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Generic Well-Posedness of Linear Optimal Control Problems without Convexity Assumptions

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Abstract: The Tonelli existence theorem in the calculus of variations and its subsequent modifications were established for integrands f which satisfy convexity and growth conditions. In our previous work a generic well-posedness result (with respect to variations of the integrand of the integral functional) without the convexity condition was established for a class of optimal control problems satisfying the Cesari growth condition. In this paper we extend this generic well-posedness result to two classes of linear optimal control problems.

Keywords: *Complete metric space; generic property; integrand; linear optimal control problem.*

Mathematics Subject Classification (2000): 49J99, 90C31.

1 Introduction

The Tonelli existence theorem in the calculus of variations [11] and its subsequent generalizations and extensions (e.g. [2, 3, 6, 9, 10]) were established for integrands f which satisfy convexity and growth conditions. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity (see [2, Ch. 16] and [1, 7, 8]).

In [13] it was shown that the convexity condition is not needed generically, and not only for the existence but also for well-posedness of the problem (with respect to some natural topology in the space of integrands). More precisely, in [13] we considered a class of optimal control problems (with the same system of differential equations, the same functional constraints and the same boundary conditions) which is identified with the corresponding complete metric space of cost functions (integrands), say $\mathcal{M}$. We did

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not impose any convexity assumptions. These integrands are only assumed to satisfy the Cesari growth condition. The main result in [13] establishes the existence of an everywhere dense G_δ-set $\mathcal{F} \subset \mathcal{M}$ such that for each integrand in $\mathcal{F}$ the corresponding optimal control problem has a unique solution.

The next steps in this area of research were done in [5, 12, 14]. In [5] we introduced a general variational principle having its prototype in the variational principle of Deville, Godefroy and Zizler [4]. A generic existence result in the calculus of variations without convexity assumptions was then obtained as a realization of this variational principle. It was also shown in [5] that some other generic well-posedness results in optimization theory known in the literature and their modifications are obtained as a realization of this variational principle. Note that the generic existence result in [5] was established for variational problems but not for optimal control problems and that the topologies in the spaces of integrands in [13] and [5] are different.

In [12] we suggested a modification of the variational principle in [5] and applied it to classes of optimal control problems with various topologies in the corresponding spaces of integrands. As a realization of this principle we established a generic existence result for a class of optimal control problems in which the constraint maps are also subject to variations as well as the cost functions [12]. In [14] we applied the variational principle obtained in [12] and established generic well-posedness results for two classes of variational problems in which the values at the end points are also subject to variations as well as the cost functions. In the present paper we establish generic well-posedness results for two classes of linear optimal control problems in which the right-hand side of the governing linear differential equations is also subject to variations.

2 Main Results

In this paper we use the following notations and definitions. Let $k \geq 1$ be an integer. We denote by $\mathrm{mes}(E)$ the Lebesgue measure of a measurable set $E \subset R^k$, by $|\cdot|$ the Euclidean norm in R^k and by $\langle \cdot, \cdot \rangle$ the scalar product in R^k. We use the convention that $\infty - \infty = 0$. For any $f \in C^q(R^k)$ we set

$$\|f\|_{C^q} = \|f\|_{C^q(R^k)} = \sup_{z \in R^k} \{|\partial^{|\alpha|} f(z)/\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}|\colon \\ \alpha_i \geq 0 \text{ is an integer}, \ i = 1, \dots, k, \ \ |\alpha| \leq q\},$$

where $|\alpha| = \sum_{i=1}^k \alpha_i$.

For each function $f\colon Y \to [-\infty, \infty]$, where Y is nonempty, we set $\inf(f) = \inf\{f(y)\colon y \in Y\}$.

In this paper we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide.) We refer to them as the weak and the strong topology, respectively. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i, d_i), $i = 1, 2$, are metric spaces with the metric d_1 and d_2 respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2), \quad (x_i, y_i) \in X \times Y, \quad i = 1, 2.$$

Let $0 \le T_1 < T_2 < \infty$ and let m, n be natural numbers. Denote by X the set of all pairs of functions (x, u), where $x\colon [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function and $u\colon [T_1, T_2] \to R^m$ is a measurable function.

To be more precise, we have to define elements of X as classes of pairs equivalent in the sense that (x_1, u_1) and (x_2, u_2) are equivalent if and only if $x_2(t) = x_1(t)$ for all $t \in [T_1, T_2]$ and $u_2(t) = u_1(t)$ for almost every $t \in (T_1, T_2)$.

For the set X we consider the metric ρ defined by

$$\rho((x_1, u_1), (x_2, u_2)) = \inf_{\epsilon > 0}\{\mathrm{mes}\{t \in [T_1, T_2]\colon |x_1(t) - x_2(t)| + |u_1(t) - u_2(t)| \ge \epsilon\} \le \epsilon\},$$
$$(x_1, u_1),\ (x_2, u_2) \in X. \tag{2.1}$$

For each $z \in R^n$, each matrix A of dimension of $n \times n$ and each matrix B of dimension $n \times m$ denote by $X(z, A, B)$ the set of all $(x, u) \in X$ such that

$$x(T_1) = z, \tag{2.2}$$
$$x'(t) = Ax(t) + Bu(t), \quad t \in (T_1, T_2) \ \text{(a.e.)}. \tag{2.3}$$

Denote by $\mathcal{M}$ the set of all functions $f\colon (T_1, T_2) \times R^n \times R^m \to R^1$ with the following properties:

(i) f is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of (T_1, T_2) and Borel subsets of $R^n \times R^m$;

(ii) $f(t, \cdot, \cdot)$ is lower semicontinuous for almost every $t \in (T_1, T_2)$;

(iii) for each $\epsilon > 0$ there exists an integrable scalar function $\psi_\epsilon(t) \ge 0$, $t \in (T_1, T_2)$, such that

$$|u| + |x| \le \psi_\epsilon(t) + \epsilon f(t, x, u) \ \text{ for all } \ (t, x, u) \in (T_1, T_2) \times R^n \times R^m;$$

(iv) for each $\epsilon, M > 0$ there exists $\delta > 0$ such that for almost every $t \in (T_1, T_2)$ the inequality $|f(t, x_1, u_1) - f(t, x_2, u_2)| \le \epsilon$ holds for each x_1, $x_2 \in R^n$ and each u_1, $u_2 \in R^m$ satisfying

$$|x_i|, |u_i| \le M, \quad i = 1, 2 \ \text{ and } \ |x_1 - x_2|, |u_1 - u_2| \le \delta;$$

(v) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that for almost every $t \in (T_1, T_2)$ the inequality

$$|f(t, x_1, u) - f(t, x_2, u)| \le \epsilon \max\{|f(t, x_1, u)|, |f(t, x_2, u)|\} + \epsilon$$

is valid for each $x_1, x_2 \in R^n$ and each $u \in R^m$ satisfying

$$|x_1|, |x_2| \le M, \quad |u| \ge \Gamma, \quad |x_1 - x_2| \le \delta;$$

(vi) there is a constant $c_f > 0$ such that $|f(t, 0, 0)| \le c_f$ for almost every $t \in (T_1, T_2)$.

The growth condition used in (iii) was proposed by Cesari [2] and its equivalents and modifications are rather common in the literature. It follows from property (i) that for any $f \in \mathcal{M}$ and any $(x, u) \in X$ the function $f(t, x(t), u(t))$, $t \in (T_1, T_2)$, is measurable. Properties (iv) and (vi) imply that for each $M > 0$ there is $c_M > 0$ such that for almost every $t \in (T_1, T_2)$ the inequality $|f(t, x, u)| \leq c_M$ holds for each $x \in R^n$ and each $u \in R^m$ satisfying $|x|$, $|u| \leq M$.

It is an elementary exercise to show that a function $f = f(t, x, u) \in C^1((T_1, T_2) \times R^n \times R^m)$ belongs to $\mathcal{M}$ if (iii) and (vi) are true and the following conditions hold:

(a) for each $M > 0$

$$\sup\{|\partial f/\partial x(t, x, u)| + |\partial f/\partial u(t, x, u)| \colon t \in (T_1, T_2),\\ x \in R^n,\ u \in R^m \text{ and } |x|, |u| \leq M\} < \infty;$$

(b) there exist an increasing function $\psi \colon [0, \infty) \to [0, \infty)$ and a bounded (on bounded subsets of $[0, \infty)$) function $\psi_0 \colon [0, \infty) \to [0, \infty)$ such that for each $(t, x, u) \in (T_1, T_2) \times R^n \times R^m$,

$$|\partial f/\partial x(t, x, u)| \leq \psi_0(|x|)\psi(|u|)$$

and

$$\psi(|u|) \leq f(t, x, u).$$

Denote by $\mathcal{M}^l$ (respectively $\mathcal{M}^c$) the set of all lower semicontinuous (respectively continuous) functions $f \in \mathcal{M}$. Now we equip the set $\mathcal{M}$ with the strong and weak topologies. For the space $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_{\mathcal{M}}(\epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} \colon |f(t, x, u) - g(t, x, u)| \leq \epsilon,\\ (t, x, u) \in (T_1, T_2) \times R^n \times R^m\}, \tag{2.4}$$

where $\epsilon > 0$. It is easy to see that the uniform space $\mathcal{M}$ with this uniformity is metrizable (by a metric $d_{\mathcal{M}}$) and complete. This uniformity generates in $\mathcal{M}$ the strong topology. Clearly $\mathcal{M}^l$ and $\mathcal{M}^c$ are closed subsets of $\mathcal{M}$ with this topology.

For each $\epsilon > 0$ we set

$$\begin{aligned} E_{\mathcal{M}w}(\epsilon) = \Big\{(f, g) \in \mathcal{M} \times \mathcal{M} \colon \quad &\text{there exists a nonnegative } \phi \in L^1(T_1, T_2) \\ \text{such that } \int_{T_1}^{T_2} \phi(t)\, dt \leq 1, \quad &\text{and for almost every } t \in (T_1, T_2), \\ |f(t, x, u) - g(t, x, u)| < \epsilon + \epsilon &\max\{|f(t, x, u)|, |g(t, x, u)|\} + \epsilon\phi(t) \\ &\text{for each } x \in R^n \text{ and each } u \in R^m\Big\}. \end{aligned} \tag{2.5}$$

From [12, Lemma 1.1] (see also Lemma 4.1 below) it follows that for the set $\mathcal{M}$, there exists a uniformity which is determined by the base $\mathcal{E}_{\mathcal{M}w}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $\mathcal{M}$ the weak topology.

For each $f \in \mathcal{M}$ define $I^{(f)} \colon X \to R^1 \cup \{\infty\}$ by

$$I^{(f)}(x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))\, dt, \quad (x, u) \in X. \tag{2.6}$$

Now we define subspaces of $\mathcal{M}$ which consist of integrands differentiable with respect to the control variable u.

Let $k \geq 1$ be an integer. Denote by $\mathcal{M}_k$ the set of all $f \in \mathcal{M}$ such that for each $(t,x) \in (T_1,T_2) \times R^n$ the function $f(t,x,\cdot) \in C^k(R^m)$. We consider the topological subspace $\mathcal{M}_k \subset \mathcal{M}$ with the relative weak topology. The strong topology on $\mathcal{M}_k$ is induced by the uniformity which is determined by the following base:

$$\begin{aligned} E_{\mathcal{M}k}(\epsilon) = \{(f,g) \in \mathcal{M}_k \times \mathcal{M}_k \colon |f(t,x,u) - g(t,x,u)| \leq \epsilon \\ \text{for all } \ (t,x,u) \in (T_1,T_2) \times R^n \times R^m \ \text{ and} \\ \|f(t,x,\cdot) - g(t,x,\cdot)\|_{C^k(R^m)} \leq \epsilon \quad \text{for all} \quad (t,x) \in (T_1,T_2) \times R^n\}, \end{aligned} \tag{2.7}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k$ with this uniformity is metrizable (by a metric $d_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^l = \mathcal{M}_k \cap \mathcal{M}^l, \quad \mathcal{M}_k^c = \mathcal{M}_k \cap \mathcal{M}^c.$$

Clearly $\mathcal{M}_k^l$ and $\mathcal{M}_k^c$ are closed sets in $\mathcal{M}_k$ with the strong topology.

Finally we define subspaces of $\mathcal{M}$ which consist of integrands differentiable with respect to the state variable x and the control variable u. Denote by $\mathcal{M}_k^*$ the set of all $f\colon (T_1,T_2)\times R^n \times R^m \to R^1$ in $\mathcal{M}$ such that for each $t \in (T_1,T_2)$ the function $f(t,\cdot,\cdot) \in C^k(R^n \times R^m)$. We consider the topological subspace $\mathcal{M}_k^* \subset \mathcal{M}$ with the relative weak topology. The strong topology in $\mathcal{M}_k^*$ is induced by the uniformity which is determined by the following base:

$$\begin{aligned} E^*_{\mathcal{M}k}(\epsilon) = \{(f,g) \in \mathcal{M}_k^* \times \mathcal{M}_k^* \colon |f(t,x,u) - g(t,x,u)| \leq \epsilon \\ \text{for all } \ (t,x,u) \in (T_1,T_2) \times R^n \times R^m \ \text{ and} \\ \|f(t,\cdot,\cdot) - g(t,\cdot,\cdot)\|_{C^k(R^{n+m})} \leq \epsilon \quad \text{for all} \quad t \in (T_1,T_2)\}, \end{aligned} \tag{2.8}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k^*$ with this uniformity is metrizable (by a metric $d^*_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^{*l} = \mathcal{M}_k^* \cap \mathcal{M}^l, \quad \mathcal{M}_k^{*c} = \mathcal{M}_k^* \cap \mathcal{M}^c.$$

Clearly $\mathcal{M}_k^{*l}$ and $\mathcal{M}_k^{*c}$ are closed sets in $\mathcal{M}_k^*$ with the strong topology.

Let $\mathcal{A}_1$ be one of the following spaces:

$$\mathcal{M}, \ \mathcal{M}^l, \ \mathcal{M}^c, \ \mathcal{M}_k, \ \mathcal{M}_k^l, \ \mathcal{M}_k^c, \ \mathcal{M}_k^*, \ \mathcal{M}_k^{*l}, \ \mathcal{M}_k^{*c}.$$

Denote by $\mathcal{A}_{21}$ the set of all matrices A of dimension of $n \times n$. For each $A = (a_{ij})_{i,j=1}^n$ set

$$\|A\| = \max\{|a_{ij}| \colon i,j = 1,\dots,n\}.$$

The space $\mathcal{A}_{21}$ is equipped with the metric d_{21} defined by

$$d_{21}(A,B) = \|A - B\|$$

where $A, B \in \mathcal{A}_{21}$.

Denote by $\mathcal{A}_{22}$ the set of all matrices A of dimension of $n \times m$. For each

$$A = (a_{ij} : \ \ i = 1,\dots,n, \ \ j = 1,\dots,m)$$

set

$$\|A\| = \max\{|a_{ij}|:\ i = 1, \dots, n, \quad j = 1, \dots, m\}.$$

The space $\mathcal{A}_{22}$ is equipped with the metric d_{22} defined by

$$d_{22}(A, B) = \|A - B\|$$

for each $A, B \in \mathcal{A}_{22}$.

Let $\mathcal{A}_{23} = R^n$ be equipped with the metric

$$d_{23}(x, y) = |x - y|, \quad x, y \in R^n.$$

Let $z \in R^n$, $\mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22}$ and let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

For each $a_2 = (A, B) \in \mathcal{A}_2$ set

$$S_{a_2} = X(z, A, B).$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $J_a : X \to R^1 \cup \{\infty\}$ by

$$J_a(x, u) = I^{(a_1)}(x, u), \quad (x, u) \in S_{a_2}, \quad J_a(x, u) = \infty, \quad (x, u) \in X \backslash S_{a_2}. \tag{2.9}$$

It follows from Propositions 4.1 and 4.2 of [12] that J_a is lower semicontinuous for all $a \in \mathcal{A}_1 \times A_2$. It is not difficult to see that for each $a \in \mathcal{A}$, $\inf(J_a)$ is finite. We will establish the following result.

Theorem 2.1 *There exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$, $\inf(J_a)$ is finite and attained at a unique point $(x_a, u_a) \in X$ and the following assertion holds:*

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(J_b)$ is finite and if $(z, v) \in X$ satisfies $J_b(z, v) \le \inf(J_b) + \delta$, then $\rho((x_a, u_a), (z, v)) \le \epsilon$ and $|J_b(z, v) - J_a(x_a, u_a)| \le \epsilon$.

Now we will state our second main result.

Let $\mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22} \times \mathcal{A}_{23}$ and let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. For each $a_2 = (A, B, z) \in \mathcal{A}_2$ we set

$$S_{a_2} = X(z, A, B).$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $\widehat{J}_a : X \to R^1 \cup \{\infty\}$ by

$$\widehat{J}_a(x, u) = I^{(a_1)}(x, u), \quad (x, u) \in S_{a_2}, \quad \widehat{J}_a(x, u) = \infty, \quad (x, u) \in X \setminus S_{a_2}.$$

It follows from Propositions 4.1 and 4.2 of [12] that $\widehat{J}_a$ is lower semicontinuous for all $a \in \mathcal{A}_1 \times A_2$. It is not difficult to see that for each $a \in \mathcal{A}$, $\inf(\widehat{J}_a)$ is finite. We will establish the following result.

Theorem 2.2 *There exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$, $\inf(\widehat{J}_a)$ is finite and attained at a unique point $(x_a, u_a) \in X$ and the following assertion holds:*

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(\widehat{J}_b)$ is finite and if $(z, v) \in X$ satisfies $\widehat{J}_b(z, v) \le \inf(\widehat{J}_b) + \delta$, then $\rho((x_a, u_a), (z, v)) \le \epsilon$ and $|\widehat{J}_b(z, v) - \widehat{J}_a(x_a, u_a)| \le \epsilon$.

3 Variational Principles

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology and denoted by τ_s. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology and denoted by τ_w. We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\overline{R} = [-\infty, \infty]$. In our study we use the following basic hypotheses about the functions.

(H1) For any $a \in \mathcal{A}$, any $\epsilon > 0$ and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} \colon d(a, b) < \epsilon\} \ne \varnothing$$

and for any $b \in \mathcal{W}$

(i) $\inf(f_b)$ is finite;
(ii) if $z \in X$ is such that $f_b(z) \le \inf(f_b) + \eta$, then $\rho(z, x) \le \gamma$ and $|f_b(z) - \alpha| \le \gamma$.

(H2) If $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in X.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $(\mathcal{A}, \tau_w)$ if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

(In a slightly different setting a similar property was introduced in [15].)

The following result was established in [12, Theorem 2.1].

Theorem 3.1 *Assume that (H1) and (H2) hold. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for f_a on (X, ρ) is strongly well posed with respect to $(\mathcal{A}, \tau_w)$.*

Now we assume that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ where $(\mathcal{A}_i, d_i)$, $i = 1, 2$, are complete metric spaces and

$$d((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2), \quad (a_1, a_2), (b_1, b_2) \in \mathcal{A}.$$

For the space $\mathcal{A}_2$ we consider the topology induced by the metric d_2 (the strong and weak topologies coincide) and for the space $\mathcal{A}_1$ we consider the strong topology which is induced by the metric d_1 and a weak topology which is weaker than the strong topology. The strong topology of $\mathcal{A}$ is the product of the strong topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$ and the weak topology of $\mathcal{A}$ is the product of the weak topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$.

Assume that with every $a \in \mathcal{A}_1$ a function $\phi_a \colon X \to R^1 \cup \{\infty\}$ is associated and with every $a \in \mathcal{A}_2$ a nonempty set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a \colon X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \quad \text{for all} \quad x \in S_{a_2}, \quad f_a(x) = \infty \quad \text{for all} \quad x \in X \setminus S_{a_2}. \tag{3.1}$$

Fix $\theta \in \mathcal{A}_2$. We use the following hypotheses.

(A1) For each $a \in \mathcal{A}$, $\inf(f_a)$ is finite and f_a is lower semicontinuous.

(A2) For each $a_1 \in \mathcal{A}_1$, each $\epsilon > 0$ and each $D > 0$ there exists a neighborhood $\mathcal{V}$ of a_1 in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{V}$ and each $x \in X$ satisfying $\min\{\phi_{a_1}(x), \phi_b(x)\} \le D$ the inequality $|\phi_{a_1}(x) - \phi_b(x)| \le \epsilon$ holds.

(A3) For each $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, each $\gamma \in (0,1)$ and each $r \in (0,1)$ there exist $\bar{a}_1 \in \mathcal{A}_1$, $\bar{x} \in S_{a_2}$, $\delta > 0$ such that $d_1(\bar{a}_1, a_1) < r$ and for each $x \in S_{a_2}$ satisfying $\phi_{\bar{a}_1}(x) \le \inf(f_{(\bar{a}_1, a_2)}) + \delta$ the inequality $\rho(x, \bar{x}) \le \gamma$ is valid.

(A4) For each $a_1 \in \mathcal{A}_1$, each $M, D > 0$ and each $\epsilon \in (0,1)$ there exists a number $\delta > 0$ such that for each $a_2 \in \mathcal{A}_2$ satisfying $d_2(a_2, \theta) \le M$, each $x \in S_{a_2}$ satisfying $\phi_{a_1}(x) \le D$ and each $\xi \in \mathcal{A}_2$ satisfying $d_2(a_2, \xi) \le \delta$ there exists $y \in S_\xi$ such that $\rho(x, y) \le \epsilon$ and $|\phi_{a_1}(x) - \phi_{a_1}(y)| \le \epsilon$.

The following result was proved in [14, Proposition 1.1].

Proposition 3.1 Assume that (A1)–(A4) hold. Then (H1) holds.

4 Proofs of Theorems 2.1 and 2.2

The following result was proved in [12, Lemma 1.1].

Lemma 4.1 *Let* $a, b \in R^1$, $\epsilon \in (0,1)$, $\Delta \ge 0$ *and let*

$$|a - b| < (1 + \Delta)\epsilon + \epsilon \max\{|a|, |b|\}.$$

Then

$$|a - b| < (1 + \Delta)(\epsilon + \epsilon^2(1-\epsilon)^{-1}) + \epsilon(1-\epsilon)^{-1} \min\{|a|, |b|\}.$$

Analogously to Proposition 4.4 of [12] we can prove the following result.

Proposition 4.1 Let $f \in \mathcal{M}$, $\epsilon \in (0,1)$ and $D > 0$. Then there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{M}$ with the weak topology such that for each $g \in \mathcal{V}$ and each $(x, u) \in X$ satisfying $\min\{I^f(x,u),\ I^g(x,u)\} \le D$ the inequality $|I^f(x,u) - I^g(x,u)| \le \epsilon$ is valid.

We preface the proofs of our main results by the following lemma.

Lemma 4.2 *Let $f \in \mathcal{M}$, $M, D > 0$ and let $\epsilon \in (0,1)$. Then there exists a number $\delta > 0$ such that for each $z \in R^n$, $A \in \mathcal{A}_{21}$, $B \in \mathcal{A}_{22}$ satisfying*

$$|z| \le M \quad and \quad \|A\|, \|B\| \le M, \tag{4.1}$$

each

$$(x,u) \in X(z,A,B) \tag{4.2}$$

which satisfies

$$I^{(f)}(x,u) \le D \tag{4.3}$$

and each $\xi \in R^n$, $P \in \mathcal{A}_{21}$ and $Q \in \mathcal{A}_{22}$ satisfying

$$|z-\xi|,\ \|A-P\|,\ \|B-Q\| \le \delta \tag{4.4}$$

there exists $(y,v) \in X(\xi,P,Q)$ such that

$$v(t) = u(t), \quad t \in (T_1,T_2) \quad a.e., \tag{4.5}$$

$$|x(t)-y(t)| \le \epsilon, \quad t \in [T_1,T_2], \tag{4.6}$$

$$|I^{(f)}(x,u) - I^f(y,v)| \le \epsilon. \tag{4.7}$$

Proof By property (iii) (see the definition of $\mathcal{M}$) there is an integrable scalar function $\psi_1(t) \ge 0$, $t \in (T_1,T_2)$, such that

$$|x| + |u| \le \psi_1(t) + f(t,x,u) \quad \text{for all} \quad (t,x,u) \in (T_1,T_2) \times R^n \times R^m. \tag{4.8}$$

Choose a positive number d_0 such that

$$d_0 > \sup\{\|e^{\tau C}\| : \ \tau \in [0, T_2 - T_1],\ C \in \mathcal{A}_{21} \ \text{ and } \ \|C\| \le M+1\}. \tag{4.9}$$

Set

$$\|\psi_1\| = \int_{T_1}^{T_2} \psi_1(t)\,dt. \tag{4.10}$$

Inequality (4.8) implies that for each $(t,x,u) \in (T_1,T_2) \times R^n \times R^m$

$$|f(t,x,u)| \le f(t,x,u) + 2\psi_1(t). \tag{4.11}$$

Choose a number

$$M_0 > 2 + M(\|\psi_1\| + D + 1). \tag{4.12}$$

We show that the following property holds:

(P) If $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1) and $(x,u) \in X(z,A,B)$ satisfies (4.3), then

$$|x(t)| \le M_0 - 2 \quad \text{for all} \quad t \in [T_1,T_2]. \tag{4.13}$$

Assume that $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1) and that $(x,u) \in X(z,A,B)$ satisfies (4.3). Then it follows from the definition of $X(z,A,B)$, (2.2), (2.3), (4.1), (4.3), (4.8), (4.10) and (4.12) that for each $t \in [T_1, T_2]$

$$\begin{aligned}
|x(t)| &\le |x(T_1)| + \left| \int_{T_1}^{t} [Ax(s) + Bu(s)]\, ds \right| \\
&\le |x(T_1)| + \|A\| \int_{T_1}^{t} |x(s)|\, ds + \|B\| \int_{T_1}^{t} |u(s)| ds \le M + M \int_{T_1}^{t} (|x(s)| + |u(s)|)\, ds \\
&\le M\left(1 + \int_{T_1}^{T_2} (|x(s)| + |u(s)|)\, ds\right) \\
&\le M\left(1 + \int_{T_1}^{T_2} f(s, x(s), u(s))\, ds + \int_{T_1}^{T_2} \psi_1(s)\, ds\right) \\
&\le M(1 + D + \|\psi_1\|) \le M_0 - 2.
\end{aligned}$$

Thus property (P) holds.

Choose a positive number

$$\epsilon_0 < \epsilon\,(T_2 - T_1 + D + 2\|\psi_1\| + 1)^{-1}/4 \tag{4.14}$$

and a positive number $\epsilon_1 < 1$ for which

$$\epsilon_1 + \epsilon_1(1-\epsilon_1)^{-1} < \epsilon_0/8. \tag{4.15}$$

In view of property (v) (see the definition of $\mathcal{M}$) there exist Γ_0, $\delta_0 > 0$ such that for almost every $t \in (T_1, T_2)$

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon_1 \max\{|f(t,x_1,u)|, |f(t,x_2,u)|\} + \epsilon_1 \tag{4.16}$$

for each $u \in R^m$ and each $x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M_0, \quad i = 1,2, \quad |u| \ge \Gamma_0, \quad |x_1 - x_2| \le 4\delta_0. \tag{4.17}$$

By property (iv) (see the definition of $\mathcal{M}$) there exists a positive number

$$\delta_1 < \min\{\delta_0, \epsilon_1, 1\} \tag{4.18}$$

such that for almost every $t \in (T_1, T_2)$ the inequality

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \le \epsilon_0 \tag{4.19}$$

holds for each $x_1, x_2 \in R^n$ and each $u_1, u_2 \in R^m$ such that

$$|x_i|, |u_i| \le M_0 + \Gamma_0 + 1, \quad i = 1,2, \quad |x_1 - x_2|, |u_1 - u_2| \le \delta_1. \tag{4.20}$$

Let $\delta_2 > 0$ satisfy

$$(\delta_2 d_0 + M\delta_2)\left(1 + D + \int_{T_1}^{T_2} \psi_1(t)\,dt\right) < \delta_1/4.$$

Choose $\delta > 0$ such that

$$\delta < \min\{1, \delta_1, \delta_2\} \tag{4.22}$$

and that for each $A, P \in \mathcal{A}_{21}$ satisfying

$$\|A\| \le M, \quad \|A - P\| \le \delta$$

and each $\tau \in [0,\, T_2 - T_1]$ the inequality

$$\|e^{\tau P} - e^{\tau A}\| \le \delta_2 \tag{4.23}$$

holds.

Assume that $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1), $(x, u) \in X$ satisfy (4.2), (4.3) and $\xi \in R^n$, $P \in \mathcal{A}_{21}$ and $Q \in \mathcal{A}_{22}$ satisfy (4.4). It follows from (4.2), (2.2) and (2.3) that

$$x(T_1) = z, \tag{4.24}$$

$$x'(t) = Ax(t) + Bu(t), \quad t \in (T_1, T_2) \quad \text{a.e.} \tag{4.25}$$

Relations (4.24) and (4.25) imply that

$$x(t) = e^{(t-T_1)A}z + \int_{T_1}^{t} e^{(t-s)A}Bu(s)\,ds, \quad t \in [T_1, T_2]. \tag{4.26}$$

In view of (4.3) and (4.8) $\int_{T_1}^{T_2} |u(t)|\,dt < \infty$. Define

$$y(t) = e^{(t-T_1)P}\xi + \int_{T_1}^{t} e^{(t-s)P}Qu(s)\,ds, \quad t \in [T_1, T_2]. \tag{4.27}$$

It is not difficult to see that

$$(y, u) \in X(\xi, P, Q). \tag{4.28}$$

It follows from (4.27), (4.26), (4.1), (4.4), (4.22), (4.9) and the choice of δ (see (4.23)) that for each $t \in [T_1, T_2]$

$$|y(t) - x(t)| = \left| e^{(t-T_1)A}z + \int_{T_1}^{t} e^{(t-s)A}Bu(s)\,ds - e^{(t-T_1)P}\xi - \int_{T_1}^{t} e^{(t-s)P}Qu(s)\,ds \right|$$

$$\le |e^{(t-T_1)P}\xi - e^{(t-T_1)P}z| + |e^{(t-T_1)P}z - e^{(t-T_1)A}z|$$

$$+\left|\int_{T_1}^{t} e^{(t-s)P}Qu(s)ds - \int_{T_1}^{T} e^{(t-s)P}Bu(s)ds\right| + \left|\int_{T_1}^{t} e^{(t-s)P}Bu(s)ds - \int_{T_1}^{t} e^{(t-s)A}Bu(s)ds\right|$$

$$\le |\xi - z| \sup\{\|e^{\tau C}x\| \colon \tau \in [0, T_2 - T_1],\ C \in \mathcal{A}_{21},\ \|C\| \le M+1\} + |z|\delta_2$$

$$+ \int_{T_1}^{t} \|e^{(t-s)P}\| \|B - Q\| |u(s)| ds + \left(\int_{T_1}^{t} \|B\| |u(s)| ds\right) \sup\{\|e^{\tau P} - e^{\tau A}\| \colon\ \tau \in [0, T_2 - T_1]\}$$

$$\le \delta d_0 + M\delta_2 + d_0\delta \int_{T_1}^{t} |u(s)| ds + \delta_2 M \int_{T_1}^{t} |u(s)| ds$$

$$\le \delta d_0 + M\delta_2 + \left(\int_{T_1}^{T_2} |u(t)| dt\right)(d_0\delta + \delta_2 M). \tag{4.29}$$

Relations (4.8) and (4.3) imply that

$$\int_{T_1}^{T_2} |u(t)|\, dt \le \int_{T_1}^{T_2} f(t, x(t), u(t))\, dt + \int_{T_1}^{T_2} \psi_1(t)\, dt \le D + \int_{T_1}^{T_2} \psi_1(t)\, dt. \tag{4.30}$$

In view of (4.29), (4.30), (4.22) and (4.21) for each $t \in [T_1, T_2]$

$$|y(t) - x(t)| \le (\delta d_0 + M\delta_2)\left(1 + D + \int_{T_1}^{T_2} \psi_1(t)\, dt\right) < \delta_1/4. \tag{4.31}$$

By property (P), (4.1), (4.2) and (4.3)

$$|x(t)| \le M_0 - 2, \quad t \in [T_1, T_2]. \tag{4.32}$$

Set

$$\Omega = \{t \in (T_1, T_2) \colon |u(t)| \ge \Gamma_0\}. \tag{4.33}$$

We will estimate

$$\int_{T_1}^{T_2} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\, dt.$$

Clearly

$$\begin{aligned}\int_{T_1}^{T_2} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\, dt &\le \int_{\Omega} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\, dt \\ &+ \int_{[T_1, T_2] \setminus \Omega} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\, dt.\end{aligned} \tag{4.34}$$

It follows from (4.33), (4.32), (4.31) and the choice of Γ_0, δ_0 (see (4.16)–(4.18)) that for almost every $t \in \Omega$

$$|f(t,x(t),u(t)) - f(t,y(t),u(t))| \leq \epsilon_1 + \epsilon_1 \max\{|f(t,x(t),u(t))|, |f(t,y(t),u(t))|\}. \tag{4.35}$$

In view of (4.35), (4.15) and Lemma 4.1 for almost every $t \in \Omega$

$$\begin{aligned}|f(t,x(t),u(t)) - f(t,y(t),u(t))| &\leq \epsilon_1 + \epsilon_1^2(1-\epsilon_1)^{-1} + \epsilon_1(1-\epsilon_1)^{-1}|f(t,x(t),u(t))| \\ &< \epsilon_0/8 + (\epsilon_0/8)|f(t,x(t),u(t))|.\end{aligned}$$

Combined with (4.8), (4.3), (4.10) and (4.14) this inequality implies that

$$\begin{aligned}\int_{\Omega} |f(t,x(t),u(t)) - f(t,y(t),u(t))|\,dt &\leq \int_{T_1}^{T_2} [\epsilon_0/8 + (\epsilon_0/8)|f(t,x(t),u(t))|\,dt \\ &\leq (\epsilon_0/8)(T_2-T_1) + (\epsilon_0/8)\int_{T_1}^{T_2} (f(t,x(t),u(t)) + 2\psi_1(t))\,dt \\ &\leq (\epsilon_0/8)(T_2-T_1) + (\epsilon_0/8)(D + 2\|\psi_1)\|) < \epsilon/8.\end{aligned} \tag{4.36}$$

It follows from the choice of δ_1 (see (4.18)–(4.20)), (4.33), (4.32) and (4.31) that for almost every $t \in (T_1,T_2)\backslash\Omega$

$$|f(t,x(t),u(t)) - f(t,y(t),u(t))| \leq \epsilon_0.$$

Together with (4.14) this implies that

$$\int_{(T_1,T_2)\backslash\Omega} |f(t,x(t),u(t)) - f(t,y(t),u(t))| \leq \epsilon_0(T_2-T_1) < \epsilon/4.$$

Combined with (4.36) and (4.31) this inequality implies that

$$|I^f(x,u) - I^f(y,u)| \leq \epsilon/2.$$

This completes the proof of Lemma 4.2.

Proofs of Theorems 2.1 and 2.2 By Theorem 3.1 and Proposition 3.1 we need only to show that the hypotheses (A1) – (A4) and (H2) hold. We have already noted in Section 2 that (A1) is valid. (H2) follows from Proposition 4.2 of [12]. Proposition 4.1 implies (A2). (A3) follows from Lemma 5.1 of [12]. Lemma 4.2 implies (A4). This completes the proofs of Theorems 2.1 and 2.2.

References

[1] Cellina, A. and Colombo, G. On a classical problem of the calculus of variations without convexity assumptions *Ann. Inst. H. Poincare, Anal. Non Linéaire* **7** (1990) 97–106.

[2] Cesari, L. *Optimization — Theory and Applications.* Springer-Verlag, New York, 1983.

[3] Clarke, F.H. *Optimization and Nonsmooth Analysis.* Wiley Interscience, 1983.
[4] Deville, R., Godefroy, R. and Zizler, V. *Smoothness and Renorming in Banach Spaces.* Longman, 1993.
[5] Ioffe, A.D. and Zaslavski, A.J. Variational principles and well-posedness in optimization and calculus of variations. *SIAM J. Control Optim.* **38** (2000) 566–581.
[6] McShane, E.J. Existence theorem for the ordinary problem of the calculus of variations. *Ann. Scoula Norm. Pisa* **3** (1934) 181–211.
[7] Mordukhovich, B.S. *Approximation Methods in Optimization and Control.* Nauka, Moscow, 1988.
[8] Mordukhovich, B.S. Existence theorems in nonconvex optimal control. In: *Calculus of Variations and Optimal Control*, CRC Press, Boca Raton, FL, 1999, 175–197.
[9] Morrey, Ch. *Multiple Integrals in the Calculus of Variations.* Springer, Berlin–Heidelberg–New York, 1967.
[10] Rockafellar, R.T. Existence and duality theorems for convex problems of Bolza. *Trans. Amer. Math. Soc.* **159** (1971) 1–40.
[11] Tonelli, L. *Fondamenti di Calcolo delle Variazioni.* Zanicelli, Bolonia, 1921–1923.
[12] Zaslavski, A.J. Generic well-posedness of optimal control problems without convexity assumptions. *SIAM J. Control Optim.* **39** (2000) 250–280.
[13] Zaslavski, A.J. Existence of solutions of optimal control problems for a generic integrand without convexity assumptions. *Nonlinear Analysis: Theory, Methods and Applications* **43** (2001) 339–361.
[14] Zaslavski, A.J. Generic well-posedness of variational problems without convexity assumptions. *J. Math. Anal. Appl.* **279** (2003) 22–42.
[15] Zolezzi, T. Well-posedness criteria in optimization with application to the calculus of variations. *Nonlinear Analysis: Theory, Methods and Applications* **25** (1995) 437–453.