Volume 5, Number 4, 2005 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 5 Number 4 2005

CONTENTS

NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 5, **No. 4, 2005**

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

 ISSN 1562-8353 Printed in Ukraine
To receive contents and abstracts by e-mail, visit our Website at: **http://www.e-ndst.kiev.ua**

Academic Periodical Press

Nonlinear Dynamics and Systems Theory
An International Journal of Research and Surveys

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. The Journal will publish original carefully refereed papers, brief notes and reviews on a wide range of nonlinear dynamics and systems theory problems. Contributions will be considered for publication in ND&ST if they have not been published previously. Before preparing your submission, it is essential that you consult our style guide; please visit our website: http://www.sciencearea.com.ua

(2) Manuscript and Correspondence. Contributions are welcome from all countries and should be written in English. Two copies of the manuscript, double spaced one column format, and the electronic version by AMSTEX, TEX or LATEX program (on diskette) should be sent directly to

Professor A.A. Martynyuk
Institute of Mechanics,
Nesterov str.3, 03057, MSP 680
Kiev-57, Ukraine
(e-mail: anmart@stability.kiev.ua).

or to one of the Editors or to a member of the Editorial Board.

The title of the article must include: author(s) name, name of institution, department, address, FAX, and e-mail; an Abstract of 50-100 words should not include any formulas and citations; key words, and AMS subject classifications number(s). The size for regular paper should be 10-14 pages, survey (up to 24 pages), short papers, letter to the editor and book reviews (2-3 pages).

(3) Tables, Graphs and Illustrations. All figures must be suitable for reproduction without being retouched or redrawn and must include a title. Line drawings should include all relevant details and should be drawn in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach a PC diskette with files of the artwork (preferably in PCX format).

(4) References. Each entry must be cited in the text by author(s) and number or by number alone. All references should be listed in their alphabetic order. Use please the following style:

Journal: [1] Poincaré, H. Title of the article. *Title of the Journal* **Vol.1**(No.1) (year) pages. [Language].
Book: [2] Liapunov, A.M. *Title of the book.* Name of the Publishers, Town, year.
Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book.* (Eds.). Name of the Publishers, Town, year, pages. [Language].

(5) Proofs and Reprints. Proofs sent to authors should be returned to the Editor with corrections within three days after receipt. Acceptance of the paper entitles the author to 10 free reprints.

(6) Editorial Policy. Every paper is reviewed by the regional editor, and/or a referee, and it may be returned for revision or rejected if considered unsuitable for publication.

(7) Copyright Assignment. When a paper is accepted for publication, author(s) will be requested to sign a form assigning copyright to Informath Publishing Group. Failure to do it promptly may delay the publication.

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published since 2001

Volume 5 Number 4 2005

CONTENTS

Founded by A.A.Martynyuk in 2001.
Registered in Ukraine Number: KB №5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562-8353 (Print), ISSN 1813-7385 (Online)) is an international journal published under the auspices of the S.P.Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and the Laboratory for Industrial and Aplied Mathematics (LIAM) at York University (Toronto, Canada). It is aimed at publishing high quality original scientific papers and surveys in area of nonlinear dynamics and systems theory and technical reports on solving practical problems. The scope of the journal is very broad covering:

SCOPE OF THE JOURNAL

Analysis of uncertain systems • Bifurcations and instability in dynamical behaviors • Celestial mechanics, variable mass processes, rockets • Control of chaotic systems • Controllability, observability, and structural properties • Deterministic and random vibrations • Differential games • Dynamical systems on manifolds • Dynamics of systems of particles • Hamilton and Lagrange equations • Hysteresis • Identification and adaptive control of stochastic systems • Modeling of real phenomena by ODE, FDE and PDE • Nonlinear boundary problems • Nonlinear control systems, guided systems • Nonlinear dynamics in biological systems • Nonlinear fluid dynamics • Nonlinear oscillations and waves • Nonlinear stability in continuum mechanics • Non-smooth dynamical systems with impacts or discontinuities • Numerical methods and simulation • Optimal control and applications • Qualitative analysis of systems with aftereffect • Robustness, sensitivity and disturbance rejection • Soft computing: artificial intelligence, neural networks, fuzzy logic, genetic algorithms, etc. • Stability of discrete systems • Stability of impulsive systems • Stability of large-scale power systems • Stability of linear and nonlinear control systems • Stochastic approximation and optimization • Symmetries and conservation laws

PUBLICATION AND SUBSCRIPTION INFORMATION

The **Nonlinear Dynamics and Systems Theory** is published four times per year in 2005. Base list subscription price per volume: US$149.00. This price is available only to individuals whose library subscribes to the journal OR who warrant that the Journal is for their own use and provide a home address for mailing. Separate rates apply to academic and corporate/government institutions. Our charge includes postage, packing, handling and airmail delivery of all issues. Mail order and inquires to:
Department of Processes Stability, S.P.Timoshenko Institute of Mechanics NAS of Ukraine, Nesterov str.,3, 03057, Kiev-57, MSP 680, Ukraine,
Tel: ++38-044-456-6140, Fax: ++38-044-456-1939, E-mail: anchern@stability.kiev.ua,
http:\\www.sciencearea.com.ua; http:\\www.e-ndst.kiev.ua

ABSTRACTING INFORMATION

Nonlinear Dynamics and Systems Theory is currently abstracted/indexed by Zentralblatt MATH and Mathematical Reviews.

Nonlinear Dynamics and Systems Theory, 5(4) (2005) 315–322

PERSONAGE IN SCIENCE

Professor Anthony N. Michel

Derong Liu*

Department of Electrical and Computer Engineering University of Illinois at Chicago, Chicago, IL 60607, USA

1 Education and Career Overview Anthony N. Michel received a B.S. degree in electrical engineering, an M.S. degree in mathematics, and a Ph.D. degree in electrical engineering, all from Marquette University, Milwaukee, WI, in 1958, 1964, and 1968, respectively. He also received a D.Sc. degree in applied mathematics from Technical University of Graz, Austria, in 1973.

Anthony N. Michel has seven years of industrial experience (one year with the U.S. Army, Corps of Engineers, and six years with AC Electronics, a Division of General Motors, both in Milwaukee). From 1968 to 1984 he was on the Electrical Engineering Faculty at Iowa State University, Ames, IA, where he was promoted from an Assistant Professor to an Associate Professor in 1969 and to a Full Professor in 1974. In 1972–1973, while on a sabbatical leave, he worked under the supervision of Professor Wolfgang Hahn at Technical University of Graz, Austria, on his D.Sc. degree in Applied Mathematics. In 1984 he joined the faculty of Electrical Engineering at the University of Notre Dame as Professor and the Department Chair. He served as Chair until 1988. In 1987 he was named Frank M. Freimann Professor of Engineering, and in 1988, he was appointed Matthew H. McCloskey Dean of Engineering. He served two terms as the Dean of the College of Engineering, from 1988 to 1998. From 1998 to December 31, 2002, he was Frank M. Freimann Professor in the Department of Electrical Engineering. Since January 1, 2003, he is Frank M. Freimann Professor of Engineering Emeritus and Matthew H. McCloskey Dean of Engineering Emeritus. He has also held visiting faculty positions at the Technical University of Vienna, Austria (1992), the Ruhr University in Bochum, Germany (1999), and the Johannes Kepler University in Linz, Austria (2004).

2 Research and Scholarly Activities In his distinguished career spanning over forty years, Anthony N. Michel has made seminal contributions in the qualitative analysis of dynamical systems, with an emphasis on stability theory. Specific areas in which he has contributed include finite-time and practical stability, Lyapunov stability of interconnected (resp., large-scale) dynamical systems, input-output properties of interconnected (resp., large-scale) systems, artificial neural networks with applications to associative memories, robust stability analysis, stability preserving mapping theory, and stability

*Corresponding author: dliu@cil.ece.uic.edu

theory of hybrid and discontinuous dynamical systems. Throughout, he has demonstrated the significance of his work with specific applications to signal processing, power systems, artificial neural networks, digital control systems, systems with state saturation constraints, and other areas.

On the topic of *finite-time* and *practical stability*, in contrast to other workers, Michel utilizes *prespecified time-varying sets* in formulating a notion of *set stability.* His Lyapunov-like results for set stability yield estimates for system trajectory behavior, obtained from the boundaries of prespecified sets [9, 10]. As a radical departure from the existing practices, this approach was subsequently adopted and extended by others.

To circumvent difficulties encountered in the analysis of *large-scale systems* with complex structure, Michel views such systems as interconnections of several simpler subsystems. The analysis is then accomplished in terms of the qualitative properties of the subsystems and the interconnecting structure. Michel advocates the use of *scalar Lyapunov functions* [1, 11, 13, 15, 20, 21] consisting of weighted sums of Lyapunov functions for the free subsystems. This approach has resulted in *significantly less conservative results* than the weak-coupling M-matrix results obtained by others who employ vector Lyapunov functions. These results in turn are applied by Michel in the *analysis and synthesis of artificial neural networks* [8, 23], and he also uses them as the basis of further results involving *computer generated norm-Lyapunov functions* which then are applied successfully in the analysis of *interconnected power systems* and *digital filters* [20]. The theory developed in this work is applicable to continuous-time and discrete-time systems, finite-dimensional and infinite-dimensional systems, and deterministic and stochastic systems [1].

Using the same philosophy as in [1, 11, 13, 15, 20, 21], Michel discovered the *first results* for the *input-output stability of interconnected systems* [12], which subsequently were expanded by many into all kinds of directions [1, 14, 16]. These results make possible the systematic analysis of *multi-loop nonlinear feedback systems* (consisting of interconnections of subsystems that satisfy, e.g., the *small gain theorem*, the *circle criterion*, the *passivity theorem*, or *Popov-like conditions*). In the same spirit, Michel established also results for the *response* (*due to periodic inputs*) *of nonlinear single-loop and multi-loop feedback systems* [17, 18], and results for the *existence*, *nonexistence*, *and stability of limit cycles* for such systems [1, 19, 22]. The proofs of the above results are rather technical and require extensive use of functional analysis results and fixed-point theorems in abstract spaces.

For his work on qualitative analysis of interconnected systems, Michel has received substantial recognition. In response to an invitation by Professor Richard Bellman, Michel co-authored with R.K. Miller the book on *qualitative analysis of large-scale dynamical systems* [1], which appeared in the Bellman Series in Mathematics in Science and Engineering (Academic Press). This book is widely referred to and has had an impact on other areas of large-scale systems (e.g., *power systems*).

Michel has also conducted extensive research in *artificial neural networks* with applications to *associative memories* [8, 23, 24, 30, 31, 35, 36]. This work, which addresses *network architectures*, *qualitative analysis*, *synthesis procedures*, and *implementation issues for several classes of continuous and discrete recurrent neural networks*, is widely referred to and one of their paradigms [24], "LSSM-linear systems in a saturated mode," has been used in the *software tool MATLAB.*

Michel has contributed significantly to robust stability analysis, most notably, for systems with *interval matrices* and *perturbed systems with perturbed equilibria.* He has established several (*Hurwitz* and *Schur*) *stability*, *controllability*, and *observability results* for linear systems with *interval plants* [27, 33, 34], while for nonlinear systems, he addresses the effects of *parameter perturbations* on the locations (and even existence) of equilibria, along with their *stability properties*, using fixed point theorems and the notion of "*extreme systems*" [28, 32, 37]. The work in [33] was *the first to provide necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices with a practical computer algorithm.* Michel has further extended the results in [28] to the *robust stability analysis of recurrent neural networks* [8, 30, 36, 37].

Michel has conducted fundamental research in qualitative analysis of dynamical systems using stability preserving mappings. He utilizes *stability preserving mappings* to develop a comparison theory for Lyapunov and Lagrange stability of *general dynamical systems* defined on metric space [5, 44], applicable to systems determined by all types of classical equations encountered in science, as well as to contemporary systems that cannot be described in this way (e.g., *discrete event systems* [29]). Some of this work has been published in Russian (in *Avtomatika i Telemekhanika*) and in a highly original book [5] (co-authored with K. Wang), where the entire Lyapunov and Lagrange stability theory is developed for general dynamical systems, making use of stability preserving mappings.

Michel's more recent research addresses *stability analysis of hybrid and discontinuous dynamical systems.* For such systems, he formulates a *general model* suitable for stability analysis (involving a notion of *generalized time*), which contains most of the hybrid and discontinuous systems considered in the literature as special cases. For this model, he establishes the *Principal Lyapunov and Lagrange stability results*, including *Converse Theorems* [7, 39, 42, 43, 45] and he applies these results in the analysis of several special classes of systems, including *switched systems* [7], *digital control systems* [7, 38], *impulsive systems* [7, 41], *pulse-width-modulated feedback control systems* [7, 46], *systems with saturation constraints* [4, 7, 25, 26, 40], and others.

Currently, Michel is working on *stability issues of infinite dimensional discontinuous dynamical systems.* In particular, he is concerned with discontinuous systems determined by *differential equations in Banach space* and by *linear and nonlinear semigroups.* Specific classes of systems that are considered in this work are those that can be described by *functional differential equations*, *Volterra integro-differential equations*, *certain classes of partial differential equations*, and others [47, 48].

Michel has played a significant role as an *educator.* His eight books [1−8] which have been well received in the systems and control community around the world, and in many instances have blazed new trails when first introduced, demonstrate his contributions as a teacher. Furthermore, his record of maintaining a highly productive research program while simultaneously serving as an effective *administrator* at Notre Dame, first as *Department Chair* (1984–1988) and then as *Dean* (1988–1998), puts him in rare company. Michel has served as *mentor* to *many outstanding graduate students.* Equal numbers of these are in *academe* and in *industry*, attesting to the fine balance Michel maintains in his research program between theory and practice. These former students have all outstanding careers. (For example, one of them was the Dean of Engineering at Washington State University.)

Anthony N. Michel has sustained a high level of significant research, mostly in control systems. His work is characterized by great depth, as exemplified by his contributions to stability theory of dynamical systems, and by great breadth, as demonstrated by the wide range of problems that he addresses. He has proved to be an excellent teacher and mentor, he has demonstrated to be an effective administrator, and he has rendered more than his share of service to his profession.

3 Service to the Profession Anthony N. Michel served as an Associate Editor of the IEEE Transactions on Circuits and Systems from 1977 to 1979, the Editor of the IEEE Transactions on Circuits and Systems from 1981 to 1983, and the President of the IEEE Circuits and Systems Society in 1989. He also served as an Associate Editor of the IEEE Transactions on Automatic Control in 1981 and 1982, an Associate Editor at Large of the IEEE Transactions on Automatic Control from 1991 to 2000, the Vice President of Technical Affairs (1994, 1995) and the Vice President of Conference Activities (1996, 1997) of the IEEE Control Systems Society. In addition, he served as an Associate Editor of IEEE Transactions on Neural Networks from 1989 to 1991. He currently serves as the Associate Editor for Book Reviews of IEEE Transactions on Automatic Control. He was Program Chair of the 1985 IEEE Conference on Decision and Control, Co-General Chair of the 1990 IEEE Symposium on Circuits and Systems, and General Chair of the 1997 IEEE Conference on Decision and Control.

4 Student Supervision Anthony N. Michel guided the work of 13 Ph.D. students at Iowa State University and 12 Ph.D. students at the University of Notre Dame. He also supervised 10 Master's degree students.

List of Doctoral Dissertations Supervised

(1) Cornick, D.E. *Numerical Optimization of Distributed Parameter Systems by Gradient Methods.* Ph.D. Dissertation, Iowa State University, 1970.
(2) Porter, D.W. *Stability of Multiple-Loop Nonlinear Time-Varying Systems.* Ph.D. Dissertation, Iowa State University, 1972.
(3) Bose, A.B. *Stability and Compensation of Systems with Multiple Nonlinearities.* Ph.D. Dissertation, Iowa State University, 1974.
(4) Oppenheimer, E.P.*Application of Interval Analysis to Problems of Linear Control Systems.* Ph.D. Dissertation, Iowa State University, 1974.
(5) Lasley, E.L. *The Qualitative Analysis of Composite Systems.* Ph.D. Dissertation, Iowa State University, 1975.
(6) Rasmussen, R.D. *Lyapunov Stability of Large-Scale Dynamical Systems.* Ph.D. Dissertation, Iowa State University, 1976.
(7) Vitacco, W.R. *Qualitative Analysis of Interconnected Dynamical Systems Containing Algebraic Loops.* Ph.D. Dissertation, Iowa State University, 1976.
(8) Gutmann, R.L. *Input-Output Stability of Interconnected Stochastic Systems.* Ph.D. Dissertation, Iowa State University, 1976.
(9) Tang, W. *Structure and Stability Analysis of Large Scale Systems using a New Graph-Theoretic Approach.* Ph.D. Dissertation, Iowa State University, 1978.
(10) Peterson, J.N. *Wind Generator Network Methodology and Analysis.* Ph.D. Dissertation, Iowa State University, 1980.
(11) Sarabudla, N.R. *Stability Analysis of Complex Dynamical Systems: Some Computational Methods.* Ph.D. Dissertation, Iowa State University, 1981.
(12) Nam, B.H. *Asymptotic Stability of Large-Scale Dynamical Systems using Computer Generated Lyapunov Functions.* Ph.D. Dissertation, Iowa State University, 1983.
(13) Erickson, K.T. *Stability Analysis of Fixed-Point Digital Filters using a Constructive Algorithm.* Ph.D. Dissertation, Iowa State University, 1983.
(14) Li, J.-H. *Qualitative Analysis and Synthesis of a Class of Neural Networks.* Ph.D. Dissertation, University of Notre Dame, 1988.
(15) Farrell, J.A. *Analysis and Synthesis Techniques for Two Classes of Nonlinear Dynamical Systems: Digital Controllers and Neural Networks.* Ph.D. Dissertation, University of Notre Dame, 1989.

(16) Sun, H.-F. *Two Problems in Finite Dimensional Dynamical Systems: Qualitative Analysis and Synthesis of a Class of Neural Networks and Linear Systems Subject to Parameter Variations.* Ph.D. Dissertation, University of Notre Dame, 1990.

(17) Gray, D.L. *New Paradigms for Feedforward and Feedback Artificial Neural Networks.* Ph.D. Dissertation, University of Notre Dame, 1990.

(18) Si, J. *Analysis and Synthesis of Discrete-Time Recurrent Neural Networks with High Order Nonlinearities.* Ph.D. Dissertation, University of Notre Dame, 1991.

(19) Yen, G. *Learning, Forgetting, and Unlearning in Associative Memories: The Eigenstructure Method and the Pseudo Inverse Method with Stability Constraints.* Ph.D. Dissertation, University of Notre Dame, 1991.

(20) Kuo, C.-H. *Robust Control Strategies for a Class of Large Scale Dynamical Systems: Contaminated Groundwater Remediation.* Ph.D. Dissertation, University of Notre Dame, 1993.

(21) Liu, D. *Qualitative Theory of Dynamical Systems with Saturation Nonlinearities.* Ph.D. Dissertation, University of Notre Dame, 1993.

(22) Ye, H. *Stability Analysis of Two Classes of Dynamical Systems: General Hybrid Systems and Neural Networks with Delays.* Ph.D. Dissertation, University of Notre Dame, 1996.

(23) Hu, B. *Qualitative Analysis of Hybrid Dynamical Systems.* Ph.D. Dissertation, University of Notre Dame, 1999.

(24) Hou, L. *Qualitative Analysis of Discontinuous Deterministic and Stochastic Dynamical Systems.* Ph.D. Dissertation, University of Notre Dame, 2000.

(25) Y. Sun, *Stability Analysis of Discontinuous Dynamical Systems.* Ph.D. Dissertation, University of Notre Dame, 2004.

5 Awards Anthony N. Michel received numerous awards in his career including the 1978 Best Transactions Paper Award of the IEEE Control Systems Society (currently called the Axelby Award) (with R.D. Rasmussen), the 1984 Guillemin-Cauer Prize Paper Award of the IEEE Circuits and Systems Society (with R. K. Miller and B.H. Nam), the 1985 Engineering Distinguished Professional Achievement Award of Marquette University, the 1993 Myril B. Reed Outstanding Paper Award of the IEEE Circuits and Systems Society (with K. Wang), the 1995 Technical Achievement Award of the IEEE Circuits and Systems Society, the 1997 Alexander von Humboldt Research Award (for Senior U.S. Scientists) from the Federal Republic of Germany, the 1998 Distinguished Member Award of the IEEE Control Systems Society, and the 2005 Distinguished Alumnus Award of Marquette University. He received an IEEE Centennial Medal in 1984, the Golden Jubilee Medal of the IEEE Circuits and Systems Society in 1999, and an IEEE Third Millennium Medal in 2000. He was a Fulbright Scholar in 1992 at the Technical University of Vienna in Austria and a Distinguished Lecturer of the IEEE Circuits and Systems Society from 1995 to 1997. He was elected Fellow of the IEEE in 1982 for contributions in the qualitative analysis of large-scale dynamic systems, and he was elected a Corresponding Member of the Russian Academy of Engineering in 1992 for contributions in qualitative analysis of dynamical systems using stability preserving mappings.

6 References Anthony N. Michel has published eight books, 30 chapters in books, 174 journal papers, and 262 conference papers. His work has been cited more than 1500 times (since 1976) in the Science Citation Index.

[1] Michel, A.N. and Miller, R.K. *Qualitative Analysis of Large Scale Dynamical Systems.* Academic Press, New York, 1977.

[2] Michel, A.N. and Herget, C.J. *Mathematical Foundations in Engineering and Science: Algebra and Analysis.* Prentice-Hall, Englewood Cliffs, NJ, 1981. (Republished after

going out of print as *Applied Algebra and Functional Analysis.* Dover Publications, New York, 1993.)

[3] Miller, R.K. and Michel, A.N. *Ordinary Differential Equations.* Academic Press, New York, 1982.

[4] Liu, D. and Michel, A.N. *Dynamical Systems with Saturation Nonlinearities: Analysis and Design.* Lecture Notes in Control and Information Sciences, Vol. 195, Springer-Verlag, Berlin, 1994.

[5] Michel, A.N. and Wang, K. *Qualitative Theory of Dynamical Systems: The Role of Stability Preserving Mappings.* Marcel Dekker, New York, 1995.

[6] Antsaklis, P.J. and Michel, A.N. *Linear Systems.* McGraw-Hill, New York, 1997.

[7] Michel, A.N., Wang, K. and Hu, B. *Qualitative Theory of Dynamical Systems: The Role of Stability Preserving Mappings* (2nd edition). Marcel Dekker, New York, 2001.

[8] Michel, A.N. and Liu, D. *Qualitative Analysis and Synthesis of Recurrent Neural Networks.* Marcel Dekker, New York, 2002.

[9] Michel, A.N. Quantitative analysis of systems: Stability, boundedness and trajectory behavior. *Archive for Rational Mechanics and Analysis* **38**(2) (1970) 107–122.

[10] Michel, A.N. and Porter, D.W. Practical stability and finite time stability of discontinuous systems. *IEEE Transactions on Circuit Theory* **19**(2) (1972) 123–129.

[11 Michel, A.N. Stability analysis of interconnected systems. *SIAM Journal on Control* **12**(3) (1974) 554–579.

[12] Porter, D.W. and Michel, A.N. Input-output stability of time-varying nonlinear multiloop feedback systems. *IEEE Transactions on Automatic Control* **19**(8) (1974) 422–427.

[13] Michel, A.N. Stability analysis of stochastic composite systems. *IEEE Transactions on Automatic Control* **20**(4) (1975) 246–250.

[14] Lasley, E.L. and Michel, A.N. Input-output stability of interconnected systems. *IEEE Trans. Autom. Control* **21**(1) (1976) 84–89.

[15] Rasmussen, R.D. and Michel, A.N. Stability of interconnected dynamical systems described on Banach spaces. *IEEE Trans. Autom. Control* **21**(8) (1976) 464–471.

[16] Miller, R.K. and Michel, A.N. L_2-stability and instability of large scale systems described by integro-differential equations. *SIAM J. Math. Anal.* **8**(3) (1977) 547–557.

[17] Miller, R.K. and Michel, A.N. On existence of periodic motions in nonlinear control systems with periodic inputs. *SIAM J. Control and Optimiz.* **18**(5) (1980) 585–598.

[18] Miller, R.K. and Michel, A.N. On the response of nonlinear multivariable interconnected feedback systems to periodic input signals. *IEEE Trans. Circ. and Systems* **27**(11) (1980) 1088–1097.

[19] Skar, S.J., Miller, R.K., and Michel, A.N. On existence and nonexistence of limit cycles in interconnected systems. *IEEE Trans. Autom. Control* **26**(10) (1981) 1153–1169.

[20] Michel, A.N., Miller, R.K. and Nam, B.H. Stability analysis of interconnected systems using computer generated Lyapunov functions. *IEEE Trans. Circ. and Systems* **29**(7) (1982) 431–440.

[21] Michel, A.N. On the status of stability of interconnected systems. Joint Special Issue on Large Scale Systems: *IEEE Trans. Automatic Control* **28**(6) (1983) 639–653; *IEEE Trans. Circ. and Systems* **30**(6) (1983) 326–340; *IEEE Trans. Systems, Man and Cybernetics* **13**(4) (1983) 439–453.

[22] Miller, R.K., Michel, A.N. and Krenz, G.S. On the stability of limit cycles in nonlinear feedback systems. *IEEE Trans. Circ. and Systems* **30**(9) (1983) 689–697.

[23] Michel, A.N., Farrell, J.A. and Porod, W. Qualitative analysis of neural networks. *IEEE Trans. Circ. and Systems* **36**(2) (1989) 229–243.

[24] Li, J.H., Michel, A.N. and Porod, W. Analysis and synthesis of a class of neural networks: Linear systems operating on a closed hypercube. *IEEE Trans. Circ. and Systems* **36**(11) (1989) 1405–1422.

[25] Liu, D. and Michel, A.N. Asymptotic stability of systems operating on a closed hypercube. *Systems & Control Letters* **19**(4) (1992) 281–285.

[26] Liu, D. and Michel, A.N. Asymptotic stability of discrete-time systems with saturation nonlinearities with applications to digital filters. *IEEE Trans. Circ. and Systems–I: Fundamental Theory and Appl.* **39**(10) (1992) 798–807.

[27] Wang, K. and Michel, A.N. On sufficient conditions for stability of interval matrices. *Systems and Control Letters* **20**(5) (1993) 345–351.

[28] Michel, A.N. and Wang, K. Robust stability: Perturbed systems with perturbed equilibria. *Systems and Control Letters* **21**(2) (1993) 155–162.

[29] Passino, K.M., Michel, A.N. and Antsaklis, P.J. Lyapunov stability of a class of discrete event system. *IEEE Trans. Autom. Control* **39**(2) (1994) 269–279.

[30] Wang, K. and Michel, A.N. Robustness and perturbation analysis of a class of artificial neural networks. *Neural Networks* **7**(2) (1994) 251–259.

[31] Liu, D. and Michel, A.N. Sparsely interconnected neural networks for associative memories with applications to cellular neural networks. *IEEE Trans. Circ. and Systems-II: Analog and Digital Signal Processing* **41**(4) (1994) 295–307.

[32] Wang, K. and Michel, A.N. Qualitative analysis of dynamical systems determined by differential inequalities with applications to robust stability. *IEEE Trans. Circ. and Systems–Part I: Fundamental Theory and Appl,* **41**(5) (1994) 377–386.

[33] Wang, K., Michel, A.N. and Liu, D. Necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. *IEEE Trans. Autom. Control* **39**(6) (1994) 1251–1255. (See also *IEEE Trans. on Autom. Control* **41**(2) (1996) 311.)

[34] Wang, K. and Michel, A.N. Necessary and sufficient conditions for the controllability and observability of a class of linear, time-invariant systems with interval plants. *IEEE Trans. Autom. Control* **39**(7) (1994) 1443–1447.

[35] Ye, H., Michel, A.N. and Wang, K. Qualitative analysis of Cohen-Grossberg neural networks with multiple delays. *Physical Review E* **51**(3) (1995) 2611–2618.

[36] Michel, A.N., Wang, K., Liu, D. and Ye, H. Qualitative limitations incurred in implementations of recurrent neural networks. *IEEE Control Systems Magazine* **15**(3) (1995) 52–65.

[37] Ye, H., Michel, A.N. and Wang, K. Robust stability of nonlinear time-delay systems with applications to neural networks. *IEEE Trans. Circuits and Systems–Part I: Fundamental Theory and Appl.* **43**(7) (1996) 532–543.

[38] Hou, L., Michel, A.N. and Ye, H. Some qualitative properties of sampled-data control systems. *IEEE Trans. Autom. Control* **42**(12) (1997) 1721–1725.

[39] Ye, H., Michel, A.N. and Hou, L. Stability theory for hybrid dynamical systems. *IEEE Trans. Autom. Control* **43**(4) (1998) 461–474.

[40] Hou, L. and Michel, A.N. Asymptotic stability of systems with saturation constraints. *IEEE Trans. Autom. Control* **43**(8) (1998) 1148–1154.

[41] Ye, H., Michel, A.N. and Hou, L. Stability analysis of systems with impulse effects. *IEEE Trans. Autom. Control* **43**(12) (1998) 1719–1723.

[42] Michel, A.N. Recent trends in the stability analysis of hybrid dynamical systems. *IEEE Trans. Circuits and Systems–Part I: Fundamental Theory and Appl.* **46**(1) (1999) 120–134.

[43] Michel, A.N. and Hu, B. Towards a stability theory of general hybrid dynamical systems. *Automatica* **35**(3) (1999) 371–384.

[44] Hou, L. and Michel, A.N. Stability preserving mapping for stochastic dynamical systems. *IEEE Trans. Autom. Control* **46**(6) (2001) 933–938.

[45] Hou, L. and Michel, A.N. Moment stability of discontinuous stochastic dynamical systems. *IEEE Trans. Autom. Control* **46**(6) (2001) 938–943.

[46] Hou, L. and Michel, A.N. Stability analysis of pulse-width-modulated feedback systems. *Automatica* **37**(9) (2001) 1335–1349.

[47] Sun, Y., Michel, A.N. and Zhai, G. Stability of discontinuous retarded functional differential equations with applications. *IEEE Trans. Autom. Control* **50**(8) (2005) 1090–1105.

[48] Michel, A.N., Sun, Y. and Molchanov, A.P. Stability analysis of discontinuous dynamical systems determined by semigroups. *IEEE Trans. Autom. Control* **50**(9) (2005) 1277–1290.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 323–344

A Parametrization Approach for Solving the Hamilton–Jacobi Equation and Application to the $\mathcal{A}_2$-Toda Lattice

M.D.S. Aliyu* and L. Smolinsky

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

Received: January 18, 2005; Revised: October 10, 2005

Abstract: Hamilton–Jacobi (HJ)-theory is an extension of Lagrangian mechanics and concerns itself with a directed search for a coordinate transformation in which the equations of motion can be easily integrated. Hamilton (1838) has developed the method for obtaining the desired transformation equations by finding a smooth function S called a *generating function* or *Hamilton's principal function*, which satisfies a certain nonlinear first-order partial-differential equation (PDE) also known as the *Hamilton–Jacobi equation* (HJE).

Unfortunately, the HJE being nonlinear is very difficult to solve; and thus, except for the case in which the variables in the equation are separable, its application remains limited. It is thus our aim in this paper to present a new approach for solving the Hamilton–Jacobi equation for a fairly large class of Hamiltonian systems and to apply it in particular to the $\mathcal{A}_2$-Toda lattice.

Keywords: *Lagrangian mechanics; Hamiltonian system; contact transformation; generating function; Hamilton–Jacobi equation.*

Mathematics Subject Classification (2000): 70H20.

1 Introduction to Hamilton–Jacobi Theory

Hamilton–Jacobi (HJ)-theory is an extension of Lagrangian mechanics and concerns itself with a directed search for a coordinate transformation in which the equations of motion can be easily integrated. The equations of motion of a given mechanical system can often be simplified considerably by a suitable transformation of variables such that all the new

*Corresponding author: dikko@math.lsu.edu

position and momentum coordinates are constants. A particular type of transformation is chosen in such a way that the new equations of motion retain the same form as in the former coordinates; such a transformation is called *canonical* or *contact* and can greatly simplify the solution to the equations of motion. Hamilton (1838) has developed the method for obtaining the desired transformation equations using what is today known as *Hamilton's principle*. It turns out that the required transformation can be obtained by finding a smooth function S called a *generating function* or *Hamilton's principal function*, which satisfies a certain nonlinear first-order partial-differential equation (PDE) also known as the *Hamilton–Jacobi equation* (HJE).

Unfortunately, the HJE being nonlinear, is very difficult to solve; and thus, it might appear that little practical advantage has been gained in the application of the HJ-theory. Nonetheless, under certain conditions, and when the Hamiltonian is independent of time, it is possible to separate the variables in the HJE, and the solution can then always be reduced to quadratures. Thus, the HJE becomes a useful computational tool only when such a separation of variables can be achieved.

The aim of this paper is two-fold. First, to give an overview of the essentials of Hamilton–Jacobi theory, namely; (i) the Hamiltonian reformulation of the equations of motion of a mechanical system; and (ii) the Hamiltonian transformation of the equations of motion. Secondly, to present an approach for solving the HJE for a fairly large class of Hamiltonian systems in which the variables in the equation may not be separable and/or the Hamiltonian is not time-independent. We apply the approach to a class of integrable Hamiltonian systems known as the Toda lattice. Computational results are presented to show the usefulness of the method.

The rest of the paper in organized as follows. In the remainder of this section, we introduce notations. In Section 2, we discuss the Hamiltonian formulation of the equations of motion of a natural mechanical system. Then we discuss Hamiltonian coordinate transformations and generating functions of the transformations in Section 3. In Section 4, we discuss the Hamilton–Jacobi equation which is the central focus of the paper. In Section 5, we review the Toda lattice as a Hamiltonian system, and discuss the method of Lax for solving the system. Then in Section 6, we discuss the main results of the paper, which is a parametrization approach for solving the HJE. We also apply the results to the $\mathcal{A}_2$-Toda lattice. Finally, in Section 7, we give conclusions.

Notation The notation is fairly standard except where otherwise stated. Moreover, R, R^n will denote respectively, the real line and the n-dimensional real vector space, $t \in R$ will denote the time parameter. Let M^n, N^n, … denote Riemannian manifolds with dimension n, which are compact. Let $TM = \bigcup_{x\in M} T_xM$, $T^\star M = \bigcup_{x\in M} T^\star_x M$ respectively denote the tangent and cotangent bundles of M with dimensions $2n$. Moreover, π_M and $\pi^\star_M$ will denote the natural projections $TM \to M$ and $T^\star M \to M$ respectively. $SO(n, M)$ and $sl(n, M)$ will denote the special orthogonal group and the lie-algebra of the special linear group of matrices over M respectively. A $C^\infty(M)$ vector-field is a mapping $f\colon M \to TM$ such that $\pi \circ f = I_M$ (the identity on M), and f has continuously differentiable partial derivatives of arbitrary order. A vector field f also defines a differential equation (or a dynamic system) $\dot{x}(t) = f(x)$, $x \in M$, $x(t_0) = x_0$.

A differential k-form ω^k_x, $k = 1, 2, \dots,$ at a point $x \in M$ is an exterior product of k-vectors from T_xM to R i.e. $\omega^k_x \colon T_xM \times \dots \times T_xM$ (k copies) $\to R$, which is a k-linear skew-symmetric function of k-vectors on T_xM. The space of all smooth k-forms on M is denoted by $\Omega^k(M)$. The $\mathcal{F}$-derivative (Frèchet derivative) of a real-valued function

$U\colon R^n \to R$ is defined as any ϱ such that $\lim\limits_{v\to 0}\dfrac{1}{\|v\|}[U(x+v)-U(x)-\langle\varrho,v\rangle]=0$, for any $v\in R^n$. For a smooth function $f\colon R^n\to R$, $f_x=\dfrac{\partial f}{\partial x}=(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})$. Further, let $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty\colon M\to R$ denote respectively, 1, 2, and ∞ norms on M, where

$$\|v(q)\|_1=\sum_i^n|v_i(q)|,\quad \|v(q)\|_2=\sum_{i=1}^n|v_i(q)|^2$$

and $\|v(q)\|_\infty=\max_i\{v_i(q)\colon i=1,\dots,n\}$ for any vector $v\colon M_q\to T_qM$. Also, if $f\colon[0,1]\to R$, then

$$\|f(s)\|_{L_p}=\left(\int_0^1|f(s)|^p\right)^{\frac{1}{p}},\quad 1\le p<\infty,$$

while $\|f(s)\|_{L_\infty}=\sup_{s\in[0,1]}|f(s)|$.

2 The Hamiltonian Formulation of Mechanics

To review the approach, let the configuration space of the system be defined by a smooth n-dimensional Riemannian manifold M. If (φ,U) is a coordinate chart, we write $\varphi=q=(q_1,\dots,q_n)$ for the local coordinates and $\dot q_i=\dfrac{\partial}{\partial q_i}$ in the tangent bundle $TM|_U=TU$. We shall be considering *natural mechanical systems* which are defined as follows.

Definition 2.1 A Lagrangian mechanical system on a Riemannian manifold is called *natural* if the Lagrangian function $L\colon T\mathcal{U}\times R\to R$, with $\mathcal{U}\subset M$ open, is equal to the difference between the kinetic energy and the potential energy of the system as

$$L(q,\dot q,t)=T(q,\dot q,t)-V(q,t), \tag{2.1}$$

where $T\colon\mathcal{U}\to R$ is the kinetic energy which is given by the quadratic form

$$T=\frac{1}{2}\langle v,v\rangle,\quad v\in T_qU$$

and $V\colon M\times R\to R$ is the potential energy of the system (which may be independent of time).

For natural mechanical systems, the kinetic energy is a positive-definite symmetric quadratic form of the generalized velocities,

$$T(q,\dot q,t)=\frac{1}{2}\,\dot q^{\mathrm T}\Psi(q,t)\dot q. \tag{2.2}$$

It is further known from Lagrangian mechanics and as can be derived using the D'Alembert's *principle of virtual work* or Hamilton's *principle of least action* [3, 7, 8], that the motion of a *holonomic conservative*[1] mechanical system satisfies Lagrange's equations of motion given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots, n. \tag{2.3}$$

Then the above equation (2.3) may always be written in the form

$$q = g(q, \dot{q}, t), \tag{2.4}$$

for some function $g\colon T\mathcal{U} \times R \to R^n$.

On the other hand, in the Hamiltonian formulation, we choose to replace all the $\dot{q}_i$ by independent coordinates, p_i, in such a way that

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \ldots, n. \tag{2.5}$$

If we let

$$p_i = h_i(q, \dot{q}), \quad i = 1, \ldots, n, \tag{2.6}$$

then the Jacobian of h with respect to $\dot{q}$, using (2.1), (2.2) and (2.5), is given by $\Psi(q)$ which is positive definite, and hence equation (2.5) can be inverted to yield

$$\dot{q}_i = g_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t), \quad i = 1, \ldots, n, \tag{2.7}$$

for some continuous functions $g_1, \ldots, g_n$. The coordinates $q = (q_1, q_2, \ldots, q_n)^{\mathrm{T}}$, in this framework, are referred to as the *generalized coordinates* and $p = (p_1, p_2, \ldots, p_n)^{\mathrm{T}}$ are the *generalized momenta*. Together, these variables form a new system of coordinates for the system known as the *phase space* of the system. If (U, φ) where $\varphi = (q_1, q_2, \ldots, q_n)$ is a chart on M, then since $p_i\colon TU \to R$, they are elements of $T^\star U$, and together with the q_i's form a system of $2n$ local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, where $p_i(q) \in T_q^\star M$, $i = 1, \ldots, n$, for the phase-space.

We now define the Hamiltonian function of the system $H\colon T^\star M \times R \to R$ as the *Legendre transform* [3, 5] of the Lagrangian function with respect to $\dot{q}$ by

$$H(q, p, t) = p^{\mathrm{T}}\dot{q} - L(q, \dot{q}, t). \tag{2.8}$$

Consider now the differential of H with respect to q, p and t as

$$dH = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} dp + \left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} dq + \frac{\partial H}{\partial t} dt. \tag{2.9}$$

The above expression must be equal to the total differential of $H = p\dot{q} - L$ for $p = \dfrac{\partial L}{\partial \dot{q}}$:

$$dH = \dot{q}^{\mathrm{T}} dp - \left(\frac{\partial L}{\partial q}\right)^{\mathrm{T}} dq - \left(\frac{\partial L}{\partial t}\right)^{\mathrm{T}} dt. \tag{2.10}$$

[1]Holonomic if the constraints on the system are expressible as equality constraints. Conservative if there exists a time-dependent potential.

Thus, in view of the independent nature of the coordinates, we obtain a set of three relationships:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}, \quad \text{and} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}.$$

Finally, applying Lagrange's equation (2.3) together with (2.5) and the preceding results, one obtains the expression for $\dot{p}$. Since we used Lagrange's equation, $\dot{q} = \dfrac{dq}{dt}$ and $\dot{p} = \dfrac{dp}{dt}$. The resulting Hamiltonian canonical equations of motion are then given by

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p, t), \tag{2.11}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}(q, p, t). \tag{2.12}$$

Thus, we have proven the following theorem.

Theorem 2.1 [3] *The system of Lagrange's equations (2.3) is equivalent to the system of $2n$ first-order Hamilton's equations (2.11), (2.12).*

In addition, for time-independent conservative systems, $H(q, p)$ has a simple physical interpretation. From (2.8) and using (2.5), we have

$$\begin{aligned} H(q,p,t) &= p^{\mathrm{T}}\dot{q} - L(q,\dot{q},t) = \dot{q}^{\mathrm{T}}\frac{\partial L}{\partial \dot{q}} - (T(q,\dot{q},t) - U(q,t)) \\ &= \dot{q}^{\mathrm{T}}\frac{\partial T}{\partial \dot{q}} - T(q,\dot{q},t) + U(q,t) \\ &= 2T(q,\dot{q},t) - T(q,\dot{q},t) + U(q,t) = T(q,\dot{q},t) + U(q,t), \end{aligned} \tag{2.13}$$

i.e., the total energy of the system. This completes the Hamiltonian formulation of the equations of motion, and can be seen as an off-shoot of the Lagrangian formulation. It can also be seen that, while the Lagrangian formulation involves n second-order equations, the Hamiltonian description sets up a system of $2n$ first-order equations in terms of the $2n$ variables p and q. This remarkably new system of coordinates gives new insight and physical meaning to the equations. However, the system of Lagrange's equations and Hamilton's equations are completely equivalent as the above theorem asserts.

Furthermore, because of the symmetry of Hamilton's equations (2.11), (2.12) and the even dimension of the system, a new structure emerges on the phase space $T^\star M$ of the system. This structure is defined by a nondegenerate closed differential 2-form $\omega^2 \in \Omega^2(M)$ which in the above local coordinates is defined as

$$\omega^2 = dp \wedge dq = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{2.14}$$

Thus, the pair $(T^\star M, \omega^2)$ form a *symplectic manifold* [1, 3, 11], and together with a C^r Hamiltonian function $H\colon T^\star M \to R$ define a Hamiltonian mechanical system. With this notation we have the following representation of a Hamiltonian system.

Definition 2.2 Let $(T^\star M, \omega^2)$ be a symplectic manifold and $H\colon T^\star M \to R$ the Hamiltonian function. Then the vector field X_H determined by the condition

$$\omega^2(X_H, Y) = dH(Y) \tag{2.15}$$

for all vector fields Y, is called the *Hamiltonian vector field* with energy function H. The tuple $(T^\star M,\, \omega^2,\, X_H)$ is called a Hamiltonian system.

Remark 2.1 It is important to note that, the nondegeneracy of ω^2 guarantees that X_H exists, and is a C^{r-1} vector field. Moreover, on a connected symplectic manifold, any two Hamiltonians for the same vector field X_H have the same differential (2.15), so differ by a constant only.

We also have the following proposition [1].

Proposition 2.1 *Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be canonical coordinates so that ω^2 is given by (2.14). Then, in these coordinates*

$$X_H = \left(\frac{\partial H}{\partial p_1},\ \ldots,\ \frac{\partial H}{\partial p_n},\ -\frac{\partial H}{\partial q_1},\ \ldots,\ -\frac{\partial H}{\partial q_n}\right) = J\cdot \nabla H$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Thus, $(q(t), p(t))$ is an integral curve of X_H if and only if Hamilton's equations (2.11), (2.12) hold.

Now suppose that a transformation of coordinates is introduced $q_i \to Q_i,\ p_i \to P_i,$ $i = 1, \ldots, n,$ defined by

$$q_i = \phi_i(Q, P, t), \tag{2.16}$$
$$p_i = \psi_i(Q, P, t) \tag{2.17}$$

such that every Hamiltonian function transforms as

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \to K(Q_1, \ldots, Q_n, P_1, \ldots, P_n, t)$$

in such a way that the new equations of motion retain the same form as in the former coordinates, i.e.,

$$\frac{dQ}{dt} = \frac{\partial K}{\partial p}(Q, P, t), \tag{2.18}$$
$$\frac{dP}{dt} = -\frac{\partial K}{\partial q}(Q, P, t). \tag{2.19}$$

Such a transformation is called *canonical* or *contact* and can greatly simplify the solution to the equation of motion, especially if Q, P are selected such that $K(\cdot,\cdot,\cdot)$ is a constant independent of Q and P. Should this happen, then Q and P will also be constants and the solution to the equations of motion are immediately at hand (given the transformation). We simply transform back to the original coordinates; under the assumption that the transformation is univalent and invertible. Hamilton (1838) has developed a method for obtaining the desired transformation equations using what is today known as *Hamilton's principle* [3, 7, 8, 10].

3 The Transformation Generating Function

A given Hamiltonian system can often be simplified considerably by a suitable transformation of variables such that all the new position and momentum coordinates (Q_i, P_i) are constants. A particular type of transformation is discussed in this section.

Accordingly, define the Lagrangian function of the system $L\colon T\mathcal{U} \times R \to R$ as the Legendre transform [3] of the Hamiltonian function by

$$L(q, \dot{q}, t) = p^{\mathrm{T}} \dot{q} - H(q, p, t). \tag{3.1}$$

Then, in the new coordinates, the new Lagrangian function is

$$\bar{L}(Q, \dot{Q}, t) = P^{\mathrm{T}} \dot{Q} - K(Q, P, t). \tag{3.2}$$

Since both $L(\cdot,\cdot,\cdot)$ and $\bar{L}(\cdot,\cdot,\cdot)$ are conserved, each must separately satisfy Hamilton's principle. However, $L(\cdot,\cdot,\cdot)$ and $\bar{L}(\cdot,\cdot,\cdot)$ need not be equal in order to satisfy the above requirement. Indeed we can write [8]

$$L(q, \dot{q}, t) = \bar{L}(Q, \dot{Q}, t) + \frac{dS}{dt}(q, p, Q, P, t) \tag{3.3}$$

for some arbitrary function $S\colon \mathcal{X} \times \mathcal{X} \times R \to R$, where $\mathcal{X} \subset T^{\star}M$ is open.

The next step is to show that, first, if such a function is known, then the transformation we seek follows directly. Secondly, that the function can be obtained by solving a certain partial differential equation.

The generating function S relates the old to the new coordinates via the equation

$$S = \int (L - \bar{L})\, dt = \sigma(q, p, Q, P, t) \tag{3.4}$$

for some function $\sigma\colon \mathcal{X} \times \mathcal{X} \times R \to R$. Thus, S is a function of $4n+1$ variables, and hence no more than four independent sets of relationships among the dependent coordinates can exist. Two such relationships expressing the old sets of coordinates in terms of the new set are given by (2.16), (2.17). Hence only two independent sets of relationships among the coordinates remain for defining S and no more than two of the four sets of coordinates may be involved. Therefore, there are four possibilities

$$S_1 = f_1(q, Q, t); \quad S_2 = f_2(q, P, t), \tag{3.5}$$

$$S_3 = f_3(p, Q, t); \quad S_4 = f_4(p, P, t). \tag{3.6}$$

Any one of the above four types of generating functions may be selected, and a transformation obtained from it. For example, if we consider the generating function S_1, taking its differential, we have

$$dS_1 = \sum_{i=1}^{n} \frac{\partial S_1}{\partial q_i}\, dq_i + \sum_{i=1}^{n} \frac{\partial S_1}{\partial Q_i}\, dQ_i + \frac{\partial S_1}{\partial t}\, dt. \tag{3.7}$$

Again, taking the differential as defined by (3.1), (3.2) (3.3), we have

$$dS_1 = \sum_{i=1}^{n} p_i \, dq_i - \sum_{i=1}^{n} P_i \, dQ_i + (K - H) \, dt. \tag{3.8}$$

Finally, using the independence of coordinates, we equate coefficients, and obtain the desired transformation equations

$$\begin{aligned} p_i &= \frac{\partial S_1}{\partial q_i}(q, Q, t) \\ P_i &= -\frac{\partial S_1}{\partial Q_i}(q, Q, t) \\ K - H &= \frac{\partial S_1}{\partial t}(q, Q, t), \quad i = 1, \ldots, n. \end{aligned} \tag{3.9}$$

Similar derivation can be applied to the remaining three types of generating functions.

4 The Hamilton–Jacobi Equation

In this section, we turn our attention to the last missing link in the Hamiltonian transformation theory; an approach for determining the transformation generating function, S. There is only one equation available for this purpose

$$H(q, p, t) + \frac{\partial S}{\partial t} = K(P, Q, t). \tag{4.1}$$

However, there are two unknown functions in this equation: S and K. Thus, the best we can do is to assume a solution for one and then solve for the other. In this regard, suppose we arbitrarily introduce the condition that K is to be *identically zero*? Under this condition, $\dot{Q}$ and $\dot{P}$ vanish; resulting in $Q = \alpha$, and $P = \beta$, constants. The inverse transformation then yields the motion $q(\alpha, \beta, t)$, $p(\alpha, \beta, t)$ in terms of these constants of integration, α and β.

Consider now generating functions of the first type. Having forced a solution on K, we must now solve the partial differential equation (PDE)

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{4.2}$$

for S, where $\frac{\partial S}{\partial q} = \left(\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\right)^{\mathrm{T}}$. This equation is known as the *Hamilton–Jacobi equation (HJE)*, and was improved and modified by Jacobi in 1838. For a given function $H(q, p, t)$, this is a first-order PDE in the unknown function $S(q, \alpha, t)$ which is customarily called *Hamilton's principal function.* We need a solution for this equation which depends on n arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ in such a way that the Jacobian determinant of $\frac{\partial S}{\partial q_i}$ with respect to (wrt) the α_j satisfies

$$\left| \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \right| \neq 0. \tag{4.3}$$

The above condition excludes the possibility in which one of the n constants α_j is additive; that is, one must have

$$S(q,\alpha,t) \neq \overline{S}(q,\alpha_1,\alpha_2,\dots,\alpha_{n-1},t) + \alpha_n. \tag{4.4}$$

A solution $S(q,\alpha,t)$ satisfying (4.3) is called a "complete solution" of the HJE (4.2), and solving the HJE is equivalent to finding the solutions of the equations of motion (2.11), (2.12). Conversely, the solution of (4.2) is nothing more than a solution of the equations (2.11), (2.12) using the method of characteristics [5, 6]. However, it is generally not simpler to solve (4.2) instead of (2.11), (2.12).

If a complete solution $S(q,\alpha,t)$ of (4.2) is known, then one has

$$\frac{\partial S}{\partial q_i} = p_i, \tag{4.5}$$

$$\frac{\partial S}{\partial \alpha_i} = -\beta_i, \quad i = 1,\dots,n. \tag{4.6}$$

Since the condition (4.3) is satisfied, the second algebraic equation above may be solved for q and the first solved for $p(\alpha,\beta,t)$. One thus has a canonical transformation from (α,β) to (q,p). And it follows from the definition of canonical transformation that the inverse transformation $\alpha = \alpha(q,p,t)$, $\beta = \beta(q,p,t)$ also is canonical.

On the other hand, if the Hamiltonian is not explicitly a function of time or is independent of time, which arises in many dynamical systems of practical interest, then the solution to (4.2) can then be formulated in the form

$$S(q,\alpha,t) = -ht + W(q,\alpha) \tag{4.7}$$

with $h = h(\alpha)$. Consequently, the use of (4.7) in (4.2) yields the following PDE in W

$$H\left(q, \frac{\partial W}{\partial q}\right) = h, \tag{4.8}$$

where h is the energy constant (if the kinetic energy of the system is homogeneous quadratic, the constant equals the total energy, E). Moreover, since W does not involve time, the new and the old Hamiltonians are equal, and it follows that $K = h$. The function W, known as *Hamilton's characteristic function*, thus generates a canonical transformation in which all the new coordinates are cyclic. Further, one may choose $h = \alpha_n$ for example, so that

$$W = W(q,\alpha_1,\dots,\alpha_{n-1},h) \tag{4.9}$$

depends on $n-1$ additional arbitrary constants besides h. Noting that the Jacobian determinant of S wrt the n arbitrary coordinates, and the n constants $\alpha_1,\dots,\alpha_{n-1}$, h may not vanish, then from (4.5), (4.6) and (4.7), we have the following system

$$\begin{aligned} &\frac{\partial W}{\partial \alpha_i} = -\beta_i, \quad i = 1,2\dots,n-1, \\ &\frac{\partial W}{\partial h} = t - \beta_n, \\ &\frac{\partial W}{\partial q} = p. \end{aligned} \tag{4.10}$$

where the term $t - \beta_n$ in the preceding equation follows directly from the fact that the system is autonomous. The above system of equations may be solved for $n-1$ components of q, say, for $q_1, q_2, \ldots, q_{n-1}$ resulting in

$$\begin{aligned} q_1 &= q_1(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \\ q_2 &= q_2(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_{n-1} &= q_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \end{aligned} \tag{4.11}$$

where the time t is replaced as the parameter q_n. These equations are then the solution for the system.

5 The Toda Lattice

The Toda lattice as a Hamiltonian system describes the motion of n particles moving in a straight line with "exponential interaction" between them. Mathematically, it is equivalent to a problem in which a single particle moves in R^n. Accordingly, let the positions of the particles at time t (in R) be $q_1(t), \ldots, q_n(t)$, respectively. We assume also that each particle has mass 1, and therefore the momentum of the i-th particle at time t is $p_i = \dot{q}_i$. Consequently, the Hamiltonian function for the *finite (or non-periodic)* lattice is defined by

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})}. \tag{5.1}$$

Thus the canonical equations for the system are given by

$$\begin{aligned} \frac{dq_j}{dt} &= p_j \quad j = 1, \ldots, n, \\ \frac{dp_1}{dt} &= -2e^{2(q_1 - q_2)}, \\ \frac{dp_j}{dt} &= -2e^{2(q_j - q_{j+1})} + 2e^{2(q_{j-1} - q_j)}, \quad j = 2, \ldots, n-1, \\ \frac{dp_n}{dt} &= 2e^{2(q_{n-1} - q_n)}. \end{aligned} \tag{5.2}$$

It may be assumed in addition that $\sum_{j=1}^n q_j = \sum_{j=1}^n p_j = 0$, and the coordinates $q_1, \ldots, q_n$ can be chosen so that this condition is satisfied. While for the periodic lattice in which the first particle interacts with the last, the Hamiltonian function is defined by

$$\widetilde{H}(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})} + e^{2(q_n - q_1)}. \tag{5.3}$$

We may also consider the infinite lattice, in which there are infinitely many particles.

Using the inverse scattering method of solving the initial value problem for the Korteweg-de Vries equation (KdV) formulated by Lax [13], the solution for the lattice can be derived using matrix formalism which led to a simplification of the equations of motion. To introduce this formalism, define the following $(n \times n)$ matrices

$$L = \begin{pmatrix} p_1 & Q_{1,2} & 0 & \cdots & 0 & 0 \\ Q_{1,2} & p_2 & Q_{2,3} & \cdots & 0 & 0 \\ 0 & Q_{2,3} & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & Q_{n-1,n} \\ 0 & 0 & 0 & \cdots & Q_{n-1,n} & p_n \end{pmatrix}, \tag{5.4}$$

$$M = \begin{pmatrix} 0 & Q_{1,2} & 0 & \cdots & 0 & 0 \\ -Q_{1,2} & 0 & Q_{2,3} & \cdots & 0 & 0 \\ 0 & -Q_{2,3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & Q_{n-1,n} \\ 0 & 0 & 0 & \cdots & -Q_{n-1,n} & 0 \end{pmatrix}, \tag{5.5}$$

where $Q_{ij} = e^{(q_i - q_j)}$. We then have the following proposition [9].

Proposition 5.1 *The Hamiltonian system for the non-periodic Toda lattice (5.2) is equivalent to the Lax equation $\dot{L} = [L, M]$, where the function L, M take values in $sl(n,R)$ and $[cdot, \cdot]$ is the Lie bracket operation in $sl(n,R)$.*

Using the above matrix formalism, the solution of the Toda system (5.2) can be derived [9, 13].

Theorem 5.1 *The solution of the Hamiltonian system for the Toda lattice is given by $L(t) = Ad(\exp tV)_I^{-1} V$, where $V = L(0)$ and I represents the identity matrix.*

The can explicitly write the solution for the case of $n = 2$. Letting $q_1 = -q$, $q_2 = q$, $p_1 = -p$ and $p_2 = p$, we have

$$L = \begin{pmatrix} p & Q \\ Q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix}, \tag{5.6}$$

where $Q = c^{-2q}$. The solution of $\dot{L} = [L,\, M]$ with

$$L(0) = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

is

$$L(t) = Ad\left[\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right]_I^{-1} \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}.$$

Now

$$\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} = \begin{pmatrix} \cosh tv & \sinh tv \\ \sinh tv & \cosh tv \end{pmatrix},$$

and hence,

$$\left[\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right]_I^{-1} = \frac{1}{\sqrt{\sinh^2 tv + \cosh^2 tv}} \begin{pmatrix} \cosh tv & \sinh tv \\ \sinh tv & \cosh tv \end{pmatrix}.$$

Therefore,

$$L(t) = \frac{v}{\sinh^2 tv + \cosh^2 tv} \begin{pmatrix} -2\sinh\, tv \cosh\, tv & 1 \\ 1 & 2\sinh\, tv \cosh\, tv \end{pmatrix},$$

which means that

$$p(t) = -v\,\frac{\sinh\, 2tv}{\cosh\, 2tv}, \qquad Q(t) = \frac{v}{\cosh\, 2tv}.$$

Furthermore, if we recall that $Q(t) = e^{-2q(t)}$, it follows that

$$q(t) = -\frac{1}{2}\log\left(\frac{v}{\cosh\, 2tv}\right) = -\frac{1}{2}\log\, v + \frac{1}{2}\log\cosh\, 2vt. \tag{5.7}$$

6 Solving the Hamilton–Jacobi Equation

It is clear from the preceding discussion that the success of the Hamiltonian approach to mechanics depends heavily on the ability to solve the HJE. Because the prospects of success are limited by the inadequate state of the mathematical art in solving nonlinear PDEs. At present, the only technique of general utility is the method of *separation of variables*. If the Hamiltonian is explicitly a function of time, then separation of variables is not readily achieved for the HJE. However, if on the other hand, the Hamiltonian is not explicitly a function of time or is independent of time, which arises in many dynamical systems of practical interest, then the HJE (4.2) degenerates to the HJE (4.8). Nevertheless, solving this resulting HJE still remains a very difficult problem in general.

In this section we propose a parametrization approach for solving the Hamilton–Jacobi equation for a fairly large class of Hamiltonian systems, and then apply the approach to the $\mathcal{A}_2$-Toda lattice as special cases. To present the approach, let the configuration space of the class of Hamiltonian systems be a smooth n-dimensional manifold M with local coordinates $q = (q_1, \dots, q_n)$, i.e. if (φ, U) is a coordinate chart, we write $\varphi = q$ and $\dot{q}_i = \dfrac{\partial}{\partial q_i}$ in the tangent bundle $TM|_U = TU$. Further, let the class of systems under consideration be represented by Hamiltonian functions $H\colon T^\star M \to R$ of the form:

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q), \tag{6.1}$$

where $(p_1(q), \dots, p_n(q)) \in T_q^\star M$, and together with $(q_1, \dots, q_n)$ form the $2n$ symplectic coordinates for the phase-space $T^\star M$ of any system in the class, while $V\colon M \to R_+$ is the potential function which we assume to be nonseparable in the variables $q_i,\ i = 1, \dots, n$. The time-independent HJE corresponding to the above Hamiltonian function is given by

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial W}{\partial q_i}\right)^2 + V(q) = h, \tag{6.2}$$

where $W\colon M \to R$ is the Hamilton's characteristic function for the system.

We then have the following theorem concerning the solution of this HJE.

Theorem 6.1 *Let M be an open subset of R^n which is simply connected and let $q = (q_1, \dots, q_n)$ be the coordinates on M. Suppose ρ, $\theta_i \colon M \to R$ for $i = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$; $\theta = (\theta_1, \cdots, \theta_{\lfloor \frac{n+1}{2} \rfloor})$; and $\zeta_i \colon R \times R^{\lfloor \frac{n+1}{2} \rfloor} \to R$ are C^2 functions such that*

$$\frac{\partial \zeta_i}{\partial q_j}(\rho(q), \theta(q)) = \frac{\partial \zeta_j}{\partial q_i}(\rho(q), \theta(q)), \qquad \forall\, i, j = 1, \dots, n, \tag{6.3}$$

and

$$\frac{1}{2} \sum_{i=1}^{n} \zeta_i^2(\rho(q), \theta(q)) + V(q) = h \tag{6.4}$$

is solvable for the functions ρ, θ. Let

$$\omega^1 = \sum_{i=1}^{n} \zeta_i(\rho(q), \theta(q)) dq_i,$$

$\omega^1 \in \Omega^p rime(M)$, and suppose C is a path in M from an initial point q_0 to an arbitrary point $q \in M$. Then

(i) *ω^1 is closed;*
(ii) *ω^1 is exact;*
(iii) *if $W(q) = \int_C \omega^1$, then W satisfies the HJE (6.2).*

Proof (i)

$$d\omega^1 = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial}{\partial q_j} \zeta_i(\rho(q), \theta(q)) dq_j \wedge dq_i,$$

which by (6.3) implies $d\omega^1 = 0$; hence, ω^1 is closed.

(ii) Since by (i) ω^1 is closed, by the simple connectedness of M (Poincaré's lemma [1]), ω^1 is also exact.

(iii) By (ii) ω^1 is exact, therefore the integral $W(q) = \int_C \omega^1$ is independent of the path C. Therefore, W corresponds to a scalar function. Furthermore, $dW = \omega^1$ and $\frac{\partial W}{\partial q_i} = \zeta_i(\rho(q), \theta(q))$, and thus substituting in the HJE (6.2) and if (6.4) holds, then W satisfies the HJE.

In the next corollary we shall construct explicitly the functions ζ_i, $i = 1, \dots, n$, in the above theorem.

Corollary 6.1 *Assume the dimension n of the system is 2, and M, ρ, θ are as in the hypotheses of Theorem 6.1, and that conditions (6.3), (6.4) are solvable for θ and ρ. Also, define the functions ζ_i, $i = 1, 2$, postulated in the theorem by $\zeta_1(q) = \rho(q) \cos\theta(q)$, $\zeta_2(q) = \rho(q) \sin\theta(q)$. Then, if*

$$\omega^1 = \sum_{i=1}^{2} \zeta_i(\rho(q), \theta(q))\, dq_i, \qquad W = \int_C \omega^1,$$

and $q\colon [0,1] \to M$ is a parametrization of C such that $q(0) = q_0$, $q(1) = q$, then

(i) *W is given by*

$$W(q,h) = \gamma \int_0^1 \sqrt{(h - V(q(s))} \left[\cos\theta(q(s))q_1^{'}(s) + \sin\theta(q(s))q_2^{'}(s)\right] ds \tag{6.5}$$

where $\gamma = \pm\sqrt{2}$ and $q_i' = \dfrac{dq_i(s)}{ds}$;

(ii) *W satisfies the HJE (6.2).*

Proof (i) If (6.3) is solvable for the function θ, then substituting the functions $\zeta_i(\rho(q), \theta(q))$, $i = 1, 2$ as defined above in (6.4), we get immediately

$$\rho(q) = \pm\sqrt{2(h - V(q))}.$$

Further, by Theorem 6.1, ω^1 given above is exact, and $W = \int_C \omega^1 dq$ is independent of the path C. Therefore, if we parametrize the path C by s, then the above line integral can be performed coordinate-wise with W given by (6.5) and $\gamma = \pm\sqrt{2}$.

(ii) follows from Theorem 6.1.

Remark 6.1 The above corollary constructs one explicit parametrization that may be used. However, because of the number of parameters available in the parametrization are limited, the above parametrization is only suitable for systems with $n = 2$. Other types of parametrizations that are suitable could also be employed.

If however the dimension n of the system is 3, then the following corollary gives a procedure for solving the HJEs.

Corollary 6.2 *Assume the dimension n of the system is 3, and M, ρ, are as in the hypotheses of Theorem 6.1. Let $\zeta_i\colon R \times R \times R \to R$, $i = 1, 2, 3$, be defined by $\zeta_1(q) = \rho(q)\sin\theta(q)\cos\varphi(q)$, $\zeta_2(q) = \rho(q)\sin\theta(q)\sin\varphi(q)$, $\zeta_3(q) = \rho(q)\cos\theta(q)$, and assume (6.3) are solvable for θ and φ, while (6.4) is solvable for ρ. Then, if*

$$\omega^1 = \sum_{i=1}^{3} \zeta_i(\rho(q), \theta, \varphi) dq_i,$$

$W = \int_C \omega^1$, and $q\colon [0,1] \to M$ is a parametrization of C such that $q(0) = q_0$, $q(1) = q$, then

(i) *W is given by*

$$\begin{aligned} W(q,h) = \gamma \int_0^1 \sqrt{(h - V(q(s)))} \Big\{ &\sin\theta(q(s))\cos\varphi(q(s))q_1^{'}(s) \\ &+ \sin\theta(q(s))\sin\varphi(q(s))q_2^{'}(s) + \cos\theta(q(s))q_3^{'}(s) \Big\} ds, \end{aligned} \tag{6.6}$$

where $\gamma = \pm\sqrt{2}$;

(ii) *W satisfies the HJE (6.2).*

Proof Proof follows along the same lines as Corollary 6.1.

Remark 6.2 Notice that, the parametrization employed in the above corollary is now of a spherical nature.

The following theorem gives bounds on the solution W and its derivatives.

Theorem 6.2 *Let $N \subset M$ be the region in which the solution W of the HJE given in Corollaries 6.1 and 6.2 exists. Then if C is a path $q\colon [0,1] \to N$ in N parametrized by $s \in [0,1]$ such that $q(0) = q_0$, $q(1) = q$ we have the following bounds on the solution and its derivatives:*

(i) $\|W(q(s), h)\|_\infty \le |\gamma|\sqrt{h}\,\|q(s)\|_{\mathcal{L}_1}$;

(ii) $\left\|\dfrac{\partial W}{\partial q}\right\|_2 = |\sqrt{2}\rho(q)/\gamma|$;

(iii) $\left\|\dfrac{\partial W}{\partial q}\right\|_\infty = |\gamma|\sqrt{h}$.

Proof (i) From (6.5) or (6.6),

$$\begin{aligned}\|W(q,h)\|_\infty &\le |\gamma| \sum_{i=1}^{n} \int_0^1 \sup_{q(s)\in N} \left|\sqrt{(h - V(q))}\right| |q_i'(s)\, dq_i(s)| \\ &\le |\gamma|\sqrt{h} \int_0^1 (|q_1'(s)\, ds| + |q_2'(s)\, ds| + \ldots + |q_n'(s)\, ds|) \\ &\le |\gamma|\sqrt{h}\|q(s)\|_{L_1}.\end{aligned}$$

(ii) Using the definition of $\partial W/\partial q_i$ given in Corollaries 6.1 and 6.2, we have

$$\left\|\frac{\partial W}{\partial q_i}\right\|_2^2 = \sum_{i=1}^{n} \left|\frac{\partial W}{\partial q_i}\right|^2 = |\sqrt{2}\rho(q)/\gamma|^2,$$

hence the result.

(iii) Follows by taking the sup over $q \in M$ of $\partial W/\partial q_i$, $i = 1, \ldots, n$.

Furthermore, the following proposition gives regularity of the solution.

Proposition 6.1 *If the functions ρ, θ_i, $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$ in Theorem 6.1 and Corollaries 6.1 and 6.2, $n = 1, 2,$ or 3 exist and the HJE (6.2) is solvable for W, then if θ_i, $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$, are C^1, then W is C^2, and consequently if θ_i, $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$, are C^r, $r \ge 1$, then W is C^{r+1}.*

Proof From the expressions (6.5), (6.6) for W, we see that ρ is a smooth function, since V is smooth. Hence, the differentiability of W depends on the differentiability of the θ_i, $i = 1, 2,$ or 3. Further, it is clear that, the integration increases the differentiability of W by 1 over that of the θ_i, $i = 1, 2,$ or 3.

We can combine Corollaries 6.1 and 6.2 for any n in the following proposition.

Proposition 6.2 *Let M be an open subset of R^n which is simply connected and let q_0 be a fixed point in M. Suppose there exists a C^1 matrix function $\mathcal{R}\colon R^l \to SO(n, R)$ for some smooth vector function $\theta = (\theta_1, \ldots, \theta_l)$, $\theta_i\colon M \to R$, $i = 1, \ldots, l$, and a C^1 vector function $\varrho(q) = [\rho(q), \ldots, \rho(q)]$, $\rho\colon M \to R$, such that the Jacobian matrix*

$$\frac{\partial}{\partial q}\mathcal{R}(\theta(q))\varrho(q) \tag{6.7}$$

is symmetric and

$$\frac{1}{2}\langle \varrho(q)|\varrho(q)\rangle + V(q) = h. \tag{6.8}$$

Let

$$\tilde{\omega}^1 = \sum_{i=1}^{n} [\mathcal{R}(\theta(q))\varrho(q)]_i dq_i$$

and suppose C is a path from q_0 to an arbitrary point $q \in M$. Then,

(i) $\tilde{\omega}^1$ *is closed;*
(ii) $\tilde{\omega}^1$ *is exact;*
(iii) *if* $\widetilde{W}(q) = \int_C \tilde{\omega}^1$, *then* $\widetilde{W}$ *satisfies the HJE (6.2).*

Proof (i)

$$d\tilde{\omega}^1 = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial}{\partial q_j}[\mathcal{R}(\theta(q))\varrho(q)]_i dq_j \wedge dq_i$$

which by (6.7) implies that $d\tilde{\omega}^1 = 0$; hence, $\tilde{\omega}^1$ is closed.

(ii) Again by simple-connectedness of M, (i) implies (ii).

(iii) By (ii) the integral $\widetilde{W}(q) = \int_C \tilde{\omega}^1$ is independent of the path, and W corresponds to a scalar function. Moreover, if $dW = \tilde{\omega}^1$ and $\partial W/\partial q_i = [\mathcal{R}(\theta(q))\varrho(q)]_i$, then substituting in the HJE (6.2) and if (6.8) holds, then W satisfies the HJE (6.2).

If the HJE (6.2) is solvable, then the dynamics of the system evolves on the n-dimensional *Lagrangian submanifold* [1, 11] $\widetilde{N}$ which is an immersed submanifold of maximal dimension, and can be locally parametrized as the graph of the function W, i.e.,

$$\widetilde{N} = \left\{ \left(q, \frac{\partial W}{\partial q}\right) : \; q \in N \subset M, \; W \text{ is a solution of HJE (6.2)} \right\}$$

as described in Section 1. Moreover, for any other solution W' of the HJE, the volume enclosed by this surface is invariant. This is stated in the following proposition.

Proposition 6.3 *Let $N \subset M$ be the region in M where the solution W of the HJE (6.2) exists. Then, for any orientation of M, the volume form of $\widetilde{N}$*

$$\omega^n = \left(\sqrt{1 + \sum_{j=1}^{n} \left(\frac{\partial W}{\partial q_j}\right)^2} \right) dq_1 \wedge dq_2 \ldots \wedge dq_n$$

is given by

$$\omega^n = (\sqrt{1 + 2(h - V(q))}) dq_1 \wedge dq_2 \ldots \wedge dq_n.$$

Proof From the HJE (6.2), we have

$$\sqrt{1+\sum_{j=1}^{n}\left(\frac{\partial W}{\partial q_j}\right)^2} = \sqrt{1+2(h-V(q)}, \quad \forall\, q \in N$$

$$\Updownarrow$$

$$\omega^n = \left(\sqrt{1+\sum_{j=1}^{n}\left(\frac{\partial W}{\partial q_j}\right)^2}\right) dq_1 \wedge \ldots \wedge dq_n = \left(\sqrt{1+2(h-V(q)}\right) dq_1 \wedge \ldots \wedge dq_n$$

$$\forall\, q \in N.$$

We now apply the above ideas to solve the HJE for the two-particle $\mathcal{A}_2$-Toda lattice. We consider the nonperiodic system described in Section 5.

6.1 Solution of the Hamilton–Jacobi equation for the $\mathcal{A}_2$-Toda system

Consider the two-particle nonperiodic Toda system (or $\mathcal{A}_2$ system) given by the Hamiltonian (5.1)

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\,(p_1^2 + p_2^2) + e^{2(q_1-q_2)}. \tag{6.9}$$

Then, the HJE corresponding to the system is given by

$$\frac{1}{2}\left\{\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2\right\} + e^{2(q_1-q_2)} = h_2. \tag{6.10}$$

The following proposition gives the solution of the above HJE corresponding to $\mathcal{A}_2$-Toda lattice.

Proposition 6.4 *Consider the HJE (6.10) corresponding to the $\mathcal{A}_2$-Toda lattice. Then a solution to the HJE is given by*

$$\begin{aligned} W(q_1', q_2', h_2) &= \cos\frac{\pi}{4}\int_1^{q_1'} \rho(q)\, dq_1 + m \sin\frac{\pi}{4}\int_1^{q_1'} \rho(q)\, dq_1 \\ &= (1+m)\left\{\frac{\sqrt{h_2 - e^{-2(b+m-1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2(b+m-1)}}}{\sqrt{h_2}}\right]}{m-1}\right. \\ &\quad \left. - \frac{\sqrt{h_2 - e^{-2b-2(m-1)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2b-2(m-1)q_1'}}}{\sqrt{h_2}}\right]}{m-1}\right\}, \quad q_1 > q_2, \end{aligned}$$

and

$$
\begin{aligned}
W(q_1', q_2', h_2) &= \cos\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 + m\sin\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 \\
&= (1+m)\left\{ \frac{\sqrt{h_2 - e^{-2(b-m+1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2(b-m+1)}}}{\sqrt{h_2}}\right]}{m-1} \right. \\
&\quad \left. - \frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2b+2(1-m)q_1'}}}{\sqrt{h_2}}\right]}{m-1} \right\}, \quad q_2 > q_1.
\end{aligned}
$$

Furthermore, a solution for the system equations (5.2) for the $\mathcal{A}_2$ with the symmetric initial conditions $q_1(0) = -q_2(0)$ and $\dot{q}_1(0) = \dot{q}_2(0) = 0$ is

$$
q(t) = -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log[\cosh 2\sqrt{h_2}(\beta - t)] \tag{6.11}
$$

where h_2 is the energy and

$$
\beta = \frac{1}{2\sqrt{h_2}}\tanh^{-1}\left(\frac{2\dot{q}_1^2(0)}{\sqrt{2h_2}}\right).
$$

Proof Applying the results of Theorem 6.1 we have

$$
\frac{\partial W}{\partial q_1} = \rho(q)\cos\theta(q), \qquad \frac{\partial W}{\partial q_2} = \rho(q)\sin\theta(q)
$$

and substituting in the HJE (6.10) we immediately get

$$
\rho(q) = \pm\sqrt{2(h_2 - e^{2(q_1-q_2)})}
$$

and

$$
\rho_{q_2}(q)\cos\theta(q) - \theta_{q_2}\rho(q)\sin\theta(q) = \rho_{q_1}(q)\sin\theta(q) + \theta_{q_1}\rho(q)\cos\theta(q). \tag{6.12}
$$

The above equation (6.12) is a first-order PDE in θ and can be solved by the method of *characteristics* [5, 6]. However, the geometry of the system allows for a simpler solution. We make the simplifying assumption that θ is a constant function. Consequently, equation (6.12) becomes

$$
\rho_{q_2}(q)\cos\theta = \rho_{q_1}(q)\sin\theta \Longrightarrow \tan\theta = \frac{\rho_{q_2}(q)}{\rho_{q_1}(q)} = -1 \Rightarrow \theta = -\frac{\pi}{4}.
$$

Thus,

$$
p_1 = \rho(q)\cos\frac{\pi}{4}, \qquad p_2 = -\rho(q)\sin\frac{\pi}{4},
$$

and integrating dW along the straightline path from $(1,-1)$ on the line

$$L: \quad q_2 = \frac{q_2'+1}{q_1'-1}q_1 + \left(1 + \frac{q_2'+1}{q_1'-1}\right) \stackrel{\text{def}}{=} mq_1 + b$$

(this follows from the configuration of the lattice) to some arbitrary point (q_1', q_2') we get

$$\begin{aligned} W(q_1', q_2', h_2) &= \cos\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 + m\sin\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 \\ &= (1+m)\Bigg\{ \frac{\sqrt{h_2 - e^{-2(b+m-1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2(b+m-1)}}}{\sqrt{h_2}}\right]}{m-1} \\ &\quad - \frac{\sqrt{h_2 - e^{-2b-2(m-1)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2b-2(m-1)q_1'}}}{\sqrt{h_2}}\right]}{m-1}\Bigg\}. \end{aligned}$$

Similarly, if we integrate from point $(-1,1)$ to (q_1', q_2'), we get

$$\begin{aligned} W(q_1', q_2', h_2) &= \cos\frac{\pi}{4}\int_{-1}^{q_1'} \rho(q)\,dq_1 + m\sin\frac{\pi}{4}\int_{-1}^{q_1'} \rho(q)\,dq_1 \\ &= (1+m)\Bigg\{ \frac{\sqrt{h_2 - e^{-2(b-m+1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2(b-m+1)}}}{\sqrt{h_2}}\right]}{m-1} \\ &\quad - \frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}}}{\sqrt{h_2}}\right]}{m-1}\Bigg\}. \end{aligned}$$

Finally, from (2.11) and (6.9), we can write

$$\dot q_1 = p_1 = \rho(q)\cos\frac{\pi}{4}, \tag{6.13}$$

$$\dot q_2 = p_2 = -\rho(q)\sin\frac{\pi}{4}. \tag{6.14}$$

Then $\dot q_1 + \dot q_2 = 0$ which implies that $q_1 + q_2 = k$, a constant, and by our choice of initial conditions, $k = 0$. Now integrating the above equations from $t = 0$ to t we get

$$\begin{aligned} \frac{1}{2\sqrt{h_2}}\tanh^{-1}\frac{\rho(q)}{\sqrt{2h_2}} &= \frac{1}{2\sqrt{h_2}}\tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}} - t, \\ \frac{1}{2\sqrt{h_2}}\tanh^{-1}\frac{\rho(q)}{\sqrt{2h_2}} &= \frac{1}{2\sqrt{h_2}}\tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}} - t. \end{aligned}$$

If we let

$$\beta = \frac{1}{2\sqrt{h_2}}\tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}},$$

then upon simplification we get

$$\begin{aligned} q_1 - q_2 &= \frac{1}{2}\log\left[h_2\left(1-\tanh^2 2\sqrt{h_2}(\beta - t)\right)\right] \\ &= \frac{1}{2}\log\left[h_2 \operatorname{sech}^2 2\sqrt{h_2}(\beta - t)\right]. \end{aligned}$$

Since $k = 0$, then $q_1 = -q_2 = -q$, and we get

$$\begin{aligned} q(t) &= -\frac{1}{2}\log\sqrt{h_2} - \frac{1}{2}\log[\operatorname{sech} 2\sqrt{h_2}(\beta - t)] \\ &= -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log[\cosh 2\sqrt{h_2}(\beta - t)]. \end{aligned}$$

Now, from (6.10) and (6.13), (6.14),

$$\rho(q(0)) = \dot{q}_1^2(0) + \dot{q}_2^2(0),$$

and in particular, if $\dot{q}_1(0) = \dot{q}_2(0) = 0$, then $\beta = 0$. Therefore,

$$q(t) = -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log(\cosh 2\sqrt{h_2}t)$$

which is of the form (5.7) with $v = \sqrt{h}$.

Next, we consider a more general solution to the HJE for the $\mathcal{A}_2$-Toda lattice. We try to solve the equation (6.12) under the fact that

$$p_1 + p_2 = \alpha \tag{6.15}$$

a constant, which follows from (5.2). Then, from the proceeding, the above equation implies that

$$\rho(q)\cos\theta(q) + \rho(q)\sin\theta(q) = \alpha. \tag{6.16}$$

Now suppose we seek a solution to (6.12) and (6.16) for $\theta(q)$ such that

$$\frac{\partial\theta(q)}{\partial q_1} = \frac{\partial\theta(q)}{\partial q_2}. \tag{6.17}$$

The above condition is satisfied if

$$\theta(q_1, q_2) = f(q_1 + q_2) \tag{6.18}$$

for some smooth function $f\colon R \to R$ of one variable, and

$$\frac{\partial\theta(q)}{\partial q_1} = \frac{\partial\theta(q)}{\partial q_2} = f'(q_1 + q_2), \tag{6.19}$$

where $f'(\cdot)$ is the derivative of the function with respect to its argument. Then substituting in (6.12) and using (6.16), we get

$$\rho_{q_2}(q)\cos f(q_1 + q_2) - \rho_{q_1}(q)\sin f(q_1 + q_2) = \alpha f'(q_1 + q_2) \tag{6.20}$$

which after substituting for $\rho_{q_1}(q)$ and $\rho_{q_2}(q)$ and making the change of variables $x = q_1 + q_2$, $y = q_1 - q_2$ becomes

$$\frac{\sqrt{2}e^{2y}}{\sqrt{h_2 - e^{2y}}}(\cos f(x) + \sin f(x)) = \alpha f'(x). \tag{6.21}$$

The above equation represents a first-order nonlinear ODE in the function $f(x)$, and can be integrated in this way

$$\int_0^x \frac{\sqrt{2}e^{2y}}{\sqrt{h_2 - e^{2y}}}\,dx = \int_0^x \frac{\alpha f'(x)}{(\cos f(x) + \sin f(x))}\,dx \tag{6.22}$$

to yield

$$f(x) = 2\tan^{-1}\left[\tanh\left(\frac{\sqrt{2}e^{2y}}{\alpha\sqrt{h_2 - e^{2y}}}\right) + 1\right]. \tag{6.23}$$

This implies that

$$\theta(q_1, q_2) = 2\tan^{-1}\left[\tanh\left(\frac{\sqrt{2}e^{2(q_1-q_2)}}{\alpha\sqrt{h_2 - e^{2(q_1-q_2)}}}\right) + 1\right]. \tag{6.24}$$

We can now obtain W by taking the line integral of $p_1(q) = \rho(q)\cos\theta(q)$ and $p_2 = \rho(q)\sin\theta(q)$ along the straightline path from $(1, -1)$ on the line

$$L\colon \quad q_2 = \frac{q_2' + 1}{q_1' - 1}q_1 - \left(1 + \frac{q_2' + 1}{q_1' - 1}\right) \overset{\text{def}}{=} mq_1 + b$$

to some arbitrary point (q_1', q_2') for $q_1 > q_2$ and from $(-1, 1)$ to (q_1', q_2') for $q_2 > q_1$. Hence we have

$$W(q, \alpha, h_2) = \int_L [\rho(q)\cos\theta(q) + m\rho(q)\sin\theta(q)]\,dq_1. \tag{6.25}$$

Using the half-angle formula, we can write

$$T(q_1) \overset{\text{def}}{=} \tan\frac{\theta(q_1)}{2} = \tanh\left(\frac{\sqrt{2}e^{2q_1(1-m)-b}}{\alpha\sqrt{h_2 - e^{2q_1(1-m)-b}}}\right) + 1, \tag{6.26}$$

$$\cos\theta(q_1) = \frac{1 - T^2(q_1)}{1 + T^2(q_1)}, \tag{6.27}$$

$$\sin\theta(q_1) = \frac{2T(q_1)}{1 + T^2(q_1)}. \tag{6.28}$$

Therefore,

$$\begin{aligned} W(q, \alpha, h_2) &= \int_1^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1 - T^2(x)}{1 + T^2(x)} + m\frac{2T(x)}{1 + T^2(x)}\right)dx \\ &= \int_1^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1 - 2mT(x) - T^2(x)}{1 + T^2(x)}\right)dx \quad \text{for} \quad q_1 > q_2 \end{aligned}$$

and

$$W(q,\alpha,h_2) = \int_{-1}^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1 - 2mT(x) - T^2(x)}{1 + T^2(x)}\right)dx \quad \text{for} \quad q_2 > q_1.$$

Unfortunately the above integrals cannot be computed in closed-form.

7 Conclusion

In this paper, we have presented a review of Hamilton–Jacobi theory and a new approach for solving the HJE for a fairly large class of Hamiltonian systems in which the variables may not be separable. The approach can also be extended to the case in which the Hamiltonian is not time-independent, and relies on finding a parametrization that allows for the equation to be solved.

The approach has been applied to the $\mathcal{A}_2$-Toda lattice, and computational results have been presented to show the usefulness of the method. It has been shown that, for the two-particle non-periodic $\mathcal{A}_2$-Toda system, the HJE can be completely integrated as expected to obtain the characteristic function and subsequently a complete solution to the equations of motion. The approach can also be applied to a fairly large class of Hamiltonian systems.

References

[1] Abraham, R. and Marsden, J. *Foundations of Mechanics.* Benjamin Cummings Pub. Co., MA, USA, 1978.
[2] Adler, M. and Van Moerbeke, P. The Toda lattice, Dynkin diagrams, singularities, and Abelian varieties. *Inventiones Math.* **103** (1991) 223–278.
[3] Arnold, V.I. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, 1985.
[4] Bogoyavlensky, O.I. On perturbation of the periodic Toda lattice. *Comm. in Math. Physics* **51** (1976) 201–209.
[5] Evans, L.C. *Partial Differential Equations.* Graduate Text in Maths., AMS, RI, USA, 1998.
[6] Fritz, J. *Partial Differential Equations.* Springer-Verlag, Applied Mathematical Sciences Series, 3rd Edition, New York, 1978.
[7] Goldstein, H. *Classical Mechanics.* Addison-Wesley, MA, USA, 1950.
[8] Groesberg, S.W. *Advanced Mechanics.* John-Wiley & Sons, Inc., NY, USA, 1968.
[9] Guest, M.A. *Harmonic Maps, Loop Groups, and Integrable Systems.* London Mathematical Society Text 38, Cambridge University Press, NY, USA, 1997.
[10] Hagedorn, P. *Nonlinear Oscillations.* Oxford Engineering Science Series, NY, USA, 1988.
[11] Marsden, J. and Ratiu, T.S. *Introduction to Mechanics and Symmetry.* Springer-Verlag, Berlin, 1999.
[12] McDaniel, A. and Smolinsky, L. The flow of the $\mathcal{G}_2$ periodic Toda lattice. *J. of Math. Physics* **38**(2) (1997) 926–945.
[13] Toda M. *Theory of Nonlinear Lattices.* Springer-Verlag, Berlin, 1981.

Nonlinear Dynamics and Systems Theory, 5(4) (2005) 345–356

Partial Functional Differential Equations and Applications to Population Dynamics

D. Bahuguna[1*] and R.K. Shukla[2]

[1]*Department of Mathematics, Indian Institute of Technology, Kanpur – 208 016, India*
[2]*Invertis Institute of Management Studies, B-186, CivilLines, Bareilly (U.P.), India*

Received: March 20, 2005; Revised: October 10, 2005

Abstract: In this paper we consider a semilinear partial functional differential equation with a nonlocal history condition arising in the study of problems in population dynamics. We reformulate it as a functional differential equation in a Banach space. Using the theory of strongly continuous and analytic semigroups we analyze the existence, uniqueness of mild, strong and classical solutions. Finally, we study the finite dimensional approximation of solutions.

Keywords: *Partial functional differential equation; strongly continuous and analytic semigroups; mild, strong and classical solutions; projections; finite dimensional approximations.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

Of concern is the following nonlocal history-valued boundary value problem for a partial functional differential equation,

$$\begin{aligned} \frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + f(w(x,t), w(x,t-\tau)), \\ & \qquad t > 0, \quad 0 < x < \pi, \\ w(0,t) &= w(\pi,t) = 0, \quad t > 0, \\ h(w|_{[-\tau,0]})(x,t) &= \phi(x,t), \quad -\tau \le t \le 0,\ \tau > 0, \quad 0 \le x \le \pi, \end{aligned} \tag{1.1}$$

*Corresponding author: d_bahuguna@yahoo.com

where w is the sought-for function in the space $\mathcal{C}_{[-\tau,T]} = C([0,\pi]\times[-\tau,T])$, for arbitrarily fixed $0 < T < \infty$, of all continuous functions endowed the supremum norm, h is a function defined from the space $\mathcal{C}_{[-\tau,0]}$ into itself, $\phi \in \mathcal{C}_{[-\tau,0]}$, the function $w|_{[-\tau,0]} \in \mathcal{C}_{[-\tau,0]}$ is the restriction of $w \in \mathcal{C}_{[-\tau,T]}$ on $[0,\pi]\times[-\tau,0]$, $a > 0$ is a constant.

Let X be the Banach space $C[0,\pi]$ of all real-valued continuous functions on $[0,\pi]$ endowed with the supremum norm

$$\|\xi\|_X = \sup_{0\le x\le\pi} |\xi(x)|, \quad \xi \in X,$$

and for $t \in [0,T]$, $0 < T < \infty$, let $\mathcal{C}_t = C([-\tau,t]; X)$, $0 < \tau < \infty$, be the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm

$$\|\psi\|_t = \sup_{-\tau\le\theta\le t} \|\psi(\theta)\|_X, \quad \psi \in \mathcal{C}_t.$$

Let $\mathcal{C}_0(\chi) = \{\bar\chi \in \mathcal{C}_0\colon\ \bar\chi(0) = \chi(0)\}$. Define a function F from $\mathcal{C}_0(\chi)$ into X by

$$F(\chi) = f(\chi(0), \chi(-\tau)), \quad \chi \in \mathcal{C}_0.$$

Then (1.1) can be written as the following nonlocal history-valued functional differential equation

$$\begin{aligned} u'(t) + Au(t) &= F(u_t), \quad t \in (0,T], \\ H(u_0) &= \phi \quad \text{on} \quad [-\tau,0], \end{aligned} \tag{1.2}$$

where A is a linear operator defined on $D(A) = \{\xi \in C[0,\pi]\colon \xi'' \in C[0,\pi],\ \xi(0) = \xi(\pi) = 0\}$ with $A\xi = -a\xi''$ for $\xi \in D(A)$, for $u \in \mathcal{C}_T$ and $t \in [0,T]$, $u_t \in \mathcal{C}_0$ given by $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau,0]$, the map H is defined from $\mathcal{C}_0$ into itself and $\phi \in \mathcal{C}_0$.

For the earlier works on existence, uniqueness and stability of various types of solutions of differential and functional differential equations with nonlocal conditions we refer to Byszewski and Akca [2], Byszewski and Lakshmikantham [4], Byszewski [5], Balachandran and Chandrasekaran [3], Lin and Liu [7] and references cited in these papers.

Our main aim is to consider various types of nonlocal history conditions H and their applications. We use the ideas and techniques used by Bahuguna [1] to study such conditions and their applications.

A few examples of H are the following. Let g be map from $\mathcal{C}_0$ into X be a map given by one of the following.

(I) Let $k \in L^1(0,\tau)$ such that $\kappa = \int_0^\tau k(s)\,ds \neq 0$. Let

$$g(\xi) = \int_{-\tau}^{0} k(-s)\xi(s)\,ds, \quad \xi \in \mathcal{C}_0.$$

(II) Let $-\tau \le t_1 < t_2 < \cdots < t_l \le 0$, $c_i \ge 0$ with $C = \sum_{i=1}^{l} c_i \neq 0$. Let

$$g(\xi) = \sum_{i=1}^{l} c_i\xi(t_i), \quad \xi \in \mathcal{C}_0.$$

(III) Let t_i and c_i be as in (II) and let $\epsilon_i > 0,\ i = 1, 2, \dots, l$. Let

$$g(\xi) = \sum_{i=1}^{l} \frac{c_i}{\epsilon_i} \int_{t_i-\epsilon_i}^{t_i} \xi(s)ds, \quad \xi \in \mathcal{C}_0.$$

If we define $\phi \in \mathcal{C}_0$ given by $\phi(\theta) \equiv x$ for all $\theta \in [-\tau, 0]$ and $H\colon \mathcal{C}_0 \to \mathcal{C}_0$ given by $H(\xi)(\theta) \equiv g(\xi)$ for all $\theta \in [-\tau, 0]$ and all $\xi \in \mathcal{C}_0$, then the condition $g(\xi) = x$ is equivalent to the condition $H(\xi) = \phi$.

Let $\chi \in \mathcal{C}_0$ be such that $H(\chi) = \phi$. The function $u \in \mathcal{C}_{\widetilde{T}},\ 0 < \widetilde{T} \le T$, such that

$$u(t) = \begin{cases} \chi(t) & t \in [-\tau, 0] \\ S(t)\chi(0) + \int\limits_0^t S(t-s)F(u_s)\,ds, & t \in [0, \widetilde{T}], \end{cases} \tag{1.3}$$

is called a *mild solution* of (1.2) on $[-\tau, \widetilde{T}]$. If a mild solution u of (1.2) on $[-\tau, \widetilde{T}]$ is such that $u(t) \in D(A)$ for a.e. $t \in [0, \widetilde{T}]$, u is differentiable a.e. on $[0, \widetilde{T}]$ and

$$u'(t) + Au(t) = F(u_t), \quad \text{a.e. on } [0, \widetilde{T}],$$

it is called a *strong solution* of (1.2) on $[-\tau, \widetilde{T}]$. If a mild solution u of (1.2) on $[-\tau, \widetilde{T}]$ is such that $u \in C^1((0, \widetilde{T}]; X)$, $u(t) \in D(A)$ for $t \in (0, \widetilde{T}]$ and satisfies

$$u'(t) + Au(t) = F(u_t), \quad t \in (0, \widetilde{T}],$$

then it is called a *classical solution* of (1.2) on $[-\tau, \widetilde{T}]$.

We first establish the existence of a mild solution $u \in \mathcal{C}_{\widetilde{T}}$ of (1.2) for some $0 < \widetilde{T} \le T$ and its continuation to the whole of $[-\tau, \infty)$. Under the additional assumption of Lipschitz continuity on ψ on $[-\tau, 0]$, we show that the mild solution u is a strong solution of (1.2) on the interval of existence and it is Lipschitz continuous. Under further additional assumption that $S(t)$ is analytic, we show that u is a classical solution of (1.2) on the interval of existence. We also show that u is unique if and only if χ satisfying $H(\chi) = \phi$ is unique. Next, we establish a global existence result. Finally, we study the finite dimensional approximation of solutions in a Hilbert space.

2 Local Existence of Mild Solutions

We first prove the following result establishing the local existence and uniqueness of a mild solution of (1.2).

Theorem 2.1 *Suppose that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \ge 0$ of bounded linear operators in X. Let $H\colon \mathcal{C}_0 \to \mathcal{C}_0$ be such that there exists a function $\chi \in \mathcal{C}_0$ such that $H(\chi) = \phi$. Let $\mathcal{C}_0(\chi) = \{\bar{\chi} \in \mathcal{C}_0\colon \bar{\chi}(0) = \chi(0)\}$. Let $F\colon \mathcal{C}_0(\chi) \to X$ satisfy a Lipschitz condition*

$$\|F(\chi_1) - F(\chi_2)\|_X \le L_F \|\chi_1 - \chi_2\|_0,$$

for all $\chi_i \in \mathcal{C}_0(\chi)$, $i = 1, 2$, *where* L_F *is a non-negative constant. Then there exists a mild solution* u *of (1.2) on* $[-\tau, T_0]$ *for some* $0 < T_0 \leq T$. *Moreover, the mild solution* u *is unique if and only if* χ *is unique.*

Proof Let $M \geq 1$ and $\omega \geq 0$ be such that $\|S(t)\|_{B(X)} \leq Me^{\omega t}$ for $t \geq 0$. Here $B(X)$ is the space of all bounded linear operators from X into itself. We choose $0 < T_0 \leq T$ be such that

$$T_0 M e^{\omega T} L_F \leq 3/4.$$

Define a map $\mathcal{F}\colon \mathcal{C}_{T_0}(\chi) \to \mathcal{C}_{T_0}(\chi)$ by

$$\mathcal{F}w(t) = \begin{cases} \chi(t) & t \in [-\tau, 0], \\ S(t)\chi(0) + \int\limits_0^t S(t-s)F(w_s)\,ds, & t \in [0, T_0]. \end{cases} \tag{2.1}$$

Here and subsequently, any function in $\mathcal{C}_T(\chi) = \{\psi \in \mathcal{C}_T\colon \psi(0) = \chi(0)\}$ is also in $\mathcal{C}_{\widetilde{T}}(\chi)$, $0 \leq \widetilde{T} \leq T$, as its restriction on the subinterval. Also, for $w_i \in \mathcal{C}_{T_0}(\chi)$, $i = 1, 2$, we have

$$\|\mathcal{F}w_1(t) - \mathcal{F}w_2(t)\|_X \leq T_0 M e^{\omega T} L_F \|w_1 - w_2\|_{T_0}.$$

Since $T_0 M e^{\omega T} L_F \leq 3/4$, $\mathcal{F}$ is a strict contraction on $\mathcal{C}_{T_0}(\chi)$ and hence has a unique fixed point $u \in \mathcal{C}_{T_0}(\chi)$.

Clearly, if $\chi \in \mathcal{C}_T$ satisfying $H(\chi) = \phi$ on $[-\tau, 0]$ is unique on $[-\tau, 0]$, then u is unique. If there are two χ and $\tilde{\chi}$ in $\mathcal{C}_0$ satisfying $H(\chi) = H(\tilde{\chi}) = \phi$ on $[-\tau, 0]$, with $\chi \neq \tilde{\chi}$ on $[-\tau, 0]$, then the corresponding solutions u and $\tilde{u}$ of (1.2) belonging to $\mathcal{C}_{T_0}(\chi)$ and $\mathcal{C}_{\tilde{T}_0}(\tilde{\chi})$ are different. This completes the proof of Theorem 2.1.

3 Global Existence of Solutions

Theorem 3.1 *Assume the hypotheses of Theorem 2.1. Then the local mild solution* u *of (1.2) exists on the whole interval* $[-\tau, \infty)$.

Proof Let $0 < T < \infty$ be arbitrarily fixed. If $T_0 < T$, consider the functional differential equation

$$\begin{aligned} v'(t) + Av(t) &= F(v_t), \quad 0 < t \leq T - T_0, \\ \widetilde{H}(v_0) &= \tilde{\phi}, \end{aligned} \tag{3.1}$$

where $\widetilde{H}\colon \mathcal{C}_0(\chi) \to \mathcal{C}_0(\chi)$ given by $\widetilde{H}\chi = \chi$ for $\chi \in \mathcal{C}_0(\chi)$ and $\tilde{\phi}(\theta) = u(T_0 + \theta)$ for $\theta \in [-\tau, 0]$. Since all the hypotheses of Theorem 2.1 are satisfied for problem (3.1), we have the existence of a mild solution $w \in \mathcal{C}_{T_1}(\chi)$, $0 < T_1 \leq T - T_0$ of (3.1). This mild solution w is unique as $\widetilde{H}$ in (3.1) is the identity map on $\mathcal{C}_0(\chi)$. We define

$$\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, T_0] \\ w(t - T_0), & t \in [T_0, T_0 + T_1]. \end{cases} \tag{3.2}$$

Then $\bar{u}$ is a mild solution of (1.2) on $[-\tau,\, T_0 + T_1]$, unique for fixed χ. Continuing this way, we get the existence of a mild solution u either on the whole interval $[-\tau, T]$

or on the maximal interval $[-\tau, t_{\max})$ of existence. In the later case we may use the arguments similar in the proof of Theorem 6.2.2 in Pazy [9, P.193–194], to conclude that $\lim\limits_{t\to t_{\max}-}\|u(t)\|_X = \infty$.

In order to show the global existence, we show that $\|u(t)\|_X \le C$ for $t \ge 0$. Let $M_1 = \max\{M, e^{\omega\tau}, (M/\omega)\|F(0)\|_X, \|\chi\|_0\}$. For $t \in [-\tau, 0]$, $e^{-\omega t}\|u(t)\|_X \le M_1$ and for $t \in [0, T)$, we have

$$e^{-\omega t}\|u(t)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.3}$$

From (3.3), for any $0 \le r \le t$, we have

$$e^{-\omega t}\|u(r)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.4}$$

Putting $r = t+\eta$, $-t \le \eta \le 0$, in (3.4), we get

$$e^{-\omega t}\|u(t+\eta)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.5}$$

Now, if $-\tau \le -t$, then

$$\begin{aligned} e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X &\le e^{-\omega t} \sup_{-\tau\le\eta\le -t} \|u(t+\eta)\|_X + e^{-\omega t} \sup_{-t\le\eta\le 0} \|u(t+\eta)\|_X \\ &\le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds, \end{aligned}$$

and for the case $-t \le -\tau$, we have

$$\begin{aligned} e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X &\le e^{-\omega t} \sup_{-t\le\eta\le 0} \|u(t+\eta)\|_X \\ &\le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \end{aligned}$$

Thus,

$$e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X \le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds.$$

Gronwall's inequality implies that

$$e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X \le 2M_1 + ML_F \int_0^t f(s) \exp\{2M\|F(0)\|_X (t-s)\}\, ds. \tag{3.6}$$

Inequality (3.6) implies that $\|u(t)\|_X$ is bounded by a continuous function and therefore $\|u(t)\|_X$ is bounded on every compact interval $[-\tau, T]$, $0 < T < \infty$. Since T is arbitrary, the global existence follows.

4 Regularity of Solutions

Theorem 4.1 *Assume the hypotheses of Theorem 2.1. If, in addition, $\chi \in \mathcal{C}_0$ satisfying $H(\chi) = \phi$ is Lipschitz continuous on $[-\tau, 0]$ and $\chi(0) \in D(A)$, then the solution u corresponding to χ is Lipschitz continuous on every compact subinterval of existence. If, in addition, X is reflexive, then u is a strong solution of (1.2) on the interval of existence and this strong solution is a classical solution of (1.2) provided $S(t)$ is an analytic semigroup.*

Proof We shall prove the result for the first case when the mild solution u exists on the whole interval. The proof can be modified easily for the second case.

We need to show the Lipschitz continuity of u only on $[0, T]$. In what follows, C_i's are positive constants depending only on R, T and $\|\chi\|_0$. Let $t \in [0, T]$ and $h \geq 0$. Then

$$\begin{aligned}\|u(t+h) - u(t)\|_X &\leq \|(S(h) - I)S(t)\chi(0)\|_X + \int_{-h}^{0} \|S(t-s)F(u_{s+h})\|_X\, ds \\ &\quad + \int_0^t \|s(t-s)[F(u_{s+h}) - F(u_s)]\|_X ds \\ &\leq C_1\left[h + \int_0^t [\|u_{s+h} - u_s\|_{\mathcal{C}_0}]\, ds\right] \\ &\leq C_1\left[h + \int_0^t \sup_{-\tau\leq\theta\leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X\right] ds.\end{aligned} \tag{4.1}$$

For the case when $-\tau \leq t < 0$ and $0 \leq t + h$ (clearly, $t + h \leq h$ in this case), we have

$$\begin{aligned}\|u(t+h) - u(t)\|_X &\leq \|(S(t+h) - I)\chi(0)\|_X + \|\chi(t) - \chi(0)\|_X \\ &\quad + \int_0^h \|S(t+h-s)F(u_s)\|_X ds \leq C_2 h.\end{aligned} \tag{4.2}$$

Combining the inequalities (4.1) and (4.2), we have for $-\tau \leq \bar{t} \leq t$,

$$\|u(\bar{t}+h) - u(\bar{t})\|_X \leq C_3\left[h + \int_0^t \sup_{-\tau\leq\theta\leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds\right]. \tag{4.3}$$

Putting $\bar{t} = t + \bar{\theta}$, $-t - \tau \leq \bar{\theta} \leq 0$, in (4.3), and taking supremum over $\bar{\theta}$ on $[-\tau, 0]$, we get

$$\sup_{-\tau\leq\theta\leq 0} \|u(t+h+\theta) - u(t+\theta)\|_X \leq 2C_3\left[h + \int_0^t \sup_{-\tau\leq\theta\leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds\right]. \tag{4.4}$$

Applying Gronwall's inequality in (4.4), we obtain

$$\|u(t+h) - u(t)\|_X \le \sup_{-\tau\le\theta\le 0} \|u(t+h+\theta) - u(t+\theta)\|_X \le C_4 h.$$

Thus, u is Lipschitz continuous on $[-\tau, T]$.

The function $\overline{F}\colon\ [0,T] \to X$ given by $\overline{F}(t) = F(u_t)$, is Lipschitz continuous and therefore differentiable a.e. on $[0,T]$ and $\overline{F}'$ is in $L^1((0,T);X)$. Consider the Cauchy problem

$$\begin{aligned} v'(t) + Av(t) &= \overline{F}(t), \quad t \in (0,T], \\ v(0) &= u(0), \end{aligned} \tag{4.5}$$

By the Corollary 2.10 on page 109 in Pazy [9], there exists a unique strong solution v of (4.5) on $[0,T]$. Clearly, $\bar{v}$ defined by

$$\bar{v}(t) = \begin{cases} u(t), & t \in [-\tau, 0] \\ v(t), & t \in [0,T], \end{cases}$$

is a strong solution of (1.2) on $[-\tau, T]$. But this strong solution is also a mild solution of (1.2) and $\bar{v} \in \mathcal{C}_T(\chi)$. By the uniqueness of such a function in $\mathcal{C}_T(\chi)$, we get $\bar{v}(t) = u(t)$ on $[-\tau, T]$. Thus u is a strong solution of (1.2). If $S(t)$ is analytic semigroup in X then we may use Corollary 3.3 on page 113 in Pazy [9] to obtain that u is a classical solution of (1.2). This completes the proof of Theorem 4.1.

5 Finite Dimensional Approximations

In this section we assume that X is a separable Hilbert space. Furthermore, we assume that in (1.2), the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset X$ into X such that $D(A)$ is dense in X, A has the pure point spectrum

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then the semigroup $S(t)$ generated by $-A$ is analytic in X. It follows that the fractional powers A^α of A for $0 \le \alpha \le 1$ are well defined from $D(A^\alpha) \subseteq X$ into X (cf. Pazy [9], pp. 69−75). $D(A^\alpha)$ is a Banach space endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|_X, \quad x \in D(A^\alpha). \tag{5.1}$$

For $t \in [0,T]$, we denote by $\mathcal{C}_t^\alpha = C([-r,t]; D(A^\alpha))$ endowed with the norm

$$\|\zeta\|_{t,\alpha} = \sup_{-r\le\eta\le t} \|\zeta(\eta)\|_\alpha, \quad \zeta \in \mathcal{C}_t^\alpha.$$

In addition, we assume the following hypotheses.

(H2) There exists a function $\chi \in \mathcal{C}_0^\alpha$ satisfying $H(\chi) = \phi$.

(H3) The map F is defined from $\mathcal{C}_0^\alpha(\chi) = \{\bar{\chi} \in \mathcal{C}_0^\alpha \colon \bar{\chi}(0) = \chi(0)\}$ into $D(A^\beta)$ for $0 < \beta \le \alpha < 1$ and there exists a non-negative constant L_F such that

$$\|F(\zeta_1) - f(\zeta_2)\|_X \le L_F \|\zeta_1 - \zeta_2\|_{0,\alpha},$$

for $\zeta_i \in \mathcal{C}_0^\alpha(\chi)$, for $i = 1, 2$.

Let X_n denote the finite dimensional subspace of X spanned by $\{u_0, u_1, \dots, u_n\}$ and let $P^n \colon X \to X_n$ be the corresponding projection operator for $n = 0, 1, 2, \dots$. Let $\chi \in \mathcal{C}_0$ be such that $H(\chi) = \phi$. Let $\bar{\chi}$ be the extension of χ by the constant value $\chi(0)$ on $[0, T]$. We set

$$T_0 = \min\left\{T, \left(\frac{3(1-\alpha)}{8L_F C_\alpha}\right)^{1-\alpha}\right\},$$

where C_α is a positive constant such that $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$ for $t > 0$.

We define

$$F_n \colon \mathcal{C}_0(\chi) \to X_n,$$

given by

$$F_n(\zeta) = P^n F(P^n \zeta), \quad \zeta \in \mathcal{C}_0(\chi),$$

where $(P^n\zeta)(\theta) = P^n \zeta(\theta)$, $-\tau \le \theta \le 0$. We denote $\psi_n = P^n \psi$ for any $\psi \in \mathcal{C}_T$.

Let $A^\alpha \colon \mathcal{C}_t^\alpha \to \mathcal{C}_t$ be given by $(A^\alpha \psi)(s) = A^\alpha(\psi(s))$, $s \in [-r, t]$, $t \in [0, T_0]$. We define a map $\mathcal{F}_n \colon \mathcal{C}_{T_0}(\chi) \to \mathcal{C}_{T_0}(\chi)$ as follows:

$$(\mathcal{F}_n \xi)(t) = \begin{cases} A^\alpha \chi_n(t), & t \in [-\tau, 0], \\ S(t) A^\alpha \chi_n(0) + \int\limits_0^t A^\alpha S(t-s) F_n(A^{-\alpha}\xi_s)\, ds, & t \in [0, T_0], \end{cases} \tag{5.2}$$

for $\xi \in \mathcal{C}_{T_0}(\chi)$.

Proposition 5.1 *There exists a unique $w_n \in \mathcal{C}_{T_0}(\chi)$ such that $F_n w_n = w_n$ on $[-r, T_0]$.*

Proof For $\xi_1, \xi_2 \in \mathcal{C}_{T_0}(\chi)$, $(\mathcal{F}_n\xi_1)(t) - (\mathcal{F}_n\xi_2)(t) = 0$ on $[-\tau, 0]$ and for $t \in [0, T_0]$, we have

$$\|(\mathcal{F}_n\xi_1)(t) - (\mathcal{F}_n\xi_2)(t)\|_X \le 2L_F C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha} \|\xi_1 - \xi_2\|_{T_0} \le \frac{3}{4} \|\xi_1 - \xi_2\|_{T_0}.$$

Taking the supremum over $[-\tau, T_0]$, it follows that $\mathcal{F}_n$ is a strict contraction on $\mathcal{C}_{T_0}(\chi)$ and hence there exits a unique $w_n \in \mathcal{C}_{T_0}(\chi)$ with $w_n = \mathcal{F}_n w_n$ on $[-\tau, T_0]$. This completes the proof of Proposition 5.1.

Let $u_n = A^{-\alpha} w_n$. Then $u_n \in \mathcal{C}_{T_0}^\alpha$ and satisfies

$$u_n(t) = \begin{cases} \chi_n(t), & t \in [-\tau, 0], \\ S(t)\chi_n(0) + \int\limits_0^t S(t-s) F_n(u_s)\, ds, & t \in [0, T_0]. \end{cases} \tag{5.3}$$

Proposition 5.2 *The sequence* $\{u_n\} \subset \mathcal{C}_{T_0}(\chi)$ *is a Cauchy sequence and therefore converges to a function* $u \in \mathcal{C}_{T_0}(\chi)$.

Proof For $n, m \in N,\ n \geq m,\ t \in [-\tau, 0]$, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|A^\alpha(\chi_n(t) - \chi_m(t))\|_X \leq \|(P^n - P^m)A^\alpha\chi(t)\|_X \to 0 \quad \text{as} \quad m \to \infty.$$

For $t \in (0, T_0]$ and n, m as above, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|(P^n - P^m)S(t)A^\alpha\chi(0)\|_X + \int_0^t \|A^\alpha S(t-s)[F_n((u_n)_s) - F_m((u_m)_s))]\|_X\, ds.$$

Now, using the fact that $F((u_m)_s) \in D(A^\beta),\ m \geq n_0$ and $0 < \alpha < \beta < 1$, we have

$$\begin{aligned}\|F_n((u_n)_s) - F_m((u_m)_s)\|_X &\leq \|(P^n - P^m)F(P^m(u_m)_s)\|_X \\ &\quad + L_F[\|(P^m - P^m)A^\alpha(u_m)_s\|_0] + L_F\|u_n - u_m\|_{s,\alpha} \\ &\leq C_1\frac{1}{\lambda_m^\beta} + C_2\|u_n - u_m\|_{s,\alpha},\end{aligned}$$

for some positive constants C_1 and C_2 independent of n and m. Thus, we have the following estimate

$$\|u_n(t) - u_m(t)\|_\alpha \leq C_0\|(P^n - P^m)A^\alpha\chi(0)\|_X + \frac{C_1 T^\gamma}{\lambda_m^\beta} + C_2\int_0^t (t-s)^\alpha\|u_n - u_m\|_{s,\alpha}\, ds, \tag{5.4}$$

where $C_0 = Me^{\omega T}$. Since $u_n - u_m = \chi_n - \chi_m$ on $[-\tau, 0]$, we have for $0 \leq \bar{t} \leq t$,

$$\|u_n(\bar{t}) - u_m(\bar{t})\|_\alpha \leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha\chi(0)\|_X + \frac{C_1 T}{\lambda_m^\beta} + C_2\int_0^{\bar{t}} (\bar{t}-s)^\alpha\|u_n - u_m\|_{s,\alpha}\, ds. \tag{5.5}$$

We put $\bar{t} = t + \eta,\ -t \leq \eta \leq 0$, to obtain

$$\|u_n(t+\eta) - u_m(t+\eta)\|_\alpha \leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha\chi(0)\|_X + \frac{C_1 T}{\lambda_m^\beta} + C_2\int_0^{t+\eta} (t+\eta-s)^\alpha\|u_n - u_m\|_{s,\alpha}\, ds. \tag{5.6}$$

Now, we put $s - \eta = \bar{s}$ to get

$$\begin{aligned}\|u_n(t+\eta) - u_m(t+\eta)\|_\alpha &\leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha\chi(0)\|_X \\ &\quad + \frac{C_1 T}{\lambda_m^\beta} + C_2\int_{-\eta}^{t} (t-\bar{s})^\alpha\|u_n - u_m\|_{\bar{s}+\eta,\alpha}\, d\bar{s} \\ &\leq \|\chi_n - \chi_m\|_{\alpha,0} + C_0\|(P^n - P^m)A^\alpha\chi(0)\| \\ &\quad + \frac{C_1 T}{\lambda_m^\beta} + C_2\int_0^{t} (t-\bar{s})^\alpha\|u_n - u_m\|_{\bar{s},\alpha}\, d\bar{s}.\end{aligned} \tag{5.7}$$

For $t \geq \tau$, we have

$$\sup_{-\tau\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha \leq \sup_{-t\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}. \tag{5.8}$$

Since $u_n(t+\eta)=\chi(t+\eta)$ for $t+\eta\leq 0$ for all $n\geq n_0$, for $0\leq t\leq\tau$, we have

$$\sup_{-\tau\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \sup_{-\tau\leq\eta\leq -t}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha+\sup_{-t\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}. \tag{5.9}$$

Combining (5.8) and (5.9), we have

$$\begin{aligned}\|u_n-u_m\|_{t,\alpha} &\leq \|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|\\ &\quad+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}.\end{aligned} \tag{5.10}$$

Application of Lemma 5.6.7 on page 159 in Pazy [9] gives the required result. This completes the proof of Proposition 5.2.

With the help of Propositions 5.1 and 5.2, we may state the following existence, uniqueness and convergence result.

Theorem 5.3 *Suppose that assumptions (H1) – (H3) hold. Then there exist functions* $u_n\in([-\tau,T_0];X_n)$, $n\in N$, *and* $u\in\mathcal{C}_{T_0}$ $(0<T_0\leq T)$ *unique for a given* $\chi\in\mathcal{C}_0$ *with* $H(\chi)=\phi$, *such that*

$$u_n(t)=\begin{cases}\chi_n(t), & t\in[-\tau,0],\\ S(t)\chi_n(0)+\int_0^t S(t-s)F_n((u_n)_s)\,ds, & t\in[0,T_0],\end{cases} \tag{5.11}$$

and

$$u(t)=\begin{cases}\chi(t), & t\in[-\tau,0],\\ S(t)\chi(0)+\int_0^t S(t-s)F(u_s)\,ds, & t\in[0,T_0],\end{cases} \tag{5.12}$$

such that $u_n\to u$ *in* $\mathcal{C}_{T_0}$ *as* $n\to\infty$, *where* $\psi_n(t)=P^n\psi(t)$ *for* $\psi\in\mathcal{C}_{T_0}$ *and* $F_n(\zeta)=P^nF(P^n\zeta)$, $\zeta\in\mathcal{C}_0$.

6 Applications

As an applicability of the theory developed in previous sections, we cite two examples of partial differential equation with retarded arguments and a nonlocal history condition. These problems are closely related to a mathematical model for population density with a time delay and self regulation (cf. [6, 10]).

Example 6.1

$$\begin{aligned}\frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + b\,w(x,t-\tau)(1-w(x,t)),\\ &\qquad\qquad t>0,\quad 0<x<\pi,\\ w(0,t) &= w(\pi,t)=0,\quad t>0,\\ h(w|_{[-\tau,0]})(x,t) &= \phi(x,t),\quad -\tau\le t\le 0,\quad \tau>0,\quad 0\le x\le\pi,\end{aligned} \tag{6.1}$$

where $w(\cdot,t)$ is the population density at time t, b is the constant rate of growth for the species. τ is a fixed positive constant and $\phi \in \mathcal{C}_{[-\tau,0]} = C([0,\pi]\times[-\tau,0])$. Let $X = C[0,\pi]$. For each t, define an operator A by

$$Au = -au'',$$

for $u \in D(A) = \{u \in C([0,\pi])\colon\ u'' \in C([0,\pi]),\ u(0)=u(\pi)=0\}$. It follows that $-A$ generates an analytic semigroup in X. The nonlinear map H can be defined as mentioned in the first section.

Let $\mathcal{C}_0(\chi)$ be the set consisting of all continuous function $\bar\chi\colon[-\tau,0]\to X$ such that $\bar\chi(0)=\chi(0)$ and define $F\colon\mathcal{C}_0(\chi)\to X$ by

$$F(\chi) = b\chi(-\tau)(1-\chi(0)),\quad \chi\in\mathcal{C}_0(\chi).$$

It is easily verified that F satisfies Lipschitz condition. The problem (6.1) now take the abstract form

$$\begin{aligned} u'(t)+Au(t) &= F(u_t),\quad t\in(0,T],\\ H(u_0) &= \phi,\quad \text{on } [-\tau,0],\end{aligned} \tag{6.2}$$

Then the theorems ensure the existence of a unique solution of the problem (6.2) (hence a unique solution of the problem (6.1)).

Example 6.2

$$\begin{aligned}\frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + b\,w(x,t)\left[1-\int_{-\tau}^{0} w_t(x,s)\,d\eta(s)\right],\\ &\qquad\qquad t>0,\quad 0<x<\pi,\\ w(0,t) &= w(\pi,t)=0,\quad t>0,\\ h(w|_{[-\tau,0]})(x,t) &= \phi(x,t),\quad -\tau\le t\le 0,\quad \tau>0,\quad 0\le x\le\pi,\end{aligned} \tag{6.3}$$

which is a population model when diffusion occurs within the population. Here $\eta(\cdot)$ is bounded, nondecreasing function on $[-\tau,0]$, $\tau\ge 0$. All other functions and maps are as described in Example 6.1.

Let $X = C([0,\pi])$. The linear operator A is defined as in the previous example. Also we define $F\colon\mathcal{C}_0(\chi)\to X$ by

$$F(\chi) = b\chi(0)\left[1-\int_{-\tau}^{0}\chi(s)\,d\eta(s)\right],\quad \chi\in\mathcal{C}_0(\chi).$$

Then clearly F satisfies Lipschitz condition and problem (6.3) transforms into the abstract form (6.2).

Since all the assumptions taken into account for establishing the existence and uniqueness results are satisfied, we can apply these results to considered problem which shows that there exists a unique solution of (6.3).

References

[1] Bahuguna, D. Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential problems. *Nonlinear Analysis* **54** (2004) 1021–1028.

[2] Byszewski, L. and Akca, H. Existence of solutions of a semilinear functional-differential evolution nonlocal problem. *Nonlinear Analysis* **34** (1998) 65 – 72.

[3] Balachandran, K. and Chandrasekaran, M. Existence of solutions of a delay differential equation with nonlocal condition. *Indian J. Pure Appl. Math.* **27** (1996) 443 – 449.

[4] Byszewski, L. and Lakshmikantham, V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.* **40** (1990) 11–19.

[5] Byszewski, L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (1991) 494 – 505.

[6] Chan, W.C. and Green, D. Stability and bifurcation in delay diffusion models. *Diff. Int. Eqns* **1** (1993) 87 –100.

[7] Lin, Y. and Liu, J.H. Semilinear integro-differential equations with nonlocal Cauchy problem. *Nonlinear Anal. Theory, Methods and Appl.* **26** (1996) 1023 –1033.

[8] Wiener, J. and Lakshmikantham, V. Excitability of a second order delay differential equations. *Nonlinear Analysis* **38** (1999) 1–11.

[9] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, Berlin, 1983.

[10] Turyn, M. A partial functional differential equation. *J. Math. Anal. Appl.* **263** (2001) 1–13.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 357–367

Exponential Stability of Perturbed Nonlinear Systems

A. Ben Abdallah[1], M. Dlala[1] and M.A. Hammami[2*]

[1] *I.P.E.I.S University of Sfax, BP 805, Sfax, Tunisia*
[2] *Faculty of Sciences of Sfax, Department of Mathematics, BP 802, Sfax 3018, Tunisia*

Received: March 18, 2005; Revised: June 13, 2005

Abstract: In this paper, we deal with the stability analysis problem of perturbed nonautonomous nonlinear systems. Uniform exponential stability is studied by using Lyapunov techniques. The question addressed is related to the restriction about the perturbed term under the assumption that the origin of the nominal system is globally exponentially stable. A new Lyapunov function is used to obtain a large class of stable dynamical systems.

Keywords: *Nonlinear systems; Lyapunov function; exponential stability.*

Mathematics Subject Classification (2000): 37B25, 34D20, 93D05, 93D15.

1 Introduction

Consider the nonautonomous perturbed system

$$\dot{x} = A(t)x + g(t, x), \tag{1}$$

where $A(n \times n)$, g are piecewise continuous in t and g is locally Lipschitz in x such that

$$g(t, 0) = 0, \quad \forall\, t \geq 0.$$

It is known [3] that, if the linearization of the nonlinear system (1) about the origin has an exponentially stable equilibrium point then the origin is an exponentially stable equilibrium for the perturbed nonlinear system and it turns out that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin of (1). For the global case, the stability analysis problem is to find sufficient conditions under which the perturbed system (1) is globally asymptotically or exponentially

*Corresponding author: MohamedAli.Hammami@fss.rnu.tn

stable with the assumption that the nominal system is globally exponentially stable. Therefore, to obtain stability of the whole system, we shall make some restrictions on the perturbed term. Suppose that the origin of the nominal system

$$\dot{x} = A(t)x \tag{2}$$

is globally exponentially stable with

$$W(t,x) = x^T P(t)x$$

as an associate Lyapunov function, where $P(t)$ is a continuous differentiable symmetric and bounded positive definite matrix, such that

$$0 < c_1 I \le P(t) \le c_2 I, \quad \forall\, t \ge 0, \tag{3}$$

which satisfies the matrix differential equation

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) = -Q(t)$$

with $Q(t)$ is continuous, symmetric and positive definite that is

$$Q(t) \ge c_3 I > 0, \quad \forall\, t \ge 0.$$

Here the constants c_1, c_2, $c_3 > 0$ and I is identical matrix.

Then calculating the derivative of W along the trajectories of the system (1) one can obtain the definiteness of $\dot{W}$ by imposing some conditions on $g(t,x)$.

For the case when

$$\|g(t,x)\| \le \eta(t)\|x\|,$$

where $\eta(t)$ is a continuous function, we obtain after taking the derivative of W along the trajectories of the whole system,

$$\dot{W}(t,x) \le -x^T Q(t)x + 2x^T P(t)g(t,x).$$

Then, one gets the following estimation on the derivative of W,

$$\dot{W}(t,x) \le (-c_3 + 2c_2\eta(t))\|x\|^2$$

which implies the global exponential stability of the equilibrium point of (1) under the condition

$$\eta(t) \le k < \frac{1}{2}\frac{c_3}{c_2}$$

with $k > 0$.

Moreover, one can obtain exponential convergence to zero for system (1) especially, where

$$g(t,x) = B(t)x$$

under the conditions $B(t)$ is continuous and

$$B(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Similar conclusions can be obtained (see [5]), where

$$\int_0^{+\infty} \|B(t)\| < \infty$$

or

$$\int_0^{+\infty} \|B(t)\|^2 < \infty.$$

Actually, the synthesis of stability of perturbed systems is based on the stability of the nominal system with $W(t,x)$ as a Lyapunov function candidate for the whole system provided that the size of the perturbation is known (see [1, 2, 4−7, 11, 12]). Panteley and Loria [8, 9] studied this problem for cascaded time-varying nonlinear systems, which can be regarded as perturbed systems, where growth conditions are given to ensure the global uniform asymptotic stability of some classes of time-varying nonlinear systems.

Our approach is to find more general classes of perturbed systems which can be globally exponentially stable by considering a new Lyapunov function which has the following form

$$V(t,x) = x^T P(t)x + \Psi(t,x),$$

where $\Psi(t,x)$ is a C^1-function which will be chosen, for some classes of systems, in such a way that $V(t,x)$ is positive definite radially unbounded and its derivative along the trajectories of (1) is negative definite. We use a cross term in the Lyapunov function, as in [10] introduced for cascade nonlinear systems, to obtain a large class of stable perturbed systems. The proposed new method is based on the non uniqueness of Lyapunov functions with a stable nominal system, which guarantees exponential stability with the requirement on the upper bound of the perturbed term. We prove that the system can be globally uniformly exponentially stable. The perturbation term is a known function which could result in general from errors in modelling, aging of parameters or disturbances. Naturally, the choice of the function $\Psi(t,x)$ depends on the perturbation term $g(t,x)$ and its smoothness is given under some restrictions on the dynamics of the system. Furthermore, we give an illustrative example in dimensional one and we show for a certain class of perturbed systems that the proposed method gives better result than the classical method.

2 Stability

In this paper the solution of a differential time-varying equation

$$\dot{x} = A(t)x + g(t,x)$$

with initial conditions $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, $x(t_0) = x_0$ is denoted $\phi(t, t_0, x_0)$.

$\dot{V}_{(\star)}(t,x)$ is the derivative of Lyapunov function $V(t,x)$ along the trajectories represented by the differential equation $(\star)$.

According to [3, 5], the equilibrium point $x = 0$ of (1) is uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0.$$

The equilibrium point $x = 0$ of (1) is globally uniformly asymptotically stable if it is uniformly stable and for any initial state $x(t_0)$, one has

$$x(t) \to 0 \quad \text{as} \quad t \to +\infty$$

uniformly in t_0, that is there exists $T = T(\varepsilon) > 0$, such that

$$\|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 + T(\varepsilon), \quad \forall\, x(t_0).$$

The equilibrium point $x = 0$ of (1) is globally exponentially stable if the following estimation holds for any initial state $x(t_0)$,

$$\|x(t)\| < \lambda_1 e^{-\lambda_2(t-t_0)}, \quad \forall\, t \geq t_0 \geq 0,$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$.

Throughout this paper, we suppose that

(A_1). There exists a continuous differentiable, symmetric, bounded, positive definite matrix $P(t)$ which satisfies (3).

(A_2). There exist a continuous function $\rho\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $k > 0$, such that

$$\forall\, t \geq 0, \quad \forall\, x \in \mathbb{R}^n, \quad \|g(t,x)\| \leq \rho(t)\|x\|$$

with

$$\rho(t) \leq k, \quad \forall\, t \geq 0$$

and

$$\int_0^{+\infty} \rho(t)dt < +\infty.$$

Note that, the quadratic function

$$W(t,x) = x^T P(t)x$$

implies by the assumption (A_1) the two following inequalities,

$$\begin{aligned} c_1\|x\|^2 \leq W(t,x) \leq c_2\|x\|^2, \\ \dot{W}_{(2)}(t,x) \leq -c_3\|x\|^2. \end{aligned} \tag{4}$$

Our goal is to seek a suitable function Ψ which is of class C^1 to compensate the perturbed term which is not always possible only for some restrictive dynamical systems. Thus, we will consider a Lyapunov function for system (1) of the form $V(t,x) = x^T P(t)x + \Psi(t,x)$, where Ψ is a C^1-function which will be chosen later such that V is definite positive function and $\dot{V}$ definite negative for some restriction on g. Notice that, continuity of the partial derivatives of the cross term can be proven for some classes of system of the form (1). Thus, if we consider the derivative of $V(t,x)$ along the trajectories of the system (1) we get

$$\dot{V}_{(1)}(t,x) = \dot{W}_{(2)}(t,x) + 2x^T P(t) \cdot g(t,x) + \dot{\Psi}(t,x).$$

The first term of the right-hand side constitute the derivative of $V(t,x)$ along the trajectories of the nominal system, which is negative definite and satisfies (4). The second term is the effect of the perturbation while the third one is the derivative of the cross term. We choose

$$\Psi(t,x) = \int_t^{+\infty} 2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))\,ds.$$

Thus, one can verify the following statement

$$2x^T P(t)\cdot g(t,x) + \dot{\Psi}(t,x) = 0$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

It follows with this choice, that

$$\dot{V}_{(1)}(t,x) = \dot{W}_{(2)}(t,x) \le -c_3\|x\|^2.$$

This yields by (A_1), the exponential stability of (1) provided that $\Psi(t,x)$ exists and it is a C^1-function or simply uniformly continuous rending $V(t,x)$ definite positive for a given perturbed function $g(t,x)$.

First, one can state the following proposition which provides a stability result.

Proposition 2.1 *If (A_1) and (A_2) are satisfied, then the origin of the system (1) is uniformly stable.*

Proof Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an initial condition. The derivative of W along the trajectories of (1) is given by

$$\begin{aligned}\dot{W}_{(1)} &= \frac{d}{ds}(W(s,\phi(s,t,x))) \\ &= \frac{\partial W}{\partial s}(s,\phi(s,t,x)) + 2\phi^T(s,t,x)P(s)A(s)\phi(s,t,x) \\ &\quad + 2\phi^T(s,t,x)P(s).g(s,\phi(s,t,x)).\end{aligned}$$

Thus,

$$\begin{aligned}\dot{W}_{(1)} &\le 2\phi^T(s,t,x)P(s).g(s,\phi(s,t,x)) \\ &\le 2c_2\rho(s)\|\phi(s,t,x)\|^2 \\ &\le 2\frac{c_2}{c_1}\,\rho(s)W(s,\phi(s,t,x))\end{aligned}$$

which implies that

$$W(s,\phi(s,t,x)) \le MW(t,x),$$

where

$$M = \exp\left\{2\,\frac{c_2}{c_1}\left(\int_0^{+\infty} \rho(u)\,du\right)\right\}.$$

We conclude that

$$\|\phi(s,t,x)\| \le \sqrt{\frac{c_2}{c_1}\,M}\,\|x\|, \qquad \forall\, s \ge t.$$

Then the equilibrium point of the system (1) is uniformly stable.

The above proposition is conceptually important because it shows the stability of the origin for all perturbations satisfying the condition (A_2).

Now, concerning the cross term, we have the following lemma.

Lemma 2.1 *Under assumptions (A_1) and (A_2), the function $\Psi(t,x)$ exists and is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$.*

Proof Observe that, using the above proposition and the fact that for all (t,x) the function $\psi(t,x)$ exists, we have each solution of (1) which starts at (t,x) is bounded for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and for all $s \geq t$.

Indeed, on the one hand

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))| \leq c_2\rho(s)\|\phi(s,t,x)\|^2$$

which gives

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))| \leq M_1\rho(s)\|x\|^2$$

which belongs to $L^1(\mathbb{R}_+)$, where $M_1 = M\,\frac{c_2^2}{c_1}$.

Thus, the integral exists for all $(t,x) \in \mathbb{R}+\times\mathbb{R}^n$ and then $\psi(t,x)$ exists.

On the other hand, the continuity of Ψ can be shown by observing that, for all $s \geq t$, the function

$$(t,x) \longmapsto \phi(s,t,x)^T P(s)g(s,\phi(s,t,x))$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$ and the fact that for all $(t,x) \in \mathbb{R}_+ \times K$, $s \geq t$, where K is a compact set in $\mathbb{R}^n$, we have

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x)| \leq M_K\rho(s).$$

The upper bound $M_K\rho(s)$ is in $L^1(\mathbb{R}_+)$ where M_K is a positive constant which depends only on K.

Next, the proposed Lyapunov function candidate for (1) must be definite positive and we will use this fact to show the exponential stability of the origin of system (1).

Theorem 2.1 *If the assumptions (A_1) and (A_2) hold, then there exist some positive constants d_1, d_2 such that*

$$d_1\|x\|^2 \leq V(t,x) \leq d_2\|x\|^2.$$

It means that, the Lyapunov function $V(t,x)$ is a decreascent function.

Proof Observe that,

$$\int_t^s \dot{W}_{(1)}(u,\phi(u,t,x))\,du = W(s,\phi(s,t,x)) - W(t,x).$$

Then, we obtain

$$W(s,\phi(s,t,x)) - W(t,x) = \int_t^s \dot{W}_{(2)}(u,\phi(u,t,x))\,du$$

$$+ \int_t^s 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))\,du.$$

Because $W(s,\phi(s,t,x))$ is bounded and $\Psi(t,x)$ exists, it means that the integral

$$\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))\,du$$

exists.

Then

$$\lim_{s\to+\infty} W(s,\phi(s,t,x)) = W_\infty(t,x)$$

exists.

It follows that,

$$\begin{aligned}
V(t,x) &= W_\infty(t,x) - \int_t^{+\infty} \dot{W}_{(2)}(u,\phi(u,t,x))\,du,\\
V(t,x) &\geq - \int_t^{+\infty} \dot{W}_{(2)}(u,\phi(u,t,x))\,du \qquad (5)\\
V(t,x) &\geq \int_t^{+\infty} c_3\|\phi(s,t,x)\|^2 ds.
\end{aligned}$$

Remark also that

$$\phi(s,t,x) = x + \int_t^s A(u)\phi(u,t,x) + g(u,\phi(u,t,x))\,du$$

which gives

$$\|\phi(s,t,x)\| \geq \|x\| - \int_t^s (L\|\phi(u,t,x)\| + \rho(u)\|\phi(u,t,x)\|)du$$

Thus,

$$\begin{aligned}
\|\phi(s,t,x)\| &\geq \|x\| - \int_t^s (L+k)\|\phi(u,t,x)\|du\\
&\geq \|x\| - \lambda(s-t)\|x\|\\
&\geq \frac{\|x\|}{2}, \quad \text{for} \quad s \in \left[t, t+\frac{1}{2\lambda}\right],
\end{aligned}$$

where

$$\lambda = (L+k)\sqrt{\frac{Mc_2}{c_1}}.$$

Hence from (5), we obtain

$$V(t,x) \geq d_1\|x\|^2.$$

Still to prove the existence of d_2, which implies in conjunction with the above expression that $V(t,x)$ is a decreascent function.

For any (t,x), we have

$$V(t,x) = W(t,x) + \Psi(t,x).$$

Thus,

$$V(t,x) \leq c_2\|x\|^2 + \int_t^{+\infty} 2|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))|ds.$$

It follows that,

$$V(t,x) \leq c_2\|x\|^2 + \int_t^{+\infty} M_1\rho(s)\|x\|^2 ds \leq c_2\|x\|^2 + M_2\|x\|^2$$
$$\leq d_2\|x\|^2.$$

Theorem 2.2 *Suppose that the assumptions (A_1), (A_2) hold and the function g is chosen in such a way that*

$$\Psi(t,x) = \int_t^{+\infty} 2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))\,ds$$

is a C^1-function, then $x=0$ is globally exponentially stable equilibrium point for (1).

Proof Still to prove that

$$\dot{\Psi}(t,x) = -2x^T P(t)g(t,x).$$

We have

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\Psi(s,\phi(s,t,x))\right)\Big|_{s=t},$$

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))ds\right)\Bigg|_{s=t}.$$

Since the solutions of (1)

$$u \longmapsto \Phi(u,t,x)$$

and

$$u \longmapsto \Phi(u,s,\Phi(s,t,x))$$

are equal for $u=s$, this implies that, for all $u \geq s \geq t \geq 0$,

$$\Phi(u,t,x) = \Phi(u,s,\Phi(s,t,x)).$$

Thus,

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))ds \right)\Bigg|_{s=t}.$$

So,

$$\dot{\Psi}(t,x) = -(2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))ds)\Big|_{s=t}.$$

Hence,

$$\dot{\Psi}(t,x) = -2\phi^T(s,t,x)P(s)g(t,x).$$

Using the fact that V is a decreascent function in conjunction with the above expression yields the global exponential stability of (1).

Finally, we give an example to illustrate the applicability of the result of this paper. Moreover, we will compare in the next section our approach with the classical one for a certain class of nonlinear system.

Example As a simple example, to compute the cross term, we consider the following scalar linear equation

$$\dot{x} = -ax + \rho(t)x, \qquad a > 0, \tag{6}$$

with $\rho(t)$ satisfies (A_2). If we choose

$$W(x) = x^2$$

as a Lyapunov function of

$$\dot{x} = -ax$$

we obtain

$$\phi(s,t,x) = \exp\left(-a(s-t) + \int_t^s \rho(u)\,du \right)x.$$

Thus,

$$\Psi(t,x) = x^2 \int_t^{+\infty} \rho(s)\exp\left(2\int_t^s \rho(u)\,du\right)e^{-2a(s-t)}ds.$$

So,

$$\Psi(t,x) = -x^2 + 2ax^2 \int_t^{+\infty} \exp\left(2\int_t^s \rho(u)\,du\right)e^{-2a(s-t)}ds.$$

It follows that, Ψ is a C^1-function and then $x = 0$ is an exponentially stable equilibrium point for (4).

3 Stability of a Certain Class of Perturbed Systems

Consider the following system

$$\dot{x} = Ax + \rho(t)B(x)x, \tag{7}$$

where $x \in \mathbb{R}^n$, $t \geq 0$, $A(n \times n)$ is a constant matrix which is supposed Hurwitz and $\rho(t)$ satisfies (A_2).

Moreover, We assume that

(A_3). $B(\cdot)$ is a C^1-function and there exists a positive constant M, such that

$$\forall\, x \in \mathbb{R}^n \quad \|B(x)\| \leq M.$$

We have the following result of stability for system (7).

Proposition 3.1 *If (A_1), (A_2) and (A_3) are satisfied, then Ψ is C^1 in $\mathbb{R}_+ \times \mathbb{R}^n$ and $x = 0$ is a globally exponentially stable equilibrium point for (7).*

Proof We denote

$$X(s) = \frac{\partial}{\partial x}\left(\Phi(s,t,x)\right)$$

and

$$Y(s) = \frac{\partial}{\partial t}\left(\Phi(s,t,x)\right), \qquad s \geq t.$$

Thus, X and Y satisfies the following two statements

$$\dot{X} = \left(A + \rho(s)\frac{\partial B}{\partial x}\left(\Phi(s,t,x)\right)\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right)X$$

with

$$X(t) = I$$

and

$$\dot{Y} = \left(A + \rho(s)\frac{\partial B}{\partial x}\left(\Phi(s,t,x)\right)\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right)Y$$

with

$$Y(t) = 0.$$

Let K be a compact set of $\mathbb{R}^n$. Because $\Phi(s,t,x)$ is uniformly bounded and $B(\cdot)$ is a C^1-function, then there exists $M_K > 0$, such that $\forall\, s \geq t \geq 0$, $\forall\, x \in K$,

$$\left\|\rho(s)\,\frac{\partial B}{\partial x}\left(\Phi(s,t,x)\right)\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right\| \leq M_K\rho(s).$$

Note that Lemma 2.1 implies that $X(s,t,x)$ and $Y(s,t,x)$ are bounded when x leaves in K.

Thus, we have

$$\Psi(t,x) = \int_t^{+\infty} \frac{\partial W}{\partial x}\left(\phi(s,t,x)\right)\rho(s)B(\phi(s,t,x))\phi(s,t,x)\,ds.$$

Because X and Y are bounded when $x \in K$, then there exist M_1 and M_2, such that

$$\left\|\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\left(\phi(s,t,x)\right)\rho(s)B(\phi(s,t,x))\phi(s,t,x)\right)\right\| \leq M_1\rho(s)$$

and

$$\left\|\frac{\partial}{\partial t}\left(\frac{\partial W}{\partial x}\left(\phi(s,t,x)\right)\rho(s)B(\phi(s,t,x))\phi(s,t,x)\right)\right\| \leq M_2\rho(s)$$

for all $s \geq t \geq 0$ and $x \in K$.

Hence, we conclude that Ψ is a C^1-function on $\mathbb{R}_+ \times \mathbb{R}^n$ and then $x = 0$ is globally exponentially stable equilibrium point of system (7).

Remark To compare the result given in this paper with the usual techniques of stability for perturbed systems, we shall consider the Lyapunov function of the nominal system as a Lyapunov function for the whole system. Let $V(t,x) = x^T Px$, where $P > 0$ is symmetric and positive definite so that

$$A^T P + PA = -Q$$

with Q symmetric and positive definite matrix. Then the derivative of $V(t,x)$ along the solutions of system (7) gives

$$\dot{V}_{(2)}(t,x) = -x^T Qx + \rho(t)x^T \Big(B^T(x)P + PB(x)\Big)x.$$

It follows that,

$$\dot{V}_{(2)}(t,x) \le \Big(-\lambda_{min}(Q) + 2\lambda_{max}(P)\rho(t)\|B(x)\|\Big)\|x\|^2,$$

$$\dot{V}_{(2)}(t,x) \le -\Big(\lambda_{min}(Q) - 2kM\lambda_{max}(P)\Big)\|x\|^2.$$

Then, if we choose

$$\lambda_{min}(Q) - 2kM\lambda_{max}(P) > 0$$

which implies that k must satisfy the following inequality

$$k < \frac{\lambda_{min}(Q)}{2M\lambda_{max}(P)}. \tag{8}$$

Hence, the system (7) is globally exponentially stable. Notice that, with our choice of Lyapunov function we don't need that the upper bound of $\rho(t)$ is limited as in (8). So, we obtain a class of stable differential system more large than by using the classical method.

References

[1] Corless, M. Guaranteed rates of exponential convergence for exponential convergence for uncertain system. *Journal of Optimization Theory and Applications* **67** (1990) 481–494.

[2] Chen, X. and Zhai, G. Robust control for a class of dynamical systems with uncertainties. *Nonlinear Dynamics and Systems Theory* **4**(3) (2004) 125–138.

[3] Hahn, W. *Stability of Motion.* Springer-Verlag, Berlin, 1967.

[4] Hammami, M.A. On the stability of nonlinear control systems with uncertainty. *Journal of Dynamical Control Systems* **7**(2) (2001) 171–179.

[5] Khalil, H.K. *Nonlinear Systems.* Mac-Millan, New York, 1992.

[6] Merkin, D.R. *Introduction to the Theory of Stability.* Springer-Verlag, Berlin, 1996.

[7] Lin, Y., Sontag, E. and Wang, Y. A smooth converse Lyapunov theorem for robust stability. *Siam Journal Control Optimization* **34**(1) (1996) 124–160.

[8] Panteley, E. and Loria, A. Global uniform asymptotic stability of cascaded non autonomous nonlinear systems. In: *Proceedings of the Fourth European Control Conference*, Louvain-La-Neuve, Belgium, No. 259, 1997.

[9] Panteley, E. and Loria, A. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. *Automatica* **37** (2001) 453–460.

[10] Sepulchre, R., Jankovic, M. and Kokotovic, P.V. Constructive Lyapunov of nonlinear cascade systems. *IEEE Transactions on Automatic Control* **41**(12) (1996) 1723–1735.

[11] Vidyasagar, M. *Nonlinear Systems Analysis.* Prentice Hall, Englewood, 1993.

[12] Wang, Y., Shi, Y. and Sasaki, H. A criterion for stability of nonlinear time-varying dynamic system. *Nonlinear Dynamics and Systems Theory* **4**(3) (2004) 217–230.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 369–379

New Stability Conditions for TS Fuzzy Continuous Nonlinear Models

M. Benrejeb[1,2], M. Gasmi[1] and P. Borne[2*]

[1] *LA.R.A , Ecole Nationale d'Ingénieurs de Tunis, BP 37, le Belvédère, 1002 Tunis, Tunisie*
[2] *LAGIS, Ecole Centrale de Lille, BP 48, 59651, Villeneuve d'Ascq, France*

Received: August 25, 2005; Revised: October 20, 2005

Abstract: Several linear and nonlinear fuzzy models stability conditions are developed in the literature. Some of them concern the linear fuzzy Takagi–Sugeno (TS) model and are based on the determination of a common positive definite matrix, solution of linear matrix inequalities.

A new explicit formulation of stability conditions and an extension to the case of nonlinear TS fuzzy continuous models are given in this paper.

The proposed criteria are based on the use of the vector norm approach associated, in the state space description, to a specific characteristic matrix form, called arrow form matrix. This representation is such that only the elements of the diagonal, those of the last row and those of the last column can be different from zero.

The obtained stability conditions, explicitly expressed by the studied models and fuzzification parameters, applicable for TS fuzzy models in particular, make the approach useful for the synthesis of stabilizing fuzzy control law.

For a class of considered Lur'e–Postnikov continuous case, the stability criterion corresponds to a simple condition on the instantaneous characteristic polynomial of the nonlinear studied system.

Keywords: *Nonlinear continuous system; TS fuzzy model; stability; arrow form matrix; vector norm; Lur'e–Postnikov system.*

Mathematics Subject Classification (2000): 93C42, 93D20.

*Corresponding author: p.borne@ec-lille.fr

1 Introduction

Takagi-Sugeno (TS) fuzzy models, proposed by Takagi and Sugeno [16] and further developed by Sugeno and Kang [15], are nonlinear systems described by a set of IF–THEN rules which gives a local linear representation of an underlining system. It is well known that such models can describe or approximate a wide class of nonlinear systems. Hence, it is important to study their stability or to synthesize their stabilizing controllers.

In fact, the stability study constitutes an important phase in the synthesis of a control law, as well as in the analysis of the dynamic behavior of a closed loop system. It has been one of the central issues concerning fuzzy control, refer to the brief survey on the stability issues given in [14].

Based on the stability conditions, model-based control of such systems has been developed for the continuous case in [5 – 7, 13, 19, 20] by using state-space models.

In recent literature, Tanaka and Sugeno [17], have provided a sufficient condition for the asymptotic stability of a fuzzy system in the sense of Lyapunov through the existence of a common Lyapunov function for all the subsystems.

This kind of design methods suffer mainly from a few limitations:

(1) one can construct a TS model if local description of the dynamical system to be controlled is available in terms of local linear models;
(2) a common positive definite matrix must be found to satisfy a matrix Lyapunov equation, which can be difficult especially when the number of fuzzy rules required to give a good plant model is large so that the dimension of the matrix equation is high;
(3) it appears that a necessary condition, for the existence of this common positive definite matrix, is that all subsystems must be asymptotically stable.

To overcome those difficulties, we propose, in this paper, to study the stability of TS fuzzy nonlinear model through the study of the convergence of a regular vector norm.

If the vector norm is of dimension one, then this is like the second Lyapunov method approach; therefore, if it is of higher dimension, then we deal with a vector-Lyapunov function [9 – 12].

The vector norm approach, based on the comparison/overvaluing principle, has a major advantage: it deals with a very large class of systems, since no restrictive assumption is made on the matrices of state equations, except that they are bounded for bounded states, in such a way that a unique continuous solution exists.

Nevertheless, although the overvaluing principle allows the simplification of the study, it also presents the corresponding drawback: overvaluation means losing information on the real behavior of the process. Thus, the cases of state equations which are the most resistant of this type of method are the ones in which replacement of coefficients by their absolute values leads to an overvaluing system which is far from reality, for instance an unstable one, whereas the initial system was stable. In many cases, this type of drawback can be bypassed by using changes of state variables leading to a good performance of the representation [2 – 4]. For instance, for continuous control, a particularly interesting case is the one in which the off-diagonal elements are naturally positive or equal to zero; in this case, the overvaluing is carried out without loss of information.

This paper is organized as follows: TS fuzzy nonlinear continuous model description is presented in Section 2. Section 3 reviews some existing stability conditions of such system. In Section 4, the vector norm approach combined with the arrow form matrix are employed to give the new stability criterion for TS fuzzy nonlinear continuous models. The case of Lur'e–Postnikov continuous system is studied in Section 5. Finally, conclusions are drawn in Section 6.

2 TS Fuzzy Nonlinear Continuous Model Description

Consider a TS fuzzy model when local description of the plant to be controlled is available in terms of nonlinear autonomous models

$$\dot{X}(t) = A_i(X)X(t) \tag{1}$$

where $X \in R^n$ describes the state vector, $A_i(\cdot)$ are matrices of appropriate dimensions, $A_i(\cdot) = \{a_{ij}(\cdot)\}$ and $a_{ij}(\cdot)\colon R^n \to R$, are nonlinear elements.

It is assumed that $X = 0$ is the unique equilibrium state of the studied system.

The above information is then fused with the available IF–THEN rules, where the i-th rule, $i = 1, \dots, r$, can have the form:

Rule i: IF $\{X(t)$ is $H_i(X)\}$ THEN $\Big\{\dot{X}(t) = A_i(\cdot)X(t)\Big\}$,

where $H_i(X)$ is the grade of the membership of the state $X(t)$.

The final output of the fuzzy system is inferred as follows:

$$\dot{X}(t) = \sum_{i=1}^{r} h_i(X)A_i(\cdot)X(t) \tag{2}$$

with, for $i = 1, \dots, r$, $0 \le h_i(X) \le 1$ and $\sum\limits_{i=1}^{r} h_i(t) = 1$.

3 Stability Conditions — Problem Statement

It is straightforward to show that a sufficient condition for asymptotic stability in the large of the equilibrium state $X = 0$ of the unforced fuzzy model, obtained by linearization of (2),

$$\dot{X}(t) = \sum_{i=1}^{r} h_i A_i X(t) \tag{3}$$

is that there exists a common symmetric positive definite matrix P such that, for $i = 1, 2, \dots, r$

$$A_i^{\mathrm{T}} P + P A_i < 0. \tag{4}$$

The necessary condition for the existence of matrix P is that each matrix must be asymptotically stable [17], i.e. all the subsystems are stable, or that matrices:

$$\sum_{j=1}^{k} A_{i_j} \tag{5}$$

where $i_j \in \{1, 2, \dots, r\}$ and $k = 2, 3, \dots, r$, are asymptotically stable [18].

The linear matrix inequality (LMI) based approaches have been used to determine the existence of a common symmetric positive definite matrix [20]. Their computation can be expensive in the case of high number of rules.

As it was shown, the stability study of the nonlinear model (2) requires the linearization of the nonlinear subsystems described by the instantaneous characteristic matrices A_i. If those matrices are in arrow form [2], stability conditions of the nonlinear system (2), as we will see in the next section, can be formulated easily.

4 New TS Fuzzy Nonlinear Model Stability Criterion

Let us consider the continuous process whose model is in the controllable form, that matrices $A_i(\cdot)$, of equation (2), are written as

$$A_i(\cdot) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & & \dots & 0 & 1 \\ -a_{i,0}(\cdot) & & \dots & & -a_{i,n-1}(\cdot) \end{bmatrix}. \tag{6}$$

A change of base under the form:

$$T = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_{n-1}^{n-2} & 0 \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_{n-1}^{n-1} & 1 \end{bmatrix} \tag{7}$$

allows the new state matrices, denoted by $F_i(\cdot)$, to be in arrow form (2)

$$F_i(\cdot) = T^{-1}A_i(\cdot)T = \begin{bmatrix} \alpha_1 & & & \beta_1 \\ & \ddots & & \vdots \\ & & \alpha_{n-1} & \beta_{n-1} \\ \gamma_{i,1}(\cdot) & \cdots & \gamma_{i,n-1}(\cdot) & \gamma_{i,n}(\cdot) \end{bmatrix}, \tag{8}$$

where

$$\beta_j = \prod_{\substack{k=1 \\ k\neq j}}^{n-1} (\alpha_j - \alpha_k)^{-1} \quad \forall\, j = 1, 2, \dots, n-1, \tag{8a}$$

$$\gamma_{i,j}(\cdot) = -P_{A_i}(\cdot, \alpha_j) \quad \forall\, j = 1, 2, \dots, n-1, \tag{8b}$$

$$\gamma_{i,n}(\cdot) = -a_{i,\,n-1}(\cdot) - \sum_{i=1}^{n-1} \alpha_i. \tag{8c}$$

$P_{A_i}(\cdot, \lambda)$ is the $A_i(\cdot)$ instantaneous characteristic polynomial such that

$$P_{A_i}(\cdot, \lambda) = \lambda^n + \sum_{l=0}^{n-1} a_{i,l}(\cdot)\lambda^l \tag{9}$$

and $\alpha_j,\ j=1,2,\dots,n-1,$ are distinct arbitrary parameters.

Let us note that the determinant of the arrow form matrix $F_i(\cdot)$ is computed as [2]

$$|F_i(\cdot)| = \left[\gamma_{i,n}(\cdot) - \sum_{j=1}^{n-1} \alpha_j^{-1}\gamma_{i,j}(\cdot)\,\beta_j\right] \prod_{k=1}^{n-1} \alpha_k. \tag{10}$$

The final output of the fuzzy system is then inferred as follows

$$\dot{Y}(t) = Q(\cdot)Y(t) \tag{11}$$

where $Y(t)$ is the new state vector such that $X(t) = TY(t)$,

$$Q(\cdot) = \sum_{i=1}^{r} h_i F_i(\cdot), \tag{11a}$$

$$Q(\cdot) = \begin{bmatrix} \alpha_1 & & & \beta_1 \\ & \ddots & & \vdots \\ & & \alpha_{n-1} & \beta_{n-1} \\ \sum\limits_{i=1}^{r} h_i\gamma_{i,1}(\cdot) & \cdots & \sum\limits_{i=1}^{r} h_i\gamma_{i,n-1}(\cdot) & \sum\limits_{i=1}^{r} h_i\gamma_{i,n}(\cdot) \end{bmatrix}. \tag{11b}$$

In such conditions, if $p(Y)$ denotes a vector norm of Y, satisfying component to component the equality

$$p(Y) = |Y| \tag{12}$$

it is possible, by the use of the aggregation techniques [2, 9], to define a comparison system (13), $Z \in R^n$, of (11)

$$\dot{Z} = M(\cdot)Z. \tag{13}$$

In this expression, the matrix $M(\cdot)$ is deduced from the matrix $Q(\cdot)$ by substituting only the off-diagonal elements by their absolute values; it can be written as

$$M(\cdot) = \begin{bmatrix} \alpha_1 & & & |\beta_1| \\ & \ddots & & \vdots \\ & & \alpha_{n-1} & |\beta_{n-1}| \\ \left|\sum\limits_{i=1}^{r} h_i\gamma_{i,1}(\cdot)\right| & \cdots & \left|\sum\limits_{i=1}^{r} h_i\gamma_{i,n-1}(\cdot)\right| & \sum\limits_{i=1}^{r} h_i\gamma_{i,n}(\cdot) \end{bmatrix}. \tag{14}$$

Noting that the non-constant elements are isolated in the last row of matrix $M(\cdot)$, then the stability condition of the continuous nonlinear system (2) can be easily deduced from the Borne and Gentina criterion [8, 11]. It comes

$$(-1)^i \Delta_i > 0, \quad i = 1, 2, \dots, n, \tag{15}$$

with Δ_i the i-th $M(\cdot)$ principal minor.

It is clear that, for $i = 1, 2, \dots, n-1$, the condition (15) is verified for $\alpha_i \in R_-$, therefore, for $i = n$ and using the relation (10), it leads to the stability condition (16).

Then, the TS fuzzy nonlinear model stability, in the continuous case, can be studied by the following proposed theorem.

Theorem 4.1 *If there exist* $\alpha_i \in R_-$, $i = 1, 2, \ldots, n-1$, $\alpha_i \neq \alpha_j$ *for all* $i \neq j$ *and* $\varepsilon \in R_+$ *such that the inequality*

$$-\sum_{i=1}^{r} h_i \gamma_{i,n}(\cdot) + \sum_{j=1}^{n-1} \left| \sum_{i=1}^{r} h_i \gamma_{i,j}(\cdot) \beta_j \right| \alpha_j^{-1} \geq \varepsilon \quad \forall\, X \in R^n \tag{16}$$

is satisfied, the equilibrium state of the studied continuous nonlinear system (3) and (7) is asymptotically globally stable.

If there exist α_j, $j = 1, 2, \ldots, n-1$, such that

$$\sum_{i=1}^{r} h_i \gamma_{i,j}(\cdot) \beta_j > 0 \quad j = 1, 2, \ldots, n-1, \tag{17}$$

the Theorem 4.1 can be simplified and the comparison system (13) can be chosen identically to (11).

Since for $Q(\cdot)$

$$\Delta_n = \sum_{i=1}^{r} h_i P_{A_i}(\cdot, 0), \tag{18}$$

$$-\sum_{i=1}^{r} h_i \gamma_{i,n}(\cdot) + \sum_{j=1}^{n-1} \alpha_j^{-1} \sum_{i=1}^{r} h_i \gamma_{i,j}(\cdot) \beta_j = \prod_{j=1}^{n-1} (-\alpha_j)^{-1} \sum_{i=1}^{r} h_i P_{A_i}(\cdot, 0). \tag{19}$$

Hence to Corollary 4.1.

Corollary 4.1 *If there exist* $\alpha_j \in R_-$, $\alpha_j \neq \alpha_k$ *for all* $j \neq k$ *and* $\varepsilon \in R_+$ *such that:*

(i) *the inequalities (17) are satisfied for all* $X \in R^n$,

(ii) $\sum_{i=1}^{r} h_i(t)\, P_{A_i}(\cdot, 0) \geq \varepsilon$ *for all* $X \in R^n$, *(20)*

the equilibrium state of the continuous system described by (2) and (6) is globally asymptotically stable.

Example 4.1. Unstable TS fuzzy model case

Given the unforced fuzzy linear system model described by (3), where $r = 2$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Obviously, the first subsystem is unstable whereas the second one is stable. However, there is no common positive definite matrix P to verify the stability condition (4).

The matrices A_1 and A_2 can be transformed to arrow form matrices F_1 and F_2, by the same change of base under the form

$$T = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \qquad F_1 = T^{-1} A_1 T = \begin{bmatrix} \alpha & 1 \\ -(\alpha^2 + \alpha - 2) & -1 - \alpha \end{bmatrix},$$

$$F_2 = T^{-1} A_2 T = \begin{bmatrix} \alpha & 1 \\ -(\alpha^2 + \alpha + 1) & -1 - \alpha \end{bmatrix}$$

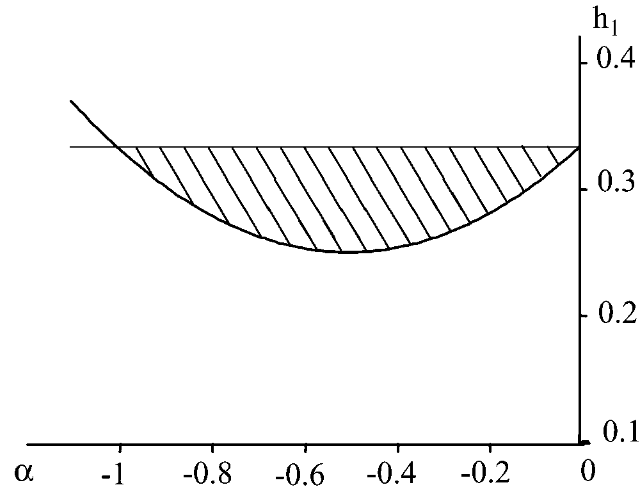

Figure 4.1. Stability domain for example 1.

where α is an arbitrary non zero parameter.

Since $h_1 + h_2 = 1$, the global fuzzy system is then described by

$$\dot{y}(t) = (h_1F_1 + h_2F_2)Y(t) = \begin{bmatrix} \alpha & 1 \\ -\alpha^2 - \alpha + 2h_1 - h_2 & -1-\alpha \end{bmatrix} Y(t).$$

The application of Theorem 4.1 leads to the following stability conditions

(i) $\alpha < 0$,

(ii) $1 + \alpha + \dfrac{|-\alpha^2 - \alpha + 2h_1 - h_2|}{\alpha} > 0$,

since $h_2 = 1 - h_1$, $h_1 \in [0;1]$, the corresponding stability domain is represented by the hatching domain in Figure 4.1.

5 Lur'e–Postnikov System Case

Consider the nonlinear system given in Figure 5.1, with $e = -C\sigma$, $\sigma = [y, y^{(1)}, \dots, y^{(n-1)}]^{\mathrm{T}}$, $\sigma \in R^n$, $C = [c_0, c_1, \dots, c_{n-2}, c_{n-1}]$ is a vector with constant elements, $c_{n-1} = 1$, $f \in \Lambda = \{f\colon R^n \to R,\ f(C\sigma)/C\sigma \equiv f^*(C\sigma),\ C\sigma \neq 0;\ f^*(\cdot) \in [\underline{f}^*, \overline{f}^*] \subset R,\ \forall \sigma \in R^n\}$ and

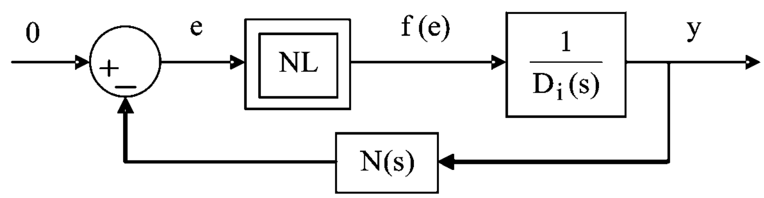

Figure 5.1. The i-th Lur'e–Postnikov model.

$$D_i(s) \;=\; s^n + \sum_{j=0}^{n-1} d_{i,j} s^j \equiv \prod_{j=1}^{n} (s - p_{i,j}), \tag{21}$$

$$N(s) = s^{n-1} + \sum_{j=0}^{n-2} c_j s^j \equiv \prod_{j=1}^{n-1} (s - z_j). \tag{22}$$

Thus, the i-th unforced Lur'e–Postnikov system can be described by

$$D_i(s)y(s) = f(e) = f^*(\cdot)e = -f^*(\cdot)N(s)y(s) \tag{23}$$

witch leads to the nonlinear differential equation

$$y^{(n)} + \sum_{j=0}^{n-1} (d_{i,j} + f^*(\cdot)c_j)y^{(j)} = 0. \tag{24}$$

With the choice of arbitrary parameters α'_j such that

$$\alpha_j = z_j, \quad j = 1, \ldots, n-1, \tag{25}$$

this system can be described [1], in the state space arrow form, by

$$F'_i(\cdot) = \begin{bmatrix} z_1 & & & \beta'_1 \\ & \ddots & & \vdots \\ & & z_{n-1} & \beta'_{n-1} \\ \gamma'_{i,1}(\cdot) & \ldots & \gamma'_{i,n-1}(\cdot) & \gamma'_{i,n}(\cdot) \end{bmatrix} \tag{26}$$

with

$$\beta'_j = \prod_{\substack{k=1\\k\neq j}}^{n-1} (z_j - z_k)^{-1}, \quad \forall\, j = 1, \ldots, n-1, \tag{26a}$$

$$\gamma'_{i,n}(\cdot) = -(d_{i,n-1} + f^*(\cdot)) - \sum_{j=1}^{n-1} z_j, \tag{26b}$$

$$\gamma'_{i,j}(\cdot) = -P'_{A_i}(\cdot, z_j), \quad \forall\, j = 1, \ldots, n-1, \tag{26c}$$

$$P'_{A_i}(\cdot, z_j) = D_i(z_j) + f^*(\cdot)\, N_i(z_j) \equiv D_i(z_j). \tag{26d}$$

The final output of the fuzzy Lur'e–Postnikov system is then inferred as (11)

$$\dot{Y}(t) = \sum_{i=1}^{r} h_i F'_i(\cdot) Y(t) \tag{27}$$

with

$$\sum_{i=1}^{r} h_i F'_i(\cdot) = \begin{bmatrix} z_1 & & & \beta'_1 \\ & \ddots & & \vdots \\ & & z_{n-1} & \beta'_{n-1} \\ \sum\limits_{i=1}^{r} h_i\gamma'_{i,1}(\cdot) & \ldots & \sum\limits_{i=1}^{r} h_i\gamma'_{i,n-1}(\cdot) & \sum\limits_{i=1}^{r} h_i\gamma'_{i,n}(\cdot) \end{bmatrix}. \tag{27a}$$

The stability conditions of the studied Lur'e–Postnikov system can be deduced by using the following proposed theorem.

Theorem 5.1 *The Lur'e–Postnikov system described by (24) is globally asymptotically stable if there exist $\varepsilon \in R_+$ such that the following conditions are verified*

$$z_j \in R_-, \; j = 1, \dots, n-1, \quad z_i \neq z_j \quad \forall\, i \neq j; \tag{28}$$

$$-\sum_{i=1}^{r} h_i(t)\gamma'_{i,n}(\cdot) + \sum_{j=1}^{n-1} (z_j)^{-1} \left| \beta'_j \sum_{i=1}^{r} h_i(t)\gamma'_{i,j}(\cdot) \right| \geq \varepsilon > 0. \tag{29}$$

Proof The non-constant elements in (27a) are isolated in the last row. Hence, the stability conditions can be easily deduced from the Theorem 4.1.

If for parameters $z_j, \; j = 1, \dots, n,$ the following condition is verified

$$\beta'_j \sum_{i=1}^{r} h_i \gamma'_{i,j}(\cdot) > 0 \tag{30}$$

the inequality (29) can then be written:

$$\prod_{i=1}^{n-1} (-z_j)^{-1} \sum_{i=1}^{r} h_i P'_{A_i}(\cdot, 0) \geq \varepsilon > 0. \tag{31}$$

Hence to Corollary 5.1.

Corollary 5.1 *If $z_j \in R_-, \; j = 1, 2, \dots, n-1, \; z_j \neq z_k, \; \forall\, j \neq k,$ and $\varepsilon \in R_+$ such that $\forall\, X \in R^n$*

(i) *the inequality (30) is satisfied;* *(32)*

(ii) $\sum\limits_{i=1}^{r} h_i P'_{A_i}(\cdot, 0) \geq \varepsilon,$ *(33)*

the Lur'e–Postnikov continuous system described by (24), (26) and (27) is globally asymptotically stable.

Example 5.1 Consider the Lur'e–Postnikov system shown in Figure 5.1 with $n = 2$, $r = 2$, $f(e) = f^*(\cdot)e$, $p_{1,1} = -1$, $p_{1,2} = -3$, $p_{2,1} = -2$, $p_{2,2} = -4$ and $z_1 = -2.5$.

From (21) and (22), one can obtain then $d_{1,1} = 4$, $d_{1,0} = 3$, $d_{2,1} = 6$, $d_{2,0} = 8$, and $c_0 = 2.5$.

According to (26), the characteristic matrices, in the arrow form, are given by

$$F'_1(\cdot) = \begin{bmatrix} z_1 & 1 \\ -(z_1^2 + 4z_1 + 3) & -4 - f^*(\cdot) - z_1 \end{bmatrix} = \begin{bmatrix} -2.5 & 1 \\ 0.75 & -1.5 - f^*(\cdot) \end{bmatrix},$$

$$F'_2(\cdot) = \begin{bmatrix} z_1 & 1 \\ -(z_1^2 + 6z_1 + 8) & -6 - f^*(\cdot) - z_1 \end{bmatrix} = \begin{bmatrix} -2.5 & 1 \\ 0.75 & -3.5 - f^*(\cdot) \end{bmatrix}.$$

The global fuzzy system is then described by

$$F'(\cdot) = \sum_{i=1}^{2} h_i F'_i(\cdot)$$

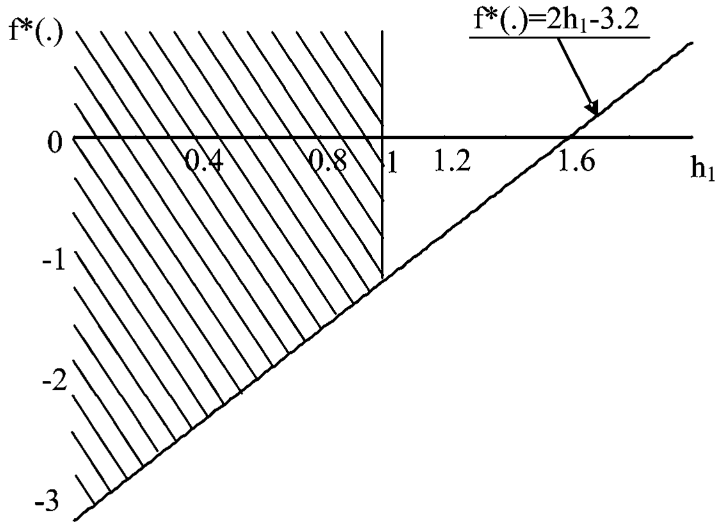

Figure 5.2. Stability domain of the studied Lur'e–Postnikov system.

such that

$$F'(\cdot) = \begin{bmatrix} z_1 & 1 \\ -(z_1^2 + (4h_1 + 6h_2)z_1 + 3h_1 + 8h_2) & -4h_1 - 6h_2 - f^*(\cdot) - z_1 \end{bmatrix}$$

or, for $z_1 = -2.5$

$$F'(\cdot) = \begin{bmatrix} -2.5 & 1 \\ 0.75 & -1.5h_1 - 3.5h_2 - f^*(\cdot) \end{bmatrix}$$

The stability condition of the global fuzzy system, using Theorem 5.1, is then given by

$$1.5h_1 + 3.5h_2 + f^*(\cdot) + \frac{0.75}{-2.5} > 0.$$

Since $h_2 = 1 - h_1$, $h_1 \in [0;\, 1]$, it can be written as

$$f^*(\cdot) > 2h_1 - 3.2$$

which is represented by the hatching domain in Figure 5.2.

6 Conclusions

The new stability conditions, formulated for nonlinear TS fuzzy continuous models case, are based on the use of the vector norm approach combined with an arrow form matrix description.

The obtained stability conditions, explicitly expressed by the studied models and fuzzification parameters, applicable for TS fuzzy models in particular, make the approach useful for the synthesis of stabilization fuzzy control law.

For an important class of Lur'e–Postnikov continuous system, the stability criterion corresponds to a simple instantaneous characteristic polynomial condition.

The considered illustrative examples showed the efficiency of the proposed new approaches.

Other similar results can be obtained easily for nonlinear TS fuzzy discrete systems.

References

[1] Benrejeb, M. Synchronisation des systèmes continus non linéaires en régime forcé. *Doctor Engineer Thesis*, USTL, Lille, France, 1976.

[2] Benrejeb, M. Sur l'analyse et la synthèse de processus complexes hiérarchisés. Application aux systèmes singulièrement perturbés. *Doctor Science Thesis*, USTL, Lille, France, 1980.

[3] Benrejeb, M., Borne, P. and Laurent, F. Sur une application de la représentation en flèche l'analyse des processus. *RAIRO Automatique* **16**(2) (1982) 133–146.

[4] Benrejeb, M. and Gasmi, M. On the use of an arrow form matrix for modeling and stability analysis of singularly perturbed nonlinear systems. *SAMS* **40** (2001) 209–225.

[5] Cao, S.G., Rees, N.W. and Feng, G. Analysis and design for a class of continuous-time fuzzy control systems. *Int. J. Control* **64** (1996) 1069–1089.

[6] Cao, S.G., Rees, N.W. and Feng, G. Quadratic stability analysis and design of continuous time fuzzy control system. *Int. J. Syst. Science* **27** (1996) 193–203.

[7] Dieulot, J.-Y. Design of stable controllers for Takagi–Sugeno systems with concentric characteristic regions. *Nonlinear Dynamics and Systems Theory* **3**(1) (2003) 65–74.

[8] Gantmacher, F.R. *Théorie des Matrices.* Editions Dunod, Paris, France, 1966.

[9] Gentina, J.C., Borne, P., Burgat, C., Bernussou, J. and Grujic, L.T. Sur la stabilité des systèmes de grande dimension. Normes vectorielles. *RAIRO Aut./Sys. Analysis and Control* **13**(1) (1979) 57–75.

[10] Grujic, L.T., Gentina, J.C. and Borne, P. General aggregation of large scale systems by vector Lyapunov functions and vector norms. *Int. J. of Control* **24**(4) (1976) 529–550.

[11] Grujic, L.T., Gentina, J.C., Borne, P., Burgat, C. and Bernussou, J. Sur la stabilité des systèmes de grande dimension. Fonctions de Lyapunov vectorielles. *RAIRO Aut./Sys. Analysis and Control* **12**(4) (1978) 319–348.

[12] Hahn, W. *Stability of the Motion.* Springer Verlag, Berlin, 1967.

[13] Rehfuess, U. and Palm, R. Design of Takagi-Sugeno controllers based on linear quadratic control. *Proc. 1st Symp. Fuzzy Logic*, Zurich, Switzerland, C10–C15, 1995.

[14] Sugeno, M. On stability of fuzzy systems expressed by fuzzy rules with singleton consequents. *IEEE Trans. Fuzzy Systems* **7**(2) (1999) 201–224.

[15] Sugeno, M. and Kang, G.T. Structure identification of fuzzy model. *Fuzzy Sets and Systems* **28** (1988) 15–33.

[16] Takagi, T. and Sugeno, M. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man Cyber* **15**(1) (1985) 116–132.

[17] Tanaka, K. and Sugeno, M. Stability analysis and design of fuzzy control systems. *Fuzzy Sets and Systems* **48** (1992) 135–156.

[18] Teixeira, M.C.M. and Zak, S.H. Stabilizing controller design for uncertain nonlinear systems using fuzzy models. *IEEE Trans. Fuzzy Systems* **7**(2) (1999) 133–142.

[19] Wang, H.O., Tanaka, K. and Griffin, M.F. Parallel distributed compensation of nonlinear systems by Takagi–Sugeno fuzzy model. *Proc. Int. Joint Conf. 4th FUZZ-IEEE / 2nd IFES*, Yokohama, Japan, (1995) 531–538.

[20] Wang, H.O., Tanaka, K. and Griffin, M.F. An approach to fuzzy control of nonlinear systems: Stability and design issues. *IEEE Trans. Fuzzy Systems* **4** (1996) 14–23.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 381–393

Feedback Stabilization of the Extended Nonholonomic Double Integrator

Fazal-ur-Rehman*

Faculty of Electronic Engineering,
GIK Institute of Engineering Sciences and Technology,
Topi, 23460, Swabi, Pakistan

Received: February 4, 2005; Revised: July 4, 2005

Abstract: This paper presents a simple control strategy of feedback stabilization for the extended nonholonomic double integrator. The strategy presents a time-varying feedback law based on the model reference approach, where the trajectory of the extended system is chosen as the model reference trajectory. The controllers are designed in such a way that after each time period , the trajectory of the nonholonomic double integrator intersects the trajectory of the model reference, which can be made asymptotically stable. The proposed feedback law is as a composition of a standard stabilizing feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop control problem stated for an abstract equation on a Lie group. This approach does not rely on a specific choice of a Lyapunov function, and does not require transformations of the model to chained forms.

Keywords: *Feedback stabilization; systems with drift; nonholonomic systems; controllability; Lie algebra; Lyapunov function.*

Mathematics Subject Classification (2000): 34K10, 34B15, 34K25.

1 Introduction

There has been much interest over the last few years in the problem of stabilization of nonholonomic systems. From practical point of view, nonholonomic systems often arise in the form of robot manipulators, mobile robots, and space and marine robots that are either designed with fewer actuators than degrees of freedom or must be able to function in the presence of actuator failures. From a theoretical stand point, there

*Corresponding author: frahman@giki.edu.pk

is considerable challenge in the synthesis of control laws for the nonholonomic systems since as pointed out in a famous paper of Brockett [6], they cannot be stabilized by continuously differentiable, time invariant, state feedback control laws. To overcome the limitations imposed by the Brockett's result, a number of approaches have been proposed for the stabilization of nonholonomic control systems to equilibrium points, see [11] for a comprehensive survey of the field. Among the proposed solutions are smooth time varying controllers [16, 17, 8, 12, 13, 15, 4], discontinues or piecewise smooth control laws [3, 5, 7, 9, 19], and hybrid controllers [5, 10, 20].

Despite the vast amount of papers published on the stabilization of nonholonomic systems, the majority has concentrated on the kinematics models of mechanical systems controlled directly by velocity inputs. Although in certain circumstances this can be acceptable, many physical systems (where forces and torques are actual inputs) will not perform well if their dynamics are neglected.

As a contribution to overcome this limitation, this paper derives a time-varying control law for the so-called the extended nonholonomic double integrator (ENDI) system. The extended nonholonomic double integrator (ENDI) system can be viewed as an extension of the so-called nonholonomic integrator [6]. Its importance stems from the fact that it captures the dynamics and kinematics of a nonholonomic system with three states and two first-order dynamics control inputs, (e.g., the dynamics of a wheeled robot subject to force and torque inputs).

This article presents a feedback stabilization control strategy based on model reference approach for ENDI. The trajectory of the extended system for ENDI model is chosen as the model reference trajectory. The extended system has equal number of inputs and state variables i.e. $m = n$ therefore can be made asymptotically stable by choosing an arbitrary Lyapunov function. This classical state feedback is then combined with a periodic continuation of a parameterized solution to an open loop steering problem for the comparison of flows of the original and extended systems. In combination with the time invariant state feedback for the extended system, the solution to this open loop problem delivers a time varying control, which provides for periodic intersection of the trajectories of the controlled extended system and the original system. For stabilizing the original system, the extended system trajectory serves as a reference. The time-invariant feedback for the extended system dictates the speed of convergence of the system trajectory to the desired terminal point, the open loop solution serves the averaging purpose in that it ensures that the "average motion" of the original system is that of the controlled extended system. The construction proposed here demonstrates that synthesis of time varying feedback stabilizers for ENDI with two control input can be viewed as a procedure of combining static feedback laws for a Lie bracket extension of the system with a solution of an open loop trajectory interception control problem.

2 The Kinematics Model of the Extended Nonholonomic Double Integrator

In [6], Brockett introduced the nonholonomic integrator system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1 - x_1 u_2,$$

where $(x_1, x_2, x_3)^{\mathrm{T}} \in R^3$ is the state vector and $(u_1, u_2)^{\mathrm{T}} \in R^2$ is a two-dimensional input. This system displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design [3, 10, 14].

The nonholonomic integrator captures (under suitable state and control transformations) the kinematics of a wheeled robot. However, the nonholonomic integrator model fails to capture the case where both the kinematics and dynamics of a wheeled robot must be taken into account. To tackle this realistic case, the nonholonomic integrator model must be extended. It is shown in [2] that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = x_1\dot{x}_2 - x_2\dot{x}_1. \tag{1}$$

By defining the state variables $x = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}} = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)^{\mathrm{T}}$, system (1) becomes as $\dot{x}_1 = x_4$, $\dot{x}_2 = x_5$, $\dot{x}_3 = x_1x_5 - x_2x_4$, $\dot{x}_4 = u_1$, $\dot{x}_5 = u_2$, which can be written in the following standard form:

$$\dot{x} = g_0(x) + g_1(x)u_1 + g_2(x)u_2, \tag{2}$$

where

$$g_0(x) = (x_4,\, x_5,\, x_1x_5 - x_2x_4,\, 0,\, 0)^{\mathrm{T}}, \quad g_1(x) = (0,0,0,1,0)^{\mathrm{T}}, \quad g_2(x) = (0,0,0,0,1)^{\mathrm{T}}.$$

As in [1], the system (2) will be referred to as the extended nonholonomic double integrator (ENDI).

The ENDI system (2) satisfies the following properties:

H1. The vector fields g_0, g_1, g_2 are real analytic and complete and, additionally,

$$g_0(0) = 0.$$

H2. The ENDI system is locally strongly accessible for any $x \in R^5$ as this satisfies the LARC (Lie algebra rank condition) for accessibility (see [18]), namely that $L(g_0, g_1, g_2)$, the Lie algebra of vector fields generated by $g_0(x)$, $g_1(x)$ and $g_2(x)$, spans R^5 at each point $x \in R^5$ that is

$$\mathrm{span}\{g_1, g_2, g_3, g_4, g_5\}(x) = R^5 \quad \text{for all} \;\; x \in R^5, \tag{3}$$

where

$$g_3(x) = [g_0(x),\, g_1(x)] = (1, 0, -x_2, 0, 0)^{\mathrm{T}}, \quad g_4(x) = [g_0(x),\, g_2(x)] = (0, 1, x_1, 0, 0)^{\mathrm{T}},$$
$$g_5(x) = [[g_0(x),\, g_1(x)],\, [g_0(x),\, g_2(x)]] = [g_3(x),\, g_4(x)] = (0, 0, 2, 0, 0)^{\mathrm{T}}.$$

H3. The controllability Lie algebra $L(g_0, g_1, g_2)$ is locally nilpotent i.e. all other Lie brackets which are not involve in accessibility rank condition are zero when evaluated at zero.

3 The Control Problem

(SP) Given a desired set point $x_{\mathrm{des}} \in R^5$, construct a feedback strategy in terms of the controls $u_i\colon R^5 \to R$, $i = 1, 2$, such that the desired set point x_{des} is an attractive set for (2), so that there exists an $\varepsilon > 0$, such that $x(t; t_0, x_0) \to x_{\mathrm{des}}$, as $t \to \infty$ for any initial condition $(t_0, x_0) \in R^+ \times B(x_{\mathrm{des}}; \varepsilon)$.

Without the loss of generality, it is assumed that $x_{\text{des}} = 0$, which can be achieved by a suitable translation of the coordinate system.

4 Basic Approach of Designing Stabilizing Control Law for ENDI

4.1 Extended system

The construction of the stabilizing feedback, presented in the next section, employs as its base a Lie bracket extension for the original system (2). This extension is a new system whose right hand side is a linear combination of the vector fields, which locally span the state space. The "coefficients" of this linear combination are regarded as "extended" controls. The extended system can be written as:

$$\dot{x} = g_0(x) + g_1(x)\nu_1 + g_2(x)\nu_2 + g_3(x)\nu_3 + g_4(x)\nu_4 + g_5(x)\nu_5. \tag{4}$$

Henceforth, equations (2) and (4) are referred to as the "original system", and the "extended system", respectively. The importance of the extended system for the purpose of control synthesis lies in the fact that, unlike the original system, it permits instantaneous motion in the "missing" Lie bracket directions g_3, g_4 and g_5.

4.2 Stabilization of the extended system

The extended system (4) can be made globally asymptotically stable if we define the following control inputs

$$\begin{aligned} \nu(x) &= (\nu_1(x),\, \nu_2(x),\, \nu_3(x),\, \nu_4(x),\, \nu_5(x))^{\mathrm{T}} \\ &= \{G(x)\}^{-1}(-x - g_0(x)) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -x_2 & x_1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -x_1 - x_4 \\ -x_2 - x_5 \\ -x_3 - x_1x_5 + x_2x_4 \\ -x_4 \\ -x_5 \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \nu(x) &= (\nu_1(x),\, \nu_2(x),\, \nu_3(x),\, \nu_4(x),\, \nu_5(x))^{\mathrm{T}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5x_2 & -0.5x_1 & 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_1 - x_4 \\ -x_2 - x_5 \\ -x_3 - x_1x_5 + x_2x_4 \\ -x_4 \\ -x_5 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_5 \\ -x_1 - x_4 \\ -x_2 - x_5 \\ \nu_5 \end{bmatrix}, \end{aligned} \tag{5}$$

where

$$\begin{gathered} \nu_5 = -\frac{1}{2}x_2(x_1 + x_4) + \frac{1}{2}x_1(x_2 + x_5) - \frac{1}{2}(x_3 + x_1x_3 - x_2x_4) \\ G(x) = (g_1(x,)\, g_2(x,)\, g_3(x,)\, g_4(x,)\, g_5(x,)). \end{gathered}$$

The existence of $\{G(x)\}^{-1}$ is guaranteed by the LARC condition.

Lemma *The extended system (4) can be made asymptotically stable by using the feedback control as given in (5).*

Proof By considering a Lyapunov function $V(x) = \frac{1}{2}\, x^{\mathrm{T}} Q x$, where Q is some symmetric and positive definite matrix, it follows that, along the controlled extended system trajectories,

$$\frac{d}{dt}V(x) = x^{\mathrm{T}} Q(g_0(x) + G(x)\{G(x)\}^{-1}(-x - g_0(*x))) = -x^{\mathrm{T}} Qx = -2V(x) < 0,$$
$$\forall\, x \in R^5 \setminus \{0\}.$$

Confirming the asymptotic stability of (4) with feedback controls (5).

The discretization of the above control in time, with sufficiently high sampling frequency $1/T$, does not prejudice stabilization in that if the feedback control (5) is substituted by the discretized control

$$\nu_i^T(x(t)) \triangleq \nu_i(x(nT)), \quad t \in [nT,\, (n+1)T), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, \dots, 5.$$

This leads to a parameterized extended system

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5, \tag{6}$$

where $a_i = \nu_i^T(x(t))$, $i = 1, 2, \dots, 5$, (which are constant over each interval $[nT$, $(n+1)T)$). For a sufficiently small T, the discretization of the extended controls preserves their stabilizing properties.

4.3 The trajectory interception problem

(TIP) Find control functions $m_i(a, t)$, $i = 1, 2$, in the class of functions which are continuous in $a = (a_1, a_2, a_3, a_4, a_5)$ and piece-wise continuous and locally bounded in t, such that for any initial condition $x(0) = x_0$ the trajectory $x^a(t; x_0, 0)$ of the extended, parameterized system (6) intersects the trajectory $x^m(t; x_0, 0)$ of the system (2) with controls m_i, $i = 1, 2$, i.e. the trajectory of the system

$$\dot{x} = g_0(x) + g_1(x)m_1(a, t) + g_2(x)m_2(a, t) \tag{7}$$

intercept with the trajectory of

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5$$

precisely at time T, so that

$$x^a(T; x_0, 0) = x^m(T; x_0, 0). \tag{8}$$

4.4 The TIP in logarithmic coordinates of flows

To solve the TIP, we employ the formalism of [21] by considering a formal equation for the evolution of flows for the system (6)

$$\dot{U}(t) = U(t)\bigg(\sum_{i=0}^{5} g_i w_i\bigg), \quad w_0 = 1, \quad U(0) = I, \tag{9}$$

and its solution can be expressed locally as

$$U(t) = \prod_{i=0}^{5} e^{\gamma_i(t) g_i}, \tag{10}$$

where the functions $\gamma_i,\ i = 0, 1, \ldots, 5,$ are the logarithmic coordinates for this flow and can be computed approximately as follows.

Equation (10) is first substituted into (9) which yields

$$\begin{aligned} g_0 w_0 + g_1 w_1 + \cdots + g_5 w_5 = \dot{\gamma}_0 g_0 + \dot{\gamma}_1 (e^{\gamma_0 Adg_0}) g_1 + \dot{\gamma}_2 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 + \ldots \\ + \dot{\gamma}_5 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5, \end{aligned} \tag{11}$$

where $(e^{AdX})Y = e^X Y e^{-X}$ and $(AdX)Y = [X,\, Y]$.

Employing the Campbell-Baker-Hausdorff formula

$$(e^{AdX})Y = e^X Y e^{-X} = Y + [X, Y] + [X,\, [X, Y]]/2! + \ldots,$$

and ignoring all other Lie brackets which are not involved in LARC equation (3). This gives

$$\begin{aligned} (e^{\gamma_0 Adg_0}) g_1 = e^{\gamma_0 g_0} g_1 e^{-\gamma_0 g_0} &= g_1 + (\gamma_0/1!)[g_0, g_1] + (\gamma_0^2/2!)[g_0,\, [g_0, g_1]] + \ldots \\ &\approx g_1 + \gamma_0 g_3. \end{aligned} \tag{12}$$

Similarly

$$\begin{aligned} (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 &= e^{\gamma_0 Adg_0} (e^{\gamma_1 Adg_1} g_2) = e^{\gamma_0 Afg_0}(g_2) \\ &\approx g_2 + \gamma_0 [g_0, g_2] = g_2 + \gamma_0 g_4, \end{aligned} \tag{13}$$

$$\begin{aligned} (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2}) g_3 &= e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (e^{\gamma_2 Adg_2} g_3) \\ &\approx e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3 + \gamma_2 [g_2, g_3]) = e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3) \\ &\approx e^{\gamma_0 Adg_0} (g_3) \approx g_3. \end{aligned} \tag{14}$$

In a similar way we can obtain

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3}) g_4 \approx g_4 + \gamma_3 g_5, \tag{15}$$

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5 \approx g_5. \tag{16}$$

Substituting (12) – (15) into equation (11) and comparing the coefficients of g_i, $i = 0, 1, \dots, 5$, yields the following approximate equations for the evolution of the logarithmic coordinates γ_i, $i = 0, 1, \dots, 5$,

$$
\begin{aligned}
\dot\gamma_0 &= 1,\\
\dot\gamma_1 &= w_1,\\
\dot\gamma_2 &= w_2,\\
\dot\gamma_3 &= -\gamma_0 w_1 + w_3,\\
\dot\gamma_4 &= -\gamma_0 w_2 + w_4,\\
\dot\gamma_5 &= \gamma_0\gamma_3 w_2 - \gamma_3 w_4 + w_5 \quad \text{with} \quad \gamma_i(0) = 0, \quad i = 0, 1, \dots, 5.
\end{aligned}
\tag{17}
$$

The TIP problem can thus be recast in the logarithmic coordinates as follows.

[TIP in LC :] On a given time horizon $T > 0$, find control functions $m_i(a, t)$, $i = 1, 2$, in the class of functions which are continuous in $a = [a_1, a_2, a_3, a_4, a_5]$, and piecewise continuous, and locally bounded in t, such that the trajectory $t \mapsto \gamma^a(t)$ of

$$\dot\gamma = M(\gamma)a, \quad \gamma(0) = 0, \tag{18}$$

intersects the trajectory $t \mapsto \gamma^m(t)$ of

$$\dot\gamma = M(\gamma)m(a, t), \quad \gamma(0) = 0, \tag{19}$$

in which $m(a, t) = [m_1(a, t),\ m_2(a, t),\ 0, 0, 0]$ at time T, so that

$$\gamma^a(T) = \gamma^m(T). \tag{20}$$

The TIP in logarithmic coordinates now takes the form of a trajectory interception problem for the following two control systems

$$
\begin{aligned}
\dot\gamma_0 &= 1,\\
\dot\gamma_1 &= w_1,\\
\dot\gamma_2 &= w_2,\\
\dot\gamma_3 &= -\gamma_0 w_1 + w_3,\\
\dot\gamma_4 &= -\gamma_0 w_2 + w_4,\\
\dot\gamma_5 &= \gamma_0\gamma_3 w_2 - \gamma_3 w_4 + w_5 \quad \text{with} \quad \gamma_i(0) = 0, \quad i = 0, 1, \dots, 5.
\end{aligned}
$$

$$
\begin{aligned}
\text{CS1}:\quad & \dot\gamma_0 = 1, & \text{CS2}:\quad & \dot\gamma_0 = 1,\\
& \dot\gamma_1 = m_1, & & \dot\gamma_1 = a_1,\\
& \dot\gamma_2 = m_2, & & \dot\gamma_2 = a_2,\\
& \dot\gamma_3 = -\gamma_0 m_1, & & \dot\gamma_3 = -\gamma_0 a_1 + a_3,\\
& \dot\gamma_4 = -\gamma_0 m_2, & & \dot\gamma_4 = -\gamma_0 a_2 + a_4,\\
& \dot\gamma_5 = \gamma_0\gamma_3 m_2, & & \dot\gamma_5 = \gamma_0\gamma_3 a_2 - \gamma_3 a_4 + a_5
\end{aligned}
\tag{21}
$$

with initial conditions $\gamma_i(0) = 0$, $i = 0, 1, \dots, 5$.

A solution to TIP is calculated by approximating the flow of $\dot x = g_0 + [g_0, g_1]$ by the flow of $\dot y = g_0 + k g_1 \sin\dfrac{2\pi t}{T}$, and the flow of $\dot x = g_0 + [[g_0, g_1], [g_0, g_2]]$ by $\dot y =$

$g_0+kg_1\sin\dfrac{2\pi t}{T}+kg_2\cos\dfrac{2\pi t}{T}$, where k is some constant. Therefore we adopt the following parameterizations of $m_i,\ i=1,2$:

$$m_1=c_1+(c_3+c_5)\sin\frac{2\pi t}{T}\quad\text{and}\quad m_2=c_2+c_4\sin\frac{2\pi t}{T}+c_5\cos\frac{2\pi t}{T},\tag{22}$$

where $c_i,\ i=1,2,\dots,5$, are found as $c_1=a_1,\ c_2=a_2,\ c_3=6.28319\,a_3/T,\ c_4=6.28319\,a_4/T$ and $c_5=6.28319\,a_5/T$, or $c_1=a_1,\ c_2=a_2,\ c_3=ka_3,\ c_4=ka_4$ and $c_5=ka_5$, where $k=6.28319/T$.

The time varying stabilizing controls for model (2), are thus given by

$$\begin{aligned}u_1&=c_1+c_3\sin\frac{2\pi t}{T}+c_5\cos\frac{2\pi t}{T},\\ u_2&=c_2+c_4\sin\frac{2\pi t}{T}+c_5\cos\frac{2\pi t}{T}.\end{aligned}\tag{23}$$

Theorem 4.1 *Suppose that a solution to the TIP problem can be found. Then, there exists an admissible time horizon $T_{\max}$ and a neighborhood of the origin R such that for any $T<T_{\max}$ the time-varying feedback controls given in (23) are asymptotically stabilizing the system (2) with the region of attraction R.*

Proof By considering a trivial Lyapunov function $V(x)=\frac{1}{2}x^{\mathrm T}x,\ x\in\mathbb{R}^5$ it follows that along the controlled system trajectories,

$$\begin{aligned}\frac{d}{dt}V(x)&=x^{\mathrm T}\dot x=x^{\mathrm T}(g_0(x)+g_1(x)u_1+g_2(x)u_2)\\ &=x^{\mathrm T}\Big(g_0(x)+g_1(x)\Big(c_1+c_3\sin\frac{2\pi t}{T}+c_5\cos\frac{2\pi t}{T}\Big)\\ &\qquad+g_2(x)\Big(c_2+c_4\sin\frac{2\pi t}{T}+c_5\cos\frac{2\pi t}{T}\Big)\Big)\\ &=x^{\mathrm T}\Big(g_0(x)+g_1(x)a_1+k_3a_3g_1(x)\sin\frac{2\pi t}{T}+k_5a_5g_1(x)\sin\frac{2\pi t}{T}\\ &\qquad+g_2(x)a_2+k_4a_4g_2(x)\sin\frac{2\pi t}{T}+k_5a_5g_2(x)\cos\frac{2\pi t}{T}\Big)\\ &\approx x^{\mathrm T}(g_0(x)+g_1(x)a_1+g_2(x)a_2+g_3(x)a_3+g_4(x)a_4+g_5(x)a_5)\\ &=x^{\mathrm T}(g_0(x)+G\nu)=-x^{\mathrm T}x<0,\end{aligned}$$

where $G=[g_1\ g_2\ g_3\ g_4\ g_5](x),\ \nu=G^{-1}\{-x-g_0(x)\}$ for all $x\in R^5\setminus\{0\}$.

Confirming the asymptotic stability of (2) with feedback controls (23).

The simulation results employing the above controls are depicted in Figures 4.1 – 4.6. In first simulation we choose $x(0)=[0.9\ \ 0.7\ \ 0.4\ \ 0.8\ \ 0.6]^{\mathrm T}$ and $T=0.9$. The results are shown in Figures 4.1 – 4.4. In 2nd simulation we choose $x(0)=[0.5\ \ 0.5\ \ 0.5\ \ 0.5\ \ 0.5]^{\mathrm T}$ and $T=0.9$. The results are shown in Figures 4.5 – 4.8.

5 Conclusion

A time varying control law is derived for the extended nonholonomic double integrator (ENDI) system that captures any kinematics completely nonholonomic model with three states and two first order dynamic control inputs, e.g., the dynamics of a wheeled robot subject to force and torque inputs. The controller yields asymptotic stability and convergence of the closed loop system to an arbitrarily small neighborhood of the origin. Simulation results captured some of the features of the proposed control laws and their performance.

References

[1] Aguiar, A.P. and Pascoal, A. Practical stabilization of the extended nonholonomic double integrator. *Proc. of MED'02 — 10th Mediterranean Conference on Control and Automation.* Lisbon, Portugal, July 2002.

[2] Aguiar, A.P. and Pascoal, A. Stabilization of the extended nonholonomic double integrator via logic based hybrid control. *SYROCO'OO — 6th International IFAC Symposium on Robot Control* (Vienna, Austria), September 2000.

[3] Astolfi, A. Discontinuous control of the Brockett integrator. *Europ. J. of Control* **4**(1) (1998) 49 – 63.

[4] Behal, A., Dawson, D., Dixon, W. and Fang, Y. Tracking and regulation control of an underactuated surface vessel with nonintegrable dynamics. *IEEE Trans. On Autom. Control* **47**(3) (2002) 495 – 500.

[5] Branicky, M.S. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. on Autom. Control* **43** (1998) 475 – 482.

[6] Brockett, R.W. Asymptotic stability and feedback stabilization. In: *Differential Geometric Control Theory.* (Eds.: R.W. Brockett, R.S. Millman, and H.J. Sussman), Birkhauser, Boston, USA, 1983, P.181–191.

[7] Canudas de Wit, C. and Berghuis, H. Practical stabilization of nonlinear systems in chained form. *Proc. 33rd IEEE Conference on Decision and Control*, (Orlando, Florida, USA), December 1994, P.3475 – 3479.

[8] Godhavn, J.M. and Egeland, O. A Lyapunov approach to exponential stabilization of nonholonomic systems in power form. *IEEE Trans. on Autom. Control* **42** (1997) 1028 – 1032.

[9] Guldner, J. and Utkin, V.I. Stabilization of nonholonomic mobile robots using Lyapunov functions for navigation and sliding mode control. *Proc. of 33-rd Conference on Decision and Control*, Orlando, Florida, USA, December 1994, P.2967 – 2972.

[10] Hespanha, J.P. Stabilization of nonholonomic integrators via logic-based switching. *Proc. 13th World Congress of IFAC.* S. Francisco, CA, USA, vol. E, June 1996, P.476 – 472.

[11] Kolmanovsky, I. and McClamroch, N.H. Developments in nonholonomic control problems. *IEEE Control Systems Magazine* **15** (1995) 20 – 36.

[12] Lucibello, P. and Oriolo, G. Robust stabilization via iterative state steering with application to chained-form systems. *Automatica* **37**(1) (2001) 71– 79.

[13] M'Closkey, R.T. and Murray, R.M. Exponential stabilization of drift less nonlinear control systems using homogeneous feedback. *IEEE Trans. on Autom. Control* **42** (1997) 614 – 628.

[14] Morgansen, K.A. and Brockett, R.W. Nonholonomic control based on approximate inversion. *Proc. of the American Control Conference*, San Diego, California, USA, June 1999, P.3515 – 3519.

[15] Morin, P. and Samson, C. Control of nonlinear chained systems: From the Routh-Hurwitz stability criterion to time-varying exponential stabilizers. *IEEE Trans. on Autom. Control* **45** (2000) 141–146.

[16] Pomet, J.B. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems and Control Letters* **18** (1992) 147–158.
[17] Samson, C. Control of chained systems: Application to path following and time-varying point-stabilization of mobile robots. *IEEE Trans. on Autom. Control* **40** (1995) 64 – 77.
[18] Sussmann, H.J. A general theorem on local controllability. *SIAM J. of Control and Optimiz.* **25**(1) (1987) 158–194.
[19] Vendittelli, M. and Oriolo, G. Stabilization of the general two trailer system. *Proc. 2000 IEEE Int. Conf. Robot. Automat.*, San Francisco, CA, 2000, P.1817 –1823.
[20] Ye, H., Michel, A.N. and Hou, L. Stability theory for hybrid dynamical systems. *IEEE Trans. on Autom. Control* **43** (1998) 461– 474.
[21] Wei, J. and Norman, E. On global representations of the solutions of linear differential equations as a product of exponentials. *Proc. of the Amer. Math. Society*, 1964, P.327 – 334.

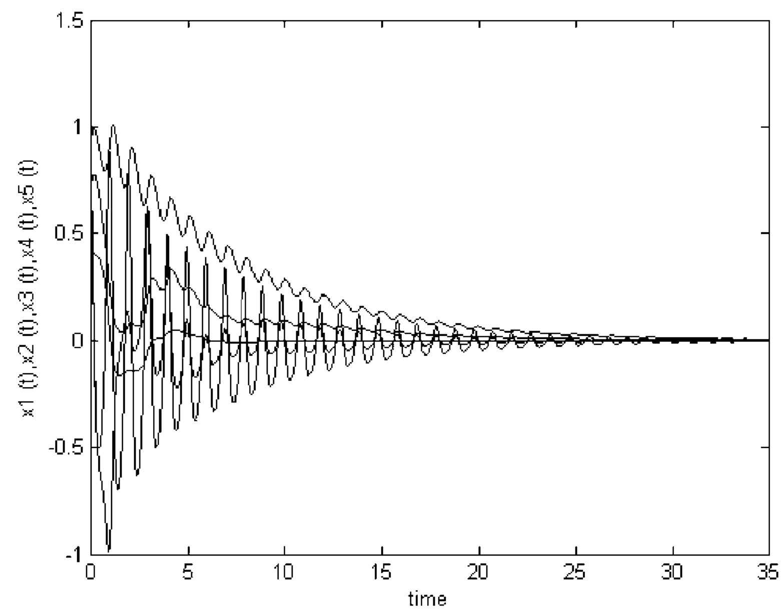

Figure 4.1. Collective Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

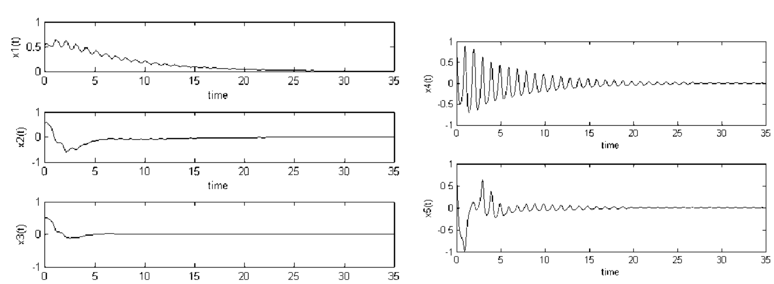

Figure 4.2. Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

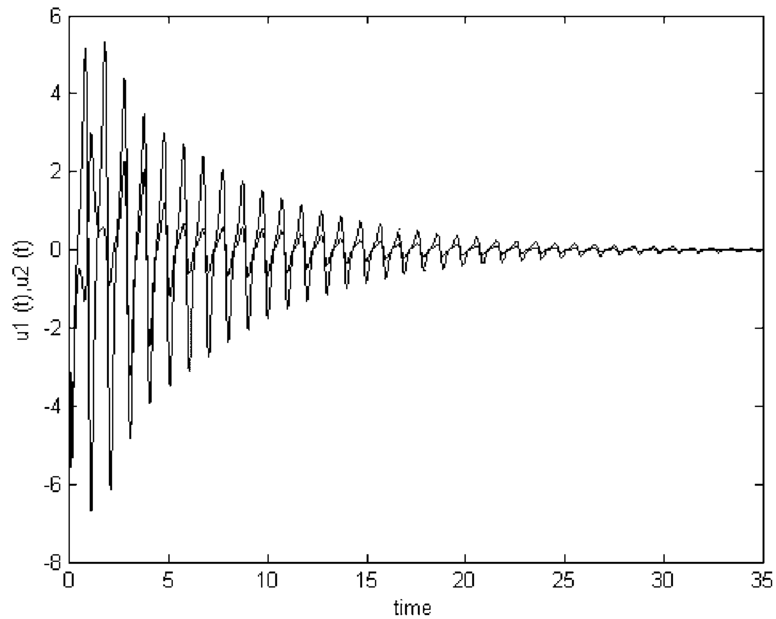

Figure 4.3. Plots of the control input $t \mapsto (u_1(t), u_2(t))$ versus time.

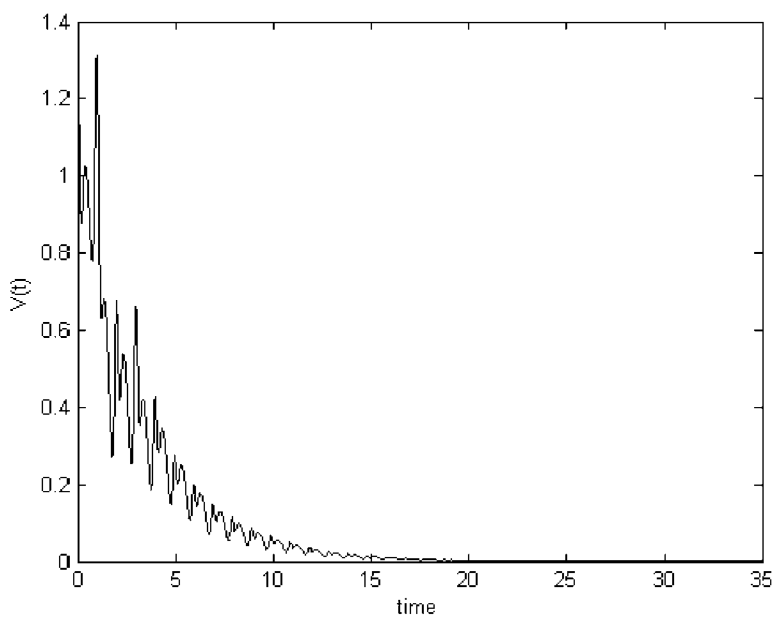

Figure 4.4. Plot of the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories versus time.

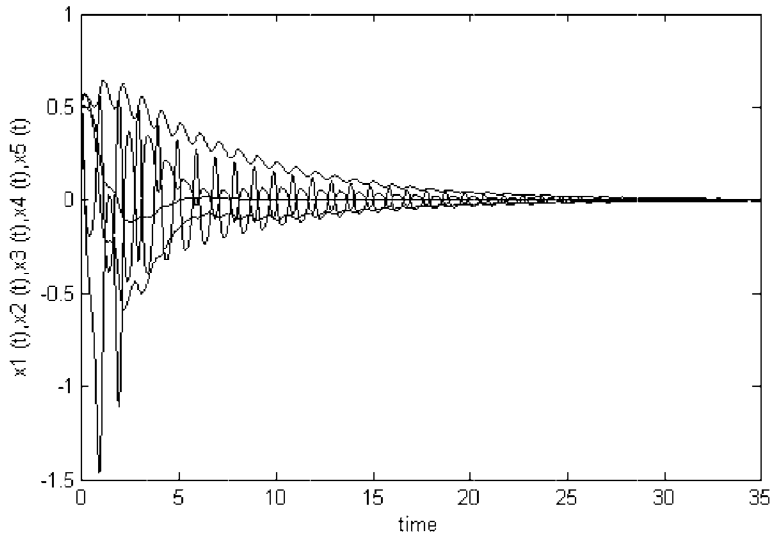

Figure 4.5. Collective plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

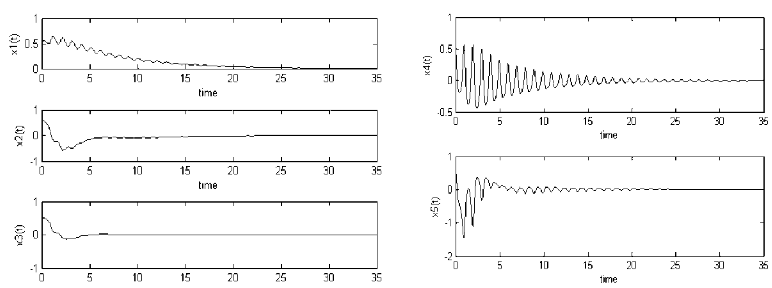

Figure 4.6. Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

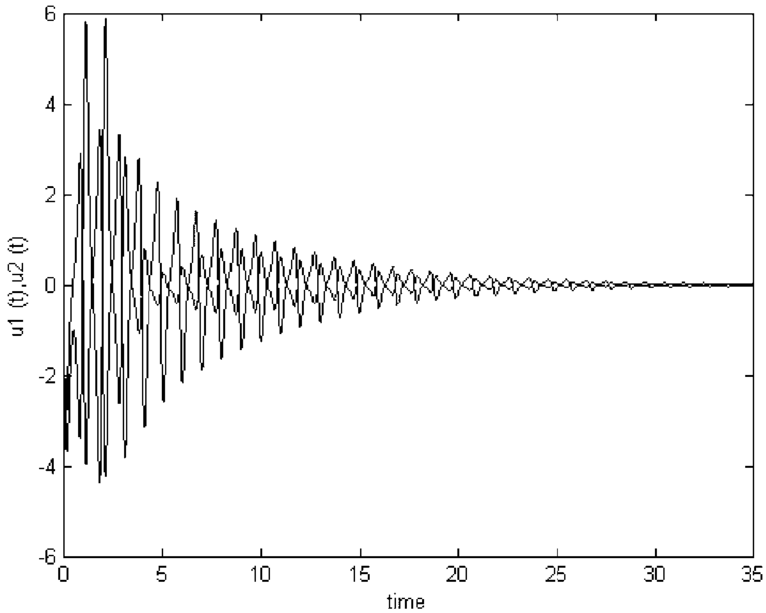

Figure 4.7. Plots of the control input $t \mapsto (u_1(t), u_2(t))$ versus time.

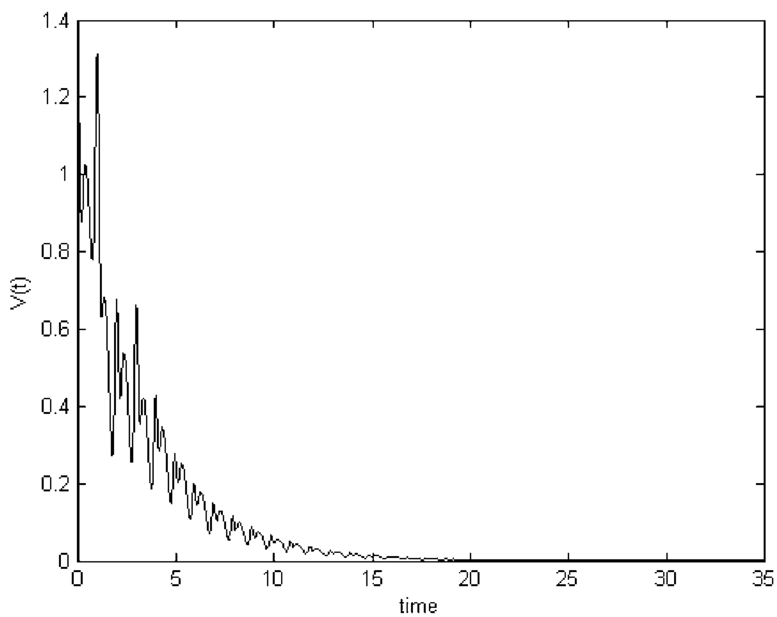

Figure 4.8. Plot of the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories versus time.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 395–405

Periodic Solution of a Convex Subquadratic Hamiltonian System

N. Kallel[1*] and M. Timoumi[2]

[1] *Institut préparatoire aux études d'ingégnieur de Sfax, Département de Mathématiques, BP 805 Sfax 3018, Tunisie*

[2] *Faculté de sciences de Monastir, Département de Mathématiques, CP 5019, Tunisie*

Received: March 1, 2005; Revised: September 10, 2005

Abstract: In this paper we study the periodic solutions of an autonomous Hamiltonian system

$$(\mathcal{H}) \qquad \dot{x} = JH'(x)$$

where H is convex and superquadratic.

We prove by using the Ambrosetti–Rabinowitz theorem and perturbation techniques that for all $T > 0$ the system $(\mathcal{H})$ has a nontrivial T-periodic solution.

Keywords: *Hamiltonian system; periodic solutions; Palais–Smale condition.*

Mathematics Subject Classification (2000): 34C25, 37J45, 70H05.

1 Introduction

In this paper we consider the Hamiltonian system:

$$(\mathcal{H}) \qquad \dot{x} = JH'(x)$$

where $H\colon\ R^{2N} \to R$ is a continuously differentiable function and

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

is the standard symplectic matrix.

In 1979, under the following assumptions:

(1) H is strictly convex,
(2) $\forall x \in R^{2N},\ H(x) \geq H(0) = 0$,
(3) $\exists\, \gamma > 2\colon\ \forall x \in R^{2N},\ H'(x)x \geq \gamma H(x)$,
(4) $\exists\, k > 0\colon\ \forall x \in R^{2N},\ H(x) \leq k|x|^{\gamma}$,

*Corresponding author: najeh.kallel@ipeis.rnu.tn

Ekeland in [1] proved that the system $(\mathcal{H})$ has for any $T>0$ at least one nonconstant T-periodic solution.

In the present paper, we try to find the same result under some more general hypotheses. Precisely, we assume that H satisfies the following hypotheses:

(H_1) H is convex;
(H_2) $\forall\, x\in R^{2N},\ x\neq 0,\ H(x)>H(0)=0$;
(H_3) there exist $\alpha>2$ and $\beta>2$ such that:

$$\forall\,(p,q)\in R^{2N},\quad H(p,q)\leq \frac{1}{\alpha}\frac{\partial H}{\partial p}(p,q)p+\frac{1}{\beta}\frac{\partial H}{\partial q}(p,q)q;$$

(H_4) There exists $l>0$ such that $\forall\,(p,q)\in R^{2N}$

$$|H_p'(p,q)|\leq l\left(1+|p|^{\alpha-1}+|q|^{\beta\frac{(\alpha-1)}{\alpha}}\right),$$
$$|H_q'(p,q)|\leq l\left(1+|q|^{\beta-1}+|p|^{\alpha\frac{(\beta-1)}{\beta}}\right);$$

(H_5) there exist $m>0,\ n>0$ such that $\forall(p,q)\in R^{2N}$

$$|H_p'(p,q)|\geq m|p|^{\alpha-1}-n.$$
$$|H_q'(p,q)|\geq m|q|^{\beta-1}-n.$$

Example 1.1 This is an example of Hamiltonian H which verifies the hypotheses $(H_1)-(H_5)$. Let $G,K\colon\ R^N\to R$ be two functions of class C^1, convex such that:

$$\forall\, x\in R^N,\quad x\neq 0,\quad G(x)>G(0)=0,\quad K(x)>K(0)=0,$$
$$\forall\, x\in R^N,\quad \frac{1}{\alpha}G'(x)x\geq G(x),\quad \frac{1}{\beta}K'(x)x\geq K(x),$$
$$\exists\, a,b>0\colon\ \forall\, x\in R^N,\quad G(x)\leq a|x|^\alpha,\quad K(x)\leq b|x|^\beta.$$

Then the Hamiltonian $H(p,q)=G(p)+K(q)$, verifies the hypotheses $(H_1)-(H_5)$.

Our main result is the following.

Theorem 1.1 *Under the hypotheses $(H_1)-(H_5)$, the system $(\mathcal{H})$ possesses for any $T>0$ a non constant T-periodic solution. Moreover, the energy h verifies the condition:*

$$h\leq\frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2}-\frac{1}{\alpha}\right)\left[\frac{4\pi}{a^2T}\right]^{\frac{\alpha}{\alpha-2}}+\left(\frac{1}{2}-\frac{1}{\beta}\right)\left[\frac{4\pi}{b^2T}\right]^{\frac{\beta}{\beta-2}}\right]+\frac{(\alpha+\beta)a^\alpha}{\alpha(\alpha\beta-\alpha-\beta)}=\bar{h}$$

with $\dfrac{a^\alpha}{\alpha}=\dfrac{b^\beta}{\beta}=\min\{H(p,q),\ |p|^\alpha+|q|^\beta=1\}.$

2 Preliminaries

Definition 2.1 Let E be a Banach space and $f\colon E \to R$ be a function of class C^1. The function f satisfies the Palais–Smale condition (PS) if every sequence (x_n) such that $(f(x_n))$ is bounded and $f'(x_n) \to 0$ as n goes to infinity, possesses a convergent subsequence.

Theorem 2.1 (Ambrosetti–Rabinowitz Theorem) [7] *Let E be a Banach space and $f\colon E \to R$ be a function of class C^1. Assume that:*

(i) *there exists $\alpha > 0$ such that:*

$$m(\alpha) = \inf\{f(x)\colon \|x\| = \alpha\} > f(0),$$

(ii) *there exists $z \in E$ such that $\|z\| \ge \alpha$ and $f(z) \le m(\alpha)$,*

(iii) *f satisfies the Palais–Smale condition (PS).*

Then there exists $\bar{x} \in E$ such that $f'(\bar{x}) = 0$ and $f(\bar{x}) \ge m(\alpha)$. Moreover

$$f(\bar{x}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E)\colon \gamma(0) = 0 \text{ et } \gamma(1) = z\}$.

We have the version of the theorem of Krasnoselskii [5].

Theorem 2.2 *Let Ω be a measurable bounded set of R^n and $f\colon \Omega \times R^N \times R^N \to \overline{R}$ be a function verifying the following condition.*

For almost every $t \in \Omega$, $f(t,\cdot,\cdot)$ is convex, of class C^1, and that for all $(x,y) \in R^N \times R^N$, $f(\cdot,x,y)$ is measurable.

Let $\alpha, \beta > 1$ be two reals, we assume that there exist $\bar{\xi} \in L^\alpha(0,T;R^N)$, $\bar{\mu} \in L^\beta(0,T;R^N)$, $\bar{u} \in L^{\alpha^*}(0,T;R^N)$, $\bar{v} \in L^{\beta^*}(0,T;R^N)$ where $\alpha^{-1} + \alpha*^{-1} = 1$, $\beta^{-1} + \beta*^{-1} = 1$, such that

$$\int_\Omega |f(t,\bar{\xi}(t),\bar{\mu}(t))|\,dt < \infty, \qquad \int_\Omega |f^*(t,\bar{u}(t),\bar{v}(t))\,dt < \infty,$$

and there exists a constant $a > 0$ such that for all $t \in \Omega$ and $(p,q) \in R^{2N}$,

$$\begin{aligned}
\left|\frac{\partial f}{\partial p}(t,p,q)\right| &\le a\max\{1,\, |p|^{\alpha-1},\, |q|^{\beta\frac{(\alpha-1)}{\alpha}}\},\\
\left|\frac{\partial f}{\partial q}(t,p,q)\right| &\le a\max\{1,\, |p|^{\alpha\frac{(\beta-1)}{\beta}},\, |q|^{\beta-1}\},
\end{aligned}$$

so the functional

$$\begin{gathered}
F\colon L^\alpha \times L^\beta \to \overline{R},\\
(p,q) \mapsto \int_\Omega f(t,p(t),q(t))\,dt
\end{gathered}$$

is of class C^1 *and*

$$[F'(p,q)](t) = \frac{\partial f}{\partial(p,q)}(t, p(t), q(t)).$$

3 Proof of Theorem 1.1

We will proceed by successive lemmas.

The hypothesis (H_3) is equivalent to the following

$$(H_6) \qquad \forall\,\lambda \geq 1, \quad \forall\,(p,q) \in R^{2N}, \quad H(\lambda^{1/\alpha}p,\, \lambda^{1/\beta}q) \geq \lambda H(p,q).$$

Let $\epsilon_0 \in \,]0, m[$ be a fixed real. For all $0 < \epsilon \leq \epsilon_0$, we consider the Hamiltonian

$$H_\epsilon(p,q) = H(p,q) + \epsilon(|p|^\alpha + |q|^\beta).$$

It's clear that H_ϵ is strictly convex and verifies $(H_2)-(H_5)$.

Set

$$\frac{a_\epsilon^\alpha}{\alpha} = \frac{b_\epsilon^\beta}{\beta} = \frac{a^\alpha}{\alpha} + \epsilon, \quad l_\epsilon = l + \epsilon, \quad m_\epsilon = m - \epsilon.$$

Lemma 3.1 *Let* α^* *and* β^* *be such that* $\frac{1}{\alpha} + \frac{1}{\alpha^*} = \frac{1}{\beta} + \frac{1}{\beta^*} = 1$, *so*

(7) H_ϵ^* *is of class* C^1;

(8) $\forall (r,s) \in R^{2N}$, $\frac{1}{\alpha^*}(H_\epsilon^*)'_r(r,s)r + \frac{1}{\beta^*}(H_\epsilon^*)'_s(r,s)s \leq {H_\epsilon}^*(r,s)$;

(9) $H_\epsilon(p,q) \geq \frac{{a_\epsilon}^\alpha}{\alpha}|p|^\alpha + \frac{{b_\epsilon}^\beta}{\beta}|q|^\beta$ *for all* $|p|^\alpha + |q|^\beta \geq 1$, $H_\epsilon(p,q) \geq \frac{a_\epsilon^\alpha}{\alpha}(|p|^\alpha + |q|^\beta - 1)$ *for all* $(p,q) \in R^{2N}$;

(10) $\forall\,(r,s) \in R^{2N}$, $H_\epsilon^*(r,s) \leq \frac{a_\epsilon^\alpha}{\alpha} + \frac{1}{\alpha^* a_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* b_\epsilon^{\beta^*}}|s|^{\beta^*}$;

(11) *there exists* $k_\epsilon > 0$ *such that* $\forall\,(p,q) \in R^{2N}$, $H_\epsilon(p,q) \leq k_\epsilon(|p|^\alpha + |q|^\beta)$;

(12) $\forall\,(r,s) \in R^{2N}$ $H_\epsilon^*(r,s) \geq \frac{1}{\alpha^* c_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* d_\epsilon^{\beta^*}}|s|^{\beta^*}$, *where* c_ϵ *and* d_ϵ *are given by* $\frac{c_\epsilon^\alpha}{\alpha} = \frac{d_\epsilon^\beta}{\beta} = k_\epsilon$.

Proof (9) Set $S = \{(p,q) \in R^{2N}\colon\ |p|^\alpha + |q|^\beta = 1\}$. For $(p,q) \in R^{2N}$ such that $|p|^\alpha + |q|^\beta \geq 1$, we set $s = |p|^\alpha + |q|^\beta$, so $(s^{-1/\alpha}p,\, s^{-1/\beta}q) \in S$ and by (H$_6$) we have

$$H_\epsilon(p,q) \geq s \min_{(p,q)\in S}\{H_\epsilon(p,q)\}.$$

For $|p|^\alpha + |q|^\beta \leq 1$, we have $H_\epsilon(p,q) \geq 0 \geq \frac{a_\epsilon^\alpha}{\alpha}(|p|^\alpha + |q|^\beta - 1)$. This is the desired result.

(7) By the inequality (9) we have for $|p|^\alpha + |q|^\beta \geq 1$

$$\frac{H_\epsilon(p,q)}{|p| + |q|} \geq \frac{a_\epsilon^\alpha}{\alpha}\left(\frac{|p|^\alpha}{|p| + |q|} + \frac{|q|^\beta}{|p| + |q|}\right)$$

and since $\alpha, \beta > 1$, so

$$\lim_{|(p,q)|\to\infty} \frac{H_\epsilon(p,q)}{|p| + |q|} = +\infty.$$

Since H_ϵ is strictly convex and of class C^1, so by a result of convex analysis the conjugate H_ϵ^* of H_ϵ is of class C^1.

(8) Let $(r,s) \in R^{2N}$ and $(p,q) = H_\epsilon^{*\prime}(r,s) = (({H_\epsilon}^*)'_r(r,s), ({H_\epsilon}^*)'_s(r,s))$, so by the Fenchel reciprocity and the hypothesis (H_3) we obtain

$$
{H_\epsilon}^*(r,s) \geq \left(1 - \frac{1}{\alpha}\right)({H_\epsilon}^*)'_r(r,s)r + \left(1 - \frac{1}{\beta}\right)({H_\epsilon}^*)'_s(r,s)s,
$$

hence the result.

(10) Let $(r,s) \in R^{2N}$, we have

$$
H_\epsilon^*(r,s) = \sup_{(p,q)\in R^{2N}} \{pr + sq - H_\epsilon(p,q)\},
$$

thus by the inequality (9)

$$
\begin{aligned}
{H_\epsilon}^*(r,s) &\leq \sup_{(p,q)\in R^{2N}} \left\{ pr + sq - \frac{a_\epsilon^\alpha}{\alpha}|p|^\alpha - \frac{b_\epsilon^\beta}{\beta}|q|^\beta + \frac{a_\epsilon^\alpha}{\alpha} \right\} \\
&\leq \frac{1}{\alpha^* a_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* b_\epsilon^{\beta^*}}|s|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}.
\end{aligned}
$$

(11) For $(p,q) \in R^{2N}$ such that $|p|^\alpha + |q|^\beta \geq 1$, there exists $\theta \in]0,1[$ such that

$$
\begin{aligned}
H_\epsilon(p,q) &= \frac{\partial H_\epsilon}{\partial p}(\theta(p,q))p + \frac{\partial H_\epsilon}{\partial q}(\theta(p,q))q \\
&\leq \left|\frac{\partial H_\epsilon}{\partial p}(\theta(p,q))\right| |p| + \left|\frac{\partial H_\epsilon}{\partial q}(\theta(p,q))\right| |q|
\end{aligned}
$$

so by the hypothesis (H_4)

$$
\begin{aligned}
H_\epsilon(p,q) &\leq l_\epsilon \left(|p| + |p|^\alpha + |p||q|^{\beta\frac{(\alpha-1)}{\alpha}} + |q| + |q|^\beta + |q||p|^{\alpha\frac{(\beta-1)}{\beta}} \right) \\
&\leq l_\epsilon \left(|p| + |q| + |p|^\alpha + |q|^\beta + \frac{|p|^\alpha}{\alpha} + \frac{|q|^\beta}{\alpha^*} + \frac{|q|^\beta}{\beta} + \frac{|p|^\alpha}{\beta^*} \right).
\end{aligned}
$$

So there exists $\tilde{k}_\epsilon > 0$ such that

$$
H_\epsilon(p,q) \leq \tilde{k}_\epsilon(|p|^\alpha + |q|^\beta).
$$

For $(p,q) \in R^{2N}$ such that , $s = |p|^\alpha + |q|^\beta \leq 1$, we have by ($\mathrm{H}_6$)

$$
\begin{aligned}
H(p,q) &\leq sH(s^{-1/\alpha},\, s^{-1/\beta}) \\
&\leq s \max_{(p,q)\in S}\{H(p,q)\} \leq k(|p|^\alpha + |q|^\beta),
\end{aligned}
$$

where $k = \max_{(p,q)\in S}\{H(p,q)\}$.

Hence, by picking $k_\epsilon = \max(\tilde{k}_\epsilon,\, k + \epsilon)$, we obtain the result.

(12) Let $(r,s)\in R^{2N}$,

$$\begin{aligned} H_\epsilon^*(r,s) &= \sup_{(p,q)\in R^{2N}}\{pr+sq-H_\epsilon(p,q)\} \\ &\geq \sup_{(p,q)\in R^{2N}}\{pr+sq-k_\epsilon(|p|^\alpha+|q|^\beta)\} \\ &\geq \frac{1}{\alpha^* c_\epsilon^{\alpha^*}}|r|^{\alpha^*}+\frac{1}{\beta^* d_\epsilon^{\beta^*}}|s|^{\beta^*}. \end{aligned}$$

Denote for μ a real ≥ 1

$$L_0^\mu=\left\{p\in L^\mu(0,T;R^N)/\int_0^T p(t)dt=0\right\}. - - - ???$$

We define on $L_0^{\alpha^*}\times L_0^{\beta^*}$ the dual action functional f_ϵ by

$$f_\epsilon(p,q)=\frac{1}{2}\int_0^T\langle J(p,q),\pi(p,q)\rangle dt+\int_0^T H_\epsilon^*(p,q)\,dt,$$

where

$$(\pi y)(t)=\int_0^t y(s)\,ds-\frac{1}{T}\int_0^T dt\int_0^t y(s)\,ds$$

is the primitive of y with zero mean.

We are interested in the search of a non trivial critical point of f_ϵ, by using the Ambrosetti–Rabinowitz theorem.

Lemma 3.2 *f_ϵ is of class C^1 and for all $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$, there exists $(\xi_\epsilon,\mu_\epsilon)\in R^N\times R^N$ such that*

$$f_\epsilon'(p,q)=-J\pi(p,q)+H_\epsilon^{*'}(p,q)+(\xi_\epsilon,\mu_\epsilon).$$

The proof is a simple application of the version of the theorem of Krasnoselskii.

Lemma 3.3 *There exist $\rho>0$ and $\gamma>0$ such that*

$$\|(p,q)\|_{L_0^{\alpha^*}\times L_0^{\beta^*}}=\rho\Rightarrow f_\epsilon(p,q)\geq\gamma.$$
$$0<\|(p,q)\|_{L_0^{\alpha^*}\times L_0^{\beta^*}}\leq\rho\Rightarrow f_\epsilon(p,q)>f_\epsilon(0,0)=0$$

Proof It's easy to verify that for all $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$ we have

$$\left|\int_0^T\prec p(t),\pi q(t)\succ dt\right|=\left|-\int_0^T\prec \pi p(t),q(t)\succ dt\right|\leq T^{\frac{1}{\alpha}+\frac{1}{\beta}}(|p|^2_{L^{\alpha^*}}+|q|^2_{L^{\beta^*}}).$$

So, by the inequality (12) for all $\epsilon\in]0,\epsilon_0]$ and $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$,

$$\begin{aligned} f_\epsilon(p,q) &\geq -T^{\frac{1}{\alpha}+\frac{1}{\beta}}(|p|^2_{L^{\alpha^*}}+|q|^2_{L^{\beta^*}})+\frac{1}{\alpha^* c_{\epsilon_0}^{\alpha^*}}|p|^{\alpha^*}_{L^{\alpha^*}}+\frac{1}{\beta^* d_{\epsilon_0}^{\beta^*}}|q|^{\beta^*}_{L^{\beta^*}} \\ &\geq -T^{\frac{1}{\alpha}+\frac{1}{\beta}}|p|^2_{L^{\alpha^*}}+\frac{1}{\alpha^* c_{\epsilon_0}^{\alpha^*}}|p|^{\alpha^*}_{L^{\alpha^*}}-T^{\frac{1}{\alpha}+\frac{1}{\beta}}|q|^2_{L^{\beta^*}}+\frac{1}{\beta^* d_{\epsilon_0}^{\beta^*}}|q|^{\beta^*}_{L^{\beta^*}} \end{aligned}$$

hence, since $\alpha^*<2,\ \beta^*<2,$ the desired result is obtained.

Lemma 3.4 *There exists* $(p_0, q_0) \in (L_0^{\alpha^*} \times L_0^{\beta^*}) \setminus \{(0,0)\}$ *such that* $f_\epsilon(p_0, q_0) = 0$.

Proof Let $Z = (p,q) \in R^{2N}$, setting $\omega(t) = Z \sin\left(-\frac{2\pi}{T}t\right) + JZ\cos\left(\frac{2\pi}{T}t\right)$, we have

$$\forall\, t \in [0,T], \quad |\omega(t)|^2 = |Z|^2 = |p|^2 + |q|^2.$$

Thus

$$\frac{1}{2}\int_0^T \prec -J\pi\omega, \omega \succ dt = -\frac{T}{4\pi}\int_0^T |\omega(t)|^2 dt = -\frac{T^2}{4\pi}|Z|^2.$$

So, it follows by the inequality (10), that for all $s \geq 0$ we have

$$f_\epsilon(s\omega) \leq -\frac{T^2}{4\pi}s^2|Z|^2 + \frac{T}{\alpha^* {a_\epsilon}^{\alpha^*}} s^{\alpha^*}|Z|^{\alpha^*} + \frac{T}{\beta^* b_\epsilon^{\beta^*}} s^{\beta^*}|Z|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}T.$$

Since $\alpha^* < 2$ and $\beta^* < 2$, we obtain the result by applying the Lemma 3.3.

Lemma 3.5 f_ϵ *verifies the Palais–Smale condition.*

Proof Let $(\omega_n)_{n\in N} = ((p_n, q_n))_{n\in N}$ a sequence of $L_0^{\alpha^*} \times L_0^{\beta^*}$ verifying $(f_\epsilon(\omega_n))_n$ is bounded and $f_\epsilon'(\omega_n)$ converges to zero as n goes to infinity. So, there exist two constants A and B such that

$$A \leq -\frac{1}{2}\int_0^T \prec J\pi\omega_n(t), \omega(t) \succ dt + \int_0^T H_\epsilon^*(\omega_n(t))\, dt \leq B, \tag{13}$$

and

$$(-\pi q_n, \pi p_n) + (({H_\epsilon}^*)_p'(\omega_n),\, ({H_\epsilon}^*)_q'(\omega_n)) + (\xi_{\epsilon,n}, \mu_{\epsilon,n}) = (\lambda_n, \eta_n) \tag{14}$$

converges to zero in $L_0^{\alpha^*} \times L_0^{\beta^*}$ as n goes to infinity.

By taking πp_n and πq_n from the expression (14) and substituting it into (13), we obtain:

$$\left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)\int_0^T H_\epsilon^*(\omega_n(t))\, dt + \frac{1}{\beta^*}\int_0^T [\prec \eta_n, q_n \succ - \prec \mu_{\epsilon,n}, q_n \succ]\, dt$$
$$-\frac{1}{\beta^*}\int_0^T \prec ({H_\epsilon}^*)_q'(\omega_n), q_n \succ dt + \frac{1}{\alpha^*}\int_0^T [\prec \lambda_n, p_n \succ - \prec \xi_{\epsilon,n}, p_n \succ]\, dt$$
$$-\frac{1}{\alpha^*}\int_0^T \prec ({H_\epsilon}^*)_p'(\omega_n), p_n \succ dt \leq \left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)B,$$

thus

$$\left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)\int_0^T H_\epsilon^*(\omega_n(t))\, dt - \int_0^T \left[\frac{1}{\alpha^*} \prec ({H_\epsilon}^*)_p'(\omega_n), p_n \succ + \frac{1}{\beta^*} \prec ({H_\epsilon}^*)_q'(\omega_n), q_n \succ\right] dt$$
$$+\frac{1}{\alpha^*}\int_0^T \prec \lambda_n, p_n \succ dt + \frac{1}{\beta^*}\int_0^T \prec \eta_n, q_n \succ dt \leq \left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)B.$$

We deduce by the inequality (8) that:

$$\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1\right)\int_0^T H_\epsilon^*(\omega_n(t))\,dt-\frac{1}{\alpha^*}|\lambda_n|_{L^\alpha}|p_n|_{L^{\alpha^*}}-\frac{1}{\beta^*}|\eta_n|_{L^\beta}|q_n|_{L^{\beta^*}} \le \left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}\right)B.$$

Hence, since $\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1>0$ and by the inequality (12), we have

$$\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1\right)\left[\frac{1}{\alpha^* c_\epsilon^*}|p|_{L^{\alpha^*}}^{\alpha^*}+\frac{1}{\beta^* d_\epsilon^{\beta^*}}|q|_{L^{\beta^*}}^{\beta^*}\right]-\frac{1}{\alpha^*}|\lambda_n|_{L^\alpha}|p|_{L^{\alpha^*}}-\frac{1}{\beta^*}|\eta_n|_{L^\beta}|q_n|_{L^{\beta^*}} \le \left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}\right)B.$$

Since $\alpha^*,\beta^*<2$ and $|\lambda_n|_{L^\alpha}\to 0,\ |\eta_n|_{L^\beta}\to 0$ as $n\to\infty$, we deduce that there exists a constant $d>0$ such that for all $n\in N$ $|p_n|_{L^{\alpha^*}},|q_n|_{L^{\beta^*}}\le d$ and up to a subsequence, we may assume that (p_n,q_n) is weakly convergent to $\omega=(p,q)$ in $L_0^{\alpha^*}\times L_0^{\beta^*}$.

Consider the set

$$D=\{-J\pi(p_n,q_n),\ n\in N\}\subset C([0,T],\,R^{2N}).$$

By (H_5), we verify that $({H_\epsilon}^{*'}(p_n,q_n))$ is bounded in $L_0^\alpha\times L_0^\beta$ and since (λ_n,η_n) goes to zero in $L_0^\alpha\times L_0^\beta$ as n goes to infinity, so by the formula (14), $(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})$ is bounded in R^{2N} and therefore we can suppose that $(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})$ converges to (ξ,μ).

Finally, since

$$H_\epsilon^{*'}(p_{n_k},q_{n_k})=(\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k}),$$

we have by the Fenchel reciprocity:

$$(p_{n_k},q_{n_k})=H_\epsilon'((\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})),$$

By (H_4) and the version of the theorem of Krasnoselskii, the map $(u,v)\longmapsto H_\epsilon'(u,v)$ defined on $L_0^\alpha\times L_0^\beta$ into $L^{\alpha^*}\times L^{\beta^*}$ is continuous. Thus the sequence $(p_{n_k},q_{n_k})=H_\epsilon'((\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k}))$ is convergent in $L^{\alpha^*}\times L^{\beta^*}$ and the lemma is proved.

The functional f_ϵ verifies all the hypotheses of the Ambrosetti–Rabinowitz theorem, consequently there exists $\bar{y}_\epsilon=(\bar{p}_\epsilon,\bar{q}_\epsilon)\in L_0^{\alpha^*}\times L_0^{\beta^*}$ such that

$$f_\epsilon'(\bar{y}_\epsilon)=0$$

and

$$f_\epsilon(\bar{y}_\epsilon)\ge\gamma.$$

By the Lemma 3.2, there exists $(\xi_\epsilon,\mu_\epsilon)\in R^{2N}$ such that

$$0=-J\pi(\bar{y}_\epsilon)+H_\epsilon^{*'}(\bar{y}_\epsilon)+(\xi_\epsilon,\mu_\epsilon),$$

which gives by the Fenchel reciprocity

$$\bar{y}_\epsilon = H'_\epsilon(J\pi(\bar{y}_\epsilon) - (\xi_\epsilon, \mu_\epsilon)).$$

Setting $\bar{x}_\epsilon = (\bar{u}_\epsilon, \bar{v}_\epsilon) = J\pi(\bar{y}_\epsilon) - (\xi_\epsilon, \mu_\epsilon)$, we have

$$\dot{\bar{x}}_\epsilon = (\dot{\bar{u}}_\epsilon, \dot{\bar{v}}_\epsilon) = J(\bar{y}_\epsilon) = JH'_\epsilon(\bar{u}_\epsilon, \bar{v}_\epsilon) = JH'_\epsilon(\bar{x}_\epsilon).$$

Thus the Hamiltonian system

$$(\mathcal{H}_\epsilon) \qquad \dot{x} = JH'_\epsilon(x)$$

possesses a T-periodic solution.

Lemma 3.6 *Let h_ϵ be the energy of the found solution $\bar{x}_\epsilon$. Then*

$$h_\epsilon \le \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2}-\frac{1}{\alpha}\right)\left[\frac{4\pi}{Ta_\epsilon^2}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2}-\frac{1}{\beta}\right)\left[\frac{4\pi}{Tb_\epsilon^2}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\,\frac{a_\epsilon^\alpha}{\alpha}. \quad (15)$$

Proof We have

$$\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) f_\epsilon(\bar{y}_\epsilon) = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left[\int_0^T \frac{1}{2} \prec H'_\epsilon(\bar{x}_\epsilon), \bar{x}_\epsilon \succ dt - \int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt\right]$$

$$= \frac{1}{\alpha}\int_0^T \prec (H_\epsilon)'_u(\bar{x}_\epsilon), \bar{u}_\epsilon \succ dt + \frac{1}{\beta}\int_0^T \prec (H_\epsilon)'_v(\bar{x}_\epsilon), \bar{v}_\epsilon \succ dt - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt$$

and by (H_3) we obtain

$$\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) f_\epsilon(\bar{y}_\epsilon) \ge \left(1-\frac{1}{\alpha}-\frac{1}{\beta}\right)\int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt,$$

which implies that

$$f_\epsilon(\bar{y}_\epsilon) \ge \frac{\alpha\beta-\alpha-\beta}{\alpha+\beta}\,h_\epsilon T.$$

On the other hand, by the Ambrosetti–Rabinowitz theorem we have

$$f_\epsilon(\bar{y}_\epsilon) = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} f_\epsilon(\gamma_\epsilon(s)),$$

where $\Gamma = \{\gamma \in C([0,T],\, L_0^{\alpha^*} \times L_0^{\beta^*})/\gamma(0) = 0 \text{ and } \gamma(1) = (p_0, q_0)\}$.

For $s \in R^+$, we set $\omega_s(t) = s\omega(t)$ where ω is defined in the proof of Lemma 3.4. We have

$$\begin{aligned}
f_\epsilon(\bar{y}_\epsilon) &\le \sup_{s\in[0,1]} f_\epsilon(s(p_0,q_0)) \le \sup_{s\ge 0} f_\epsilon(s\omega)\\
&\le \sup_{s\ge 0}\left\{-\frac{T^2}{4\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}}s^{\alpha^*}|Z|^{\alpha^*} + \frac{T}{\beta^* b_\epsilon^{\beta^*}}s^{\beta^*}|Z|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}T\right\}\\
&\le \sup_{s\ge 0}\left\{\frac{-T^2}{8\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}}s^{\alpha^*}|Z|^{\alpha^*}\right\} + \frac{a_\epsilon^\alpha}{\alpha}T\\
&\quad + \sup_{s\ge 0}\left\{-\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\beta^* b_\epsilon^{\beta^*}}s^{\beta^*}|Z|^{\beta^*}\right\}.
\end{aligned}$$

Setting

$$\varphi(s) = -\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}} s^{\alpha^*}|Z|^{\alpha^*}, \quad \psi(s) = -\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\beta^* b_\epsilon^{\beta^*}} s^{\beta^*}|Z|^{\beta^*}.$$

So φ attains its maximum at

$$\bar{s} = \left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{1}{2-\alpha^*}} \frac{1}{|Z|},$$

and ψ attains its maximum at

$$\bar{\bar{s}} = \left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{1}{2-\beta^*}} \frac{1}{|Z|}.$$

A simple computation gives

$$\varphi(\bar{s}) = T\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}}$$

and

$$\psi(\bar{\bar{s}}) = T\left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}},$$

so

$$f_\epsilon(\bar{y}_\epsilon) \le T\left[\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{a_\epsilon^\alpha}{\alpha}T.$$

Consequently

$$h_\epsilon \le \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\frac{a_\epsilon^\alpha}{\alpha}$$

and the Lemma 3.6 is proved.

Lemma 3.7 *The set* $E = \{\bar{x}_\epsilon\colon\ 0 < \epsilon \le \epsilon_0\}$ *is relatively compact in* $C([0,T], R^{2N})$.

Proof We have for all $\epsilon \in]0, \epsilon_0]$,

$$0 < \frac{a^\alpha}{\alpha} < \frac{a_\epsilon^\alpha}{\alpha} < \frac{a^\alpha}{\alpha} + \epsilon_0.$$

Thus, by (15), there exists $R \in R_+^*$ such that

$$H(\bar{x}_\epsilon(t)) \le R$$

for all $t \in [0,T]$ and $\epsilon \in]0, \epsilon_0]$.

Since $\lim\limits_{|x|\to\infty} H(x) = +\infty$, so there exists $\lambda \in R_+^*$ such that for all $t \in [0,T]$ and $\epsilon \in]0, \epsilon_0]$ $\bar{x}_\epsilon(t) \in B(0,\lambda)$. Consequently, for all $t \in [0,T]$, the set $E(t)$ is relatively compact in R^{2N}.

On the other hand, since H' is continuous, there exists $\eta > 0$ independent of ϵ such that for all $\epsilon \in]0, \epsilon_0]$ and $t, t' \in [0, T]$, $\|\bar{x}_\epsilon(t) - \bar{x}_\epsilon(t')\| \leq \eta |t - t'|^{1/2}$. Thus E is equicontinuous. Hence, by the theorem of Ascoli, E is relatively compact in $C([0, T], R^{2N})$.

So, we may extract from E a subsequence $(\bar{x}_{\epsilon_n})$, $\epsilon_n \to 0$, which is convergent uniformly in $[0, T]$. Let $\bar{x} = (\bar{u}, \bar{v})$ be its limit; we have

$$\begin{aligned}\dot{\bar{x}}_{\epsilon_n} = JH'_{\epsilon_n}(\bar{x}_{\epsilon_n}) &= J(H'(\bar{x}_{\epsilon_n}) + \epsilon_n(\alpha|\bar{u}_{\epsilon_n}|^{\alpha-2}\bar{u}_{\epsilon_n},\, \beta|\bar{v}_{\epsilon_n}|^{\beta-2}\bar{v}_{\epsilon_n}))\\ &\to JH'(\bar{u}, \bar{v}) \quad \text{uniformly,}\end{aligned}$$

which implies that

$$\dot{\bar{x}} = JH'(\bar{x}).$$

So it's clear that $H^*_{\epsilon_n}(\bar{p}_{\epsilon_n}, \bar{q}_{\epsilon_n})$ is convergent uniformly to $H^*(\dot{\bar{x}})$ and

$$0 < \gamma \leq \lim_{n\to\infty} f_{\epsilon_n}(\bar{y}_{\epsilon_n}) = f(\dot{\bar{x}}).$$

Since $f(0,0) = 0$, so $\dot{\bar{x}} \neq 0$ and $\bar{x}$ is not constant.

Finally, we have $\lim\limits_{n\to\infty} a_{\epsilon_n} = a$ and $\lim\limits_{n\to\infty} b_{\epsilon_n} = b$, thus $\lim\limits_{n\to\infty} \bar{h}_{\epsilon_n} = \bar{h}$ and so $h = H(\bar{x}) \leq \bar{h}$.

References

[1] Aubin, J.P. and Ekelend, I. *Applied Nonlinear Analysis.* Wiley, 1986.

[2] Aze, D. and Blot, J. *Systèmes Hamiltoniens: Leurs Solutions Périodiques.* Cedic, 1982.

[3] Ekeland, I. and Temam, R. *Analyse Convexe et Problèmes Variationnels.* Paris, 1974.

[4] Felmer, P.L. Periodic solutions of super-quadratic Hamiltonian systems. *J. Diff. Eqns* **102** (1993) 188 – 207.

[5] Krasnosel'skii, M.A. *Topological Methods in the Theory of Nonlinear Integral Equations.* Pergamon Press, 1964.

[6] Kunze, M. Periodic solutions of a singular Lagrangian system related to dispersion-managed fiber communication devices. *Nonlinear Dynamics and Systems Theory* **1**(2) (2001) 159–168.

[7] Mawhin, J. and Willem, M. *Critical Point Theory and Hamiltonian System.* Springer–Verlag, 1987.

[8] Rabinowitz, P.H. Minimax methods in critical point theory with application to differential equations. *CBMS Reg. Conf. Ser. in Math.* 65, Amer. Math. Soc., Providence, R.I., 1986.

[9] Timoumi, M. Oscillations de systèmes Hamiltoniens surquadratiques non coercitifs. *Demonstratio Mathématica* **XXVII**(2) (1994) 293 – 300.

Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 407–417

Satellite Maneuvers Using the Hénon's Orbit Transfer Problem: Application to Geostationary Satellites

A.F.B.A. Prado*

Instituto Nacional de Pesquisas Espaciais,
São José dos Campos, 12227-010, Brazil

Received: January 12, 2004; Revised: April 4, 2005

Abstract: The main objective of the present paper is to study minimum fuel maneuvers to change the position of a spacecraft in orbit around the Earth. The control used is a bi-impulsive maneuver, where the first impulse is applied in the initial position of the satellite to send it to a transfer orbit that will cross the desired final position of the spacecraft. Both initial and final position of the satellite belongs to the same Keplerian orbit. The goal is to find the transfer that has the minimum total increment in velocity and that performs the desired maneuver.

Keywords: *Astrodynamics; orbital maneuvers; bi-impulsive control.*

Mathematics Subject Classification (2000): 70M20, 70H12.

1 Introduction

In this paper, the problem of transfer orbits from one body back to the same body (known in the literature as the Henon's problem) is used to study maneuvers that has the goal of changing the position of a satellite, in the sense of sending it to a different point (true anomaly) of the same orbit. The net result is a relocation of the satellite in the same orbit. The problem of transfer orbits from one body back to the same body has been under investigation for a long time. Hénon [6] originally developed a timing condition for orbits that allow a spacecraft to leave a massless body M_2, go in an orbit around the primary M_1 and meet M_2 again, after a certain time. This was treated as the problem of consecutive collision orbits in the restricted three body problem. Several authors then worked on improvements of this problem. Hitzl [7] and Hitzl and Hénon [8, 9] studied stability and critical orbits. Perko [12] derived a proof of existence and a timing condition

*Corresponding author: prado@dem.inpe.br

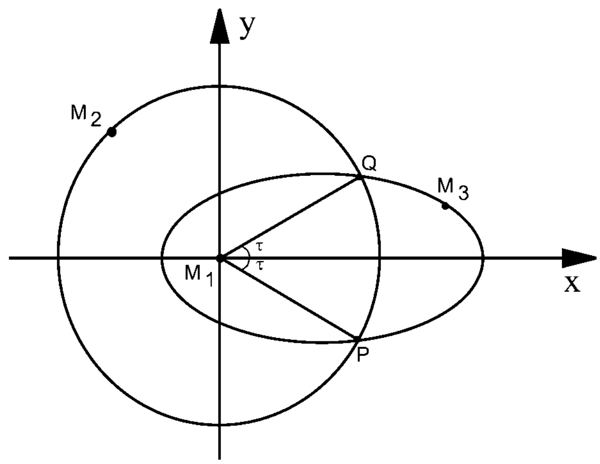

Figure 2.1. Orbit transfer from M_2 back to M_2.

for what was shown later to be a special case of Hénon's work. Results for the perturbed case where μ (mass of M_2 divided by the mass of M_1) is not zero also appeared in the literature. Some examples are the papers published by Gomez and Olle [3, 4] and Bruno [1]. Howell [10] and Howell and Marsh [11] extended Hénon's results for the case where the orbit of M_2 is elliptic.

In the present research this problem is formulated as that of an orbit transfer, as done previously in Prado [13], which can be solved with Gooding's implementation of the Lambert's problem [5]. In the approach used here, the second body M_2 is a fixed point in the orbit of the spacecraft and not a real body, but this nomenclature is used to facilitate the comparison with the results obtained from the consecutive collision orbits problem approach. Both cases, with the circular or elliptic orbits for the spacecraft are considered in the present research. The implementation developed here is generic with respect to the angle that the spacecraft has to be shifted. These transfer orbits are studied in terms of the ΔV and the time required for the transfer. The ΔVs are plotted against the transfer time for several cases and a family of transfer orbits with very small ΔV (on the order of 0.001 in canonical units, a system of units where the gravitational constant of M_1, the angular velocity of the spacecraft and the distance between M_1 and the spacecraft are all unity) is shown to exist in almost all cases studied. These orbits are studied in detail. They consist of a family of slightly different orbits (when compared to the orbit of M_2) that meet all the requirements to provide the transfer desired. A relocation of a geostationary satellite is shown as an example of a practical application of this theory.

2 Formulation of the Problem

Let M_1 be the main body of the system (the Earth, in the example used here) and M_2 be a fixed point in a circular or elliptic orbit around M_1. The massless spacecraft M_3 leaves the point M_2 from a position denoted by P $(t = -\tau)$, follows an orbit around M_1 and meets again with M_2 at a point Q $(t = \tau)$. The basic equations of the Kepler problem apply. The canonical system of units is used. Figure 2.1 shows a sketch of the transfer.

The solution to be found is the coordinate of the point P as a function of the transfer time. The solution is not unique, and a graph including many solutions was published by Hénon [6]. He plotted $\underline{\eta}/\pi$ (where $\underline{\eta}$ is the redefined "eccentric anomaly" of the point P) against τ/π (where τ is half of the transfer time). Another problem that is considered in the present research is the calculation of the ΔV and the time required for each of these transfers, in a search for transfer orbits with small ΔV. The solution consists of plots of the ΔV against the time required for the transfer (both in canonical units). A detailed study of the transfer orbits with small ΔV is included.

2.1 Lambert's problem formulation

A different approach used in the present research formulates Hénon's problem as a Lambert's problem. The Lambert's problem can be defined as [5]:

"An (unperturbed) orbit, about a given inverse-square-law center of force is to be found connecting two given points, P and Q, with a flight time Δt $(= t_2 - t_1)$ that has been specified. The problem must always have at least one solution and the actual number, which is denoted by N, depends on the geometry of the problem — it is assumed, for convenience and with no loss of generality, that t is positive."

Using this formulation, Hénon's problem can be defined in the following way: "Find an unperturbed orbit for M_3, around M_1, which leaves the point P at $t = -\tau$ and goes to point Q at $t = \tau$". Since M_2 is assumed to have zero mass, it has no participation in the equations of motion of the system. Its only use is to relate the time τ with the eccentric anomaly η, in such a way that M_3 has the same position as M_2 at P and Q at the times $t = -\tau$ and $t = \tau$, respectively.

3 Mathematical Formulation

In terms of mathematical formulation, Hénon's problem formulated as a Lambert's problem can be described as follows. The following information is available:

1. The position of M_3 at $t = -\tau$ (point P). It can be specified by the radius vector R_1 and the angle $-\tau$. R_1 can be related to $-\tau$ by using the equation $R_1 = a(1-\mathrm{e}^2)/(1+\mathrm{e}\cos(-\tau))$ for the orbit of M_2, since M_2 and M_3 occupy the same position at $t = -\tau$.

2. The position of M_3 at $t = \tau$ (point Q). It can be specified by the radius vector R_2 and the angle τ. R_2 can be related to τ by using the same equation used in the above paragraph.

3. The total time for the transfer, $\Delta t = 2\tau$. Remember that the angular velocity of the system is unity, so τ can be considered to be the time as well as the angle.

4. The total angle the spacecraft must travel to go from P to Q, that is called ϕ. For the case where the orbit of M_3 is elliptic this variable has several possible values. First of all, there are two possible choices for the transfer: the one that uses the direction of the shortest possible angle between P and Q (that is called the "short way"), and the one that uses the direction of the longest possible angle between these two points (that is called the "long way"). Which one is the shortest or the longest depends on the value of τ. After considering these two choices, it is also necessary to consider the possibilities of multi-revolution transfers. In this case, the spacecraft leaves P, makes one or more complete revolutions around M_1, and then goes to Q. Then, by combining

these two factors, the possible values for ϕ are: $2\tau + 2m\pi$ and $2(\pi - \tau) + 2m\pi$, where m is an integer that represents the number of complete revolutions during the transfer. There is no upper limit for m, and this problem has an infinite number of solutions. In the case where the orbit of M_3 is parabolic or hyperbolic, ϕ has a unique value. The multi-revolution transfer does not exist anymore (the orbit is not closed), and the only direction of transfer that has a solution is the one that makes the spacecraft goes in a retrograde orbit passing by periapse at $t = 0$.

The information needed (the solution of the Lambert's problem) is the Keplerian orbit that contains the points P and Q and requires the given transfer time $\Delta t = 2\pi$ for a spacecraft to travel between these two points. This solution can be specified in several ways. The velocity vectors at P or Q are two possible choices, since the corresponding position vectors are available. The Keplerian elements of the transfer orbit is also another possible set of coordinates to express the solution of this problem. In the implementation developed here, all three sets of coordinates are obtained, since all of them are useful later.

To obtain the ΔVs, the following steps are taken:

1. Find the radial and transverse velocity components of M_2 at P and Q. They are also the velocity components of M_3 just before the first impulse and just after the second impulse, respectively, since they match their orbits at these points. They are obtained from the equations [2]:

$$V_r = \frac{\mathrm{e}\sin(\nu)}{\sqrt{a(1-\mathrm{e}^2)}}, \tag{1}$$

$$V_t = \frac{1+\mathrm{e}\cos(\nu)}{\sqrt{a(1-\mathrm{e}^2)}}, \tag{2}$$

where V_r and V_t are the radial and transverse components of the velocity vector, a and e are the semi-major axis and the eccentricity of the transfer orbit and ν is the true anomaly of the spacecraft.

2. Find an unperturbed orbit for M_3 that allows it to leave the point P at $t = -\tau$ and arrive at point Q at $t = \tau$. This orbit is found by solving the associate Lambert's problem, as explained in the next section. At this point the total time for this transfer, 2τ is already known.

3. Find the velocity components at these points (P and Q) in the transfer orbit determined above. They are the velocity components for M_3 just after the first impulse and just before the second impulse. They are provided by Gooding's Lambert routine [5].

4. With the velocity components just after and just before both impulses it is possible to calculate the magnitude of both impulses (ΔV_1 and ΔV_2) and add them together to get the total impulse required (ΔV) for the transfer.

4 Gooding's Implementation of the Lambert's Problem

The solution of the Lambert's problem, as defined in the previous paragraphs, has been under investigation for a long time. The approach to solve this problem is to set up a set of non-linear equations (from the two-body problem) and start an iterative process to find an orbit that satisfies all the requirements. There is no closed-form solution available for this problem. The major difficulty is to choose the best set of equations and parameters for iterations to guarantee that convergence occurs in all cases. The routine used in this research is due to Gooding [5]. He chooses $\pm\sqrt{1-s/2a}$ as the parameter for convergence, where a is the semi-major axis of the transfer orbit and s the semi-perimeter of the triangle formed by P, Q and M_1. He also makes several substitutions of variables, trying to find the best set of equations to guarantee convergence in all cases. His implementation is able to find all the possible solutions of the Lambert's problem, including "long way", "short way" and "multi-revolution" transfers. He gives the velocity vectors at P and Q and the Keplerian elements of the transfer orbit in his solution.

Including all phases of the present research, Gooding's routine has been called about 3 million times with no failure detected.

5 Results

In this section some results are shown in the problem of finding the ΔVs required for the transfers to be able to get the transfers with the minimum consumption. Plots of $(\Delta V) \times (\tau/\pi)$ were made for thousands of possible transfer orbits. Five orbits for M_2 around M_1 are used:

(1) The circular orbit with $a = 1$.
(2) The elliptic orbit with e $= 0.4$ and $a = 1$, with M_2 passing by periapse at $t = 0$.
(3) The elliptic orbit with e $= 0.4$ and $a = 1$, with M_2 passing by apoapse at $t = 0$.
(4) The elliptic orbit with e $= 0.97$ and $a = 1$, with M_2 passing by periapse at $t = 0$.
(5) The elliptic orbit with e $= 0.97$ and $a = 1$, with M_2 passing by apoapse at $t = 0$.

The results for orbits 1, 2 and 4 are shown in Figures 5.1 – 5.3. The vertical axis shows the total ΔV in canonical units and the horizontal axis shows τ/π, where τ is half of the transfer time. Only elliptic transfer orbits are included in these plots, since the hyperbolic or parabolic transfer orbits are too expensive, in terms of ΔV (always more than 1.6), to be useful. In these figures, τ/π varies from 0 to 14 and the maximum number of complete revolutions allowed for M_3, while in its transfer orbit, is also 14. This means that we restrict ourselves to the orbits contained in a square region with side 14 ($0 \le \tau/\pi \le 14$ and $0 \le \nu/\pi \le 14$).

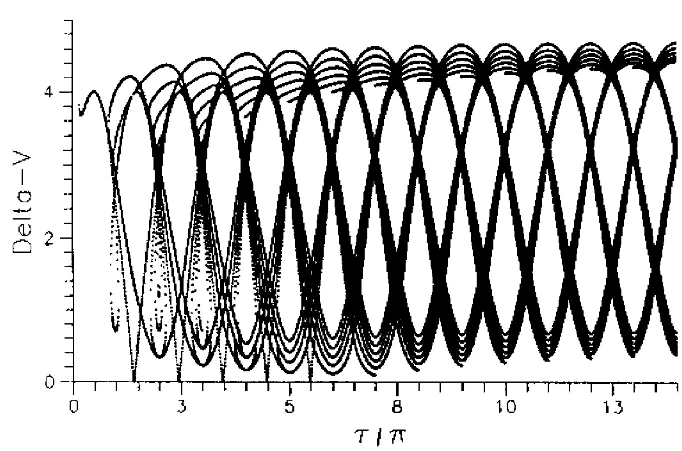

Figure 5.1. (ΔV) *vs* (τ/π) for Orbit 1 for M_2.

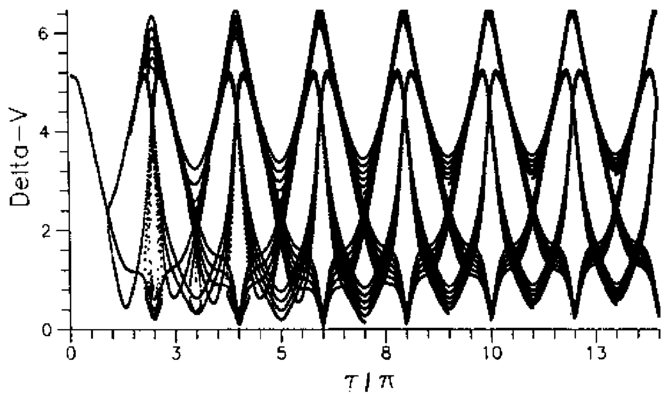

Figure 5.2. (ΔV) *vs* (τ/π) for Orbit 2 for M_2.

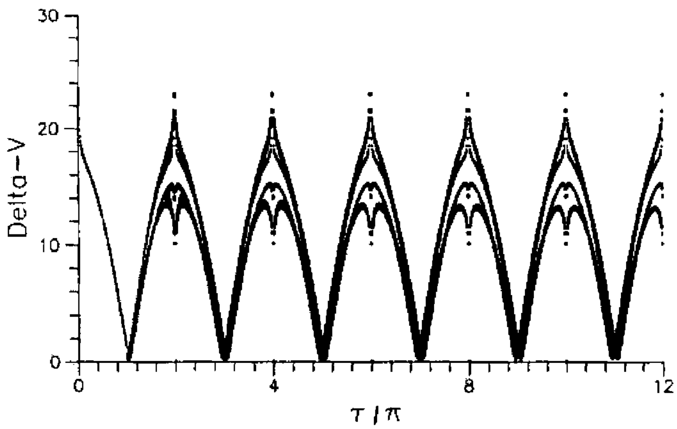

Figure 5.3. (ΔV) *vs* (τ/π) for Orbit 4 for M_2.

An examination of those figures shows the existence of points (orbits) with very small ΔV. They appear in several locations in the plot and they reveal a whole family of small ΔV transfer orbits. In all cases studied in this research, this family appears in the "short transfer time" part of the graph (small τ). A more detailed plot of $(\Delta V) vs (\tau/\pi)$ is shown in Figure 5.4. It includes only the orbits where $\Delta V \leq 0.5$ and it is restricted to orbit 1 (circular orbit) only. Plots for the orbits 3 and 5 are similar to the plots for orbits 2 and 4, respectively, and are omitted in the present text to save space. It is possible to see that the local minimums increase with time after $\tau/\pi = 6$. An investigation for τ/π varying from zero to 200 (and with the maximum number of complete revolutions for M_3 equal to 200) was done, and no more orbits with $\Delta V \leq 0.1$ were found.

Table 5.1 shows the main characteristics of the orbits with $\Delta V \leq 0.1$ found in the circular and elliptic cases. It is interesting to see that for the circular case (see the part e $= 0$ in Table 5.1) most of the orbits appear in pairs, with almost identical values of τ/π. A good example is the pair formed by the first two orbits in Table 5.1: $\tau/\pi = 1.400$ and $\tau/\pi = 1.410$. In each pair one orbit has the periapse in a positive abscissa and the other one has the periapse in a negative abscissa. In this Table the orbit of M_2 is assumed to

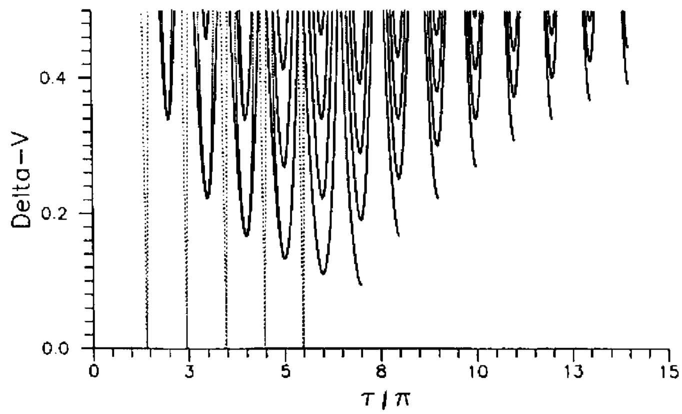

Figure 5.4. (ΔV) *vs* (τ/π) for $\Delta V \leq 0.5$ (Orbit 1 for M_2).

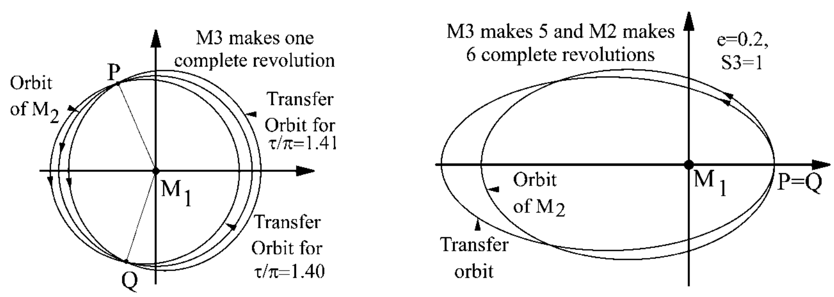

Figure 5.5. Some transfer orbits with small ΔV.

be elliptic with several values for the eccentricity. Both cases, M_2 at periapse at $t = 0$ and M_2 at apoapse at $t = 0$ are considered. Figure 5.5 shows some of those orbits.

Table 5.1 and Figure 5.5 show the mechanism of the majority of these transfer orbits. They consist of orbits with slightly different semi-major axis and eccentricity (compared with the orbit of M_2) and they have a periapse coincident with the periapse of the orbit of M_2. They have mean angular velocity (n) such that $2\tau(1-n) = \pm 2\pi$. Then, after M_3 makes m complete revolutions in its transfer orbit, M_2 makes $m+1$ or $m-1$ complete revolutions in its own orbit and they can meet each other at the common periapse, after the time 2τ.

Here τ is half of the transfer time in canonical units, $\underline{\nu}$ is redefined true anomaly, $\underline{\eta}$ is redefined eccentric anomaly, a is semi-major axis of the transfer orbit, e is eccentricity of the transfer orbit, $S3 = 1$ if M_2 is at periapse at $t = 0$ and -1 if it is at apoapse, $L = 1$ for "short way" transfer, 0 for "long way" transfer, $P = 1$ if periapse is in a positive abscissa, 0 if in a negative abscissa, $S = 1$ if transfer is direct, 0 if transfer is retrograde, $A = 1$ if M_3 pass by the periapse at $t = 0$, 0 if it pass by the apoapse, ΔV is Velocity increment in meters/second.

Table 5.1. Transfer orbits with $\Delta V \leq 0.1$ for the circular and elliptic case.

	τ/π	a	e	η/π	ν/π	L	P	S	A	ΔV
e=0	1.400	0.993	0.0216	1.406	1.400	1	1	1	1	0.0417
	1.410	1.003	0.0105	1.406	1.410	1	0	1	0	0.0204
	2.440	0.997	0.0167	2.445	2.440	1	0	1	0	0.0331
	2.450	1.002	0.0149	2.445	2.450	1	1	1	1	0.0295
	3.460	0.999	0.0036	3.461	3.460	1	1	1	1	0.0072
	3.470	1.003	0.0279	3.461	3.470	1	0	1	0	0.0555
	4.460	0.997	0.0310	4.469	4.460	1	0	1	0	0.0618
	4.470	1.000	0.0005	4.469	4.470	1	1	1	1	0.0010
	5.470	0.998	0.0169	5.475	5.470	1	1	1	1	0.0336
	5.480	1.001	0.0146	5.475	5.480	1	0	1	0	0.0292
	6.990	1.108	0.9777	5.991	6.990	0	0	1	1	0.0955
e2=0.1 S3=-1	1.410	1.4386	1.0025	0.1085	1.4729	1	0	1	0	0.0453
	2.440	2.4133	0.9979	0.1125	2.3793	1	0	1	0	0.0435
	3.460	3.4930	0.9995	0.0962	3.5238	0	0	1	0	0.0404
	4.470	4.4380	1.0002	0.0975	4.4078	1	0	1	0	0.0398
	5.480	5.5072	1.0011	0.1142	5.5436	0	0	1	0	0.0500
	7.000	6.0000	1.1082	0.1879	6.0000	0	0	1	1	0.0869
e2=0.1 S3=+1	1.400	1.3747	0.9962	0.1132	1.3420	1	1	1	1	0.0411
	2.440	2.4772	0.9970	0.0829	2.5036	0	1	1	1	0.0503
	3.460	3.4293	0.9999	0.1009	3.3982	1	1	1	1	0.0389
	4.470	4.5018	1.0000	0.1003	4.5337	0	1	1	1	0.0402
	5.470	5.4435	0.9989	0.1148	5.4078	1	1	1	1	0.0479
e2=0.2,S3=-1	7.000	6.0000	1.1082	0.2782	6.0000	0	0	1	1	0.0793
e2=0.2, S3=1	6.000	5.0000	1.1292	0.2916	5.0000	1	1	1	0	0.0917
e2=0.5, S3=-1	5.000	4.0000	1.1604	0.5691	4.0000	1	0	1	1	0.0789
e2=0.5 S3=+1	4.000	3.0000	1.2114	0.5873	3.0000	1	1	1	0	0.0993
	4.000	5.0000	0.8618	0.4198	5.0000	1	1	1	0	0.0939
	6.000	5.0000	1.1292	0.5572	5.0000	1	1	1	0	0.0655
e2=0.6 S3=+1	4.000	3.0000	1.2114	0.6698	3.0000	1	1	1	0	0.0863
	4.000	5.0000	0.8618	0.5358	5.0000	1	1	1	0	0.0810
	6.000	5.0000	1.1292	0.6458	5.0000	1	1	1	0	0.0568
e2=0.7 S3=-1	3.000	2.0000	1.3104	0.7711	2.0000	1	0	1	1	0.0985
	3.000	4.0000	0.8255	0.6366	4.0000	1	0	1	1	0.0897
	5.000	4.0000	1.1604	0.7415	4.0000	1	0	1	1	0.0577
	7.000	5.0000	1.2515	0.7603	5.0000	1	0	1	0	0.0837
e2=0.7 S3=+1	4.000	3.0000	1.2114	0.7524	3.0000	1	1	1	0	0.0728
	6.000	4.0000	1.3104	0.7711	4.0000	1	1	1	1	0.0985
	4.000	5.0000	0.8618	0.6519	5.0000	1	1	1	0	0.0679
	6.000	5.0000	1.1292	0.7343	5.0000	1	1	1	0	0.0478

6 Practical Applications

To show one possible practical application for these orbits, this theory is applied in a transfer for a satellite from one point in a circular geostationary orbit to another point in the same orbit (a point 180 degrees ahead of the initial point is used as an example, but the scheme proposed here can be used for any transfer angle desired). This problem is very important nowadays. Its solution can be used to transfer a geosynchronous satellite, to use it above a point with different longitude on Earth. Figure 6.1 shows this situation.

Figure 6.2 shows the $(\Delta V) vs (\tau/\pi)$ for elliptic transfer orbits. Hyperbolic transfer

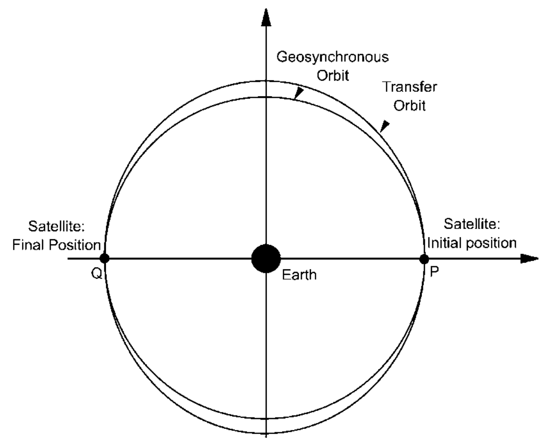

Figure 6.1. Orbit transfer for a geosynchronous satellite.

orbits are also available, but they have ΔV too large to be useful. It is assumed that the change in longitude desired for the satellite is 180 degrees. Table 6.1 shows the whole family of small ΔV orbits. Under the assumption that the orbital velocity of the satellite is 3075 m/s [14] and its orbital period is 1 day, Table 6.1 shows the real values of ΔV and 2τ (total time required for the transfer). The mechanism used by these transfers is to insert M_3 in an elliptic transfer orbit that have a periapse coincident with the periapse of the orbit of M_2. These transfer orbits have a mean angular velocity (n) smaller than 1, such that $(1-n)2\tau = \pi$. Then, in the same time that M_3 makes m revolutions in its transfer orbit, M_2 makes $m + (1/2)$ revolutions in its own orbit and M_3 meets with a point 180 degrees ahead of its initial point at Q.

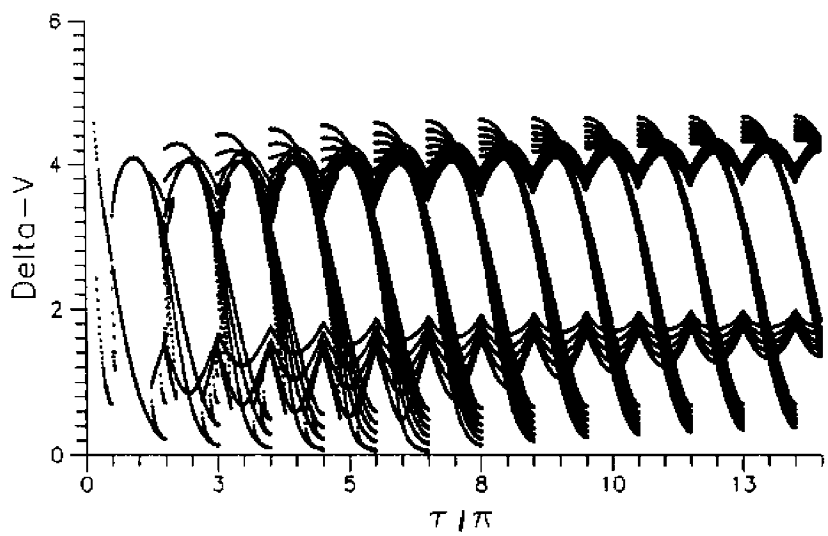

Figure 6.2. (ΔV) *vs* (τ/π) to transfer a geosynchronous satellite (Elliptic Transfer Orbits).

The same comment about other multi-revolution possible transfer orbits with a lower ΔV made in the previous cases are valid here. In this case M_2 does not exist as a real body. It is only a reference point in orbit and, in consequence, its mass is really zero. For this reason, this example fits very well the model used and the results found here are expected to be in close agreement with the real world.

Table 6.1 Transfer orbits with $\Delta V \leq 0.1$ for the transfer in the geosynchronous orbit.

τ/π	$\underline{\eta}/\pi$	a	e	$\underline{\nu}/\pi$	L	P	S	A	ΔV_c	ΔT	ΔV
3.500	3.0000	1.1081	0.0976	3.0000	0	1	1	0	0.095	3.49	292
3.500	4.0000	0.9149	0.0931	4.0000	0	0	1	0	0.095	3.49	292
4.500	4.0000	1.0816	0.0755	4.0000	0	0	1	1	0.074	4.49	228
4.500	5.0000	0.9322	0.0727	5.0000	0	1	1	1	0.074	4.49	228
5.500	5.0000	1.0656	0.0616	5.0000	0	1	1	0	0.061	5.49	188
5.500	6.0000	0.9437	0.0597	6.0000	0	0	1	0	0.061	5.49	188
6.500	6.0000	1.0548	0.0520	6.0000	0	0	1	1	0.051	6.49	157

The symbols are the same ones used in the previous tables.

7 Conclusions

The problem previously called "consecutive collision orbits" in the three-body problem is formulated as a problem of transfer orbits from one body back to the same body. Using this approach, Hénon's problem became a special case of the Lambert's problem.

Gooding's implementation of the Lambert's problem [5] is used to solve this problem with great success.

The ΔVs and the transfer time required for these transfers are calculated. Among a large number of transfer orbits, a small family is found, such that the ΔV required for the transfer is very small. These orbits and their properties are shown in detail.

A practical applications for these orbits are studied in detail: a transfer for a satellite from a point in a circular geosynchronous orbit to another point in this same orbit, 180 degrees ahead of its initial point.

The possibilities of transfers like this one is open for several types of missions and the algorithm developed here can be used to relocate a satellite to a different position in one orbit.

Acknowledgments

The author is grateful to the Foundation to Support Research in the São Paulo State (FAPESP) for the research grant received under Contract 2003/03262-4 and to CNPq (National Council for Scientific and Technological Development) — Brazil for the contract 300221/95-9.

References

[1] Bruno, A.D. On periodic flybys of the Moon. *Celestial Mechanics* **24** (1981) 255–268.

[2] Danby, J.M.A. *Fundamentals of Celestial Mechanics.* Willmann-Bell Inc., Richmond, VA, 1988.

[3] Gomez, G. and Ollé, M. Second species solutions in the circular and elliptic restricted three-body problem I: Existence and asymptotic approximations. *Celestial Mechanics* **52** (1991) 107–146.

[4] Gomez, G. and Ollé, M. Second species solutions in the circular and elliptic restricted three-body problem II: Numerical explorations. *Celestial Mechanics* **52** (1991) 147–166.

[5] Gooding, R.H. A Procedure for the solution of Lambert's orbital boundary-value problem. *Celestial Mechanics* **48** (1990) 145–165.

[6] Hénon, M. Sur les orbites interplanétaires qui rencontrent deux fois la Terre. *Bull. Astron.* **3** (1968) 377–402.

[7] Hitzl, D.L. Generating orbits for stable close encounter periodic solutions of the restricted problem. *AIAA Journal* **15** (1977) 1410–1418.

[8] Hitzl, D.L. and Hénon, M. Critical generating orbits for second species periodic solutions of the restricted problem. *Celestial Mechanics* **15** (1977) 421–452.

[9] Hitzl, D.L. and Hénon, M. The stability of second periodic orbits in the restricted problem ($\mu = 0$). *Acta Astronautica* **4** (1977) 1019–1039.

[10] Howell, K.C. Consecutive collision orbits in the limiting case $\mu = 0$ of the elliptic restricted problem. *Celestial Mechanics* **40** (1987) 393–407.

[11] Howell, K.C. and Marsh, S.M. A general timing condition for consecutive collision orbits in the limiting case $\mu = 0$ of the elliptic restricted problem. *Celestial Mechanics* **52** (1991) 167–194.

[12] Perko, L.M. Periodic orbits in the restricted three-body problem: existence and asymptotic approximation. *Siam J. Appl. Math.* **27** (1974) 200–237.

[13] Prado, A.F.B.A. *Optimal Transfer and Swing-By Orbits in the Two- and Three-Body Problems.* Ph.D. Dissertation, Dept. of Aerospace Engineering and Engineering Mechanics, Univ. of Texas, Austin, TX, 1993.

[14] Wertz, J.R. and Larson, W.J. *Space Mission Analysis and Design.* Kluwer Academic Publishers, Dordrecht, Netherlands, 1991.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Published since 2001

Contents of Volume 5, 2005

Volume 5 Number 1 March 2005

Volume 5 Number 2 June 2005

Volume 5 Number 3 September 2005

Volume 5 Number 4 December 2005