaciais,
São José dos Campos, 12227-010, Brazil

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Abstract: This paper considers the problem of optimal maneuvers to insert a satellite in a constellation. The main idea is to assume that a satellite constellation is given, with all the Keplerian elements of the satellite members having known values. Then, it is necessary to maneuver a new satellite from a parking orbit until its position in the constellation. The control available to perform this maneuver is the application of two impulses (instantaneous change in the velocity of the spacecraft) to the satellite and the objective is to perform this maneuver with minimum fuel consumption. The maneuver that changes the angular position of a satellite keeping all the other Keplerian elements constant is also considered.

Keywords: *Orbital maneuver; astrodynamics; impulsive control; satellite constelation.*

Mathematics Subject Classification (2000): 70F15, 70M20, 93C99.

1 Introduction

To solve the problem of optimal maneuvers to insert a satellite in a constellation, two basic types of maneuvers are simulated: the planar ones, where the initial and final orbits belong to the same plane, and the three-dimensional ones, where they belong to different planes. The initial conditions to solve this problem are the orbits of the spacecraft in the parking and in the final orbits, including the information required to specify its positions in the orbits (the true anomaly or any other equivalent quantity) and the minimum and

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†Corresponding author: prado@dem.inpe.br

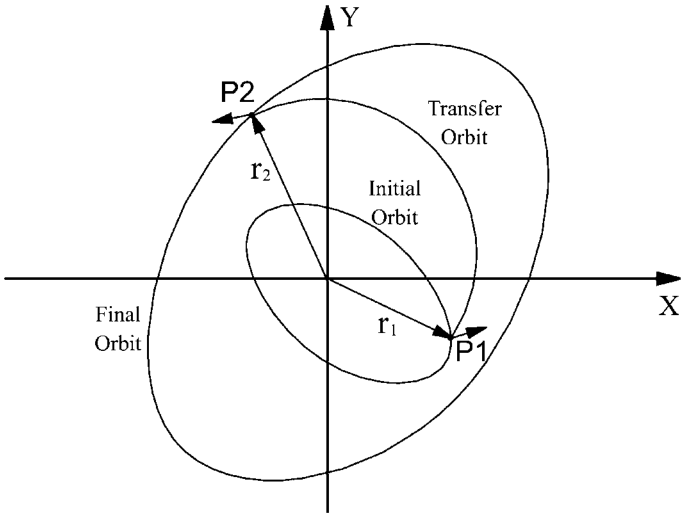

Figure 2.1. Scheme of the transfer.

maximum time of flight for the maneuver. The solution that is searched is the transfer orbit that satisfies all the initial conditions and that require the minimum total impulse (the addition of the magnitudes of the two impulses applied). To obtain the solution of this problem, the Lambert Problem associated with each particular transfer is formulated and solved. This approach is similar to the one used in Prado [1] to study the rendezvous maneuver. The Lambert Problem can be defined has the problem of finding a Keplerian orbit that passes through two given points and that requires a specified time of flight for a spacecraft to travel between those two given points. This problem is then solved using the algorithm developed by Gooding [2]. With the solutions given by this routine, it is possible to calculate the magnitude of both impulses that have to be applied. Several test cases (planar and non-planar) are solved to verify the algorithm developed. The total impulse required is then plotted as a function of the time specified for the transfer. Single and multi-revolution maneuvers are simulated. With those plots it is possible to choose transfer orbits that can satisfy both requirements of minimization of fuel expenditure and constraints on the time for the duration of the maneuver.

2 Statement of the Problem

In this section, the problem and the method of solution used are clearly defined. Figure 2.1 shows a sketch of the maneuver. A spacecraft is in an initial Keplerian orbit. The objective of the maneuver is to move this spacecraft from its initial orbit to a transfer orbit that intercepts the final orbit desired for the spacecraft, that is a second (final) Keplerian orbit. From the moment of the interception the spacecraft has to follow the final orbit.

There are many alternatives to solve this problem. For the present research, the model assumed for the control of the spacecraft is a bi-impulsive thrust, where the first impulse is applied at a time t_0, in a such way that the spacecraft will meet the target point at the time t_f; and the second impulse is applied at the time t_f, to put the spacecraft in its final orbit. Figure 2.1 shows an example of a direct single revolution transfer, where the spacecraft meets the target before making a complete revolution around the attracting body. Transfers where one or more revolutions are completed by the spacecraft before arriving at the destination are also possible and are considered in the present paper. The question that is considered here is how (magnitude and direction) and when to perform those two impulses to obtain the maneuver that has the minimum fuel consumption (minimum total ΔV). To answer this question, the following procedure was developed. The initial and final orbits are given, as an input of the procedure. The information on the position of the spacecraft in both initial and final orbits (the true anomaly or some equivalent quantity) is also required and given. Then, the following parameters are specified: the initial time t_0 of the maneuver, a value for the lower limit for the transfer time $(t_f - t_0)$, a value for the upper limit for the transfer time, a value for the increment of the transfer time, the number of revolutions of the spacecraft before meeting the desired point of the final orbit. With those parameters, an algorithm with the following steps is applied:

i) the lower limit for the transfer time is taken as the transfer time Δt;

ii) the Cartesian elements of the spacecraft at the initial time of the maneuver t_0 are calculated, using two-body celestial mechanics. This position is called $\underline{r}_i$ and this velocity $\underline{v}_i$;

iii) the Cartesian elements of the spacecraft at the final time of the maneuver $t_f = t_0 + \Delta t$ is calculated, using two-body celestial mechanics. This position is called $\underline{r}_f$ and this velocity $\underline{v}_f$;

iv) a value for the integer number of revolutions K of the spacecraft (number of complete orbits that the spacecraft makes during the maneuver) is assumed. Then, with $\underline{r}_i$, $\underline{v}_i$, $\underline{r}_f$, $\underline{v}_f$, Δt and K all the input data to solve the Lambert Problem is available. The original Lambert Problem is one of the most important and popular topics in celestial mechanics. Several important authors worked on it, trying to find better ways to solve the numerical difficulties involved [2−7]. It can be defined as: "A Keplerian orbit, about a given gravitational center of force is to be found connecting two given points (P_1 and P_2) in a given time Δt." The solution of the Lambert Problem gives the transfer orbit, the transfer time and the ΔV required. The Lambert Problem may have none, one or two solutions;

v) then, a step of time is added to the transfer time and the algorithm goes back to the step ii, with the new transfer time Δt.

This procedure is repeated until the upper limit for the transfer time is reached. It is also assumed several values for the number K of revolutions of the spacecraft.

3 Results

To study the optimal maneuvers several simulations were performed, using the algorithm described in the last section. The initial orbit is always a circular parking orbit with semi-major axis of 7700 km. This orbit is considered equatorial to study three-dimensional maneuvers and to have the same inclination of the final orbit when a planar maneuver is considered. For the final orbit two different cases are considered. The first one is the orbit of the GPS satellites, that are circular with semi-major axis of 20160 km and inclination of 55 degrees. The second one is the orbit of the satellites that belong to the Russian constellation Molniya, that are satellites that stay in frozen orbits with

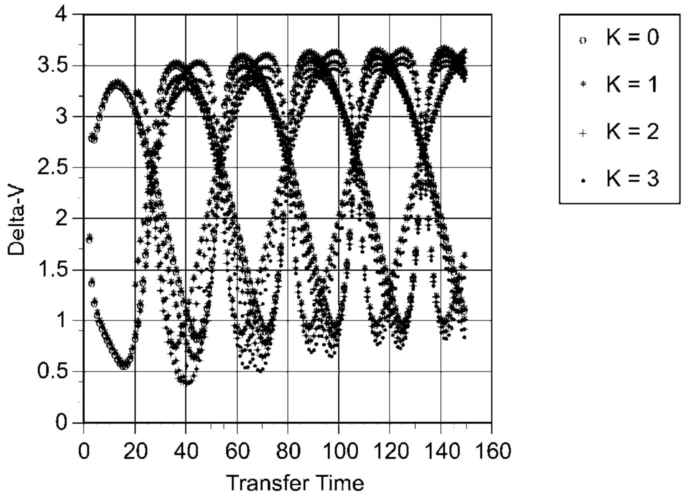

Figure 3.1. ΔV *vs.* transfer time for the first maneuver.

high eccentricity, to make them stay a long time in the apoapsis to be able to be useful satellites for communication. Their orbit have semi-major axis of 26560 km, eccentricity of 0.72 and inclination of 63.435 degrees. All the values of the variables are expressed in canonical units, except for the angles, that are expressed in degrees. The canonical units are dimensionless.

In the first case, it is simulated a very simple example, where the maneuver is planar and the two orbits are circular. Then, in the canonical system of units (using 7700 km as the unit for the measurements), the input data are:

$$a_i = 1.0; \quad \mathrm{e}_i = 0.0; \quad i_i = 55.0; \quad \Omega_i = 0.0; \quad \omega_i = 0.0; \quad T_i = 0.0;$$
$$a_f = 2.6182; \quad \mathrm{e}_f = 0.0; \quad i_f = 55.0; \quad \Omega_f = 0.0; \quad \omega_f = 0.0; \quad T_f = 0.0.$$

The nomenclature used here and in the rest of this paper is: a – semi-major axis, e – eccentricity, i – inclination of the orbit, Ω – argument of the ascending node, ω – argument of the periapsis, T – the time of the passage by the periapsis. The subscript "i" stands for the initial orbit of the spacecraft and the subscript "f" stands for the final orbit of the spacecraft.

The transfer time is constrained to the interval $0.1 \leq \Delta t \leq 150$ and the step of time is 1.0. It is assumed that $0 \leq K \leq 3$. Figure 3.1 shows the results obtained for this case: total ΔV *vs.* transfer time for each value of K.

For the second maneuver, a more generic case of a planar maneuver is used, where the final orbit is elliptic and its argument of the periapsis is also constrained. The two orbits are:

$$a_i = 1.0; \quad \mathrm{e}_i = 0.0; \quad i_i = 63.435; \quad \Omega_i = 0.0; \quad \omega_i = 0.0; \quad T_i = 0.0;$$
$$a_f = 3.4494; \quad \mathrm{e}_f = 0.72; \quad i_f = 63.435; \quad \Omega_f = 0.0; \quad \omega_f = 270.0; \quad T_f = 0.0.$$

This maneuver represents a transfer from our parking orbit to an orbit used by the Molniya constellation. The transfer time is also constrained to the interval $0.1 \leq \Delta t \leq$

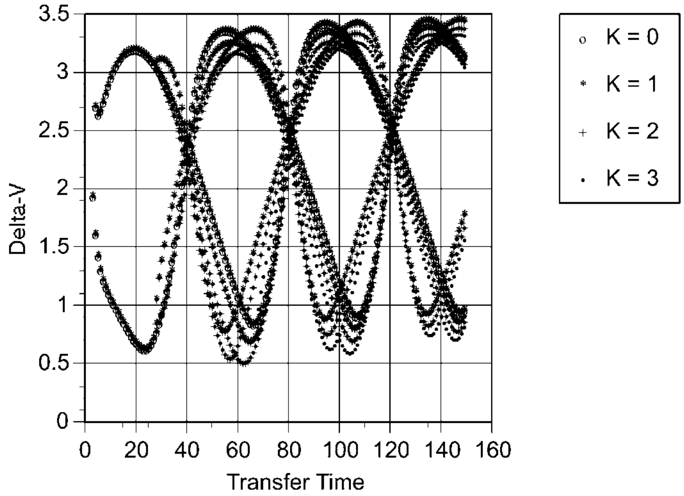

Figure 3.2. ΔV *vs.* transfer time for the second maneuver.

150 and the step of time is 1.0. It is assumed that $0 \le K \le 3$. Figure 3.2 shows the results obtained for this case: total ΔV *vs.* transfer time for each value of K.

For the third maneuver two non-coplanar orbits are used. This maneuver makes the satellite to start in an equatorial orbit and travel to the orbit of the GPS constellation. In this way, the two orbits are:

$$a_i = 1.0; \quad \mathrm{e}_i = 0.0; \quad i_i = 0.0; \quad \Omega_i = 0.0; \quad \omega_i = 0.0; \quad T_i = 0.0;$$
$$a_f = 2.6182; \quad \mathrm{e}_f = 0.0; \quad i_f = 55.0; \quad \Omega_f = 0.0; \quad \omega_f = 0.0; \quad T_f = 0.0.$$

The transfer time is also constrained to the interval $0.1 \le \Delta t \le 150$ and the step of time is 1.0. It is assumed that $0 \le K \le 3$. Figure 3.3 shows the results obtained for this case: total ΔV *vs.* transfer time for each value of K.

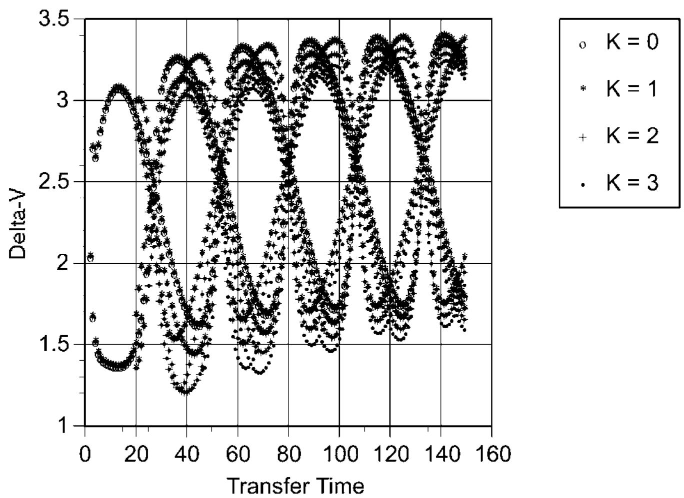

Figure 3.3. ΔV *vs.* transfer time for the third maneuver.

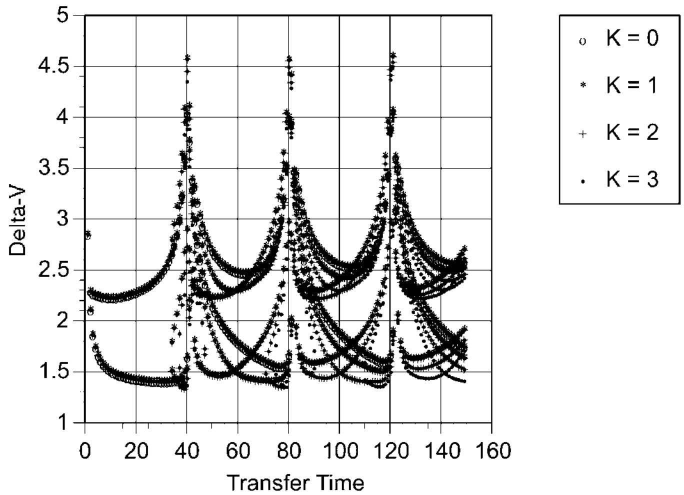

Figure 3.4. ΔV *vs.* transfer time for the fourth maneuver.

For the fourth maneuver, a three-dimensional transfer between the parking orbit and the Molniya orbit is used. The two orbits are:

$$a_i = 1.0; \quad \mathrm{e}_i = 0.0; \quad i_i = 0.0; \quad \Omega_i = 0.0; \quad \omega_i = 0.0; \quad T_i = 0.0;$$
$$a_f = 3.4494; \quad \mathrm{e}_f = 0.72; \quad i_f = 63.4350; \quad \Omega_f = 0.0; \quad \omega_f = 270.0^\circ; \quad T_f = 0.0.$$

The transfer time is also constrained to the interval $0.1 \leq \Delta t \leq 150$ and the step of time is 1.0. It is assumed that $0 \leq K \leq 3$. Figure 3.4 shows the results obtained for this case: total ΔV *vs.* transfer time for each value of K.

For the fifth maneuver, a relocation of 20 degrees for a satellite that belongs to a GPS constellation is used. It means that the satellite has to change its orbital position in 20 degrees, keeping all the other Keplerian elements constant. Then, the two orbits are:

$$a_i = 2.6182; \quad \mathrm{e}_i = 0.0; \quad i_i = 55.0; \quad \Omega_i = 0.0; \quad \omega_i = 0.0; \quad T_i = 0.0;$$
$$a_f = 2.6182; \quad \mathrm{e}_f = 0.0; \quad i_f = 55.0; \quad \Omega_f = 0.0; \quad \omega_f = 20.0; \quad T_f = 0.0.$$

The transfer time is also constrained to the interval $0.1 \leq \Delta t \leq 150$ and the step of time is 1.0. It is assumed that $0 \leq K \leq 3$. Figure 3.5 shows the results obtained for this case: total ΔV *vs.* transfer time for each value of K.

There are several characteristics that come from those simulations. First of all, it is possible to see that the solutions appear in pairs (two values of ΔV for a given transfer time). The values for the ΔV for a given family of transfer orbits oscillate with the increase of the transfer time. The first minimum of each family is usually the global minimum, what means that, after the best geometry for the maneuver is achieved, any extra time added for the maneuver does not generate a reduction in the fuel consumed. Another characteristic visible in those plots is that when K increases, the beginning of the curve shift to the right. This result is expected, because when revolutions are added to the maneuver, the minimum time required to get a solution has to increase.

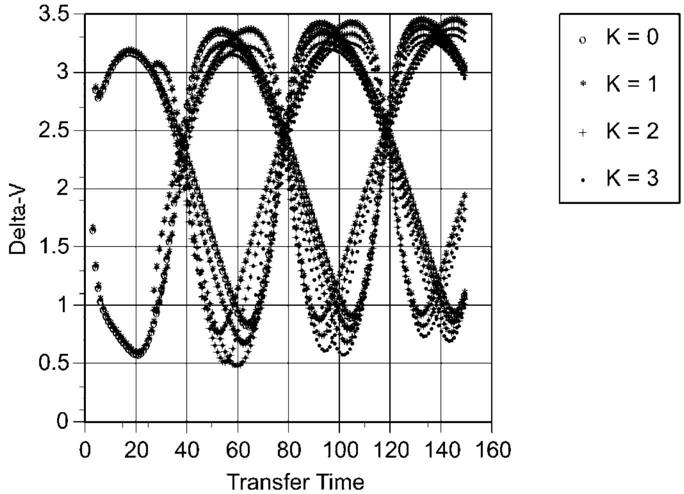

Figure 3.5. ΔV *vs.* transfer time for the fifth maneuver.

4 Conclusions

An algorithm to solve the optimal maneuver (in terms of minimum consumption of fuel) to transfer a satellite from a parking orbit to a fixed position in a final orbit with two impulses was derived and used for several maneuvers. This algorithm was explained in details and it includes single and multi-revolution transfers. Then, several cases using planar and non-planar maneuvers were solved. Several figures showed the fuel consumed *vs.* time for the transfer. Those results are useful to mission designers, because they can help to design transfer orbits to include a satellite in a given constellation.

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Optimal Control of Nonlinear Systems with Controlled Transitions

Ming-Qing Xiao*

Department of Mathematics, Southern Illinois University, Carbondale, IL62901/USA

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Abstract: This paper studies the optimum stochastic control problem with piecewise deterministic dynamics. The controls enter through the system dynamics as well as the transitions for the underlying Markov chain process, and are allowed to depend on both the continuous state and the current state of the Markov chain. The paper shows that the feedback optimal control relies on the viscosity solutions of a finite set of coupled Hamilton-Jacobi-Bellman (HJB) equations. Explicit control structures are provided by using the concept of subdifferential of a continuous function.

Keywords: *Markov process; Hamilton–Jacobi–Bellman equations; viscosity solutions; β-stochastically stabilizable.*

Mathematics Subject Classification (2000): 49J15, 49J52, 49J55, 93E03.

1 Introduction

In this paper, we consider a dynamical system which is nonlinear in the state and linear in the piecewise continuous control u_1:

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t),\theta(t)) + B(x(t),\theta(t))u_1(t), \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $x \in R^n$, x_0 is a fixed (known) initial state, u_1 is a control, taking values in a bounded set $U_1 \subset R^r$, and $\theta(t)$ is a controlled, continuous time Markov process, taking values in a finite state space $\mathcal{S}$, of cardinality s. Transitions from state $i \in \mathcal{S}$ to $j \in \mathcal{S}$ occur at a rate controlled by a second controller, who chooses at time t an action $u_2(t)$

*Corresponding author: mxiao@math.siu.edu

from a finite set $U_2(i)$ of actions available at state i. Let $U_2 = \cup_{i\in\mathcal{S}} U_2(i)$. The controlled rate matrix (of transitions within $\mathcal{S}$) is

$$\Lambda = \{\lambda_{i,a,j}\}, \quad i,j \in \mathcal{S}, \quad a = u_2(t) \in U_2(i), \tag{1.2}$$

where henceforth we drop the "commas" in the subscripts of λ. The λ_{iaj}'s are real numbers such that for any $i \neq j$, and $a \in U_2(i)$, $\lambda_{iaj} \geq 0$, and for all $a \in U_2(i)$ and $i \in \mathcal{S}$, $\lambda_{iai} = -\sum_{j\neq i} \lambda_{iaj}$. Fix some initial state i_0 of the controlled Markov chain $\mathcal{S}$, and the final time t_f (which may be infinite). We consider the class of policies $\mu_k \in \mathcal{U}_k$ for controller $(k = 1, 2)$, whose elements (taking values in U_k) are of the form

$$u_k(t) = \mu_k(t, x(t), \theta(t)), \quad t \in [0, t_f). \tag{1.3}$$

For the finite-horizon case, μ_k is taken to be piecewise continuous in the first argument and local Lipschitz in second argument and measurable in the third argument. In the infinite-horizon case, the dependence of μ_k on t is dropped, but otherwise it is defined the same way. Define $\mathcal{X} = R^n \times \mathcal{S}$ to be the combined state space of the system and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ to be the class of admissible multi-strategies $\mu = (u_1, u_2)$, appropriately defined depending on whether t_f is finite or infinite. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. Define a running cost $L\colon \mathcal{X} \times \mathcal{U}_1 \to [0, \infty)$ as

$$L(x, i, u_1) = Q(x, i) + \langle u_1, R(x, i) u_1 \rangle, \tag{1.4}$$

where the definitions of Q and R will be made precise later in Section 2.1.

To any fixed initial state (x_0, i_0) and a multi-strategy $\mu \in \mathcal{U}$, there corresponds a unique probability measure $P^{\mu}_{x_0,i_0}$ on the canonical probability space of the states and actions of the players, equipped with the standard Borel σ-algebra. Denote by $E^{\mu}_{x_0,i_0}$ the expectation operator corresponding to $P^{\mu}_{x_0,i_0}$.

For each fixed initial state (x_0, i_0), multi-strategy $\mu \in \mathcal{U}$, and a finite horizon of duration t_f, the discounted (expected) cost function is defined as

$$J_\beta(0; x_0, i_0, \mu; t_f) = E^{\mu}_{x_0,i_0}\left\{ g(x(t_f), \theta(t_f)) e^{-\beta t_f} + \int_0^{t_f} e^{-\beta t} L(x(t), \theta(t), u_1(t))\, dt \right\}, \tag{1.5}$$

where g is a terminal cost function whose condition will be specified in next section, $\beta \geq 0$ is the discount factor, and the expectation is over the joint process $\{x, \theta\}$. For t_f infinite, a corresponding discounted cost function is defined as:

$$J_\beta(0; x_0, i_0, \mu) = E^{\mu}_{x_0,i_0}\left\{ \int_0^{\infty} e^{-\beta t} L(x(t), \theta(t), u_1(t))\, dt \right\}. \tag{1.6}$$

We further denote the *cost-to-go* from any time-state pair $(t; x, i)$, under a multi-strategy $\mu \in \mathcal{U}$ by

$$J_\beta(t; x, i, \mu; t_f) = E^{\mu}_{x,i}\left\{ g(x(t_f), \theta(t_f)) e^{-\beta(t_f - t)} + \int_t^{t_f} e^{-\beta(\tau - t)} L(x(\tau), \theta(\tau), u_1(\tau))\, d\tau \right\} \tag{1.7}$$

and

$$J_\beta(t;x,i,\mu) = E^{\mu}_{x,i}\left\{ \int_t^{\infty} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau \right\}, \tag{1.8}$$

for the finite-horizon and infinite-horizon cases, respectively. The optimal *value functions* are then defined by, respectively,

$$V(t;x,i;t_f) = \inf_{\mu_1\in\mathcal{U}_1}\inf_{\mu_2\in\mathcal{U}_2} J_\beta(t;x,i,\mu;t_f) \tag{1.9}$$

and

$$V(x,i) = \inf_{\mu_1\in\mathcal{U}_1}\inf_{\mu_2\in\mathcal{U}_2} J_\beta(t;x,i,\mu) \tag{1.10}$$

for $i = 1,2,\dots,s$. Dynamic programming arguments (for background on the approach that can be used here, see [2, 3]), lead to the two coupled HJB equations (2.11) and (2.12), corresponding to the finite and infinite-horizon cases, respectively. More precisely, if these equations admit unique *viscosity solutions* on R^n, then $V(t;x,i;t_f)$ and $V(t;x)$ thus defined constitute the optimal value functions for the finite-horizon and infinite-horizon cases, respectively. Xiao and Başar have showed, under the assumptions given in next section, that the coupled HJB equations (for finite and infinite-horizon, respectively) admit viscosity solutions, and moreover the viscosity solutions are unique if $\beta > 0$ [4].

In this paper, we study the structures of the optimal controllers for the nonlinear system (1.1). The major challenge here is that this is not a standard optimal control problem in which only a single HJB equation is considered (f.g. see [6−8]). The optimal control considered in this paper is related to a system of coupled HJB equations. Based on the results obtained in [4], we show that the optimal feedback control is determined by the subdifferential of the viscosity solutions of (2.11) in finite horizon case, and (2.12) in infinite horizon case. Explicit expression of the optimal feedback controller is obtained in terms of the subdifferential of viscosity solutions of (2.11) or (2.12).

The remainder of this paper is structured as follows. In Section 2, we provide the necessary assumptions for the system (1.1), and the definitions of viscosity solutions of the coupled HJB equations (2.11) and (2.12). In Section 3, we give a detail discussion of the structure of the optimal controller by using the concept of viscosity solution and the concept of subdifferential of a continuous function. The pair of optimal feedback controller, one is the control which enters through the system dynamics, and the another is the control of transitions for the underlying Markov chain process, is explicitly given in this section. As an illustration, in Section 4 we apply the result to the linear quadratic case. The paper ends with the concluding remarks.

2 Assumptions and Definitions

2.1 Assumptions

Assumption 1 For each i, f is an n-vector function, and there exists a constant $L_f \ge 0$ such that

$$\sup_i\{|f(x,i) - f(y,i)|\} \le L_f|x-y|, \quad x \in R^n,$$

where $|\cdot|$ stands for the Euclidean norm of R^n.

Assumption 2 For each i, $B(x,i)$ is an $n \times r$ matrix, and

$$\sup_i \{\|B(x,i)\|\} \le C_{b1}, \quad \sup_i\{\|B(x,i) - B(y,i)\|\} \le C_{b2}|x-y|, \quad \forall\, x,y \in R^n,$$

for some positive constants C_{b1} and C_{b2}, where $\|\cdot\|$ stands for the matrix's norm.

Assumption 3 For each i, $Q(\cdot,i)\colon\ R^n \to [0,+\infty)$, with

$$0 \le \sup_i\{Q(x,i)\} \le C_q|x|^2, \quad \forall\, x \in R^n,$$

for some $C_q > 0$.

Assumption 4 For each i, $R(x,i)$ is an $n \times n$ matrix with $R(x,i) = R(x,i)^{\mathrm{T}} > 0$, for all $x \in R^n$, and

$$\sup_i\{\|R(x,i)\|\} \le C_r, \quad \sup_i\{\|R(x,i) - R(y,i)\|\} \le C_r'|x-y|, \quad x,y \in R^n,$$

for some $C_r, C_r' > 0$, and there exists $L_R > 0$ such that

$$\sup_i\{\|R^{-1}(x,i) - R^{-1}(y,i)\|\} \le L_R|x-y|, \quad \forall\, x,y \in R^n.$$

Assumption 5 For $i \ne j$, $0 \le \lambda_{iaj} \le C_\lambda$, where C_λ is a positive constant, and

$$\lambda_{iai} + \sum_{j\ne i} \lambda_{iaj} \equiv 0, \quad 1 \le i \le s.$$

Assumption 6 For each i and any $g(\cdot,i)\colon\ R^n \to [0,\infty)$,

$$\sup_i\{|g(x,i)|\} \le (1+C_g)|x|^2,$$
$$\sup_i\{|g(x,i) - g(y,i)|\} \le C_g'(1+|x|+|y|)|x-y|$$

for all $x,y \in R^n$, where C_g, C_g' are positive constants.

Assumption 7 β is a positive real number.

Assumption 8 For any $z \in R^n$, there exists a nondecreasing function $\omega\colon\ \{0\} \cup R^+ \to \{0\} \cup R^+$ such that $\omega(0) = 0$, $\lim\limits_{\rho\to+\infty} \omega(\rho)/\rho = +\infty$ and

$$\langle z,\, B(x,i)R^{-1}(x,i)B(x,i)^{\mathrm{T}} z\rangle \ge \omega(|z|), \quad \forall\, x \in R^n, \quad \forall\, i \in \mathcal{S},$$

where $B(x,i)^{\mathrm{T}}$ represents the transpose of $B(x,i)$.

Throughout the paper, the following conventions will be adopted, unless otherwise indicated:

(1) u_2 and a are used interchangeably to denote the second control;
(2) by an abuse of notation $\mu_1(t)$ will be used to denote $\mu_1(x(t),\theta(t))$.

2.2 Two coupled Hamilton–Jacobi–Bellman (HJB) equations

Let Ω be a nonempty open set of R^n. We here introduce two coupled Hamilton-Jacobi-Bellman (HJB) equations.

(I) Finite horizon:

$$\begin{gathered}\beta V(t,x,i) + \sup_{u_1\in U_1, u_2\in U_2(i)} \Big[-\mathcal{A}^{(u_1,u_2)}V(t,x,i) - L(x,i,u_1)\Big] = 0 \ \text{ in } \ (0,t_f]\times\Omega;\\ V(t_f,x,i) = g(x,i) \ \text{ on } \ \Omega, \quad i=1,2,\dots,s, \quad \text{where } s \text{ is a positive integer,}\end{gathered} \tag{2.11}$$

and $U_1 \subset R^r$, $U_2(i)$ is a finite set for each $i \in \mathcal{S} = \{1,2,\dots,s\}$.

(II) Infinite horizon:

$$\beta V(x,i) + \sup_{u_1\in R^r, u_2\in U_2(i)} \Big[-G^{(u_1,u_2)}V(x,i) - L(x,i,u_1)\Big] = 0 \ \text{ in } \ \Omega \tag{2.12}$$

for $i = 1,2,\dots,s$, where again $U_1 \subset R^r$ and $U_2(i)$ is a finite set for each $i \in \mathcal{S}$.

Here, the operators $\mathcal{A}$ and G are defined, respectively, as follows for each $u_1 \in R^r$, $a \in U_2(i)$, $i \in \mathcal{S}$:

$$\mathcal{A}^{(u_1,a)}V(t,x,i) = \frac{\partial V(t,x,i)}{\partial t} + [D_xV(t,x,i)]\cdot F(x,u_1,i) + \sum_{j\in\mathcal{S}} \lambda_{iaj}V(t,x,j) \tag{2.13}$$

and

$$G^{(u_1,a)}V(x,i) = [D_xV(x,i)]\cdot F(x,u_1,i) + \sum_{j\in\mathcal{S}} \lambda_{iaj}V(x,j), \tag{1.14}$$

with

$$F(x,u_1,i) = f(x,i) + B(x,i)u_1, \quad L(x,i,u_1) = Q(x,i) + \langle u_1, R(x,i)u_1\rangle. \tag{2.15}$$

Definition 2.1 Let $\overline{V}$ be a vector function

$$\overline{V} = (V(\cdot,\cdot,1),\, V(\cdot,\cdot,2),\, \dots,\, V(\cdot,\cdot,s))\colon \ ([0,t_f]\times\Omega)^s \to R^n.$$

We say that

(1) $\overline{V}$ is a *viscosity subsolution* of (2.11), if for any i, $V(\cdot,\cdot,i)$ is upper semi-continuous and

$$\begin{gathered}\beta\Phi(t_0,x_0,i) + \sup_{(u_1,u_2)} \Big[-\mathcal{A}^{(u_1,u_2)}\Phi(t_0,x_0,i) - L(x_0,i,u_1)\Big] \le 0 \quad \text{on} \quad \Omega,\\ \Phi(t_f,x,i) \le g(t_f,x,i) \quad \text{on} \quad \Omega,\end{gathered}$$

whenever $\Phi(\cdot,\cdot,i) \in C^1([0,t_f]\times\Omega)$ is such that $V(t,x,i) - \Phi(t,x,i)$ attains a local maximum at (t_0,x_0) with $\Phi(t_0,x_0,j) = V(t_0,x_0,j)$ for each $j \in \mathcal{S}$, and $(u_1,u_2) \in U_1\times U_2$;

(2) $\overline{V}$ is a *viscosity supersolution* of (2.11) if for any i, $V(\cdot,\cdot,i)$ is lower semi-continuous and

$$\beta\Phi(t_0,x_0,i)+\sup_{(u_1,u_2)}\Big[-\mathcal{A}^{(u_1,u_2)}\Phi(t_0,x_0,i)-L(x_0,i,u_1)\Big]\geq 0 \quad \text{on} \quad \Omega,$$

$$\Phi(t_f,x,i)\geq g(t_f,x,i),$$

whenever $\Phi(\cdot,\cdot,i)\in C^1([0,t_f]\times\Omega)$ is such that $V(t,x,i)-\Phi(t,x,i)$ attains a local minimum at (t_0,x_0) with $\Phi(t_0,x_0,j)=V(t_0,x_0,j)$ for each $j\in\mathcal{S}$, and $(u_1,u_2)\in U_1\times U_2$;

(3) $\overline{V}$ is a *viscosity solution* of (2.11) if $\overline{V}$ is both a viscosity supersolution and a viscosity subsolution.

The notion of a viscosity solution for (2.12) can be introduced analogously. The following theorem is from [4].

Theorem 2.1 *Let the control space $\mathcal{U}_1$ be bounded. Under the Assumptions 1 – 8 given above, the coupled Hamilton–Jacobi–Bellman (HJB) equations (2.11) (resp. (2.12)) admit unique viscosity solutions on $[0,t_f]\times\Omega$ (resp. Ω). The viscosity solutions are the optimal value functions given by (1.9) (resp. (1.10)).*

3 Construction of the Optimum Stochastic Control

We discuss in this section the derivation of the optimal control law of the system (1.1). We first introduce the notations of a superdifferential and a subdifferential of a continuous function.

Definition 3.1 Let $V\in C([0,t_f]\times R^n)$ and $(t,x)\in[0,t_f]\times R^n$. Then

(1) the superdifferential, $D^+V(t,x)$, of V at (t,x) is

$$D^+V(t,x)=\left\{(q,p)\in R^{n+1}\colon \limsup_{(s,y)\to(t,x)}\frac{V(s,y)-V(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|x-y|}\leq 0\right\};$$

(2) The subdifferential, $D^-V(t,x)$, of V at (t,x) is

$$D^-V(t,x)=\left\{(q,p)\in R^{n+1}\colon \limsup_{(s,y)\to(t,x)}\frac{V(s,y)-V(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|x-y|}\geq 0\right\}.$$

Remark 3.1 It is easy to see that when V is differentiable at (t,x), we have

$$D^+V(t,x)=D^-V(t,x)=\left(\frac{\partial}{\partial t}V(t,x),D_xV(t,x)\right).$$

Lemma 3.1 *The following propositions hold*

(1) $\{V(\cdot\,;\cdot\,,i;t_f)\}_{i=1}^s$ *is a viscosity subsolution of (2.11) in* $[0,t_f]\times\Omega$ *if and only if for each* $i\in\mathcal{S}$

$$-q+\beta V(t;x,i;t_f)-\inf_{\mu_2}\left\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(t;x,i;t_f)\right\}+H_i(x,p)\le 0 \tag{3.16}$$

for any $(q,p)\in D^+V(t;x,i;t_f)$,

(2) $\{V(\cdot\,;\cdot\,,i;t_f)\}_{i=1}^s$ *is a viscosity supersolution of (2.11) in* $[0,t_f]\times\Omega$ *if and only if for each* $i\in\mathcal{S}$

$$-q+\beta V(t;x,i;t_f)-\inf_{\mu_2}\left\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(t;x,i;t_f)\right\}+H_i(x,p)\ge 0 \tag{3.17}$$

for any $(q,p)\in D^-V(t;x,i;t_f)$.

In both cases, H_i *is defined to be where*

$$H_i(x,p)=-\langle p,\, f(x,i)\rangle-Q(x,i)+\frac{1}{4}\langle p,\, B(x,i)R^{-1}(x,i)B(x,i)^{\mathrm{T}}p\rangle.$$

Proof We prove the first part, as the proof of the second part is similar.

Suppose that (3.16) holds. Let $\varphi(\cdot,\cdot,i)\subset C^1([0,t_f]\times R^n)$ be such that (t_0,x_0) is a local maximizer of $V(\cdot\,;\cdot,i;t_f)-\varphi(\cdot,\cdot,i)$ for some i with $V(t_0;x_0,k;t_f)=\varphi(t_0,x_0,k)$, $k=1,2,\dots,s$. Since $\varphi(\cdot,\cdot,i)\in C^1([0,t_f]\times R^n)$ for each i, it yields

$$\begin{aligned}\varphi(t,x,i)&=\varphi(t_0,x_0,i)+\varphi_t(t_0,x_0,i)(t-t_0)+\varphi_x(t_0,x_0,i)(x-x_0)\\&\quad+o(|t-t_0|)+o(|x-x_0|).\end{aligned}$$

Hence for (t,x) sufficiently close to (t_0,x_0), that (t_0,x_0) is a local maximizer of $V(\cdot\,;\cdot,i;t_f)-\varphi(\cdot,\cdot,i)$ leads to

$$\begin{aligned}V(t;x,i;t_f)&\le\varphi(t_0,x_0,i)+\varphi_t(t_0,x_0,i)(t-t_0)+\varphi_x(t_0,x_0,i)(x-x_0)\\&\quad+o(|t-t_0|)+o(|x-x_0|).\end{aligned}$$

Now let $p=\varphi_x(t_0,x_0,i)$ and $q=\varphi_t(t_0,x_0,i)$. Then (3.16) implies that

$$-\varphi_t(t_0,x_0,i)+\beta\varphi(t_0;x_0,i;t_f)-\inf_{\mu_2}\left\{\sum_{j\in\mathcal{S}}\lambda_{iaj}\varphi(t_0;x_0,i;t_f)\right\}+H_i\left(x_0,\varphi_x(t_0,x_0,i)\right)\le 0.$$

Thus $\{V(\cdot\,;\cdot,i;t_f)\}_{i=1}^s$ is a viscosity subsolution of (2.11).

Conversely, let $(q,p)\in D^+V(t;x,i;t_f)$. When (s,y) is sufficiently close to (t,x), according to the definition of superdifferential, we have

$$V(s;y,i;t_f)\le V(t;x,i;t_f)+q(s-t)+p\cdot(y-x)+o(|s-t|)+o(|y-x|).$$

Introduce test functions

$$\varphi(s,y,i)=V(t;x,i;t_f)+q(s-t)+p\cdot(y-x)+g_1(|s-t|)+g_2(|x-y|)$$

for $i=1,2,\dots,s$, and $g_j\colon[0,\infty)\to[0,\infty)$, $j=1,2$, are nondecreasing functions such that $g_j(r)=o(r)$ and $\left.\dfrac{d}{dr}g_j(r)\right|_{r=0}=0$ (for construction of such functions, see [1] or [2]). Hence by such choice of g_1, g_2, one can see that in fact (t,x) is a local strict maximizer of $V(\cdot\,;\cdot,\cdot;i)-\varphi(\cdot,\cdot,i)$ and

$$\varphi_x(t,x,i)=p,\quad \varphi_t(t,x,i)=q,$$

as a result of which (3.16) holds by the definition of viscosity subsolution.

Definition 3.2 An admissible feedback controller $u_1(t) = \mu_1(t,x)$ for system (1.1) is a nonlinear mapping $F\colon\ R^+ \times R^n \to R^r$ such that $\mu_1(t,x) = F(t,x)$ and the following system

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t),\theta(t)) + B(x(t),\theta(t))F(t,x(t)) \\ x(0) &= x_0 \\ \theta(0) &= i_0 \end{aligned} \tag{3.18}$$

has at least one solution defined on $(0,\infty)$ in the Carathéodory sense, i.e., absolutely continuous functions verifying (3.18) almost everywhere.

Definition 3.3 We say that the system (1.1) is β-stochastically stabilizable if, for all finite $x_0 \in R^n$ and $i_0 \in \mathcal{S}$ with $x(0) = x_0$, $\theta(0) = i_0$, there exists an admissible feedback control $\mu(t,x(t),\theta(t))$ such that

$$\lim_{t_f\to\infty} E_{x,i} \int_0^{t_f} \left[e^{-\beta t} L(x(t,\theta(t),\mu),\theta(t),\mu)\right] dt < \infty.$$

Now we are in the position for construction of the optimal control.

Theorem 3.1 *Assume that (2.11) has a viscosity solution* $\{V(\cdot;\cdot,i;t_f)\}_{i=1}^{s}$. *Suppose that for each* i *there exists a* p *with* $(q,p) \in D^-V(t;x,i;t_f)$ *such that*

$$\mu_1^*(x,i) = -\frac{1}{2}\, R^{-1}(x,i)B^{\mathrm{T}}(x,i)p(t,x,i) \tag{3.19}$$

is an admissible feedback controller for system (1.1). Let $\mu_2^*(t,x,i) = f(t,x,i)$ *be the argument such that* $\Big\{\sum_{j\in\mathcal{S}} \lambda_{iaj} V(t;x,i;t_f)\Big\}$ *reaches its infimum (which always exists because the set* U_2 *is a finite set and* U_1 *is a bounded set). Then* (μ_1^*,μ_2^*) *is an optimal feedback control for system (1.1) under the cost function*

$$J_\beta(t;x,i,\mu;t_f) = E_{x,i}^{\mu}\left\{ g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\, d\tau \right\}. \tag{3.20}$$

Proof According to Theorem 2.1, we know that

$$V(t;x,i;t_f) = \inf_{\mu\in\mathcal{U}} E_{x,i}^{\mu}\left\{ g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\, d\tau \right\} \tag{3.21}$$

and thus $V(t;x,i;t_f)$ is absolutely continuous with respect to t for each fixed x. For $(q,p) \in D^-V(t;x,i;t_f)$, $(t,x) \in [0,t_f]\times\Omega$, by the definition of subdifferential, one can see that

$$V(s;y,i;t_f) \ge V(t;x,i;t_f) + q(s-t) + p\cdot(y-x) + o(|s-t|) + o(|y-x|) \tag{3.22}$$

when (s,y) is sufficiently close to (t,x). Similar to the proof of Lemma 3.1, we define a C^1 function

$$\psi(s,y,i) = V(t;x,i;t_f) + q(s-t) + p\cdot(y-x) + g_1(|s-t|) + g_2(|y-x|) \tag{3.23}$$

where $g_j\colon [0,\infty)\to[0,\infty)$, $j=1,2$, are nondecreasing functions such that $g_j(r)=o(r)$ and $dfracddrg(r)|_{r=0}=0$. It is now ready to see that $V(s;y,i;t_f)-\psi(s,y,i)$ has a local strict minimizer at (t,x). The definition of viscosity subsolution leads to

$$-\psi_t(t,x,i) + \beta\psi(t,x,i) - \inf_{\mu_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}\psi(t,x,i)\Big\} + H_i(x,\psi_x(t,x,i)) \ge 0. \tag{3.24}$$

According to the definition of H_i, the above inequality can be written as

$$\begin{aligned}&\psi_t(t,x,i) + L(x,i,u_1) - \Big|R^{1/2}(x,i)u_1 + \frac{1}{2}R^{-1/2}B^{\mathrm T}(x,i)p(t,x,i)\Big|^2\\ &\psi_x(t,x,i)[f(x,i)+B(x,i)u_1] - \beta\psi(t,x,i) + \inf_{\mu_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}\psi(t,x,i)\Big\} \le 0.\end{aligned} \tag{3.25}$$

According to Dynkin's formula (see [2]), we know that

$$E_{x,i}e^{-(\beta t_f - t)}\psi(t_f,x(t_f),\theta(t_f)) - \psi(t,x,i) = \int_t^{t_f}\mathcal{A}^{(\mu_1,\mu_2)}e^{-\beta(\tau-t)}\psi(\tau,x(\tau),\theta(\tau))\,d\tau.$$

Integrating (3.25) from t to t_f one obtains

$$\begin{aligned}V(t;x,i;t_f) \ge E^{\mu}_{x,i}&\Big\{g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)}L(x,\theta,u_1)\,d\tau\Big\}\\ &-\int_t^{t_f} e^{-\beta(\tau-t)}|R^{1/2}(x,i)u_1 + R^{-1/2}B^{\mathrm T}(x,i)p(\tau,x,i)|^2\,d\tau.\end{aligned}$$

If we set

$$\begin{aligned}u_1 &= \mu_1^*(t,x,i) = -\frac{1}{2}R^{-1}(x,i)B^{\mathrm T}(x,i)p(t,x,i)\\ u_2 &= \mu_2^*(t,x,i) = f(t,x,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(t;x,i;t_f)\Big\}\end{aligned}$$

then we have

$$\begin{aligned}V(t;x,i;t_f) &= E^{(\mu_1^*,\mu_2^*)}_{x,i}\Big\{g(x(t_f),\theta(t_f))e^{-\beta t_f} + \int_t^{t_f} e^{-\beta\tau}L(x(\tau),\theta(\tau),\mu_1^*)\,dt\Big\}\\ &= \inf_{\mu\in\mathcal{U}} E^{\mu}_{x,i}\Big\{g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)}L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau\Big\}\end{aligned}$$

since $V(t;x,i;t_f)$ is the optimal value function according to Theorem 2.1, and this completes the proof of the theorem.

For the infinite horizon case we have a similar result:

Theorem 3.2 *Assume that (2.12) has viscosity solution* $\{V(\cdot,i)\}_{i=1}^{s}$. *Suppose that for each* i *there exists a* p *with* $(q,p)\in D^{-}V(x,i)$ *such that*

$$\mu_1^*(x,i) = -\frac{1}{2}\,R^{-1}(x,i)B^{\mathrm{T}}(x,i)p(x,i) \tag{3.26}$$

is an admissible feedback controller for system (1.1). Let

$$\mu_2^*(x,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(x,i)\Big\}. \tag{3.27}$$

Then (μ_1^*,μ_2^*) *is a pair of optimal feedback controls for system (1.1) under the cost function*

$$J_\beta(t;x,i,\mu) = E_{x,i}^{\mu}\Big\{\int_t^\infty e^{-\beta(\tau-t)}L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau\Big\}, \tag{3.28}$$

Proof According to Theorem 2.1, the value function $V(x,i)$ given by (1.10) is the viscosity solution of (2.12). For any $t_f\in(t,\infty)$, $V(x,i)$ is also the (steady-state) solution of the Cauchy problem

$$\beta V(t;x,i;t_f) + \sup_{(u_1,u_2)}\Big[\mathcal{A}^{(u_1,u_2)}V(t;x,i;t_f) - L(x,i,u_1)\Big] = 0 \quad \text{in} \quad (t,t_f)\times\Omega,$$
$$V(t_f;x,i;t_f) = V(x,i) \quad \text{on} \quad R^n.$$

Now by applying the previous theorem it yields that

$$\mu_1^*(x,i) = -\frac{1}{2}\,R^{-1}(x,i)B^{\mathrm{T}}(x,i)p(x,i), \tag{3.29}$$

$$\mu_2^*(x,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(x,i)\Big\} \tag{3.30}$$

is a pair of optimal feedback controllers in the time duration $(0,t_f]$ for any $t_f>t$, this leads to

$$V(x,i) \ge E_{x,i}\Big\{\int_t^\infty e^{-\beta(\tau-t)}L(x(\tau),\theta(\tau),\mu_1^*(\tau))\,d\tau\Big\}, \tag{3.31}$$

while Theorem 2.1 implies that the above inequality should be an equality. This yields the conclusion of the theorem.

Remark 3.2 Theorem 3.2 implies that if (2.12) admits a viscosity solution and μ_1^* given in the theorem is an admissible feedback control, then (μ_1^*,μ_2^*) is β-stochastically stabilizable for (1.1).

4 Linear Quadratic Case

In order to make the outcome of the paper more transparent, let us consider the scalar linear-quadratic problem. Let $n=1$ and

$$f(x,i) = A(i)x, \quad B(x,i) = B(i), \quad Q(x,i) = Q(i)x^2, \quad R(x,i) = R(i),$$

and $g(x,i) = Q_{t_f}(i)x^2$ for $i = 1,2,\dots,s$. In this case (2.11) admits a unique viscosity solution V. Moreover, V is convex with respect to x and Lipschitz with respect to t (c.f. [5]). One can show that $x \to p(t,x,i)$ in this case is linear, thus self-adjoint on R, and therefore

$$V(t;x;i;t_f) = P(t,i)x^2 \tag{4.32}$$

for any $x \in R$. Substituting (4.32) into (2.11), we obtain a system of coupled ordinary differential equations

$$\begin{aligned} &-\frac{\partial}{\partial t}P(t,i) + \beta P(t,i) - 2A(i)P(t,i) - Q(i) \\ &+ P^2(t,i)B^2(i)R^{-1}(i) - \inf_{u_2}\sum_{j\in\mathcal{S}}\lambda_{iaj}P(t,j) = 0, \\ &P(t_f,i) = Q_{t_f}, \end{aligned} \tag{4.33}$$

for $i = 1,2,\dots,s$. The solution of (4.33) now is in the sense that $P(t,i)$ is absolutely continuous and satisfies (4.33) almost everywhere on $[0,t_f]$. According to Theorem 3.1, the pair of the optimal feedback control $\mu = (\mu_1,\mu_2)$ is given by

$$\mu_1^*(t,x,i) = -R^{-1}(i)B(i)P(t,i)x, \tag{4.34}$$

$$\mu_2^*(t,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}P(t,j)\Big\}. \tag{4.35}$$

Similarly, in the infinite horizon case, for each $i \in \mathcal{S}$

$$V(x,i) = P(i)x^2 \tag{4.36}$$

where $(P(1),P(2),\dots,P(s))$ satisfies a system of algebraic coupled equations

$$\beta P(i) - 2A(i)P(i) - Q(i) + P^2(i)B^2(i)R^{-1}(i) - \inf_{u_2}\sum_{j\in\mathcal{S}}\lambda_{iaj}P(j) = 0 \tag{4.37}$$

for $i = 1,2,\dots,s$. In this case the optimal feedback control $\mu = (\mu_1,\mu_2)$ is given by

$$\mu_1^*(x,i) = -R^{-1}(i)B(i)P(i)x, \tag{4.38}$$

$$\mu_2^*(i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}P(j)\Big\} \tag{4.39}$$

from Theorem 3.2.

5 Concluding Remarks

In this paper, we study the optimum stochastic control problem, in which the controls enter through the system dynamics as well as the transitions for the underlying Markov chain process, and are allowed to depend on both the continuous state and the current state of the Markov chain. The structure of the optimal controller is obtained in this paper which therefore makes possible to construct the optimal control by the approach of numerical method.

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Convergence of Solutions to a Class of Systems of Delay Differential Equations

Taishan Yi and Lihong Huang*

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P. R. China

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Abstract: This paper is concerned with a delay differential system which can be regarded as a mathematical model of compartmental system with pipes and time delays. It is shown that every solution of such a differential system tends to a constant vector as $t \to \infty$. The obtained results improve and extend some existing ones in the literature.

Keywords: *Convergence; delay differential equation; compartmental system.*

Mathematics Subject Classification (2000): 34C12, 39A11.

1 Introduction

Recently, there has been much attention in the study of the asymptotic behavior of solutions for the following scalar delay differential equation

$$\frac{dx(t)}{dt} = -F(x(t)) + F(x(t-r)), \tag{1.1}$$

where $r > 0$ is a constant, and $F\colon R \to R$ is continuous. System (1.1), which has been used to model a variety of phenomena such as some population growth, the spread of epidemics, the dynamics of capital stocks, *etc.* has been discussed extensively in the literature (see, for example, [2–5, 7, 8, 10, 12–14, 17]), in which various approaches including the first integral, invariance principle of Lyapunov–Razumikhin type, *etc.* have been applied to conclude that every solution of system (1.1) tends to a constant. However, most of the study deals with the problem of convergence of solutions of system (1.1) under the assumption that F is either strictly increasing or locally Lipschitz continuous and nondecreasing. To the best of our knowledge, there exist no results for the asymptotic behavior of system (1.1) with F only assumed to be nondecreasing. Meanwhile, we stress

*Corresponding author: lhhuang@hnu.cn

the fact that the aforementioned approaches seem to fail to be applicable to system (1.1) when F is only assumed to be nondecreasing and hence a different analysis is needed in this case. This situation motivates us to study further system (1.1) or its more general case with new methods or techniques based on the assumption that F is only required to be nondecreasing in this work.

More precisely, in this paper we are concerned with the following system of delay differential equations

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -F(x_1(t)) + F(x_2(t-r_2)),\\ \frac{dx_2(t)}{dt} &= -F(x_2(t)) + F(x_1(t-r_1)),\end{aligned} \tag{1.2}$$

where r_1 and r_2 are positive constants, F is continuous and nondecreasing on R. System (1.2) can be used as a mathematical model of compartmental system with pipes and time delays, where $x_i(t)$ denotes the amount of the material in the i-th compartment at time t, r_i denotes the transit time for the material flow to pass through the pipe, $F(x_i(t))$ denotes the rate of flow of material loss of the i compartment, and $F(x_i(t-r_i))$ denotes the rate of material flows from the i-th compartment into the j-th compartment through a pipe, $i \neq j,\ i,j = 1,2$. Compartmental models are frequently used in, e.g., theoretical epidemiology, physiology, population dynamics, the analysis of ecosystems, and chemical reaction kinetics. For more details, we refer to the work of Anderson [1], Györi [9, 10] and Györi and Wu [11]. The main goal of the present paper is to show that every solution of system (1.2) approaches a constant vector by using monotonicity arguments. To this end, we begin by describing some monotonicity properties possessed by system (1.2) with the help of the comparison principles for delay differential equations developed by Smith [15]. Then, we introduce the notion of the admitting closed interval with respect to F and present some important properties of system (1.2) by making use of the notion. Finally, based on the above preparations, we prove our main results, which improve and extend the corresponding results in the aforementioned literature.

The paper is organized as follows. In Section 2, we introduce some necessary notations and establish some preliminary results, which are important in the proofs of our main results. Our main results are presented in Section 3.

2 Preliminary Results

In this section, some important properties of system (1.2) will be presented, which are of importance in proving our main results in Section 3.

Throughout this paper, we will use R_+ to denote the set of all nonnegative real numbers and R_+^2 denote the set of all nonnegative vectors in R^2. Define

$$C = C([-r_1, 0], R) \times C([-r_2, 0],\ R)$$

as the Banach space equipped with a supremum norm. Define

$$C_+ = C([-r_1, 0], R_+) \times C([-r_2, 0],\ R_+).$$

It follows that C_+ is an order cone in C and hence, C_+ induces a closed partial ordered relation on C. For any $\varphi, \psi \in C$ and $A \subseteq C$, the following notations will be used:

$\varphi \le \psi$ iff $\psi - \varphi \in C_+$, $\varphi < \psi$ iff $\varphi \le \psi$ and $\varphi \ne \psi$, $\varphi \ll \psi$ iff $\psi - \varphi \in \operatorname{Int} C_+$, $\varphi \le A$ iff $\varphi \le \psi$ for any $\psi \in A$, $\varphi < A$ iff $\varphi < \psi$ for any $\psi \in A$, $\varphi \ll A$ iff $\varphi \ll \psi$ for any $\psi \in A$. Notations such as "$\ge$", "$>$" and "$\gg$" have the natural meanings.

In what follows, we assume that $\varphi \in C$ and use $x_t(\varphi)$ $(x(t,\varphi))$ to denote the solution of (1.2) with the initial data $x_0(\varphi) = \varphi$. For any $x \in R^2$, let us define $\widehat{x} = ((\widehat{x})_1, (\widehat{x})_2)$, where $(\widehat{x})_i(\theta) = x_i$, $\theta \in [-r_i, 0]$, $i = 1, 2$.

Lemma 2.1 *For any constants K, t_0 and x_0, the initial value problem*

$$\begin{aligned} \frac{dx(t)}{dt} &= -F(x(t)) + K, \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

exists a unique solution $x(t, t_0, x_0)$ on $[t_0, \infty)$.

Proof From the Peano theorem, we know that solutions of the initial value problem (2.1) locally exist. Again, since F is nondecreasing, it follows from [6] that right-hand solutions of the initial value problem (2.1) are also unique. Hence, $x(t, t_0, x_0)$ exists and is unique on $[t_0, \eta)$ for some positive constant η, where $[t_0, \eta)$ denotes the maximal right-interval of existence of $x(t, t_0, x_0)$. We will show that $\eta = +\infty$. Otherwise, $\overline{\lim\limits_{t\to\eta^-}} |x(t, t_0, x_0)| = +\infty$. We next distinguish several cases to finish the proof.

Case 1. There exists $t_1 \in [t_0, \eta)$ such that $-F(x(t_1, t_0, x_0)) + K = 0$. Then let

$$\widetilde{x}(t) = \begin{cases} x(t, t_0, x_0), & t_0 \le t \le t_1, \\ x(t_1, t_0, x_0), & t \ge t_1. \end{cases}$$

It follows that $\widetilde{x}(t)$ satisfies (2.1) and hence, $x(t, t_0, x_0) \equiv \widetilde{x}(t)$. But this contradicts $\eta < +\infty$.

Case 2. $-F(x(t, t_0, x_0)) + K < 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly decreasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \le x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \ge -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \ge (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim\limits_{t\to\eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

Case 3. $-F(x(t, t_0, x_0)) + K > 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly increasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \ge x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \le -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \le (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim\limits_{t\to\eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

The proof of the lemma is complete.

Lemma 2.2 *Let $0 < r \in R$ be given and $d \in C([t_0, t_0 + r])$. Then, for any constant x_0, the initial value problem*

$$\begin{aligned} \frac{dx}{dt}(t) &= -F(x(t)) + d(t), \\ x(t_0) &= x_0, \end{aligned}$$

exists a unique solution $x(t, t_0, x_0)$ on $[t_0, t_0 + r]$.

Proof Lemma 2.2 follows by applying the standard technique of differential inequalities and Lemma 2.1.

Lemma 2.3 *Let $\varphi \in C$. Then $x_t(\varphi)$ exists and is unique on* $[0, +\infty)$.

Proof Let $\tau = \min\{r_1, r_2\}$. We will show that $x_t(\varphi)$ exists and is unique on $[0, \tau]$. To see this, let $d_1(t) = F(\varphi_2(t - r_2))$ and $d_2(t) = F(\varphi_1(t - r_1))$ for any $t \in [0, \tau]$. Consider the following system

$$\frac{dx}{dt}_i(t) = -F(x_i(t)) + d_i(t),$$
$$x_i(0) = \varphi_i(0),$$

where $i \in \{1, 2\}$. By Lemma 2.2, $x_i(t)$ exists and is unique on $[0, \tau]$. Hence, $x_i(t, \varphi)$ exists and is unique on $[0, \tau]$, that is, $x_t(\varphi)$ exists and is unique on $[0, \tau]$. It follows from induction that $x_t(\varphi)$ exists and is unique on $[0, +\infty)$. The proof of the lemma is now complete.

Before continuing, it is convenient to introduce the following notations and establish some convention. Set
$$E_F = \{e \in R^2 \colon\ F(e_1) = F(e_2)\}.$$

Define $O(\varphi) = \{x_t(\varphi) \colon\ t \geq 0\}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in C, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$. If $O(\varphi)$ is bounded, define

$$\omega(\varphi) = \bigcap_{t \geq 0} \overline{O(x_t(\varphi))},$$

i. e., $\omega(\varphi) = \{\psi \in C \colon$ there exists a subsequence $t_k \to +\infty$ such that $x_{t_k}(\varphi) \to \psi\}$. It follows that $\omega(x)$ is nonempty, compact, invariant and connected.

We make the following key definition.

Definition 2.1 Let

$$s(\alpha) = \sup\{\beta \in R \colon\ F(\beta) = F(\alpha)\} \quad \text{and} \quad i(\alpha) = \inf\{\beta \in R \colon\ F(\beta) = F(\alpha)\}.$$

$[a, b]$ is called *an admitting closed super-interval with respect to* F if $F(a) = F(b)$ and $a = i(a)$; $[a, b]$ is called *an admitting closed sub-interval with respect to* F if $F(a) = F(b)$ and $b = s(b)$.

Lemma 2.4 *Let $\varphi, \psi \in C$ with $\psi \geq \varphi$. Then $x_t(\psi) \geq x_t(\varphi, F)$ for $t \in R_+$. Moreover, we have the following*

(1) *Assume that $e \in E$, $\varphi \geq \hat{e}$ and $i \in \{1, 2\}$. If $e_i < s(e_i)$ and $\varphi_i(0) > e_i$, then $x_i(t, \varphi) > e_i$ for $t \geq 0$.*
(2) *Assume that $e \in E$, $\varphi \leq \hat{e}$ and $i \in \{1, 2\}$. If $e_i > i(e_i)$ and $\varphi_i(0) < e_i$, then $x_i(t, \varphi) < e_i$ for $t \geq 0$.*

Proof The first assertion of the lemma follows from [15, Proposition 2.1].

We next will prove conclusion (1), conclusion (2) can be proved similarly. It follows that $x_t(\varphi) \geq \hat{e}$ for $t \geq 0$. Without loss of generality, we may assume that $i = 1$. From (1.2), we get

$$\begin{aligned} x_1'(t, \psi) &= -F(x_1(t, \psi)) + F(x_2(t - r_2, \psi)) \\ &\geq -F(x_1(t, \psi)) + F(e_2) = -F(x_1(t, \psi)) + F(e_1). \end{aligned}$$

Consider the following auxiliary system

$$\begin{aligned} y'(t) &= -F(y(t)) + F(e_1), \\ y(0) &= \min\{x_1(0,\varphi), s(e_1)\}. \end{aligned}$$

Then, from Lemma 2.1, we know that $y(t) \equiv y(0) > e_1$ for $t \geq 0$. Thus, by the comparison theorem for ordinary differential equations in Walter [16], we have

$$x_1(t,\varphi) \geq y(t) > e_1 \quad \text{for} \quad t \geq 0.$$

This completes the proof.

Now, we are in a position to present an important properties of system (1.2).

Lemma 2.5 *Let $[a,b]$ be an admitting closed super-interval with respect to F. Then the following conclusions hold:*

(1) *if $e_1 \in [a,b)$ and $e_2 = a$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi,F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (e_1 + \widehat{M,\, e_2} - \varepsilon_M)$;*

(2) *if $e_1 = a$ and $e_2 \in [a,b)$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi,F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (e_1 - \widehat{\varepsilon_M,\, e_2} + M)$.*

Proof We only prove conclusion (1), the other conclusion can be proved similarly. Without loss of generality, we may assume that $F(a) = 0$. Let $M > 0$. In view of Lemma 2.4, we may assume that $M < b - e_1$. Let $r(x) = M - x + F(e_2 - x)r_2$. Then $\lim\limits_{x\to 0^+} r(x) = M > 0$. Hence, there exists $\varepsilon_M > 0$ such that $r(\varepsilon_M) > M/2$. Let $\xi = (e_1 + \widehat{M,\, e_2} - \varepsilon_M) \in C$. Then by Lemma 2.4, $x(t,\xi,F) \leq (e_1 + M,\, e_2)$ for $t \geq 0$. In what follows, define

$$t_1 = \inf\{t > 0\colon\ x_1(t,\xi,F) \leq e_1\} \quad \text{and} \quad t_2 = \inf\{t > 0\colon\ x_2(t,\xi,F) = e_2\}.$$

We will show that $t_1 \leq t_2 + r_2$. If not, then $t_1 > t_2 + r_2$. From (1.2), we obtain

$$\frac{dx_2(t)}{dt} = -F(x_2(t)) + F(x_1(t - r_1)).$$

Therefore,

$$x_2(t_2) = e_2 \quad \text{and} \quad \frac{dx_2(t)}{dt} = -F(x_2(t)) \quad \text{for} \quad t \in [t_2,\, t_1 + r_1].$$

By Lemma 2.1, we have $x_2(t) = e_2$ for any $t \in [t_2,\, t_1 + r_1]$. From (1.2) again, we obtain

$$\frac{dx_1(t)}{dt} = -F(x_1(t)) + F(x_2(t - r_2)).$$

Hence,

$$x_1(t_2 + r_2) > e_1 \quad \text{and} \quad \frac{dx_1(t)}{dt} = -F(x_1(t)) \quad \text{for} \quad t \in [t_2 + r_2,\, t_1 + r_1 + r_2].$$

Thus, by Lemma 2.1, we have

$$x_1(t) = x_1(t_2 + r_2) > e_1 \quad \text{for} \quad t \in [t_2 + r_2,\, t_1 + r_1 + r_2].$$

Therefore, $x_1(t_1) > e_1$, a contradiction to the definition of t_1. We next distinguish two cases to finish the proof.

Case 1. $t_1 = +\infty$. It follows that $t_2 = +\infty$. Thus,

$$x_1(t) > e_1 \quad \text{and} \quad x_2(t) < e_2 \quad \text{for} \quad t \in R_+.$$

Therefore, from system (1.2), it follows that $x_1(t)$ is decreasing and $x_2(t)$ is increasing on R_+. Hence, there exist $e_1', e_2' \in R$ such that $x_1(t) \to e_1'$ and $x_2(t) \to e_2'$. So, we have $e_1' \geq e_1$ and $e_2' \leq e_2$. In view of Definition 2.1 and the fact that $e_2 = a$, we obtain $e_2' = e_2$. We will show that $e_1' > e_1$. Otherwise, $e_1' = e_1$. From (1.2), it follows that

$$\begin{aligned} \frac{dx_1'(t)}{dt} &= -F(x_1(t)) + F(x_2(t - r_2)), \\ \frac{dx_2'(t)}{dt} &= -F(x_2(t)) + F(x_1(t - r_1)). \end{aligned}$$

Thus,

$$\begin{aligned} x_1(t) - (e_1 + M) &= \int_0^t F(x_2(s - r_2))\, ds, \\ x_2(t) - (e_2 - \varepsilon_M) &= -\int_0^t F(x_2(s))\, ds. \end{aligned}$$

Letting $t \to \infty$, we have

$$\begin{aligned} -M &= \int_{-r_2}^{0} F(x_2(s))ds + \int_0^\infty F(x_2(s))\, ds, \\ \varepsilon_M &= -\int_0^\infty F(x_2(s))\, ds. \end{aligned}$$

Therefore, $M + F(e_2 - \varepsilon_M)r_2 - \varepsilon_M = 0$, a contradiction to the choice of ε_M.

Case 2. $t_1 < \infty$. Then, from (1.2), it follows that

$$\begin{aligned} \frac{dx_1'(t)}{dt} &= -F(x_1(t)) + F(x_2(t - r_2)), \\ \frac{dx_2'(t)}{dt} &= -F(x_2(t)) + F(x_1(t - r_1)). \end{aligned}$$

Thus,

$$\frac{dx_1(t)}{dt} = F(x_2(t - r_2)) \quad \text{for} \quad t \in [0, t_1], \quad \text{and} \quad \frac{dx_2(t)}{dt} = -F(x_2(t)) \quad \text{for} \quad t \in [0,\, t_1 + r_1].$$

Hence,

$$x_1(t_1) - x_1(0) = \int_0^{t_1} F(x_2(s-r_2))\,ds, \quad \text{and} \quad x_2(t_1-r_2) - x_2(0) = -\int_0^{t_1-r_2} F(x_2(s))\,ds.$$

Therefore, $x_1(t_1) - x_1(0) = \int_0^{t_1-r_2} F(x_2(s))ds + F(e_2 - \varepsilon_M)r_2$. It follows that

$$x_1(t_1) - x_1(0) = x_2(0) - x_2(t_1 - r_2) + F(e_2 - \varepsilon_M)r_2.$$

Consequently,

$$e_1 \geq x_1(t_1) \geq e_1 + M + e_2 - \varepsilon_M - e_2 + F(e_2 - \varepsilon_M)r_2,$$

that is, $M - \varepsilon_M + F(e_2 - \varepsilon_M)r_2 \leq 0$, a contradiction to the choice of ε_M. The proof of the lemma is now complete.

Arguing as in the proof of Lemma 2.5, we can get the following result.

Lemma 2.6 *Let $[a,b]$ is an admitting closed sub-interval with respect to F. Then the following conclusions hold:*

(1) *if $e_1 \in [a,b)$ and $e_2 = a$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (\widehat{e_1 + M, e_2 - \varepsilon_M})$;*

(2) *if $e_1 = a$ and $e_2 \in [a,b)$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (\widehat{e_1 - \varepsilon_M, e_2 + M})$.*

In what follows, we assume that $\varphi \in C$. If $O(\varphi)$ is bounded, define

$$D_\varphi^+ = \{e \in E_F \colon \hat{e} \leq \omega(\varphi)\} \quad \text{and} \quad D_\varphi^- = \{e \in E_F \colon \hat{e} \geq \omega(\varphi)\}.$$

We are now in a position to state another lemma.

Lemma 2.7 *D_φ^+ contains the maximum element, that is, $\sup D_\varphi^+ \in D_\varphi^+$. Hence, there exists $e^* \in D_\varphi^+$ such that $e^* \geq D_\varphi^+$.*

Proof Since $O(\varphi)$ is bounded, $\omega(\varphi)$ is compact. Hence, there exists $\alpha \in R$ such that

$$\widehat{(\alpha, \alpha)} \leq \omega(\varphi).$$

Let $D = \{e \in D_\varphi^+ \colon (\alpha, \alpha) \leq e\}$. Then D is compact. It follows that D contains the maximal element and we denote it by $e^* = (e_1^*, e_2^*)$. Next we will show that $\sup D = e^*$. If not, then there exist $e_1, e_2 \in R$ such that $(e_1, e_2) \in D$ and $(e^* - (e_1, e_2)) \notin R_+^2$. Without loss of generality, we may assume that $e_1^* > e_1$ and $e_2^* < e_2$. By the definition of D, we obtain

$$\widehat{(e_1^*, e_2)} \leq \omega(\varphi) \quad \text{and} \quad F(e_1^*) = F(e_2).$$

Therefore,

$$\widehat{(e_1^*, e_2)} \in D \quad \text{and} \quad (e_1^*, e_2^*) < (e_1^*, e_2),$$

a contradiction to the definition of e^*. It follows that $\sup D_\varphi^+ = e^*$. This completes the proof.

Arguing as in the proof of Lemma 2.7, we can get the following result.

Lemma 2.8 *D^-_φ contains the minimum element, that is, $\inf D^-_\varphi \in D^-_\varphi$. Hence, there exists $e^* \in D^-_\varphi$ such that $e^* \le D^-_\varphi$.*

3 Main Results

The purpose of this section is to show that every solution of (1.2) tends to a constant vector as $t \to \infty$, which is our main result in this paper.

Lemma 3.1 *Assume that $\varphi \in C$. Then $O(\varphi)$ is bounded. Hence, $\omega(\varphi)$ is compact.*

Proof Lemma 3.1 follows immediately from Lemma 2.4 and system (1.2).

Lemma 3.2 *Let $\varphi \in C$ and $e^* = \sup D^+_\varphi$. If $\omega(\varphi)\backslash\{\widehat{e^*}\} \ne \phi$, then $e_1^* = e_2^* = s(e_2^*)$.*

Proof By way of contradiction, if this is not true, then there exists $i \in \{1,2\}$ such that $e_i < s(e_2^*)$. We next distinguish several cases to finish the proof.

Case 1. $e_1^* < s(e_2^*)$ and $e_2^* < s(e_2^*)$.

By the invariance of $\omega(\varphi)$, we may assume that there exists $\psi \in \omega(\varphi)$ such that $\psi_1(0) > e_1^*$. From the conclusion (1) of Lemma 2.4, it follows that

$$(x_{r_1}(\psi))_1(\theta) > e_1^*, \quad \theta \in [-r_1, 0].$$

Thus, we can choose $M > 0$ such that

$$e_1^* + 3M < s(e_2^*) \quad \text{and} \quad (x_{r_1}(\psi)) \ge (\widehat{e_1^* + 3M,\, i(e_2^*)}).$$

Let $a = i(e_1^*)$, $b = s(e_1^*)$, $e_1 = M + e_1^*$ and $e_2 = a$. Then, for the above $M > 0$ and the admitting closed super-interval $[a, b]$, by Lemma 2.5 (1), there exists $\varepsilon_M > 0$ such that

$$\varliminf_{t\to\infty} x_t(\eta) \ge \widehat{(e_1, e_2)}, \quad \eta \ge (\widehat{e_1 + M,\, e_2 - \varepsilon_M}).$$

From the choice of $M > 0$, it follows that $x_{r_1}(\psi) \gg (\widehat{e_1 + M, e_2 - \varepsilon_M})$. By the definition of $\omega(\varphi)$ again, there exists $t_1 > 0$ such that $x_{t_1}(\varphi) \ge (\widehat{e_1 + M,\, e_2 - \varepsilon_M})$. Hence, $\varliminf_{t\to\infty} x_t(\varphi) \ge \widehat{(e_1, e_2)}$. Thus,

$$\omega(\varphi) \ge \widehat{(e_1, e_2)} = (\widehat{e_1^* + M,\, e_2^*}).$$

Again, from the choice of $M > 0$, it follows that $(\widehat{e_1^* + M,\, e_2^*}) \in E_F$. But this contradicts the fact that $e^* = \sup D^+_\varphi$.

Case 2. $e_1^* < s(e_2^*)$ and $e_2^* = s(e_2^*)$.

We claim that for any $\psi \in \omega(\varphi)$, $\psi_1(\theta) = e_1^*$, $\theta \in [-r_1, 0]$. If not, then, by the invariance of $\omega(\varphi)$, there exists $\psi \in \omega(\varphi)$ such that $\psi_1(0) > e_1^*$. Arguing as in the proof of Case 1, we can prove that this is a contraction. Therefore, our claim is true. Since $\omega(\varphi) \setminus \{e^*\} \ne \phi$, it follows from the above claim and the invariance of $\omega(\varphi)$ that there exists $\psi \in \omega(\varphi)$ such that $\psi_2(0) > e_2^*$. From (1.2), we obtain

$$x_1'(t, \psi) = -F(x_1(t, \psi)) + F(x_2(t - r_2,\, \psi)).$$

Thus, $F(e_1^*) = F(x_2(t - r_2, \psi))$. Hence, $F(\psi_2(0)) = F(e_1^*)$. From the Definition 2.1, it follows that

$$\psi_2(0) \le s(e_1^*) = s(e_2^*) = e_2^*,$$

which yields a contradiction. (*Please, make a correction of this statement!*)

Case 3. $e_1^* = s(e_2^*)$ and $e_2^* < s(e_2^*)$.

Arguing as in the proof of Case 2, we can conclude that this is a contraction.

Therefore, $e_1^* = e_2^* = s(e_2^*)$. This completes the proof.

Lemma 3.3 *Let $\varphi \in C$ and $e^{**} = \inf D_\varphi^-$. If $\omega(\varphi) \setminus \{\widehat{e^{**}}\} \ne \phi$, then $e_1^{**} = e_2^{**} = i(e_2^{**})$.*

Proof The proof of the lemma is similar to that of Lemma 3.2 and thus is omitted.

The main result of this paper is the next theorem.

Theorem 3.1 *Let $\varphi \in C$. Then there exists $e^* \in E_F$ such that $\omega(\varphi) = \{\widehat{e^*}\}$.*

Proof Let $e^* = \sup D_\varphi^+$ and $e^{**} = \inf D_\varphi^-$. We will show that $\omega(\varphi) = \{\widehat{e^*}\}$. Otherwise, $\omega(\varphi) \setminus \{e^*\} \ne \phi$ and $\omega(\varphi) \setminus \{e^{**}\} \ne \phi$. Hence, by Lemmas 3.2 and 3.3, we obtain

$$e_1^* = e_2^* = s(e_1^*) \quad \text{and} \quad e_1^{**} = e_2^{**} = i(e_1^{**}).$$

Thus, $e_1^* < e_1^{**}$. Observe that for any $\psi \in \omega(\varphi)$, we know that $\widehat{e^{**}} - \psi,\ \psi - \widehat{e^*} \notin \operatorname{Int} C_+$. We next assume that $\psi \in \omega(\varphi)$. Then by the invariance of $\omega(\varphi)$, there exists a *full orbit* of the solution semiflow of (1.2) in $\omega(\varphi)$ through ψ, and we below will use $x_t(\psi)$ to denote such a *full orbit*. Hence, $x(t, \psi)$ is continuously differentiable in its first arguments $t \in R$. Let $x(t) = x(t, \psi)$, $t \in R$. We next distinguish several cases to finish the proof.

Case 1. There exist $t_1, t_2 \in [-r_1, 0]$ such that $x_1(t_1) = e_1^*$ and $x_1(t_2) = e_1^{**}$.

It follows that $\dfrac{dx_1(t_1)}{dt} = \dfrac{dx_1(t_2)}{dt} = 0$. From (1.2), we get

$$F(x_2(t_1 - r_2)) = F(e_1^*) \quad \text{and} \quad F(x_1(t_2 - r_2)) = F(e_1^{**}).$$

Thus, by the definition of 2.1, we have (*Please, make a correction of this statement!*)

$$x_2(t_1 - r_2) = e_1^* \quad \text{and} \quad x_2(t_2 - r_2) = e_1^{**}.$$

Similarly, we can get

$$x_1(t_1 - r_1 - r_2) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1 - r_2) = e_1^{**}.$$

Therefore, by induction, we can get

$$\begin{aligned} x_1(t_1 - k(r_1 + r_2)) &= e_1^*, \\ x_1(t_2 - k(r_1 + r_2)) &= e_1^{**}, \\ x_2(t_1 - r_2 - k(r_1 + r_2)) &= e_1^*, \\ x_2(t_2 - r_2 - k(r_1 + r_2)) &= e_1^{**}. \end{aligned}$$

Without loss of generality, we may assume that $t_1 < t_2$. Let

$$a_k = \int_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} F(x_1(s))\,ds \quad \text{and} \quad b_k = \int_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} F(x_2(s))\,ds.$$

Integrating (1.2), we get

$$\int_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} \frac{dx_1(s)}{dt}\,ds = \int_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} (-F(x_1(s)) + F(x_2(s-r_2)))\,ds$$

and

$$\int_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} \frac{dx_2(s)}{dt}\,ds = \int_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} (-F(x_2(s)) + F(x_1(s-r_1)))\,ds.$$

Thus,

$$e_1^{**} - e_1^* = -a_k + b_k \quad \text{and} \quad e_1^{**} - e_1^* = -b_k + a_{k+1}.$$

That is, $2(e_1^{**} - e_1^*) = a_{k+1} - a_k$. Summarizing up in the above equation as k goes from 1 to n, we get

$$\sum_{k=1}^{n} 2(e_1^{**} - e_1^*) = \sum_{k=1}^{n} (a_{k+1} - a_k).$$

Hence,

$$2n(e_1^{**} - e_1^*) = a_{n+1} - a_1 \le 2r_1 F(e_1^{**}),$$

which yields a contradiction by letting $n \to +\infty$.

Case 2. There exist $t_1, t_2 \in [-r_2, 0]$ such that $x_2(t_1) = e_1^*$ and $x_2(t_2) = e_1^{**}$.

Using a similar argument as that of Case 1, we can show that this is also a contradiction.

Case 3. There exist $t_1 \in [-r_1, 0]$ and $t_2 \in [-r_2, 0]$ such that $x_1(t_1) = e_1^*$ and $x_2(t_2) = e_1^{**}$.

Then, from (1.2), we know that

$$F(x_2(t_1 - r_2)) = F(x_1(t_1)) \quad \text{and} \quad F(x_1(t_2 - r_1)) = F(x_2(t_2)).$$

Thus,

$$x_2(t_1 - r_2) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1) = e_1^{**}.$$

Without loss of generality, we may assume that $t_1 < t_2$. Then,

$$0 \le t_1 - (t_2 - r_1) = t_1 + r_1 - t_2 \le r_1, \quad x_1(t_1) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1) = e_1^{**}.$$

Using a similar argument as that of Case 1, we can show that this is also a contradiction.

Case 4. There exist $t_1 \in [-r_1, 0]$ and $t_2 \in [-r_2, 0]$ such that $x_1(t_1) = e_1^{**}$ and $x_2(t_2) = e_1^*$.

Likewise, by using a similar argument as that of Case 3, it is easily shown that this is a contradiction.

Therefore, we can now conclude that $\omega(\varphi) = \{\widehat{e^*}\}$. This completes the proof.

If $r_1 = r_2 = r$ and consider the synchronized solutions of (1.2) with $x(t) = y(t) = \varphi(t)$ for $t \in [-\max\{\tau_1, \tau_2\}, 0]$, then, as an application of Theorem 3.1, we get the following result for system (1.1).

Corollary 3.1 *Every solution of system (1.1) tends to a constant as* $t \to \infty$.

Remark 3.1 If F in (1.1) is strictly increasing on R, then Corollary 3.1 has been proved by [13]. If, however, F is only assumed to be nondecreasing on R, then the result of Corollary 3.1 is actually new. For example, consider the case where

$$F(t) = \begin{cases} x^3, & t > 0, \\ 0, & -1 \le t \le 0, \\ (x+1)^3, & t < -1, \end{cases}$$

or

$$F(t) = \begin{cases} x-1, & t > 1, \\ 0, & -1 \le t \le 1, \\ x+1, & t < -1, \end{cases}$$

in (1.1), Corollary 3.1 can be applied to (1.1) while the corresponding result of [13] fails since, in this case, F is not strictly increasing on R.

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Existence of Nonoscillatory Solution of Third Order Linear Neutral Delay Difference Equation with Positive and Negative Coefficients

Xiao-Zhu Zhong[1], Hai-Long Xing[1], Yan Shi[2*], Jing-Cui Liang[1] and Dong-Hua Wang[1]

[1]*School of Sciences, Yanshan University, Qinghuangdao 066004, China*
[2]*School of Information Science, Kyushu Tokai University, Toroku, Kumamoto 862-8652, Japan*

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Abstract: In this paper, by using fixed point theorem, the problem of existence of the nonoscillatory solution for a class of neutral delay difference equations with both positive and negative coefficients has been investigated. Under the assumption of third order, a sufficient condition is proposed for the existence of the nonoscillatory solution. Further studies on the underlying problem have also been conducted.

Keywords: *Neutral delay difference equation; oscillation; positive and negative coefficients.*

Mathematics Subject Classification (2000): 35D05, 35E05.

1 Introduction

In recent years, there are many scholars who have devoted their researches to the differential equations with positive and negative coefficients and obtained some interesting results, see for example, [1−8] and the references therein. At the same time, the research on difference equations with positive and negative coefficients is getting people's attention and is becoming a new field of research [9−12]. In [12] the existence of positive solution of the second order difference equation with positive and negative coefficients was studied. In this paper we consider the third order equation

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2) = 0 \tag{1}$$

*Corresponding author: shi@ktmail.ktokai-u.ac.jp

where $p \in R$, $\tau \in \{1, 2, \dots\}$, $\delta_1, \delta_2 \in \{0, 1, 2, \dots\}$, $\{R_1(n)\}$, $\{R_2(n)\}$ are positive real sequences, and satisfy

$$\sum_{n=1}^{+\infty} n^2 R_i(n) < +\infty, \quad i = 1, 2. \tag{2}$$

The related conclusion in [12] is generalized in this paper to the case of third order equations. A sufficient condition for the existence of the positive solution of the equation (1) is obtained.

For simplicity, we list basic conceptions and symbols as follows:

Δ symbols for the forward difference operator, say $\Delta y(n) = y(n+1) - y(n)$;

Z symbols for the integer set and R for the real numbers set.

Assume $a \in Z$ and let $N(a) = \{a, a+1, \dots\}$, $N = N(0)$. For any given $a, b \in Z$ and $a \le b$, let $N(a, b) = \{a, a+1, \dots, b\}$.

The solution of the difference equation (1) is called eventually positive if there exists a positive integer M such that $x(n) > 0$ for $n \in N(M)$. If there exists a positive integer M such that $x(n) < 0$ for $n \in N(M)$, then is called eventually negative.

The solution of the difference equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

2 Main Results and Proofs

Theorem 2.1 *Suppose*

(i) $\sum\limits_{n=1}^{+\infty} n^2 R_i(n) < +\infty, \quad i = 1, 2, \quad n \in N(n_0);$ (3)

(ii) *there exists a positive integer T_1 which is sufficiently large such that, for any $\alpha > 0$, when $n > T_1$, we have*

$$\alpha R_1(n) - R_2(n) \ge 0; \tag{4}$$

(iii) $p \ne \pm 1.$ (5)

Then the equation (1) has an eventually positive solution.

Proof Let L_∞ denote the set of all the bounded real sequences $x = \{x(n)\}$ on $N(n_0)$, define the norm $\|x\| = \sup x(n)$, then L_∞ forms a Banach space. There are four situations to be contemplated:

Case 1: $0 \le p < 1$.

From (3) and (4), we select a positive integer $n_1 \ge \max\{T_1, n_0 + \delta\}$ which is large enough, where $\delta = \max\{\tau, \delta_1, \delta_2\}$, such that

$$\sum_{n=n_1}^{+\infty} n^2 [R_1(n) + R_2(n)] < 1 - p, \tag{6}$$

$$0 \le \sum_{n=n_1}^{+\infty} n^2 [M_2 R_1(n) - M_1 R_2(n)] \le p - 1 + M_2, \tag{7}$$

where M_1, M_2 are positive constants and satisfy

$$1 - M_2 < p \le \frac{1 - M_1}{1 + M_2}. \tag{8}$$

Let

$$A = \{x \in L_\infty \colon\ M_1 \le x(n) \le M_2,\ \ n \in N(n_0)\}. \tag{9}$$

It is clear that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon\ A \to L_\infty$ as following:

$$Tx(n) = \begin{cases} 1 - p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_1, \\ Tx(n_1), & n_0 \le n < n_1. \end{cases} \tag{10}$$

Now we shall prove that T is a self-mapping on A where there are two situations to be contemplated:

Case 1-a: $n \ge n_1$.

For any $x \in A$, from (9), (10), we find that

$$\begin{aligned} Tx(n) &= 1 - p - px(n-\tau) + \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\le 1 - p + \sum_{s=n}^{+\infty} s^2(R_1(s)M_2 - R_2(s)M_1), \end{aligned}$$

therefore from (7), we have

$$Tx(n) \le 1 - p + p - 1 + M_2 = M_2. \tag{11}$$

From (4) we have

$$\begin{aligned} &\sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\quad = \sum_{s=n}^{+\infty} C^2_{s+2-n}x(s-\delta_2)\left[\frac{x(s-\delta_1)}{x(s-\delta_2)}R_1(s) - R_2(s)\right] \ge 0. \end{aligned}$$

Hence we also have

$$Tx(n) \ge 1 - p - px(n-\tau).$$

Since $0 \le p < 1$, from (8) and (9), we get

$$Tx(n) \ge 1 - p - pM_2 \ge M_1. \tag{12}$$

Case 1-b: $n_0 \le n < n_1$.

For any $x \in A$, from (10) we know that

$$Tx(n) = Tx(n_1)$$

and from (11) and (12) we obtain

$$M_1 \le Tx(n_1) \le M_2.$$

Hence we have

$$M_1 \le Tx(n) \le M_2.$$

Considering the two cases of a and b, for any $x \in A,$ we have

$$M_1 \le Tx \le M_2.$$

Hence, $Tx \in A$, namely T is a self-mapping on A.

In what follows, we shall prove that T is a contraction mapping on A where there are also two situations to be contemplated:

Proof of Case 1-a. $n \ge n_1$.

For any $x_1, x_2 \in A$, we have

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| = \Bigg| &- px_1(n-\tau) + \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\ &+ px_2(n-\tau) - \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Bigg| \\ &\le |-px_1(n-\tau) + px_2(n-\tau)| + \sum_{s=n}^{+\infty} C_{s+2-n}^2 R_1(s)|x_1(s-\delta_1) - x_2(s-\delta_1)| \\ &+ \sum_{s=n}^{+\infty} C_{s+2-n}^2 R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|. \end{aligned}$$

Hence from (5), we have

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\le \left[p + \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s) + R_2(s))\right] \|x_1 - x_2\| \\ &\le \left[p + \sum_{s=n}^{+\infty} s^2 (R_1(s) + R_2(s))\right] \|x_1 - x_2\|. \end{aligned}$$

Then from (6), there exists $0 < q_1 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_1 \|x_1 - x_2\|.$$

Proof of Case 1-b. $n_0 \le n < n_1$.

From (10), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_1) - Tx_2(n_1)| \le q_1 \|x_1 - x_2\|.$$

In both the cases of a and b, for any $x_1, x_2 \in A$, $n \ge n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \le q_1 \|x_1 - x_2\|.$$

So T is a contraction mapping on A.

On summarizing the above cases we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely $Tx = x$, where $x = x(n)$ satisfies

$$Tx(n) = \begin{cases} 1 - p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \geq n_1, \\ Tx(n_1), & n_0 \leq n \leq n_1. \end{cases}$$

From this, the fixed point $x(n)$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2) = 0.$$

Hence this fixed point $x(n)$ is a positive solution of the equation (1).

Case 2: $1 < p$.

From (3) and (4), we select a positive integer $n_2 > t_1 > n_0$ which is large enough and satisfies

$$n_2 + \tau = n_0 + \max\{\delta_1, \delta_2\}$$

such that

$$\frac{1}{p}\sum_{n=n_2}^{+\infty} n^2[R_1(n) + R_2(n)] < 1 - \frac{1}{p}, \tag{13}$$

$$\sum_{n=n_2}^{+\infty} n^2[M_4R_1(n) - M_3R_2(n)] \leq 1 - p + pM_4, \tag{14}$$

where M_1, M_2 are positive constants and satisfy

$$(1 - M_3)p \geq 1 + M_4, \quad p(1 - M_4) < 1. \tag{15}$$

Let

$$A = \{x \in L_\infty\colon\ M_3 \leq x(n) \leq M_4,\ n \in N(n_0)\}. \tag{16}$$

It is clear that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon\ A \to L_\infty$ as follow:

$$Tx(n) = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+\tau}^{+\infty} C_{s+2-n-r}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \geq n_2, \\ Tx(n_2), & n_0 \leq n \leq n_2. \end{cases} \tag{17}$$

In the following, we shall prove that T is a self-mapping on A. Here there are still two situations to be discussed:

Case 2-a: $n \geq n_2$.

For any $x \in A$, from (16), (17) and $p > 1$ we find that

$$Tx(n) = 1 - \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C_{s+2-n-\tau}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2))$$

$$\leq 1 - \frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s)M_4 - R_2(s)M_3)$$

and therefore, from (14) ,we have

$$Tx(n) \leq 1 - \frac{1}{p} + \frac{1}{p}(1 - p + M_4) = M_4. \tag{18}$$

Since from (4), we have

$$\sum_{s=n+\tau}^{+\infty} C_{s+2-n-\tau}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2))$$
$$= \sum_{s=n+\tau}^{+\infty} C_{s+2-n-\tau}^{2} x(s-\delta_2)\left[\frac{x(s-\delta_1)}{x(s-\delta_2)}R_1(s) - R_2(s)\right] \geq 0.$$

Hence we also have

$$Tx(n) \geq 1 - \frac{1}{p} - \frac{1}{p}x(n+\tau).$$

Since $p > 1$, from (15) and (16), we get

$$Tx(n) \geq 1 - \frac{1}{p} - \frac{1}{p}M_4 \geq M_3. \tag{19}$$

Case 2-b: $n_0 \leq n < n_2$.
For any $x \in A$, from (17) we find that

$$Tx(n) = Tx(n_2).$$

Then, from (18) and (19) we obtain

$$M_3 \leq Tx(n_2) \leq M_4.$$

Hence

$$M_3 \leq Tx(n) \leq M_4.$$

Based on the two cases of a and b, for any $x \in A$, we have

$$M_3 \leq Tx \leq M_4.$$

Hence $Tx \in$A, namely, T is a self-mapping on A.

In what follows, we shall prove that T is a contraction mapping on A where following two situations need to be discussed.

Proof of Case 2-a. $n \geq n_2$.

For any $x_1, x_2 \in A$, we have

$$
\begin{aligned}
&|Tx_1(n) - Tx_2(n)| \\
&\quad = \Big| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\
&\qquad + \frac{1}{p}x_2(n+\tau) - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Big| \\
&\quad \le \Big| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}x_2(n+\tau)\Big| + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}R_1(s)|x_1(s-\delta_1) \\
&\qquad - x_2(s-\delta_1)| + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|.
\end{aligned}
$$

So from (5) we have

$$
\begin{aligned}
|Tx_1(n) - Tx_2(n)| &\le \left[\frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s) + R_2(s))\right]\|x_1 - x_2\| \\
&\le \left[\frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s) + R_2(s))\right]\|x_1 - x_2\|.
\end{aligned}
$$

Then, from (13), there exists $0 < q_2 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_2\|x_1 - x_2\|.$$

Proof of Case 2-b. $n_0 \le n < n_2$.
From (17), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_2) - Tx_2(n_2)| \le q_2\|x_1 - x_2\|.$$

Considering the cases of a and b, for any $x_1, x_2 \in A$, $n \ge n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \le q_2\|x_1 - x_2\|.$$

So T is a contraction mapping on A.Based on the above discussion we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$
Tx(n) = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+\tau}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_2, \\ Tx(n_2), & n_0 \le n < n_2. \end{cases}
$$

Therefore this fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n)+px(n-\tau)]+R_1(n)x(n-\delta_1)-R_2(n)x(n-\delta_2)=0.$$

Hence this fixed point $\{x(n)\}$ is a positive solution of the equation (1).

Case 3: $-1<p<0$.

From (3) and (4), we select a positive integer $n_3 \geq \max\{T_1, n_0+\delta\}$ where $\delta = \max\{\tau, \delta_1, \delta_2\}$, such that

$$\sum_{n=n_3}^{+\infty} n^2[R_1(n)+R_2(n)] < p+1, \tag{20}$$

$$0 \leq \sum_{n=n_3}^{+\infty} n^2[M_6R_1(n)-M_5R_2(n)] \leq (p+1)(M_6-1), \tag{21}$$

where M_5 and M_6 are positive constants and satisfy

$$0<M_5\leq 1<M_6. \tag{22}$$

Let

$$A=\{x\in L_\infty\colon\ M_5\leq x(n)\leq M_6,\ \ n\in N(n_0)\}. \tag{23}$$

It is obvious that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon\ A\to L_\infty$ as follow:

$$Tx(n)=\begin{cases} 1+p-px(n-\tau)+ & \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^2(R_1(s)x(s-\delta_1)-R_2(s)x(s-\delta_2)), & n\geq n_3, \\ Tx(n_3), & n_0\leq n<n_3. \end{cases} \tag{24}$$

We shall prove that T is a self-mapping on A where the following two situations are to be discussed.

Case 3-a: $n\geq n_3$.

For any $x\in A$, from (23), (24), we find that

$$\begin{aligned} Tx(n) &= 1+p-px(n-\tau)+\sum_{s=n}^{+\infty} C_{s+2-n}^2(R_1(s)x(s-\delta_1)-R_2(s)x(s-\delta_2)) \\ &\leq 1+p-pM_6+\sum_{s=n}^{+\infty} s^2(R_1(s)M_6-R_2(s)M_5) \end{aligned}$$

and therefore, from (21), we have

$$Tx(n)\leq 1+p-pM_6+(1+p)(M_6-1)=M_6. \tag{25}$$

From (4), we have

$$\sum_{s=n}^{+\infty} s^2(R_1(s)M_6-R_2(s)M_5)\geq 0.$$

Hence, from (22) and (23), we have

$$\begin{aligned} Tx(n) &\geq 1 + p - pM_5 = (1+p) - (1+p)M_5 + M_5 \\ &= (1+p)(1-M_5) + M_5 = M_5. \end{aligned} \tag{26}$$

Case 3-b: $n_0 \leq n < n_3$.
For any $x \in A$, from (24) we find that

$$Tx(n) = Tx(n_3).$$

Then, from (25) and (26) we obtain

$$M_5 \leq Tx(n_3) \leq M_6.$$

Hence

$$M_5 \leq Tx(n) \leq M_6.$$

In both cases of a and b, for any $x \in A$, we have

$$M_5 \leq Tx \leq M_6,$$

namely, $Tx \in A$. Hence, T is a self-mapping on A. Now we shall prove that T is a contraction mapping on A under the two situations bellow.

Proof of Case 3-a. $n \geq n_3$.
For any $x_1, x_2 \in A$, we have

$$\begin{aligned} &|Tx_1(n) - Tx_2(n)| \\ &\quad = \Big| - px_1(n-\tau) + \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\ &\quad + px_2(n-\tau) - \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))| \\ &\quad \leq |-px_1(n-\tau) + px_2(n-\tau)| + \sum_{s=n}^{+\infty} C^2_{s+2-n}R_1(s)|x_1(s-\delta_1) \\ &\quad - x_2(s-\delta_1)| + \sum_{s=n}^{+\infty} C^2_{s+2-n}R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|. \end{aligned}$$

Hence from (5), the following inequality is hold

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\leq \left[p + \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s) - R_2(s))\right] \|x_1 - x_2\| \\ &\leq \left[p + \sum_{s=n}^{+\infty} s^2(R_1(s) - R_2(s))\right] \|x_1 - x_2\|. \end{aligned}$$

According to (20), there exists $0 < q_3 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_3 \|x_1 - x_2\|.$$

Proof of Case 3-b. $n_0 \le n < n_3$.
From (20), (24) we have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_3) - Tx_2(n_3)| \le q_3 \|x_1 - x_2\|.$$

In both cases of a and b, for any $x_1, x_2 \in A$, when $n \ge n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \le q_3 \|x_1 - x_2\|.$$

So T is a contraction mapping on A. Based on the Banach contraction mapping principle we know that there exist a fixed point x of T on A, say, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$Tx(n) = \begin{cases} 1 + p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_3, \\ Tx(n_3), & n_0 \le n < n_3. \end{cases}$$

Thus this fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2).$$

Hence, this fixed point $\{x(n)\}$ is a positive solution of the equation (1).

Case 4: $p < -1$

From (3) and (4), we select a positive integer $n_4 > T_1 > n_0$ which is large enough to satisfy

$$n_4 + \tau \ge n_0 + \max\{\delta_1, \delta_2\}$$

such that

$$\sum_{n=n_4}^{+\infty} n^2[R_1(n) + R_2(n)] < -p - 1, \tag{27}$$

$$\sum_{n=n_4}^{+\infty} n^2[M_8 R_1(n) - M_7 R_2(n)] \le (p+1)(M_7 - 1) \tag{28}$$

where M_7, M_8 are positive constants and satisfy

$$0 < M_7 < 1 < M_8. \tag{29}$$

Let

$$A = \{x \in L_\infty :\ M_7 \le x(n) \le M_8,\ n \in N(n_0)\}. \tag{30}$$

It is obvious that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon A \to L_\infty$ as follow:

$$Tx(n) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+r}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_4, \\ Tx(n_4), & n_0 \le n < n_4. \end{cases} \tag{31}$$

We shall prove that T is a self-mapping on A under the following two situations.

Case 4-a: $n \ge n_4$.

For any $x \in A$, from (28), (30), (31) and $p < -1$ we find that

$$\begin{aligned} Tx(n) &= 1 + \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)M_8 - R_2(s)M_7) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s)M_8 - R_2(s)M_7) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}(p+1)(M_7 - 1) = M_7. \end{aligned} \tag{32}$$

Since from (4), we have

$$\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ = \sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}x(s-\delta_2)\left[\frac{x(s-\delta_1)}{x(s-\delta_2)}R_1(s) - R_2(s)\right] \ge 0.$$

Hence, from (29), (30), we have

$$\begin{aligned} Tx(n) &= 1 + \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\le 1 + \frac{1}{p} - \frac{1}{p}M_8 \le M_8. \end{aligned} \tag{33}$$

Case 4-b: $n_0 \le n < n_4$.

For any $x \in A$, from (31) we find that

$$Tx(n) = Tx(n_4).$$

Then, from (32) and (33) we obtain

$$M_7 \le Tx(n_4) \le M_8.$$

Hence, we have

$$M_7 \le Tx(n) \le M_8.$$

In both two cases of a and b, for any $x \in A$, we have

$$M_7 \le Tx \le M_8.$$

Hence $Tx \in A$, namely, T is a self-mapping on A. We shall prove that T is a contraction mapping on A as bellow.

Proof of Case 4-a. $n \ge n_4$.
For any $x_1, x_2 \in A$, we have

$$\begin{aligned}
&|Tx_1(n) - Tx_2(n)| \\
&\qquad = \Bigg| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\
&\qquad + \frac{1}{p}x_2(n+\tau) - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Bigg| \\
&\qquad \le \Bigg| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}x_2(n+\tau)\Bigg| - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 R_1(s)|x_1(s-\delta_1) - x_2(s-\delta_1)| \\
&\qquad - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)| \\
&\qquad \le \Bigg[-\frac{1}{p} - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s) + R_2(s))\Bigg]\|x_1 - x_2\|
\end{aligned}$$

and from (27), we know

$$\frac{1}{p} - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s) + R_2(s)) < 1.$$

Then, there exists $0 < q_4 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_4\|x_1 - x_2\|.$$

Proof of Case 4-b. $n_0 \le n < n_4$.
From (31), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_4) - Tx_2(n_4)| \le q_4\|x_1 - x_2\|.$$

In both cases of a and b, for any $x_1, x_2 \in A$, $n \ge n_0$, we have

$$|Tx_1 - Tx_2| \le q_4\|x_1 - x_2\|.$$

So T is a contraction mapping on A. Based on the above analysis, we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$Tx(n) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+r}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \geq n_4, \\ Tx(n_4), & n_0 \leq n < n_4. \end{cases}$$

From this, the fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2x(n-\delta_2) = 0.$$

Hence, this fixed point $\{x(n)\}$ is a positive solution of the equation (1). Therefore the theorem is proved.

3 Conclusions

Under the conditions of third order, this paper studies the existence of the nonoscillatory solution of the neutral delay difference equation with positive and negative coefficients and gets a sufficient condition for the existence of the nonoscillatory solution. We can find that the similar results of the second order difference equation in the literature [12] have been successfully extended to the third order one. This will naturally urge us to consider whether the high order one has the similar results. When we study this problem, the way applied in this paper can be helpful to us.

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