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Existence of Nonoscillatory Solution of High-Order Nonlinear Difference Equation

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Abstract: In this paper, the existence of the nonoscillatory solution to the equation of a class of high-order nonlinear neutral delay difference is investigated. By using fixed point theorem, a sufficient condition is proposed for the existence of eventually positive solution.

Keywords: *Difference equation; oscillation; positive solution.*

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1 Introduction

In the computer designing and the ecomodeling, a class of neutral difference equation is proposed. In recent years, the oscillatory behavior of neutral difference equations was intensively studied, and some good results were obtained [1–4]. Now we consider the nonlinear high-order difference equation

$$\Delta^m(x_n - px_{n-\tau}) + q_n f(x_{n-\sigma}) = 0, \tag{1}$$

where m is a positive odd number; $n \in N = \{0, 1, 2, \dots\}$, $p \in R$; for $n \in N$, $q_n \in R^+$, $\sigma \in N$, $\tau \in N \backslash \{0\}$, $\mu = \max\{\tau, \sigma\}$, $f \in C(R, R)$ satisfying that $xf(x) > 0$ for $x \neq 0$ and for $\forall\, x, y \in R$,

$$|f(x) - f(y)| \le L|x - y| \tag{2}$$

where L is a positive constant. The case of $p = 1$ was studied in [5], the case of the equation (1) of even order was studied in [6]. In this paper, by using fixed point theorem, the case of the equation (1) of odd order is studied under the condition of $p \neq \pm 1$, and

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a sufficient condition for the existence of the positive solution to the equation (1) is obtained.

The real sequence $\{y_n\}_{n=-\mu}^{\infty}$ is called a solution to the equation (1) if each item y_n satisfies the equation (1) for $n \geq 0$. It is called an eventually positive solution if $y_n > 0$ for sufficiently large n.

2 Main Results and Proofs

Theorem 2.1 *Suppose $p \neq \pm 1$, $f(x)$ satisfy the condition (2) and*

$$\sum_{n=0}^{+\infty} n^{m-1} q_n < +\infty. \tag{3}$$

Then there is a bounded eventually positive solution to the equation (1).

Proof Let L_∞ denote a Banach space of all the bounded real sequences $x = \{x_n\}_{n=N-\mu}^{\infty}$ and define the norm $\|x\| = \sup\limits_{n \geq N-\mu} |x_n|$. We introduce the following notation

$$u^{(l)} = \prod_{i=0}^{l-1} (u-i), \quad u \geq l, \tag{4}$$

and let $u^{(0)} = 1$. There are four situations to be discussed:

Case 1: $0 \leq p < 1$.

Let $M = \max\{f(t)\colon 1 \leq t \leq 2\}$, $A = \max\{M, L\}$. (3)implies that there exists a positive integer N which is large enough, such that

$$\frac{A}{(m-1)!} \sum_{k=N}^{+\infty} k^{m-1} q_k \leq \frac{1-p}{2}. \tag{5}$$

Define a subset $\Omega = \{x \in L_\infty : 1 \leq x_n \leq 2,\ n \geq N-\mu\}$ on L_∞. Then Ω is a bounded closed convex subset on L_∞. Define a mapping $T\colon \Omega \to L_\infty$ as following:

$$(Tx)_n = \begin{cases} 1 - p + p x_{n-\tau} + \sum\limits_{k=n}^{+\infty} \frac{(k-n+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), & n \geq N, \\ (Tx)_N, & N - \mu \leq n < N. \end{cases} \tag{6}$$

Next, we show that T is a continuous operator. Let $x^i \in \Omega$, $i = 1, 2 \ldots,$ such that $\lim\limits_{i \to +\infty} \|x^i - x\| = 0$. As Ω is a closed subset, so $x \in \Omega$. For $n \geq N$, from (6) one obtains

$$\begin{aligned} |(Tx^i)_n - (Tx)_n| &\leq p|x^i_{n-\tau} - x_{n-\tau}| + \sum_{k=n}^{+\infty} \frac{(k-n+1)^{(m-1)}}{(m-1)!} q_k |f(x^i_{k-\sigma}) - f(x_{k-\sigma})| \\ &\leq p\|x^i - x\| + \sum_{k=n}^{+\infty} \frac{(k-n+1)^{(m-1)}}{(m-1)!} q_k |f(x^i_{k-\sigma}) - f(x_{k-\sigma})|. \end{aligned}$$

As $f(x)$ is continuous, based on the Lebesgue dominated convergence theorem, one can obtain

$$\lim_{i\to+\infty}\sup_{n\geq N}\|(Tx^i)_n-(Tx)_n\|=0.$$

Obviously, the above formula is also hold for $N-\mu\leq n<N$. So it is easy to deduce

$$\lim_{i\to+\infty}\|Tx^i-Tx\|=0.$$

Namely T is a continuous operator.

For $\forall\, x\in\Omega$, when $n\geq N$, we can get from (6)

$$(Tx)_n=1-p+px_{n-\tau}+\sum_{k=n}^{+\infty}\frac{(k-n+1)^{(m-1)}}{(m-1)!}\,q_kf(x_{k-\sigma})\leq 1-p+2p+A\sum_{k=n}^{+\infty}\frac{k^{m-1}}{(m-1)!}\,q_k.$$

Thus from (5) we can immediately obtain

$$(Tx)_n\leq 1-p+2p+\frac{1-p}{2}=\frac{3+p}{2}<2.$$

Similarly, from (6) we also have

$$(Tx)_n=1-p+px_{n-\tau}+\sum_{k=n}^{+\infty}\frac{(k-n+1)^{(m-1)}}{(m-1)!}\,q_kf(x_{k-\sigma})>1-p+p=1.$$

Obviously, when $N-\mu\leq n<N$, $1<(Tx)_n<2$ also holds. Hence $T\Omega\in\Omega$, namely T is a self-mapping on Ω.

In what follows, we will prove that T is a contraction mapping on Ω.

For $\forall\, x,y\in\Omega$, when $n\geq N$, from the definition of T one obtains

$$\begin{aligned}|(Tx)_n-(Ty)_n|&\leq|x_{n-\tau}-x_{n-\tau}|+L\sum_{k=n}^{+\infty}\frac{(k-n+1)^{(m-1)}}{(m-1)!}\,q_k|x_{k-\sigma}-y_{k-\sigma}|\\&\leq\|x-y\|\left(p+A\sum_{k=N}^{+\infty}\frac{(k-n+1)^{(m-1)}}{(m-1)!}\,q_k\right).\end{aligned}$$

Then it follows from (5) that

$$|(Tx)_n-(Ty)_n|\leq\|x-y\|\left(p+\frac{1-p}{2}\right)=\frac{1+p}{2}\,\|x-y\|.$$

Obviously, when $N-\mu\leq n<N$, $|(Tx)_n-(Ty)_n|\leq\frac{1+p}{2}\|x-y\|$ also holds. Hence, we have

$$\|Tx-Ty\|=\sup_{n\geq N-\mu}|(Tx)_n-(Ty)_n|\leq\frac{1+p}{2}\,\|x-y\|$$

when $0\leq p<1$, $0<\frac{1+p}{2}<1$, so T is a contraction mapping on Ω.

We can conclude from the Banach contraction mapping principle that there exists a fixed point $x\in\Omega$, such that $Tx=x$. In what follows, we will prove that the fixed point x is a bounded positive solution to the equation (1).

As a matter of fact, when $n \geq N-\mu,\ x_n \geq 1 > 0,$ each item of the fixed point satisfies

$$x_n = \begin{cases} 1-p+px_{n-\tau}+\sum\limits_{k=n}^{+\infty} \dfrac{(k-n+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), & n \geq N, \\ x_N, & N-\mu \leq n < N, \end{cases}$$

so

$$x_n - px_{n-\tau} = 1-p+\sum_{k=n}^{+\infty} \frac{(k-n+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), \quad n \geq N.$$

From (4), we can deduce that

$$(k-n)^{(m-1)} - (k-n+1)^{(m-1)} = (1-m)(k-n)^{(m-2)}.$$

Hence

$$\Delta(x_n - px_{n-\tau}) = -\sum_{k=n}^{+\infty} \frac{(k-n)^{(m-2)}}{(m-2)!} q_k f(x_{k-\sigma}), \quad n \geq N.$$

From (4), we can also deduce that

$$(k-n-1)^{(m-2)} - (k-n)^{(m-2)} = (2-m)(k-n-1)^{(m-3)}.$$

Then

$$\Delta^2(x_n - px_{n-\tau}) = \sum_{k=n}^{+\infty} \frac{(k-n-1)^{(m-3)}}{(m-3)!} q_k f(x_{k-\sigma}), \quad n \geq N.$$

In general, we can have

$$\Delta^i(x_n - px_{n-\tau}) = (-1)^i \sum_{k=n}^{+\infty} \frac{(k-n-i+1)^{(m-i-1)}}{(m-i-1)!} q_k f(x_{k-\sigma}), \quad n \geq N,$$

where $u^{(0)} = 1,\ i = 1, 2, \ldots, m-1$. Because m is an odd number, we get

$$\Delta^m(x_n - px_{n-\tau}) = -q_n f(x_{n-\sigma}).$$

So the fixed point x is a bounded positive solution to the equation (1). Thus the proof of the situation (1) is completed.

Case 2: $p > 1$.

Let $M = \max\{f(t)\colon \frac{p-1}{2} \leq t \leq p\},\ A = \max\{M, L\}$. (3) implies that there exists a positive integer N which is large enough, such that

$$\frac{A}{(m-1)!} \sum_{k=N}^{+\infty} k^{m-1} q_k \leq \frac{p-1}{2}.$$

Define a subset $\Omega = \{x \in L_\infty\colon \frac{p-1}{2} \leq x_n \leq p,\ n \geq N-\mu\}$ on L_∞. Then Ω is a bounded closed convex subset on L_∞. Define a mapping $T\colon \Omega \to L_\infty$ as following:

$$(Tx)_n = \begin{cases} p-1+\frac{1}{p}x_{n+\tau}-\frac{1}{p}\sum\limits_{k=n+\tau}^{+\infty} \frac{(k-n-\tau+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), & n \geq N, \\ (Tx)_N, & N-\mu \leq n < N. \end{cases}$$

Case 3: $-1 < p < 0$.

Let $M = \max\{f(t)\colon\ 1+p \le t \le 2\}$, $A = \max\{M, L\}$. (3) implies that there exists a positive integer N which is large enough, such that

$$\frac{A}{(m-1)!}\sum_{k=N}^{+\infty} k^{m-1} q_k \le \frac{1+p}{2}.$$

Define a subset $\Omega = \{x \in L_\infty\colon\ p+1 \le x_n \le 2,\ n \ge N-\mu\}$ on L_∞. Then Ω is a bounded closed convex subset on L_∞. Define a mapping $T\colon\ \Omega \to L_\infty$ as following:

$$(Tx)_n = \begin{cases} 1 - p + p x_{n-\tau} + \sum\limits_{k=n}^{+\infty} \dfrac{(k-n+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), & n \ge N, \\ (Tx)_N, & N-\mu \le n < N. \end{cases}$$

Case 4: $p < -1$.

Let $M = \max\{f(t)\colon\ \frac{(p+1)^2}{p(p-1)} \le t \le \frac{2(p+1}{p-1}\}$, $A = \max\{M, L\}$. (3) implies that there exists a positive integer N which is large enough, such that

$$\frac{A}{(m-1)!}\sum_{k=N}^{+\infty} k^{m-1} q_k \le \frac{(p+1)^2}{p}.$$

Define a subset $\Omega = \{x \in L_\infty\colon\ \frac{(p+1)^2}{p(p-1)} \le x_n \le \frac{2(p+1}{p-1},\ n \ge N-\mu\}$ on L_∞. Then Ω is a bounded closed convex subset on L_∞. Define a mapping $T\colon\ \Omega \to L_\infty$ as following:

$$(Tx)_n = \begin{cases} 1 + \frac{1}{p} + \frac{1}{p} x_{n+\tau} - \frac{1}{p} \sum\limits_{k=n+\tau}^{+\infty} \frac{(k-n-\tau+1)^{(m-1)}}{(m-1)!} q_k f(x_{k-\sigma}), & n \ge N, \\ (Tx)_N, & N-\mu \le n < N. \end{cases}$$

The proofs of case 2, 3, 4 are similar to that of case 1, so they are omitted.

3 Conclusions

For the high-order equation (1), the case of $p = 1$ was studied in [5], the case of even order equation was studied in [6]. The result obtained here is an extension of works in [5] and [6], that is to say, the case of odd order equation is studied under the condition of $p \ne \pm 1$. By using fixed point theorem, a sufficient condition for the existence of the positive solution to the odd order equation (1) is obtained. Thus all the cases of equation (1), say of both the odd order and the even order, have been studied. So the result in this paper is not only simpler than that in [6] but also more general than it.

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