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A Fredholm Operator and Solution Sets to Evolution Systems

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Abstract: In this paper we deal with the Peano phenomenon for general initial boundary-value problems of quasilinear evolution systems with arbitrary even order space derivatives. The nonlinearity is a continuous or continuously Frechét differentiable function. Qualitative and quantitative structure of solution sets is studied by the theory of proper, Fredholm and Nemitskiĭ operators. These results can be applied to the different technical and natural science models.

Keywords: *Evolution systems; an initial boundary-value problem; a linear Fredholm operator; a proper and coercive operator; a bifurcation point; a surjectivity.*

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0 Introduction

The Peano phenomenon of the existence of a solution continuum of the initial value problem for ordinary differential systems is well-known. This phenomenon has been studied by many authors in [3 – 5, 8, 17, 27]. The structure of solution sets for second order partial differential problems was observed in the authors papers [12, 13].

In this paper we shall study generic properties of quasilinear initial boundary-value problems for evolution systems of an even order with the continuous or continuous differentiable nonlinearities and the general boundary value conditions. In special Hölder spaces we use the Nikoľskiĭ decomposition theorem from [29, P. 233] for linear Fredholm operators, the global inversion theorem of [9, 6] and [7, PP. 42 – 43] and the Ambrosetti

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solution quantitive results from [2, P. 216]. In the consideration on surjectivity the generalized Leray–Schauder condition is employed which is similar to that one in [20]. In the case of nonlinear Fredholm operators we use the main Quinn and Smale theorem from [22] and [24].

The present results allow us to observe different problems describing dynamics of mechanical processes (bendding, vibration), phisycal-heating processes, reaction-diffusion processes in chemical and biological technologies or in the ecology.

1 The Formulation of Problem, Assumptions and Spaces

The set $\Omega \subset R^n$ for $n \in N$ means a bounded domain with the boundary $\partial\Omega$. The real number T will be positive and $Q = (0,T] \times \Omega,\ \Gamma = (0,T] \times \partial\Omega$. If the multiindex $k = (k_1, \dots, k_n)$ with $|k| = \sum\limits_{i=1}^{n} k_i$, then we use the notation D_x^k for the differential operator $\dfrac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$ and D_t for $\dfrac{\partial}{\partial t}$. If the module $|k| = 0$ then D_x^k means an identity mapping. The symbol $cl\,M$ means the closure of the set M in R^n.

In this paper we consider the general system of $p \geq 1$ nonlinear differential equations (parabolic or non-parabolic type) of an arbitrary even order $2b$ (b is a positive integer) with p unknown functions in the column vector form $(u_1, \dots, u_p)^{\mathrm{T}} = u\colon cl\,Q \to R^p$. Its matrix form is given as follows:

$$A(t,x,D_t,D_x)u + f(t,x,\overline{D}_x^{\gamma}u) = g(t,x) \quad \text{for} \quad (t,x) \in Q, \tag{1.1}$$

where

$$A(t,x,D_t,D_x)u = D_t u - \sum_{|k|=2b} a_k(t,x) D_x^k u - \sum_{0 \leq |k| \leq 2b-1} a_k(t,x) D_x^k u,$$

and $\overline{D}_x^{\gamma}u$ is a vector function whose components are derivatives $D_x^{\gamma}u_l$ with the different multiindices $0 \leq |\gamma| \leq 2b-1$ for $l = 1, \dots, p$.

The system of boundary conditions is given by the vector equation with the bp components

$$B(t,x,D_x)u\Big|_{cl\,\Gamma} = (B_1(t,x,D_x)u, \dots, B_{b\,p}(t,x,D_x)u)^{\mathrm{T}}\big|_{cl\,\Gamma} = 0 \tag{1.2}$$

in which

$$B_j(t,x,D_x)u = \sum_{0 \leq |k| \leq r_j} b_{jk}(t,x) D_x^k u$$

for an integer $0 \leq r_j \leq 2b-1$ and $j = 1, \dots, bp$.

Further the initial value homogeneous condition

$$u(0,x) = 0 \quad \text{for} \quad x \in \bar{\Omega} \tag{1.3}$$

is considered.

Here the given functions are the following mappings: $a_k = (a_k^{hl})_{h,l=1}^{p}\colon cl\,Q \to R^{p^2}$ for $0 \leq |k| \leq 2b$ are $(p \times p)$-matrix functions; $b_{jk} = (b_{jk}^1, \dots, b_{jk}^p)\colon cl\,\Gamma \to R^p$ for $0 \leq |k| \leq r_j,\ j = 1, \dots, b_p$ are row vector functions; $f = (f_1, \dots, f_p)^{\mathrm{T}}\colon cl\,Q \times R^{\kappa} \to R^p$

and $g = (g_1, \dots, g_p)^{\mathrm{T}} : cl\, Q \to R^p$ are column vector functions, where κ is a positive integer given by the inequality

$$\kappa \le \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \dots + \binom{n+|\gamma|-2}{|\gamma|-1} + \binom{n+|\gamma|-1}{|\gamma|} \right] p.$$

Under several supplementary assumptions, problem (1.1) – (1.3) defines homeomorphism between some Hölder spaces. Now, we formulate these suppositions.

(P) A δ-uniform parabolic condition holds for system (1.1) in the sense of J.G. Petrovskiiĭ, $\delta > 0$.

The system (1.1) and boundary condition (1.2) are connected by

(C) a δ^+-uniform complementary condition with $\delta^+ > 0$ and

(Q) a compatibility condition.

The coefficients of the operator $A(t, x, D_t, D_x)$ from (1.1) and of $B(t, x, D_x)$ from (1.2) and the boundary $\partial\Omega$ satisfy

($\mathrm{S}^{l+\alpha}$) a smoothness condition for a nonnegative integer l and a number $\alpha \in (0, 1)$.

We shall be employed with the Banach spaces of continuously differentiable functions $C^l_x(cl\, Q, R^p)$ and $C^{l/2b,l}_{t,x}(cl\, Q, R^p)$ and the Hölder spaces $C^{l+\alpha}_x(cl\, Q, R^p)$, $C^{(l+\alpha)/2b,l+\alpha}_{t,x}(cl\, Q, R^p)$ for a nonnegative integer l and $\alpha \in (0, 1)$.

For the exact definition of conditions (P), (C), (Q), ($S^{l+\alpha}$) see [19, PP. 12–21] and for the definition of spaces see [19, PP. 8–12] or [11].

The homeomorphism result for (1.1) – (1.3) can be formulated as follows:

Proposition 1.1 (see [19, P. 21] and [15, PP. 182–183]) Let the conditions (P), (C) and (S^α) be satisfied for $\alpha \in (0, 1)$. Necessary and sufficient conditions for the existence and uniqueness of the solution

$$u \in C^{(2b+\alpha)/2b,2b+\alpha}_{t,x}(cl\, Q, R^p)$$

of linear problem (1.1)–(1.3) for $f = 0$ is

$$g \in C^{\alpha/2b,\alpha}_{t,x}(cl\, Q, R^p)$$

and the compatibility condition (Q).

Moreover, there exists a constant $c > 0$ independent of g such that

$$c^{-1}\|g\|_{\alpha/2b,\alpha,Q,p} \le \|u\|_{(2b+\alpha)/2b,2b+\alpha,Q,p} \le c\|g\|_{\alpha/2b,\alpha,Q,p}$$

2 General Results

In this part we remind some notions and assertions from the nonlinear functional analysis applied in the fundamental lemmas and theorems.

Throughout this paper we shall assume that X and Y are Banach spaces either both over the real or complex field.

In the Zeidler books [31, PP. 365–366] and [32, PP. 667–668] we find definitions of the linear and nonlinear Fredholm operator.

The following proposition gives the necessary and sufficient condition for a linear operator to be Fredholm.

Proposition 2.1 (S.M. Nikoľskiĭ [29, P. 233]) A linear bounded operator $A\colon X \to Y$ is Fredholm of the zero index iff $A = C+T$, where $C\colon X \to Y$ is a linear homeomorphism and $T\colon X \to Y$ is a linear completely continuous operator.

In the theory and applications of nonlinear operators, the notions as a proper, *σ-proper*, closed, coercive operator (for definitions see books [31] and [32]) are very frequent. Their significant application gives the following statements.

Proposition 2.2 (the Ambrosetti theorem [2, P. 216]) Let $F \in C(X,Y)$ be a proper mapping. Then the cardinal number $\operatorname{card} F^{-1}(q)$ of the set $F^{-1}(q)$ is constant and finite (it may be zero) for every q taken from the same component (nonempty and connected subset) of the set $Y \setminus F(\Sigma)$. Here Σ means a closed set of all points $u \in X$ at which F is not locally invertible.

A relation between the local invertibility and homeomorphism of X onto Y gives the global inverse mapping theorem.

Proposition 2.3 (R. Cacciopoli [9], E. Zeidler [31, P. 174]) Let $F \in C(X,Y)$ be a locally invertible mapping in X. Then F is a homeomorphism of X onto Y iff F is proper.

The following propositions give necessary and sufficient conditions for the proper mapping.

Proposition 2.4 (see [31, P. 176], [23, P. 49] and [27, P. 20]) Let $F \in C(X,Y)$.

(i) If F is proper, then F is a nonconstant closed mapping.
(ii) If $\dim X = +\infty$ and F is a nonconstant closed mapping, then F is proper.

Proposition 2.5 (see [23, PP. 58–59], [31, P. 498] and [27, P. 20]) Suppose that $F\colon X \to Y$ and $F = F_1 + F_2$, where

(i) $F_1 : X \to Y$ is a continuous proper mapping on X and
(ii) $F_2 : X \to Y$ is complete continuous.

Then

(i) the restriction of the mapping F to an arbitrary bounded closed set in X is a proper mapping;
(ii) if moreover, F is coercive, then F is a proper mapping.

Now we can formulate some sufficient conditions for the surjectivity of an operator.

Proposition 2.6 (see [27, PP. 24 and 27]) Let X be a real Banach space. Suppose

(i) $P = I - f\colon X \to X$ is a condensing field, where $I\colon X \to X$ is the identity,
(ii) P is coercive,
(iii) there exists a strictly solvable field $G = I - g\colon X \to X$ and $R > 0$ such that for all solutions $u \in X$ of the equation

$$P(u) = kG(u)$$

and for all $k < 0$ the estimation $\|u\|_X < R$ holds.

Then the following statements are true:

(i) P is a proper mapping,
(ii) P is strictly surjective,
(iii) $\operatorname{card} F^{-1}(q)$ is constant, finite and nonzero for every q from the same connected component of the set $Y \setminus F(\Sigma)$. For Σ see Proposition 2.2.

The definition of a condensing field is understood in the sense given in [10, P. 69]. For the definition of a strict solvable field and strict surjective field see in [29].

Remark 2.1 It is clear that an operator F is strictly surjective, then it is surjective and if F is strictly solvable, then it is also solvable. Moreover, if F is strictly surjective, then it is strictly solvable, too.

Proposition 2.7 (the Schauder invariance of domain theorem [31, P. 705]) Let $F\colon (M \subseteq X) \to X$ be continuous and locally compact perturbation of identity on the open nonempty set M in the Banach space X. Then

(i) if F is locally injective on M so F is an open mapping;
(ii) if F is injective on M so F is a homeomorphism from M onto the open set $F(M)$.

For the compact perturbation of C^1–Fredholm operator we shall use the following proposition.

Proposition 2.8 (E. Zeidler [32, P. 672]) Let $A\colon D(A) \subset X \to Y$ be a C^1–Fredholm operator on the open set $D(A)$ and $B\colon D(A) \to Y$ be a compact mapping from the class C^1. Then $A+B\colon D(A) \to Y$ is a Fredholm (possible nonlinear) operator with the same index as A at each point of $D(A)$.

In the following propositions we use the notion of *a regular, singular, critical point* of an operator and *a regular, singular values* of operators. The reader finds these definitions in [32, P. 668] or [31, P. 184].

Also, we need a residual set. A subset of a topological space Z is called *residual* iff it is a countable intersection of dense and open subsets of Z.

By the Baire theorem in any complete metric space or locally compact Hausdorff topological space, a residual set is dense in this space.

The most important theorem for nonlinear Fredholm mappings is due to S. Smale [24, P. 862] and Quinn [22]. It is also in [7, PP. 11–12].

Proposition 2.9 (a Smale–Quinn Theorem) If $F\colon X \to Y$ is a Fredholm mapping (possible nonlinear) of the class $C^k(X,Y)$ in the Frechét sense and either

(i) X has a countable basis (S. Smale) or
(ii) F is σ-proper (Quinn),

then the set R_F of all regular values of F is residual in Y. Moreover, if F is proper, then R_F is open and dense set in Y.

A necessary and sufficient condition for a local diffeomorphism (see [31, p. 171]) is given in the following proposition.

Proposition 2.10 (a Local Inverse Mapping Theorem, [31, p. 172]) Let $F\colon U(u_0) \subset X \to Y$ be a C^1-mapping in the Frechét sense. Then F is a local C^1-diffeomorphism at u_0 iff u_0 is a regular point of F.

Proposition 2.11 ([23, P. 89]) Let $\dim Y \geq 3$ and $F\colon X \to Y$ be a Fredholm mapping of the zero index. If $u_0 \in X$ is an isolated singular point of F, then F is locally invertible at u_0.

To illustrate the following results we shall need estimations of a Green $p \times p$-matrix for linear problem (1.1)–(1.3).

Lemma 2.1 *Let the assumptions* (P), (C), (S^α) *be satisfied for* $\alpha \in (0,1)$. *Then we have for the Green matrix* G *of linear problem (1.1) – (1.3) with* $f = 0$

$$|D_t^{k_0} D_x^k G(t,x;\tau,\xi)| \le c(t-\tau)^{-\mu} \|x-\xi\|_{R^n}^{2b\mu-(n+2bk_0+|k|)} E \tag{2.1}$$

for $0 \le 2bk_0 + |k| \le 2b$ *and* $\mu \le (n + 2bk_0 + |k|)/2b$, *thereby* $0 \le \tau < t \le T$ *and* $x, \xi \in cl\,\Omega$, $x \ne \xi$. *The positive constant* c *does not depend on* t, x, τ, ξ *and* E *means the* $p \times p$*-matrix consisting only of units,* $r = 2b/(2b-1)$.

Proof Since $n+2bk_0+|k|-2b\mu \ge 0$ and $\|x-\xi\|_{R^n} < \operatorname{diam}\Omega$ so for $0 < \delta \le t-\tau \le T$ we obtain (2.1) by the estimation (see [15, PP. 182–183])

$$\begin{aligned}
|D_t^{k_0} D_x^k G(t,x;\tau,\xi)| &\le c_1 (t-\tau)^{-\frac{n+2bk_0+|k|}{2b}} \exp\{-c_2 \frac{\|x-\xi\|_{R^n}^r}{(t-\tau)^{1/(2b-1)}}\} \\
&\le c_1 (t-\tau)^{-\mu} \|x-\xi\|_{R^n}^{2b\mu-(n+2bk_0+|k|)} \\
&\times [\, \|x-\xi\|_{R^n}^{2b}/(t-\tau)\,]^{(n+2bk_0+|k|-2b\mu)/2b} \exp\{-c_2[\, \|x-\xi\|_{R^n}^{2b}/(t-\tau)\,]^{1/(2b-1)}\} E.
\end{aligned}$$

If $0 < t-\tau < \delta$ with respect to

$$\lim_{y\to+\infty} y^u \exp\{-cy^v\} = 0$$

for every $u, v \in R$ and $c > 0$, we get estimation (2.1).

Remark 2.2 For any $x = (x_1, \dots, x_n) \in R^n$ the inequalities

$$c_n \sum_{i=1}^n |x_i| \le \|x\|_{R^n} \le \sum_{i=1}^n |x_i| \tag{2.2}$$

hold, if $c_n \in (0, 1/(\sqrt{2})^{n-1})$, $n \in N$, does not depend of x.

Remark 2.3 Also, we see that the mild solution $u \in C_x^{|\gamma|}(cl\,Q, R)$ of problem (1.1)–(1.3) satisfies the column vector integro-differential equation

$$\begin{aligned}
u(t,x) &= \int_0^t d\tau \int_\Omega G(t,x;\tau,\xi)\,[\,g(\tau,\xi) - f(\tau,\xi,\overline{D}^\gamma u(\tau,\xi))\,]\,d\xi =: \\
&= (Su)(t,x) \quad \text{for} \quad (t,x) \in cl\,Q
\end{aligned} \tag{2.3}$$

for $0 \le |\gamma| \le 2b-1$ and on the contrary the solution $v \in C_x^{|\gamma|}(cl\,Q, R^p)$ satisfying (2.3) is a mild solution of (1.1) – (1.3).

3 Operator Formulation and Fundamental Lemmas

Consider the following operators:

(i)

$$A \colon X \to Y, \tag{3.1}$$

where

$$(Au)(t,x) = A(t,x,D_t,D_x)u(t,x) = D_t u(t,x) - \sum_{0\le |k|\le 2b} a_k(t,x)D_x^k u(t,x)$$

for $(t,x)\in cl\,Q,\ u\in X$,

$$X = \{u\in X_\rho;\, B_j(t,x,D_x)u|_\Gamma = 0, \quad j=1,2,\dots,bp,$$
$$u(0,x) = 0 \quad \text{for} \quad x\in cl\,Q\} \subset C(cl\,Q,\, R^p).$$

Here

$$X_\rho \subset C_{t,x}^{(2b+\alpha)/2b,2b+\alpha}(cl\,Q, R^p)$$

is the Banach space of continuous functions $u\colon cl\,Q\to R^p$ with the continuous derivatives $D_x^k u$ for $|k| = 1,\dots,2b$ and $D_x^{k_0}D_x^k u$ for $1\le 2bk_0+|k|\le 2b$ on $cl\,Q$ and with the finite norm

$$\|u\|_{X_\rho} = \max_{l=1,\dots,p}\Big[\sum_{0\le 2bk_0+|k|\le 2b}\ \sup_{(t,x)\in cl\,Q}\Big|D_t^{k_0}D_x^k u_l(t,x)\Big| + \langle D_t u_l\rangle_{x,\alpha,Q}^y$$
$$+ \sum_{|k|=2b}\langle D_x^k u_l\rangle_{x,\alpha+\rho,Q}^y + \langle D_t u_l\rangle_{t,\alpha/2b,Q}^s$$
$$+ \sum_{|k|=1}^{2b-1}\langle D_x^k u_l\rangle_{t,(2b+\alpha-|k|)/2b,Q}^s + \sum_{|k|=2b}\langle D_x^k u_l\rangle_{t,(\alpha+\rho)/2b,Q}^s\Big],$$

where $\rho>0$ and $\alpha+\rho<1$. Further

$$Y = TX \subset C_{t,\,x}^{\alpha/2b,\,\alpha}(cl\,Q, R^p)$$

for $\alpha\in(0,1)$ with the norm

$$\|u\|_Y = \max_{l=1,\dots,p}\Big[\sup_{(t,x)\in cl\,Q}|u_l(t,x)| + \langle u_l\rangle_{x,\alpha,Q}^y + \langle u_l\rangle_{t,\alpha/2b,Q}^s\Big].$$

We understand

$$\langle v\rangle_{t,\mu,Q}^s = \sup_{\substack{(t,x),(s,x)\in cl\,Q\\ t\ne s}} \frac{|v(t,x)-v(s,x)|}{|t-s|^\mu},$$

$$\langle v\rangle_{x,\mu,Q}^y = \sup_{\substack{(t,x),(t,y)\in cl\,Q\\ x\ne y}} \frac{|v(t,x)-v(t,y)|}{\|x-y\|_{R^n}^\mu}.$$

for $v\colon cl\,Q\to R$.

(ii) The *Nemitskiĭ operator*

$$N\colon X\to Y, \tag{3.2}$$

where

$$(Nu)(t,x) = (f\circ u)(t,x) = f(t,x,\overline{D}_x^{\gamma}u(t,x))$$

for $(t,x)\in cl\,Q,\ u\in X$.

(iii) The operator

$$F\colon X\to Y, \tag{3.3}$$

where

$$(Fu)(t,x) = (Au)(t,x) + (Nu)(t,x) \quad \text{for} \quad (t,x)\in cl\,Q, \quad u\in X.$$

Together with the solution sets of given problem (1.1) − (1.3) we shall search for the bifurcation points sets.

Definition 3.1 (i) A couple $(u,g) \in X \times Y$ will be called the *bifurcation point* of (1.1) – (1.3) iff u is a solution of this problem and there exists a sequence $\{g_k\}_{k \in N} \subset Y$ such that $\lim_{k \to \infty} g_k = g$ in Y and initial boundary value problem (1.1) – (1.3) with $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = u$ in X.

(ii) The set of all solutions $u \in X$ of (1.1) – (1.3) (or the set of all functions $g \in Y$) such that (u,g) is a bifurcation point of (1.1) – (1.3) will be called the *domain of bifurcation* (the *bifurcation range*) of (1.1) – (1.3).

Example 3.1 The point $(u_r, 0) \in X \times Y$ for $r \in \langle 0, T\rangle$ is a bifurcation point of the Neumann problem (parabolic and non-parabolic)

$$\frac{\partial u}{\partial t} = \pm \frac{\partial^2 u}{\partial x^2} + f(t,x,u), \quad (t,x) \in (0,T\rangle \times \Omega = Q \subset R^2, \tag{3.1*}$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \qquad t \in \langle 0, T\rangle, \tag{3.2*}$$

$$u(0,x) = 0, \qquad x \in \overline{\Omega} \tag{3.3*}$$

for $f(t,x,u) = |u|^{1/2} - au$, $a > 0$. Here for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0, & \text{if} \quad (t,x) \in \langle 0, r\rangle \times \overline{\Omega}, \\ \dfrac{1}{\alpha^2}\Big(1 - \exp\Big\{-\dfrac{a}{2}(t-r)\Big\}\Big)^2, & \text{if} \quad (t,x) \in (r, T\rangle \times \overline{\Omega}. \end{cases}$$

The functions $u_0(t,x) = \frac{1}{\alpha^2}(1 - \exp\{-at/2\})^2$, $u_T(t,x) = 0$ are solutions of the given problem, too.

Really, there is the zero sequence $\{g_k\}_{k\in N}$ of the right hand side of (1.1) for which there exist two different sequences of solutions

$$\{u_k\}_{k\in N} = \Big\{u_{\frac{r(k+1)}{k+2}}\Big\}_{k\in N} \quad \text{and} \quad \{v_k\}_{k\in N} = \Big\{v_{\frac{rk}{k+1}} v\Big\}_{k\in N}$$

with the same limit $u_r \in X$.

The following equivalence result is true.

Lemma 3.1 (i) *The function $u \in X$ is a solution of initial boundary-value problem (1.1) – (1.3) for $g \in Y$ iff $Fu = g$.*

(ii) *The couple $(u,g) \in X \times Y$ is a bifurcation point of (1.1) – (1.3) iff $Fu = g$ and u is a point at which F is not locally invertible, i.e. $u \in \Sigma$.*

Proof The first assertion is clear.

If (u,g) is a bifurcation point of (1.1) – (1.3), then with respect to Definition 3.1 we get $F(u) = g$, $F(u_k) = g_k = F(v_k)$, $u_k \neq v_k$. Thus F is not locally injective at u. Hence, F is not locally invertible at u, i.e. $u \in \Sigma$. On the contrary, if F is not locally invertible at u and $F(u) = g$, then F is not locally injective at u. Hence, it follows that the couple $(u,g) \in X \times Y$ is a bifurcation point of (1.1) – (1.3). The second assertion is proved.

The following lemma gives sufficient conditions under which the operator A is a Fredholm type.

Assumption A.1 There exists a linear homeomorphism $H\colon X \to Y$ with

$$Hu = D_t u - H(t,x,D_x)u, \quad u \in X,$$

where

$$H(t,x,D_x)u = \sum_{|k|=2b} h_k(t,x)D_x^k u + \sum_{0\le|k|\le 2b-1} h_k(t,x)D_x^k u$$

satisfies $(\mathrm{S}^{\alpha+\rho})$ for $\alpha\in(0,1)$, $\rho>0$, $\alpha+\rho<1$.

Lemma 3.2 *Let the operator A from (3.1) satisfy the smoothness hypothesis* $(\mathrm{S}^{\alpha+\rho})$, $\alpha\in(0,1)$, $\rho>0$, $\alpha+\rho<1$ *(it has not to satisfy the conditions* (P), (C), (Q)*). Further let Assumption A.1 hold.*

Then

(i) $\dim X = +\infty$;

(ii) *the operator $A\colon X\to Y$ is a linear bounded Fredholm operator of the zero index.*

Proof (i) The equation

$$\dim C_0^\infty(Q,R) = +\infty$$

and the inclusion

$$C_0^\infty(Q,R)\subset X$$

imply $\dim X = +\infty$.

(ii) Since the coefficients a_k for $0\le|k|\le 2b$ are continuous on the compact set $cl\,Q$, there is a positive constant $K>0$ such that

$$\|Au\|_Y \le K(\|D_t u\|_Y + \sum_{0\le|k|\le 2b}\|D_x^k u\|_Y) = K\|u\|_X$$

for all $u\in X$, whence the operator A is bounded on X.

If the operator A is a homeomorphism, then statement (ii) is clear.

If A is not the homeomorphism, then by the Nikol̆skiĭ decomposition theorem from Proposition 2.1, it is sufficient to show that

$$Au = Hu + (H(t,x,D_x) - A(t,x,D_x))u = Hu + Tu,$$

thereby the mapping $T\colon X\to Y$ is the linear completely continuous operator. It will be proved by generalized Ascoli - Arzelà theorem from [21, P. 31].

From the hypothesis $(\mathrm{S}^{\alpha+\rho})$, the equi-boundedness of

$$Tu = \sum_{|k|=2b}(h_k(t,x)-a_k(t,x))D_x^k u + \sum_{0\le|k|\le 2b-1}(h_k(t,x)-a_k(t,x))D_x^k u$$

holds at the bounded set $S\subset X$, i.e. there is a constant $K_1(n,\alpha,T,\Omega)>0$ such that $\|Tu\|_Y\le K_1\|u\|_X$ for all $u\in S$.

Now for the equi-continuity of the set $TS\subset Y$ we have to prove the inequality (for every element u_l, $l=1,\dots,p$, of $u=(u_1,\dots,u_p)$)

$$|(Tu)_l(t,x)-(Tu)_l(s,y)| + \frac{|(Tu)_l(t,x)-(Tu)_l(t,y)|}{\|x-y\|_{R^n}^\alpha} + \frac{|(Tu)_l(t,x)-(Tu)_l(s,x)|}{|t-s|^{\alpha/2b}} < \varepsilon$$

for all $u \in S$ and (t,x), (s,y), (t,y), $(s,x) \in cl\,Q$, $x \neq y$, $t \neq s$ for which the norms $\|x-y\|_{R^n}$ and $|t-s|$ are sufficiently small, $\varepsilon > 0$.

With respect to $(\mathrm{S}^{\alpha+\rho})$ we obtain for the first member of the previous inequality

$$\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(s,y)| \\
&\quad \leq \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)|\,|D_x^k u_l(t,x)| \\
&\quad + \sum_{|k|=2b} |h_k(s,y) - a_k(s,y)|\,|D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\quad + \sum_{0\leq|k|\leq 2b-1} |h_k(s,y) - a_k(s,y)|\,|D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\quad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)| \\
&\quad + K_3 \sum_{|k|=2b} |D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\quad + K_3 \sum_{0\leq|k|\leq 2b-1} |D_x^k u_l(t,x) - D_x^k u_l(s,y|,
\end{aligned}$$

where K_2, K_3 are positive constants dependent only on n, α, T, Ω. For $|t-s| < \delta$, $\|x-y\|_{R^n} < \delta$ with a sufficiently small $\delta > 0$ the every member of the last inequality is smaller than fixed arbitrary $\varepsilon > 0$. (Since $u \in S \subset X$, the number δ does not depend on u.)

For the second member we get by the condition $(\mathrm{S}^{\alpha+\rho})$ and using the mean value theorem

$$\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(t,y)|\,\|x-y\|_{R^n}^{-\alpha} e \\
&\quad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(t,y)|\,\|x-y\|_{R^n}^{-\alpha} \\
&\quad + K_3 \sum_{|k|=2b} \left|D_x^k u_l(t,x) - D_x^k u_l(t,y)\right|\,\|x-y\|_{R^n}^{-\alpha} \\
&\quad + K_3 \sum_{0\leq|k|\leq 2b-1} \left|D_x^k u_l(t,x) - D_x^k u_l(t,y)\right|\,\|x-y\|_{R^n}^{-\alpha} \\
&\quad \leq K(2\|x-y\|_{R^n}^{\rho} + \|x-y\|^{1-\alpha})
\end{aligned}$$

By the similar way we have for the third member

$$\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(s,x)| \cdot |t-s|^{-\alpha/2b} \\
&\quad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,x)|\,|t-s|^{-\alpha/2b} \\
&\quad + K_3 \sum_{|k|=2b} \left|D_x^k u_l(t,x) - D_x^k u_l(s,x)\right|\,|t-s|^{-\alpha/2b} \\
&\quad + K_3 \sum_{0\leq|k|\leq 2b-1} \left|D_x^k u_l(t,x) - D_x^k u_l(s,x)\right|\,|t-s|^{-\alpha/2b}
\end{aligned}$$

$$\le K\left(2|t-s|^{\rho/2b} + |t-s|^{1-\alpha/2b} + \sum_{|k|=1}^{2b-1} |t-s|^{1-|k|/2b}\right).$$

By these three estimations the assertion (ii) is proved.

Remark 3.1 Necessary and sufficient conditions for the existence of a linear homeomorphism $H\colon X \to Y$ from the assumption (A.1) are given in Proposition 1.1. Concretely, for example, $Hu = \dfrac{\partial u}{\partial t} - \Delta u$, $u \in X$.

Corollary 3.1 *Let $\mathcal{L}$ mean the set of all linear differential operators $A = D_t - A(t,x,D_x)\colon X \to Y$ satisfying the hypothesis $(S^{\alpha+\rho})$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$. Then for each $A \in \mathcal{L}$ the initial boundary-value homogeneous problem $Au = 0$, (1.2), (1.3) has a nontrivial solution or any $A \in \mathcal{L}$ is a linear bounded Fredholm operator of the zero index.*

Proof Really, if there exists an operator $A \in \mathcal{L}$ such that the problem $Au = 0$, (1.2), (1.3) has only trivial solution, then A is homeomorphism X onto Y (see Proposition 1.1). Then by Lemma 3.2 all operators of $\mathcal{L}$ are Fredholm of the zero index.

Assumption N.1 The vector function $f \in C(cl\,Q \times R^{\kappa}, R^p)$ satisfies the following local grown vector condition

$$|\,f(t,x,u^\gamma) - f(s,y,v^\gamma)\,| \le L\left[\,|t-s|^{\beta_1} + \|x-y\|_{R^n}^{\beta_2} + \sum_{l=1}^{p}\sum_{0\le|\gamma|\le 2b-1} |u_l^\gamma - v_l^\gamma|^{\beta_{\gamma,l}}\,\right] J$$

for (t,x,u^γ), (s,y,v^γ) from a compact subset of R^κ and $\beta_1 > \alpha/2b$, $\beta_2 > \alpha$, $\beta_{\gamma,l} > \alpha/(\alpha+\rho)$, $0 \le |\gamma| \le 2b-1$, $l = 1,\dots,p$, where $L > 0$.

Lemma 3.3 *Suppose Assumption N.1 holds. Then the Nemitskiĭ operator $N\colon X \to Y$ from (3.2) is completely continuous on X.*

Proof For any bounded set $S \subset X$ the N is equi-bounded in Y. Indeed, for all $u \in S$ using (N.1) the norm

$$\|Nu\|_Y \le \max_{l=1,\dots,p}\Bigg[\sup_{(t,x)\in cl\,Q} |f_l(t,x,\overline{D}_x^\gamma u(t,x))|$$
$$+ L \sup_{\substack{(t,x),(t,y)\in cl\,Q\\ x\neq y}} \frac{\|x-y\|_{R^n}^{\beta_2} + \sum\limits_{l=1}^{p}\sum\limits_{0\le|\gamma|\le 2b-1} |D_x^\gamma u_l(t,x) - D_x^\gamma u_l(t,y)|^{\beta_{\gamma,l}}}{\|x-y\|_{R^n}^{\alpha}}$$
$$+ \sup_{\substack{(t,x),(s,x)\in cl\,Q\\ t\neq s}} \frac{|t-s|^{\beta_1} + \sum\limits_{l=1}^{p}\sum\limits_{0\le|\gamma|\le 2b-1} |D_x^\gamma u_l(t,x) - D_x^\gamma u_l(s,x)|^{\beta_{\gamma,l}}}{|t-s|^{\alpha/2b}}\Bigg]$$

Hence, it is bounded by a positive constant $K(\Omega, T, L, \alpha, \beta_1, \beta_2, \beta_{\gamma,l})$.

Also, for $|t-s|^2 + \|x-y\|^2_{R^n} < \delta^2$ with a sufficiently small $\delta > 0$ we get the equicontinuity of N. It is sufficient to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$|(Nu)_l(t,x) - (Nu)_l(s,y)| + \frac{|(Nu)_l(t,x) - (Nu)_l(t,y)|}{\|x-y\|^\alpha_{R^n}} + \frac{|(Nu)_l(t,x) - (Nu)_l(s,x)|}{|t-s|^{\alpha/2b}} < \varepsilon$$

is true for all $u \in S$, if both t, s and x, y to be sufficiently near and $l = 1, \dots, p$.

Assumption F.1 For each bounded set $S \subset Y$ there is a constant $K^a > 0$ such that for all solutions $u \in X$ of (1.1)–(1.3) with $g \in S$ the inequality

$$\|u\|_{a,\,Q} = \max_{l=1,\dots,p} \sum_{0 \le |k| \le a} \sup_{(t,x) \in cl\,Q} |D^k_x u_l(t,x)| \le K^a \tag{3.4}$$

holds for $a = \max\{|\gamma|, r\}$. Here r is an integer $0 \le r \le 2b-1$ for which the coefficients of operators A and H from (3.1) and (A.1), respectively satisfy the relations $a_k = h_k$ for $|k| = r+1, \dots, 2b$ and $a_k \ne h_k$ for at least one multiindex k with $|k| = r$ on $cl\,Q$.

Lemma 3.4 *Let* $(S^{\alpha+\rho}$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho > 1)$, (A.1), (N.1) *and an almost coercivity condition of Assumption F.1 be satisfied. Then*

(i) *F from (3.3) is coercive at X.*

(ii) *F is proper and continuous at X.*

Proof (i) We need to prove that if the set $S \subset Y$ is bounded in Y, then the set of arguments $F^{-1}(S) \subset X$ is bounded in X.

By (3.4) and the Assumption F.1 it follows that the set $F^{-1}(S)$ is bounded in the norm $\|\cdot\|_{a,\,Q}$. Hence and by Assumption N.1 one obtains the estimation $\|Nu\|_Y \le K_4$ for all $u \in F^{-1}(S)$. From Lemma 3.2 (ii) also $\|Au\|_Y \le \|Fu\|_Y + \|Nu\|_Y \le K_5$ for any $u \in F^{-1}(S)$, where K_4, K_5 are positive constants.

On the other hand, Assumption A.1 ensures the existence and uniqueness of the solution $u \in X$ of the linear equation $Hu = y$ for any $y \in Y$ and (see the Green representation of solution from (2.3) and [15, PP. 182–183] and estimation (2.1)) the estimation

$$\|u\|_X \le K_6 \|y\|_Y, \quad K_6 > 0, : u \in F^{-1}(S) \tag{3.5}$$

is true.

Then for $u \in F^{-1}(S)$ we have

$$Hu = Au + \sum_{0 \le |k| \le 2b} (a_k(t,x) - h_k(t,x)) D^k_x u.$$

With respect to (S^α) and Assumption F.1

$$\|y\|_Y = \|Hu\|_Y \le \|Au\|_Y + \sum_{0 \le |k| \le r} \|a_k - h_k\|_Y \, \|D^k_x u\|_Y \le$$
$$K_5 + K_7 \|u\|_{r,\,Q} \le K_5 + K_7 \|u\|_{a,\,Q} \le K_5 + K_7 K^a, \quad K_7 > 0.$$

Hence and by (3.5)

$$\|u\|_X \le K_6(K_5 + K_7 K^a), \quad u \in F^{-1}(S).$$

(ii) Since $\dim X = +\infty$ and A is a nonconstant and closed mapping on X, then by Proposition 2.4 (ii) it is proper on X. From Lemma 3.3 the operator N is completely continuous on X. From (i) of this lemma F is coercive on X. The Proposition 2.5 (ii) concludes the proof of (ii) and the proof of Lemma 3.4.

In the following lemmas we shall consider the continuous nonlinearity f. Conditions for the continuous F-differentiability of the Nemitskiĭ operator N give the following lemma.

Assumption N.2 For $l = 1,\dots,p$ and the multiindices β with the modulus $0 \le |\beta| \le 2b-1$,

$$\frac{\partial f}{\partial v_{\beta,\, l}} \in C(cl\, Q \times R^\kappa, R^p)$$

where κ represents the number of all components in the vector function $\overline{D}_x^\beta u$ from (1.1).

Lemma 3.5 *Let the Nemitskiĭ operator* $N\colon X \to Y$ *satisfy Assumptions N.1 and N.2. Then*

(i) *the operator* N *is continuously Frechét differentiable on* X*, i.e.* $N \in C^1(X,Y)$*;*
(ii) *if moreover* $(\mathrm{S}^{\alpha+\rho})$ *for* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$ *holds, then* $F \in C^1(X,Y)$.

Proof (i) We need to prove that Frechét derivative $N'\colon X \to L(X,Y)$ defined by the vector equation

$$N'(u)h(t,x) = \sum_{\substack{0\le|\beta|\le 2b-1\\ \mathrm{card}\{\beta,l\}=\kappa\\ l=1,\dots,p}} \frac{\partial f}{\partial v_\beta}[t,x,\overline{D}_x^\gamma u(t,x)] D_x^\beta h_l(t,x) \tag{3.6}$$

is continuous on X for every $u,h \in X$. Here $\beta = (\beta_1,\dots,\beta_n)$ represents every multiindex $\gamma = (\gamma_1,\dots,\gamma_n)$ appearing in the nonlinearity f. It is sufficient to show for every fixed $v \in X$ the implication:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon, v) > 0 \quad \forall\, u \in X, \quad \|u - v\|_X < \delta \Rightarrow \|N'u - N'v\|_{L(X,Y)} < \varepsilon,$$

i.e.

$$\sup_{h\in X,\, \|h\|_X \le 1} \|N'(u)h - N'(v)h\|_Y < \varepsilon \tag{3.7}$$

Let us take an arbitrary $\varepsilon > 0$ and $u \in X$ such that $\|u-v\|_X < \delta$, i.e. $|D_t u_l(t,x) - D_t v_l(t,x)| < \delta$ and $|D_x^k u_l(t,x) - D_x^k v_l(t,x)| < \delta$ for all multiindices $0 \le |k| \le 2b$ on $cl\, Q$. Hence with respect to the uniform continuity of $\frac{\partial f}{\partial v_{\beta,l}}$ for $0 \le |\beta| \le 2b-1$, $l = 1,\dots,p$, on every compact of $cl\, Q \times R^\kappa$ we get the vector inequality

$$|N'(u)h(t,x) - N'(v)h(t,x)| \le \sum_{\substack{0\le|\beta|\le 2b-1\\ \mathrm{card}\{\beta\}=\kappa\\ l=1,\dots,p}} \left| \frac{\partial f}{\partial v_{\beta,l}}[t,x,\overline{D}_x^\gamma u(t,x)] - \frac{\partial f}{\partial v_{\beta,l}}[t,x,\overline{D}_x^\gamma v(t,x)] \right| \left| D_x^\beta h_l(t,x) \right| < \varepsilon J$$

for $\|h\|_X \le 1$ and all $(t,x) \in cl\, Q$. It finishes the proof of (3.7).

(ii) We easily see that Fréchet derivative $F'\colon X \to L(X,Y)$ is defined by the vector equation

$$F'(u)h(t,x) = D_t h(t,x) - \sum_{0\le|k|\le 2b} a_k(t,x) D_x^k h(t,x) + N'(u)h(t,x)$$

for $u, h \in X$. Hence and by (i) we get $F \in C^1(X,Y)$.

Lemma 3.6 *Let the hypotheses* $(\mathrm{S}^{\alpha+\rho})$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$, *(A.1), (N.1) and (N.2) be satisfied. Then* $F = A + N\colon X \to Y$ *is a nonlinear Fredholm operator of the zero index on* X.

Proof According to Lemma 3.2 the operator $A\colon X \to Y$ is a linear continuous and C^1–Fredholm mapping of the zero index. By the statement of Lemma 3.3 the operator $N\colon X \to Y$ is compact. By Lemma 3.5 it belongs to the class C^1. Then Proposition 2.8 implies that F is a nonlinear Fredholm operator with the zero index.

4 The Solution Set for Continuous Nonlinearities

The first results for that proper mapping F give the following theorem.

Theorem 4.1 *Let hypotheses* $(\mathrm{S}^{\alpha+\rho})$ *for* $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$, *and Assumptions A.1, N.1 hold. Then*

(a) *for any compact set of the right hand sides* $g \in Y$ *of (1.1) the corresponding set of all solutions of (1.1) – (1.3) is a countable union of compact sets;*

(b) *for* $u_0 \in X$ *there exists a neighborhood* $U(u_0)$ *of* u_0 *and* $U(F(u_0))$ *of* $F(u_0) \in Y$ *such that for each* $g \in U(F(u_0))$ *there is an unique solution of (1.1) – (1.3) iff the operator* F *is locally injective at* u_0*;*

(c) *let moreover (F.1) hold. Then for any compact set of the right hand sides* $g \in Y$ *from (1.1), the set of all solutions of (1.1) – (1.3) is compact (possible empty).*

Proof (a) Since $F = A + N$ (see (3.3)) by the decomposition of $A = C + T$ (Proposition 2.1) we have $F = C + (T + N)$, where C is a continuous and proper mapping X onto Y (see Proposition 2.4), A is a Fredholm operator of the zero index, T and N are completely continuous mappings. Since X is a countable union of closed balls in X, so with respect to Proposition 2.5 (i) the operator F is σ-proper (continuous). Lemma 3.1 (i) implies assertion (a).

(b) Suppose that F is injective in a neighborhood $U(u_0)$ of $u_0 \in X$. From the decomposition (for H see Lemma 3.2)

$$F = H + (T + N)$$

we obtain $H^{-1}F = I + H^{-1}(T+N)$ which is a completely continuous and injective perturbation of the identity $I\colon X \to Y$ in $U(u_0)$. According to Proposition 2.7 (i) the set $H^{-1}F(U(u_0))$ is open in X and the restriction $H^{-1}F|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $H^{-1}F(U(u_0))$. Therefore F is locally invertible at u_0. Again by Lemma 3.1 (i) we obtain (b).

(c) By Lemma 3.4 (ii) the operator $F\colon X \to Y$ is proper which implies the given assertion and includes the proof of Theorem 4.1.

We have the following theorem on further qualitative and quantitative properties of the set solutions of (1.1) – (1.3).

Theorem 4.2 *Let hypotheses* ($\mathrm{S}^{\alpha+\rho}$) *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 be satisfied. For solutions of (1.1) – (1.3) the following statements are true:*

(d) *the set of solution for each* $g \in Y$ *is compact (possible empty);*
(e) *the set* $R(F) = g \in Y$ *such that there exists at least one solution* $u \in X$ *of (1.1) – (1.3) is closed and connected in* Y*;*
(f) *the domain of bifurcation* D_b *is closed in* X *and the bifurcation range* R_b *is closed in* Y*. The set* $F(X \setminus D_b)$ *is open in* Y*;*
(g) *if* $Y \setminus R_b \neq \varnothing$*, then each component of* $Y \setminus R_b$ *is a nonempty open set (i.e. domain);*
(h) *if* $Y \setminus R_b \neq \varnothing$*, the number* n_g *of solutions is finite and constant (it may be zero) on each component of* $Y \setminus R_b$*, i.e.* n_g *is the same nonnegative integer for each* g *belonging to the same component of* $Y \setminus R_b$*;*
(i) *if* $R_b = \varnothing$*, then the given problem has a unique solution* $u \in X$ *for each* $g \in Y$ *and this solution continuously depends on* g *as a mapping from* Y *onto* X*;*
(j) *if* $R_b \neq \varnothing$*, then the boundary* $\partial F(X \setminus D_b)$ *is a subset of* $F(D_b) = R_b$ $(\partial F(X \setminus D_b) \subset F(D_b))$.

Proof The assertion (d) follows directly from Theorem 4.1 (c).

(e) Take the sequence $\{g_n\}_{n\in N} \subset R(F) \subset Y$ converging to $g \in Y$ as $n \to \infty$. By (d) there is a compact set of all solutions $\{u_\gamma\}_{\gamma\in I} \subset X$ (here I means an index set) of the equations $F(u) = g_n$ for $n = 1, 2, \dots$. Thus there exists a subsequence $\{u_{n_k}\}_{k\in N} \subset \{u_\gamma\}_{\gamma\in I}$ converging to $u \in X$ and $F(u_{n_k}) = g_{n_k} \to g$ in Y as $n \to \infty$. Since the mapping F is proper (Lemma 3.4 (ii)) by Proposition 2.4 (i) it is closed, whence $F(u) = g$, i.e. $g \in R(F)$. The set $R(F)$ is closed. $R(F) = F(X)$ is connected as a continuous image of the connected set X.

(f) According to Lemma 3.1 (ii) $D_b = \Sigma$ and $R_b = F(D_b) = F(\Sigma)$. Since $X \setminus \Sigma$ is an open set then D_b is closed in X and its continuous image R_b is a closed set in Y.

Since, $X \setminus D_b = X \setminus \Sigma$ is the set of all points at which the mapping F is locally invertible, to each $u_0 \in X \setminus D_b$ there is a neighborhood $U_1(F(u_0)) \subset F(X \setminus D_b)$. It means, the set $F(X \setminus D_b)$ is open.

(g) The set $Y \setminus R_b = Y \setminus F(D_b) \neq \varnothing$ is open in Y. Then each its component is nonempty and open, too.

(h) This directly follows from Proposition 2.2.

(i) By $R_b = \varnothing$ is $D_b = \varnothing$ and the mapping F is locally invertible in X. Proposition 2.5 (ii) asserts that F is a proper mapping. Then from the global inverse mapping theorem (Proposition 2.3) implies F is homeomorphism X onto Y.

(j) From Lemma 3.1 (ii) $D_b = \Sigma$ and by (f) D_b and $F(D_b)$ are closed. Then $\partial F(X \setminus D_b) = \partial F(D_b) \subset F(D_b)$.

This finishes the proof of the theorem.

The following two theorems are on the surjectivity of (1.1) – (1.3).

Theorem 4.3 *Under the assumptions* ($\mathrm{S}^{\alpha+\rho}$), $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 each of the following conditions is sufficient for the solvability of problem (1.1) – (1.3) for each* $g \in Y$*:*

(k) *for each* $g \in R_b$ *there is a solution* $u \in X \setminus D_b$ *of (1.1) – (1.3);*
(l) *the set* $Y \setminus R_b$ *is connected and there is* $g \in R(F) \setminus R_b$ *(for* $R(F)$ *see Theorem 4.2 (e)).*

Proof First of all we see that conditions (k) and (l) are mutually equivalent to the conditions

(k′) $F(D_b) \subset F(X \setminus D_b)$

and

(l′) $Y \setminus R_b$ is a connected set and $F(X \setminus D_b) \setminus R_b \neq \varnothing$,

respectively.

From the proof of Theorem 4.2 (f) we have $D_b = \Sigma$.

(k) From (k′) we have $F(X) = F(D_b) \cup F(X \setminus D_b) = F(X \setminus D_b)$. So $R(F) = F(X)$ is closed and connected in Y (Theorem 4.2 (e)) as well as open set in Y (see Theorem 4.2 (f)). Thus $R(F) = Y$ which implies the surjectivity of F.

(l) By (h) of Theorem 4.2, $\operatorname{card} F^{-1}(\{g\})$ is a constant $k \geq 0$ for every g from the same component of $Y \setminus R_b$.

If $k = 0$ for all $g \in Y \setminus R_b$ such that $F(X) = R_b$, whence $F(X \setminus D_b) \subset R_b$. It is a contradiction with (l′).

Assumptions S.1 There exists a constant $K^a > 0$ such that all solutions $u \in X$ of the initial boundary-value problem for the equation

$$Hu + \mu(Au - Hu + Nu) = 0, \quad \mu \in (0,1)$$

with data (1.2), (1.3) fulfil inequality (3.4) from Lemma 3.4. H is the linear homeomorphism from Assumption A.1.

Theorem 4.4 *Let* ($S^{\alpha+\rho}$) *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 hold together with the hypothesis S.1.*

Then

(m) *problem (1.1) – (1.3) has at least one solution for each* $g \in Y$*;*

(n) *the number* n_g *of solutions (1.1) – (1.3) is finite, constant and different from zero on each component of the set* $Y \setminus R_b$ *(for all* g *belonging to the same component of* $Y \setminus R_b$*).*

Proof (m) It is sufficient to prove the surjectivity of $F\colon X \to Y$. By Lemma 3.2 (see the proof of (ii)) we can write

$$F = A + N = H + (T + N)$$

The mapping

$$H^{-1}F = I + H^{-1}(T + N)\colon\ X \to X$$

is a completely continuous and condensing field (see [31, P. 496]).

Let $S \subset X$ be a bounded set. Then $H(S)$ is a bounded set in Y. From the coercivity of F (see Lemma 3.4 (i)) the set $F^{-1}[H(S)] = (H^{-1}F)^{-1}(S)$ is bounded at X. Hence $H^{-1}F$ is coercive.

Now we show that condition (iii) from Proposition 2.6 is satisfied for the condensing and coercive field $P = H^{-1}F$. Take the strictly solvable field $G(u) = u$. Then the equation $P(u) = kG(u)$ implies

$$(H^{-1}F)(u) = ku.$$

Hence we get for $u \in X$ and $k < 0$

$$Hu + (1-k)^{-1}[Au - Hu + Nu] = 0$$

where $(1-k)^{-1} \in (0,1)$. With respect to Assumption S.1

$$\|u\|_{a,Q} \le K^a$$

for $a = \max\{|\gamma|, r\}$, where $|\gamma| = 0, 1, \dots, 2b-1$ and $0 \le r \le 2b-1$ are fixed. Using the same method as in Lemma 3.4 (i) we obtain for all solutions of

$$(H^{-1}F)u = ku$$

the estimation $\|u\|_X \le K_8$, $K_8 > 0$. By Proposition 2.6 we have the strict surjectivity of $H^{-1}F$ and so F. This proves (m).

(n) From the surjectivity of F on X it follows $n_g \neq 0$. The other assertions of (n) follow from Theorem 4.2 (h).

Example 4.1 The simple example illustrating results of this part can be the initial boundary-value problem for the system of p equations.

$$\frac{\partial u_l}{\partial t} - K_l \frac{\partial^2 u_l}{\partial x^2} + f_l(u) = 0, \quad (t,x) \in \langle 0, T\rangle \times \Omega \subset R \times R,$$

where $l = 1, \dots, p$ with the conditions

$$\frac{\partial u_l}{\partial x}(t,0) = \frac{\partial u_l}{\partial x}(t,1) = 0, \quad t \in \langle 0, T\rangle,$$
$$u_l(0,x) = 0, \; x \in \, cl\,\Omega.$$

We take $K_l > 0$ and

$$f_l(u) = \begin{cases} u_l^{1/2}, & \text{if } u_l \in \langle 0, a\rangle, \\ a^{1/2}, & \text{if } u_l \in \langle a, \infty), \\ 0, & \text{if } u_l \le 0, \end{cases}$$

for $l = 1, \dots, p$. Assumption A.1 is satisfied by Proposition 1.1. The condition N.1 can be verified by elementary calculus. The supposition F.1 follows from equation (2.3) and Green matrix estimations (2.1). The condition $(S^{\alpha+\rho})$ holds for $0 < \alpha < 1/2$, $1/2 < \rho < 1$ and $\alpha + \rho < 1$ (for example $\alpha = 1/5$, $\rho = 3/5$).

5 The Solution Set for C^1-nonlinearities

With respect to the C^1-,differentiability of the operator N from (3.2) we prove here several stronger results than in Chapter 4 for the solutions of (1.1)–(1.3).

Theorem 5.1 *Suppose that* $(S^{\alpha+\rho})$ *for* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$ *and Assumptions A.1, N.1, N.2 and F.1 are satisfied and* R_b *means the bifurcation range of (1.1)–(1.3) from Definition 3.1. Then the set* $Y \setminus R_b$ *is open and dense in* Y *and thus the bifurcation range* R_b *of initial boundary-value problem (1.1)–(1.3) is nowhere dense in* Y.

Proof The openness of $Y \setminus R_b$ follows from the statement (f) of Theorem 4.2.

From previous lemmas the operator $A\colon X \to Y$ is a linear continuous Fredholm mapping of the zero index and the Nemitskiĭ operator $N\colon X \to Y$ is compact and $N \in C^1(X,Y)$.

For every $u \in X$ the linear operator $N'\colon X \to Y$ from (3.6) is completely continuous on X. By the Nikoľskiĭ decomposition theorem (see Proposition 2.1) the operator $F'(u) = A + N'(u)\colon X \to Y$ is a linear Fredholm mapping of the zero index for each $u \in X$. By Lemma 3.5 (ii) there is $F \in C^1(X,Y)$ and by Lemma 3.6 the F is a nonlinear Fredholm operator of the zero index.

According to the Banach open mapping theorem (see [30, P. 77]) the mutual equivalence is true: $F'(u)$ is a linear homeomorphism iff it is a bijective mapping. Since $F'(u)$ for every $u \in X$ is a linear Fredholm mapping of the zero index so $F'(u)$ is bijective iff it is injective (in this case the injectivity implies surjectivity, see Proposition 8.14 (1) from [31, P. 366]). We see that $u \in X$ is a singular point of the Fredholm operator F iff u is a critical point of F.

From Proposition 2.10 we obtain that set Σ (of all points $u \in X$ for which F is not locally invertible) is a subset of all critical point F. Then, evidently Σ is a subset of all singular points S of F, i.e. $\Sigma \subset S$. Hence we get for the set of regular values R_F of the operator F the relations

$$R_F = Y \setminus F(S) \subset Y \setminus F(\Sigma) \subset Y \setminus R_b \subset Y,$$

where $R_b \subset F(\Sigma)$ is a bifurcation range of F.

Since $F\colon X \to Y$ is nonconstant closed mapping with $\dim X = \infty$, by Proposition 2.4 we obtain that F is a proper mapping. By Proposition 2.9 (the Quinn version) the set R_F is residual, open and dense in Y. Hence $Y \setminus R_b$ is dense in Y, too. With respect to Lemma 3.1 (ii) we can conclude the proof.

In the following results we shall deal with the linear problem in $h \in X$

$$Ah(t,x) + \sum_{\substack{0\le|\beta|\le 2b-1\\ \mathrm{card}\{|\beta|\}=\kappa}} \frac{\partial f}{\partial v_\beta}[t,x,D_x^\gamma u(t,x)]D_x^\beta h(t,x) = g(t,x) \tag{5.1}$$

for $(t,x) \in Q$ and some fixed $u \in X$ with condition (1.2), (1.3). The left side of equation (5.1) represents the Frechét derivative $F'(u)h$ of the operator $F = A + N\colon X \to Y$.

Theorem 5.2 *Let the hypotheses* $\mathrm{S}^{\alpha+\rho}$ *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1, N.2 and F.1 are satisfied. Then*

(o) *the number solutions of (1.1) – (1.3) is constant and finite (it may be zero) on each connected component of the open set* $Y \setminus F(S)$, *i.e. for any g belonging to the same connected component of* $Y \setminus F(S)$. *Here* S *means the set of all critical points of the operator* $F = A + N\colon X \to Y$;

(p) *let* $u_0 \in X$ *be a regular solution of (1.1) – (1.3) with the right hand side* $g_0 \in Y$. *Then there exists a neighborhood* $U(g_0) \subset Y$ *of* g_0 *such that for any* $g \in U(g_0)$ *initial-boundary value problem (1.1) – (1.3) has one and only one solution* $u \in X$. *This solution continuously depends on* g.
The associated linear problem (5.1), (1.2), (1.3) for $u = u_0$ *has a unique solution* $h \in X$ *for any* g *from a neighborhood* $U(g_0)$ *of* $g_0 = F(u_0)$. *This solution continuously depends on* g;

(q) *denote by G the set of all right hand side $g \in Y$ of equation (1.1) for which the corresponding solutions $u \in X$ of (1.1) – (1.3) are its critical points. Then G is closed nowhere dense in Y;*

(r) *if the singular points set of (1.1) – (1.3) is empty, then this problem has unique solution $u \in X$ for each $g \in Y$. It continuously depends on the right hand side g.*

Proof (o) In the proof of Theorem 5.1 we have shown that the set of all singular points of F is equal to the set of all critical points of F. Then the Ambrosetti theorem (see Proposition 2.2) implies the statement (o).

(p) Since $u_0 \in X \setminus S$, where S is a set of all singular (in our case all critical) points then by Proposition 2.10 the mapping F is a local C^1-diffeomorphism at u_0. This proves first part of (p) for (1.1) – (1.3).

From F as the C^1-diffeomorphism follows that $F' \in C(X,Y)$, $(F^{-1})' \in C(X,Y)$, where $F'(u)h$ is the left hand side of (5.1) and $(F^{-1})'(Fu) = (F'(u))^{-1}$ for every $u \in X$. Hence linear problem (5.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X$ for any $g \in U(g_0)$ with $g_0 = F(u_0)$. This solution continuously depends on all right hand side g. The proof of (p) is completed.

(q) In our case the equality $G = F(S)$ holds, where S is the set of all critical (all singular) points of F. By the Smale–Quinn theorem (Proposition 2.9) we obtain the expected results.

(r) By Proposition 2.10, the operator $F\colon X \to Y$ is a local C^1-diffeomorphism at any point $u \in X$. Hence follows the last assertion.

Assumption H.1 Linear homogeneous problem (5.1), (1.2), (1.3) (for $g = 0$) has only zero solution $h = 0 \in X$ for any $u \in X$.

By the point (p) of Theorem 5.2 we obtain the following corollary.

Corollary 5.1 *Let the hypotheses of Theorem 5.2 and Assumption H.1 hold. Then initial boundary-value problem (1.1) – (1.3) has a unique solution $u \in X$ for any $g \in Y$. Moreover, linear problem (5.1), (1.2), (1.3) has a unique solution $h \in X$ for any $u \in X$ and the right hand side $g \in Y$ of (5.1). This solution continuously depends on g.*

Corollary 5.2 *Let the assumptions of Theorem 5.2 be satisfied. Then we have:*

(s) *if the set S of all singular (in our case all critical) points of F is nonempty, then $\partial F(X \setminus S) \subset F(S)$;*

(t) *if $F(S) \subset F(X \setminus S)$, then problem (1.1) – (1.3) has the solution $u \in X$ for any $g \in Y$, i.e. $R(F) = Y$ (F is a surjectivity of X onto Y);*

(u) *if $Y \setminus F(S)$ is connected and $X \setminus S \neq \varnothing$, then $R(F) = Y$ (the solvability of (1.1) – (1.3) for any $g \in Y$).*

Proof By Theorem 5.2 (q) the set $F(S)$ is closed in Y and by Proposition 2.9 $F(X \setminus S)$ is open in Y. Hence we have the equations

$$F(X) = F(S) \cup F(X \setminus S) = F(S) \cup \overline{F(X \setminus S)} = \overline{F(X)} \tag{5.2}$$

which implies that $F(X)$ is a closed set.

(s) Since $F \in C^1(X,Y)$ we get $\Sigma \subset S$, as in Theorem 5.1. Hence and by Theorem 4.2 (i)

$$\partial F(X \setminus S) \subset \partial F(X \setminus \Sigma) \subset F(\Sigma) \subset F(S).$$

(t) From the first equation of (5.2) we have $F(X) = F(X \setminus S)$ and so $R(F)$ is an open as well as a closed subset of the connected space Y. Thus $R(F) = Y$.

(u) Since $Y \setminus F(S)$ is connected, and by Proposition 2.2 we obtain the card $F^{-1}(\{g\}) =$ const $= k \geq 0$ for each $g \in Y \setminus F(S)$.

If $k = 0$, then $F(X) = F(S)$ and $F(X \setminus S) \subset F(S)$ and this is a contradiction with $X \setminus S \neq \varnothing$. Thus $k > 0$.

Assumption H.2 Each point $u \in X$ is either a regular point or an isolated critical point of problem (1.1) – (1.3).

Theorem 5.3 *Suppose that hypotheses* $(\mathrm{S}^{\alpha+\rho})$ *with* $\alpha \in (0, 1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1, N.2, F.1 and H.2 hold. Then for every* $g \in Y$ *there exists one solution* $u \in X$ *of (1.1) – (1.3). It continuously depends on* g.

Proof The associated operator $F\colon X \to Y$ is a proper C^1-Fredholm mapping of the zero index. By Proposition 2.10 F is a local C^1-diffeomorphism at a regular point of F. In the isolated singular point, by Proposition 2.11 F is locally invertible. Since F is proper, the global inverse mapping theorem (see Proposition 2.3) implies the statement of this problem.

Example 5.1 Example 4.1 illustrates the results of Chapter 5 for $f_l(u) = \sin\left(\sum_{i=1}^{l} u_i^2\right)$.

6 Conclusion

The studied models describe different natural science phenomena (a reaction-diffusion and environment models, a diffusive waves in fluid dynamics — the Burges equation, the wave propagation in a large number of biological and chemical systems — the Fisher equation, a nerve pulse propagation in nerve fibers and wall motion in liquid crystals).

We can apply the Fredholm theory to hyperbolic equations modeling different nonlinear vibration problems, to a nonlinear dispersion (the nonlinear Klein–Gordan equation), a propagation of magnetic flux and the stability of fluid notions (the nonlinear Sine–Gordan equation) and so on.

References

[1] Agranovič, M.S. and Višik, M. I. Elliptic problems with a parameter and parabolic general type problems. *Uspechi Mat. Nauk* **19**(3) (1964) 53 – 161. [Russian].

[2] Ambrosetti, A. Global inversion theorems and applications to nonlinear problems. *Conferenze del Seminario di Matematica dell' Università di Bari, Atti del* 3° *Seminario di Analisi Funzionale ed Applicazioni, A Survey on the Theoretical and Numerical Trends in Nonlinear Analysis*, Gius. Laterza et Figli, Bari, 1976, P. 211 – 232.

[3] Andres, J., Gabor, G. and Górniewicz, L. Boundary value problems on infinite intervals. *Trans. Amer. Math. Soc.* **351** (2000) 4861 – 4903.

[4] Andres, J., Gabor, G. and Górniewicz, L. Topological structure of solution sets to multivalued asymptotic problems. *Z. Anal. Anw.* **19** (2000) 35 – 60.

[5] Aronszajn, N. Le correspondant topologique de l'unicité dans la théorie des équations différentielles. *Ann. of Math.* **43** (1942) 730 – 738.

[6] Banach, S. and Mazur, S. Űber mehrdeutige stetige Abbildungen. *Studia Math.* **5** (1934) 174 – 178.

[7] Borisovič, Ju.G., Zvjagin, V.G. and Sapronov, Ju.G. Nonlinear Fredholm mappings and the Leray–Schauder theory. *Uspechi Mat. Nauk* **XXXII**(4) (1977) 3 – 54. [Russian].

[8] Browder, F.E. and Gupta, Ch.P. Topological degree and nonlinear mappings of analytic type in Banach spaces. *J. Math. Anal. Appl.* **26** (1969) 390 – 402.

[9] Cacciopoli, R. Un principio di inversione per le corrispondenze funzionali e sue applicazioni alle equazioni alle derivate parziali. *Rend. Accademia Naz. Lincei* **VI**(16) (1932).

[10] Deimling, K. *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, Heidelberg, 1985.

[11] Ďurikovič, V. An initial-boundary value problem for quasi-linear parabolic systems of higher order. *Ann. Polon. Math.* **XXX** (1974) 145 – 164.

[12] Ďurikovič, V. and Ďurikovičová, Ma. Some generic properties of nonlinear second order diffusional type problem. *Arch. Math. (Brno)* **35** (1999) 229 – 244.

[13] Ďurikovič, V. and Ďurikovičová, Mo. Sets of solutions of nonlinear initial-boundary value problems. *Topological Methods in Nonlinear Analysis, J. of the Juliusz Schauder Center* **17** (2001) 157 – 182.

[14] Eideľman, S.D. *Parabolic systems.* Nauka, Moscow, 1964. [Russian].

[15] Eideľman, S.D. and Ivasišen, S.D. The investigation of the Green's matrix for homogeneous boundary value problems of a parabolic type. *Trudy Moskov. Mat. Obshch.* **23** (1970) 179 – 234. [Russian].

[16] Fujita, H. On some nonexistence and nonuniqueness theorems for nonlinear parabolic equation. *Proc. Symp. Pure Math., v. 28, pt. 1. Nonlinear Functional Analysis.* Amer. Math. Soc., Providence, R. J. 1970.

[17] Górniewicz, L. *Topological Fixed Point Theory of Multivalued Mappings.* Kluwer Acad. Publ., Dordrecht–Boston–London, 1999.

[18] Henry, D. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin–Heidelberg–New York, 1981.

[19] Ivasishen, S.D. *Green Matrices of Parabolic Boundary Value Problems.* Vyscha Shkola, Kiev, 1990. [Russian].

[20] Marteli, M. and Vignoli, A. A generalized Leray–Schauder condition. *Lincei Rend. Sc. fis. mat. e nat.* **LVII** (1974) 374 – 379.

[21] Martin, R.H. *Nonlinear Operator and Differential Equations in Banach Spaces.* Yohn Villey and Sons, New York–London–Sydney–Toronto, 1975.

[22] Quinn, F. Transversal approximation on Banach manifolds. *Proc. Sympos. Pure Math. (Global Analysis)* **15** (1970) 213 – 223.

[23] Sadyrchanov, R.S. *Selected Questions of Nonlinear Functional Analysis.* Publishers ELM, Baku, 1989. [Russian].

[24] Smale, S. An infinite dimensional version of Sard's theorem. *Amer. J. Math.* **87** (1965) 861 – 866.

[25] Solonikov, V.A. Estimations of L_p solutions for elliptic and parabolic systems. *Trudy Math. Inst. im. V.A. Steklova AN SSSR* **102** (1969) 446 – 472.

[26] Solonikov, V.A. On Boundary value problem for linear parabolic differential systems of the general type. *Trudy Math. Inst. im. V.A. Steklova AN SSSR* **83** (1965) 3 – 162. [Russian].

[27] Šeda, V. Fredholm mappings and the generalized boundary value problem. *Differ. and Integr. Eqns* **8** (1995) 19 – 40.

[28] Taylor, A.E. *Introduction of Functional Analysis.* John Wiley and Sons, Inc., New York, 1967.

[29] Trenogin, V.A. *Functional Analysis.* Nauka, Moscow, 1980. [Russian].

[30] Yosida, K. *Functional Analysis.* Springer-Verlag, Berlin–Heidelberg–New York, 1966.

[31] Zeidler, E. *Nonlinear Functional Analysis and its Applications I, Fixed-Point Theorems.* Springer-Verlag, Berlin–Heidelberg–Tokyo, 1986.

[32] Zeidler, E. *Nonlinear Functional Analysis and its Applications II/B, Nonlinear Monoton Operators.* Springer-Verlag, Berlin–Heidelberg–London–Paris–Tokyo, 1990.