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Optimal Reconfiguration of Spacecraft Formations Using a Variational Numerical Method

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Abstract: One of the key issues when working with formations of spacecraft is how to reconfigure the formation in order to change its orientation, its pointing or just to arrive to a given pattern. In this paper we treat these reconfiguration tasks as an optimal problem and set out the problem using the finite element method. Although the methodology is general, and suits to many different types of problems, the examples that have been considered focus in some basic maneuvers of the TPF and Darwin missions about the L_2 Lagrange point of the Earth-Sun system.

Keywords: *Formation flight; optimal reconfiguration; finite element method; spacecraft formations.*

Mathematics Subject Classification (2000): 70M20, 70G75, 65K10, 65L60.

1 Introduction

In the last few years, the interest in constellations of spacecraft and formation flight has been increasing. One of the major applications of this technique is for remote sensing missions, where using the formation it is possible to increase the resolution as a virtual antenna, resulting in a much larger one than using a single spacecraft. Examples of this procedure are missions such as Darwin of the ESA and TPF of the NASA (see [5, 3]).

Among others, one of the problems one must face when working with formations of spacecraft are the reconfiguration maneuvers. For instance, several situations where the need of reconfiguration of the formation appears are the following:

- There are some basic maneuvers that a formation must be capable to perform, such as expansions and contractions, or simply to change the pattern to perform specific tasks.

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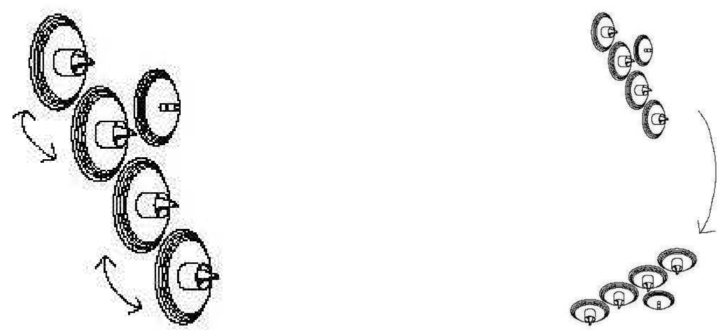

Figure 1.1: Two examples of reconfiguration of spacecraft. In the left-hand side, we change the position of the inner and outer spacecraft of the TPF formation. In the right-hand side we have a constellation pointing to a certain target and we reconfigure it to point to another one.

- The lifetime of a formation finishes when a spacecraft ends its fuel. Many times will be mandatory to equilibrate the consume of fuel of all the spacecraft to extend the lifetime of the formation. An example of this situation is the TPF formation, where the outer spacecraft consume more fuel than the inner ones. To exchange the position of the inner and outer spacecraft, as in Figure 1.1 may be a solution to the problem.

- In interferometry missions, the formation of spacecraft usually will have to point to many targets. It is necessary to reconfigure the formation in order to point to the next goal as it is represented in Figure 1.1.

- In some cases, due to the number of spacecraft of the formation, the deployment phase might follow after a rendezvous of several motherships. Deployments can be treated as special cases of reconfigurations where the satellites depart from different locations and configure a final pattern.

The objective of this work is to compute reconfigurations of the spacecraft in a systematic way and taking into account collision avoidance during the execution of the maneuvers. Earlier approaches with different methodologies can be found in the literature. For instance C.R. McInnes creates a local topology based on artificial potential functions (see [2]). The method is also used in the guidance of robots which move avoiding fixed objects or in the guidance of a robotic arm (see [7]). Singh and Hadaegh also treated the problem as an optimization one, modeling the trajectory with cubic splines (see [6]). However our proposed research in the use of finite element methods looks very promising due to the huge amount of knowledge on this area. The finite element method implements in a systematic and general way giving a full methodology. There is no need to look or adjust functions or parameters in special situations and different levels of approximation can be attained. Moreover it has a complete mathematical foundation in behind assuring nice properties such as convergence to solutions and adaptability.

2 The methodology

In the results we present, we assume that the spacecraft are in a L_2 Halo orbit of 120,000 km of z-amplitude in the Sun-Earth system. The model we use for the computations is the RTBP, but the procedure is easily generalized to any other one or to free space. Since we work with formations of a diameter of few hundred meters, the size of the formation with respect to the orbit is very small and it is feasible to use the linearized equations about the nonlinear orbit.

In this paper, the problem we consider is how to reconfigure a formation of N spacecraft in a selected time T. The formation will evolve in the vicinity of a given point on the halo orbit. Let us denote by X_i the position and velocity of the i-th spacecraft of the formation with respect to this point on the nominal halo orbit. The governing equations for the formation are:

$$\left\{ \begin{array}{l} \dot{X}_i(t) = A(t)X_i(t) + U_i(t), \\ X_i(0) = X_i^0, \quad X_i(T) = X_i^T, \end{array} \right. \tag{1}$$

in the time interval $[0, T]$, for $i = 1 \dots N$. Here $A(t)$ is the Jacobian matrix of the equations of motion about the halo orbit and $U_i(t)$ is the control law to be applied to the i-th satellite, so it is of the form $U_i(t) = (0, 0, 0, u_i^x(t), u_i^y(t), u_i^z(t))^t$. The final goal is to find optimal controls, $U_1, \dots, U_N$, subjected to certain restrictions on mutual distances (i.e. Euclidean norms in the position components of $X_i(t)$) or being the satellites on a certain surface, manifold, etc. Restrictions can be also time dependent.

The finite element method could be applied directly to equations (1), but in order to work with the simplest equations, we introduce a change of coordinates which casts $A(t)$ into its Jordan form which is

$$\begin{pmatrix} \lambda_1(t) & & & & & \\ & -\lambda_1(t) & & & & \\ & & 0 & \lambda_2(t) & & \\ & & -\lambda_2(t) & 0 & & \\ & & & & 0 & \lambda_3(t) \\ & & & & -\lambda_3(t) & 0 \end{pmatrix}.$$

This change of variables reduces (1) into a new set of equations each one them of the form

$$\left\{ \begin{array}{l} \ddot{x}(t) + \lambda(t)\, \dot{x}(t) + \tau(t)\, x(t) = u(t), \\ x(0) = x_0, \qquad x(T) = x_T, \\ \dot{x}(0) = v_0, \qquad \dot{x}(T) = v_T. \end{array} \right. \tag{2}$$

where $\lambda(t)$ and $\tau(t)$ are computed from a corresponding $\lambda_i(t)$ set. At this point it is also worth to mention that if one wants to consider the motion of the formation in free space, which is common in many studies of formation flight, we need only to take $\lambda(t) = \tau(t) \equiv 0$.

Assuming the time interval $[0, T]$ splitted in a given number, M, of smaller intervals (elements), $t_0 = 0,\, t_1, \dots, t_{M-1}, t_M = T$, we apply the Galerkin finite element method in time to the equations (2) by means of considering products by weight functions $w(t)$ and the usual weak form on each element is computed from the expression (see [8]):

$$\int_{t_k}^{t_{k+1}} w(t)\, (\ddot{x}(t) + \lambda(t)\, \dot{x}(t) + \tau(t)\, x(t)) dt = \int_{t_k}^{t_{k+1}} w(t)\, u(t) dt, \quad k = 0, \dots, M-1.$$

As it is well known, depending on the order of the elements used in the procedure one obtains different linear systems of equations associated with them. In case of considering a linear element, the system is

$$\begin{pmatrix} K_{11}^k & K_{12}^k \\ K_{21}^k & K_{22}^k \end{pmatrix} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} \Delta v_k \\ \Delta v_{k+1} \end{pmatrix},$$

which essentially states a relation between the nodal values x_k and x_{k+1}, which are related to the positions of the reconfiguration trajectory, and the delta-v's, Δv_k and Δv_{k+1}, applied in the nodal places of the element. We note that at this moment x_k, x_{k+1}, Δv_k and Δv_{k+1} are unknowns and the K_{ij}^k are 3×3 known matrices which are systematically computed following Galerkin's method. Finally, assembling the elementary equations we obtain the relations between the all the nodal positions and delta-v's. For each one of the spacecraft of the formation $(i = 1 \dots N)$ we obtain a system of the form

$$\begin{pmatrix} K_{22}^{0} + K_{11}^{1} & K_{12}^{1} & 0 & & \\ K_{21}^{1} & K_{22}^{1} + K_{11}^{2} & K_{12}^{2} & & \\ \ddots & \ddots & \ddots & & \\ & K_{21}^{M-3} & K_{22}^{M-3} + K_{11}^{M-2} & K_{12}^{M-2} & \\ & 0 & K_{21}^{M-2} & K_{22}^{M-2} + K_{11}^{M-1} & \end{pmatrix} \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,M-2} \\ x_{i,M-1} \end{pmatrix} +$$

$$+ \begin{pmatrix} K_{21}^{0} x_{i,0} \\ 0 \\ \vdots \\ 0 \\ K_{12}^{M-2} x_{i,T} \end{pmatrix} = \begin{pmatrix} \Delta v_{i,1} \\ \Delta v_{i,2} \\ \vdots \\ \Delta v_{i,M-2} \\ \Delta v_{i,M-1} \end{pmatrix}, \tag{3}$$

for the interior nodes, while for the boundary ones we incorporate what is known as essential boundary conditions and results in

$$\Delta v_{i,0} = K_{11}^0\, x_{i,0} + v_{i,0} + K_{12}^0\, x_{i,1}, \quad \Delta v_{i,M} = K_{21}^{M-1}\, x_{i,M-1} + K_{2,2}^{M-1}\, x_{i,M} - v_{i,T}.$$

From now on we treat the problem as an optimal control problem, where the functional we minimize is a penalty on the delta-v of the spacecraft. Collision avoidance and any other type of requirements enters in the method as restriction functions in the $x_{i,k}$ and $v_{i,k}$.

The objective function

The solution to our reconfiguration problem must be found attending essentially to the fuel consumption of the satellites. For this purpose we have selected an objective function of the form

$$J(\Delta v_{1,0}, \dots, \Delta v_{N,M}) = \sum_{i=0}^{N} J_i, \quad \text{with} \quad J_i(\Delta v_{i,0}, \dots, \Delta v_{i,M}) = \sum_{k=0}^{M} \rho_{i,k} ||\Delta v_{i,k}||^2, \tag{4}$$

(here $|| * ||$ denotes Euclidean norm), because it is directly related with the fuel expenditure, moreover derivatives are easily computed.

We consider parameters $\rho_{i,k}$ because in some way can be selected to equilibrate the fuel consumption of the spacecraft. For instance, incrementing the values of ρ corresponding to a particular satellite the penalty function takes into account that for that particular satellite fuel consumption is more critical. We also note that parameters ρ can depend on time (through the subscript k), this fact may be used to penalize the delta-v in certain time intervals.

Collision avoidance and other restrictions

In the reconfiguration planning it is essential to avoid collision between spacecraft. A minimum security distance between them must be required while maneuvering. In our procedure this distance can be chosen constant for general purpose reconfigurations or variable in time for special situations. For instance, during deployment a variable security distance is needed because at the beginning of the maneuver the satellites are closer than the usual safety distance demanded for reconfigurations.

As we previously stated, collision avoidance enters in the variational method as constraints in the position coordinates of $X_i(t)$ of (1). Via the change of coordinates used to obtain (2) and the finite element discretization, the constraints translate into conditions on the $x_{i,k}$ nodal variables. The finite element discretization is used to compute the distance between each spacecraft on each element and to check that this distance is greater than the security distance.

In a similar way many other restrictions can be applied (provided that there exist compatibility with the requirements). For instance some other cases that we have been studying are the following

- It is possible to maintain a formation in a region determined by a geometrical condition, such as an sphere, a paraboloid or a plane in an optimal way.
- It is also possible to keep the geometrical condition during a formation reconfiguration. For instance satellites can be forced to move in a plane or in a sphere while maneuvering for the reconfiguration.
- We can impose that particular spacecraft do not perform maneuvers during certain time intervals. In this case restrictions are set fixing the values of some $\Delta v_{i,k}$ (to zero in this example).
- The procedure can be used to keep the formation pointing continuously to a selected goal.
- Satellites can be restricted to maintain only certain relative distances between them. For instance to keep an equilateral triangle or tetrahedron. When the relative distances do not depend on time, the formation will evolve like a solid.
- Moreover in all these cases the final position of the spacecraft of the formation can be selected fixed (as in the formulation given by equations (1)), restricted to certain conditions or free.

Computing the initial seed for the iterative procedure

The computation of the optimal value of the objective function (4) involves an iterative process which needs an initial seed. In our approach this initial seed is computed using the uncoupled systems (3) and without taking into account the restrictions. This is, the initial guess does not have to be compatible with distance requirements between spacecraft or other type of constraints.

We have chosen to minimize the each one of the values given by

$$\bar{J}_i = \sum_{k=0}^{M} ||\Delta v_{i,k}||^2, \quad i = 1 \dots N.$$

Note that we do not use the parameter ρ to find the initial seed. If we denote by $K\, x_i + b_i = \Delta v_i$, the system (3) the function $\bar{J}_i$ casts into the form

$$\bar{J}_i = (K\, x_i + b_i)^T (K\, x_i + b_i) + (\Delta v_0)^2 + (\Delta v_{_M})^2.$$

Then it is easy to see that the trajectory of the i-th satellite which minimizes $\bar{J}_i$, and it is represented by the nodal values x_i, is obtained solving a linear system

$$(K^T K + C_i)x_i + (K^T b_i + d_i) = 0,$$

where C is an sparse matrix and d is an sparse vector.

Other computational issues

In order to find reconfiguration paths, we have implemented a specific program in C which returns the optimal trajectory for a given discretization of finite elements in time. The program obtains the trajectory in an iterative way. Usually we start with a few number of elements (typically 6 elements in the first step) and then we refine it by more or less doubling the number of elements at each iteration.

Using a slow computer such as a Pentium 3, 1.5 GHz, the CPU time to find the initial optimal six element trajectory from the initial seed is less than 20 seconds for a formation of 5 or 6 satellites. Doubling the number of elements from 50 to 100 for the same formation requires less than 40 seconds. Of course these CPU times, specially for the initial iterations, depend strongly on the characteristics of the reconfiguration demanded, but these ones are in general good indicators.

In case that problems of convergence had appeared in the first iterations we could also have used continuation methods, for instance with respect to the security distance, but it has not been necessary in all the examples that we have tried.

3 Some examples of reconfigurations

To illustrate the procedure we have selected three examples, two of them related with current missions of the NASA and ESA agencies. The TPF Mission (Terrestrial Planet Finder) is one of the masterpieces of the NASA Origins Program. Its goal is the detection and characterization of Earth-like planets that orbit nearby stars (see [3], [1]). The TPF configuration is currently considered to be formed by five spacecraft contained in a plane. Four spacecraft are evenly distributed in a baseline of approximately 100 m. The fifth one, the collector, forms an equilater triangle with the two interior satellites of the baseline (see Figure 3.1).

The Darwin Mission is a project of the ESA with a similar objective of the TPF Mission. The Darwin configuration (see [5]) is formed by seven spacecraft contained in a plane. Six of them are on the vertices of a regular hexagon of radius about 100 m and the seventh one is located at the baricenter (see again Figure 3.1).

In the examples, and only for illustration purposes, of we have also computed the delta-v on/off necessary to reconfigure the formation. We note however that the on/off control does not take into account collision avoidance which is a major requirement in the following examples, where the satellites will irremediable collide using the on/off technique.

Changing inner-outer position in the TPF formation

In the TPF formation, the exterior spacecraft have a bigger fuel consumption than the interior ones (see [4]). To compensate the difference, we can consider changing their position at some moment in the lifetime of the mission. In Table 3.1 we present the cost, in terms of total delta-v (cm/s), for each satellite to accomplish the task. Relative trajectories and the profile of delta-v consumption is represented in Figure 3.2.

Figure 3.1: Representations of the TPF (left-hand side) and Darwin (right-hand side) configurations.

Satellite	1	2	3	4	5	Total (cm/s)
Δv	1.65	1.63	1.65	1.67	0.19	6.79
Δv on-off	0.23	0.23	0.23	0.23	0.00	0.92

Table 3.1: Change the position of the consecutive interior and exterior spacecraft of TPF in 8 hours. The formation is considered about a halo orbit of 120000 km of z-amplitude. The results corresponds to a discretization of each trajectory in 50 linear elements.

Exchanging positions of several spacecraft

In this example (see Figure 3.3), we have 4 small spacecraft in a square of length 40 meters and another one in the center of the it. Again the satellites are about a halo orbit of 120000 km of z-amplitude being the central one on the halo orbit. The example consist in switching the satellites located in the opposite vertices, while the central one returns to the same place after letting the other ones pass near the center. Total costs are displayed in Table 3.2.

Rendezvous and formation deployment

We consider the Darwin configuration to perform an example of rendezvous and deployment. We start with two groups of 3 and 4 spacecraft separated by a distance of 1000 m. The example is again about a halo orbit of 120000 km of z-amplitude about the L_2 Sun-Earth libration point. It turns out that the optimum place for the rendezvous is the relative point located in the center of mass of the initial configuration. We show the trajectory and the profiles of the delta-v expenditures in Figure 3.4. Total amounts of Δv are given in Table 3.3.

Satellite	1	2	3	4	5	Total (cm/s)
Δv	1.17	1.02	1.21	0.93	0.28	4.61
Δv on-off	0.39	0.39	0.39	0.39	0.00	1.56

Table 3.2: Delta-v expenditure to change the position of four satellites located in opposite corners of an square of 40 m. Reconfiguration time has been set to 8 hours. In this example 100 linear elements have been used.

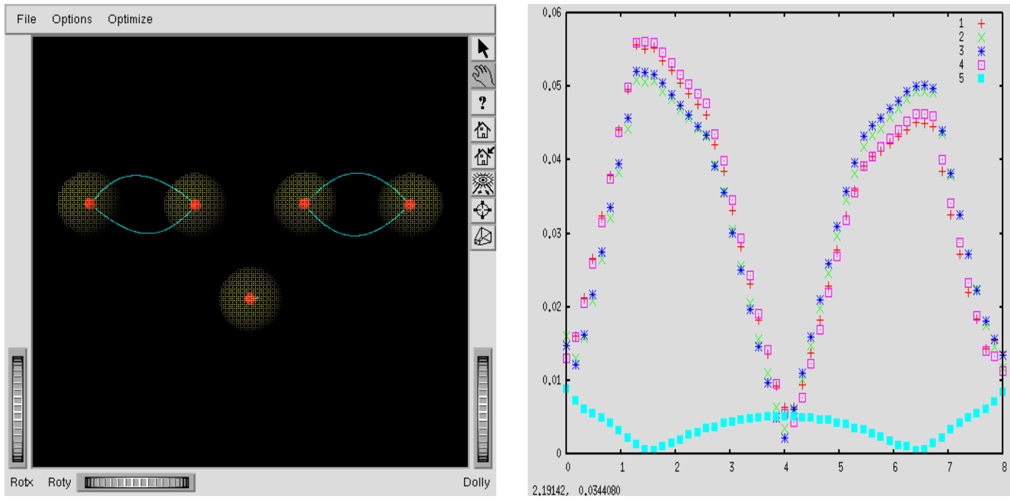

Figure 3.2: In the left-hand side plot we have the trajectory obtained when we change the consecutive interior and exterior spacecraft in the TPF configuration. Satellites are represented in the center of the collision avoidance spheres which cannot intersect during the maneuver (the radius of the sphere is half of the security distance considered). In the right hand side plot we show the profile of the Δv expenditure for each satellite. The corresponding total amounts are given in Table 3.1

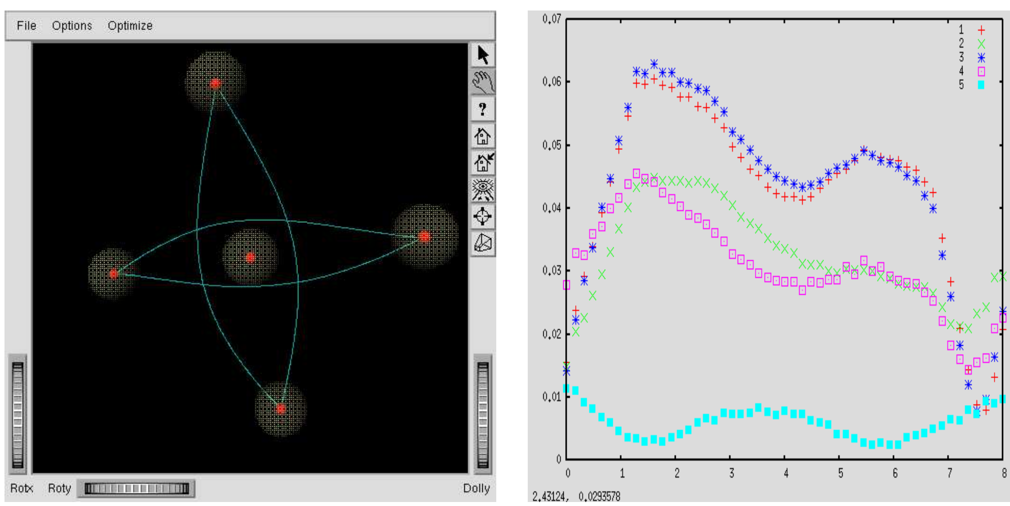

Figure 3.3: In the left-hand side plot we have the trajectory of the spacecraft when we change the position of the opposite spacecraft in a square. In the right hand side plot we have the profile of the Δv expenditure for each spacecraft. Total amounts of Δv are given in Table 3.2.

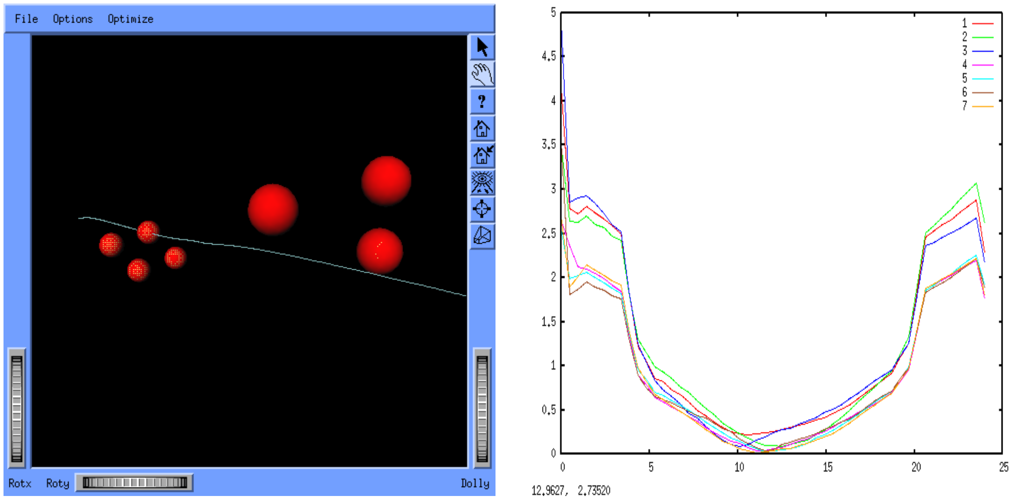

Figure 3.4: In the left-hand side we plot a snapshot of a rendezvous trajectory for the Darwin mission. In the right hand side plot we show the profile of the Δv expenditure for each spacecraft. Total amounts of Δv are given in Table 3.3

Satellite	1	2	3	4	5	6	7	Total (cm/s)
Δv	1.83	1.83	1.83	1.33	1.33	1.33	1.33	10.81
Δv on-off	1.32	1.32	1.32	0.99	0.99	0.99	0.99	7.92

Table 3.3: Δv cost corresponding to the rendezvous example for the Darwin formation. The two groups of satellites depart from 1000 m apart and perform rendezvous in one day. The example uses 50 linear elements for each satellite.

4 Conclusions

In this paper we present a technique for reconfigurations of spacecraft formations based in the use of the finite element method and optimal control. The finite element method provides a systematic approach to the discretization of the problem which tends to a low thrust continuous solution when the mesh of elements is refined. This approach has been presented in several examples concerning the TPF and Darwin missions with satisfactory results but much more general situations can be dealt using the methodology.

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References

[1] Beichman, C., Gómez, G., Lo, M.W., Masdemont, J.J. and Romans, L. Searching for Life with the Terrestrial Planet Finder: Lagrange Point Options for a Formation Flying Interferometer. *Advances in Space Reseach* **34** (2004) 637–644.

[2] McInnes, C.R. Autonomous proximity manoeuvring using artificial potential functions. *ESA Journal* **17**(2) (1993) 159–169.

[3] The TPF Science Working Group. *The Terrestrial Planet Finder: A NASA Origins Program to Search for Habitable Planets*. JPL Publication 99-3, 1999.

[4] Gómez, G., Lo, M.W., Masdemont, J.J. and Museth, K. Simulation on Formation flight near Lagrange Points for the TPF mission. *Advances in the Astronautical Sciences* **109** (2002) 61–75.

[5] The Science and Technology Team of Darwin and Alcatel Study Team. *Darwin. The Infrared Space Interferometer. Concept and Feasibility Study Report*. European Space Agency, ESA-SCI(2000)12, iii+218, July 2000.

[6] Singh, G. and Hadaegh, F. Autonomous Path-Planning for Formation-Flying Applications. In: *International Symposium on Space Flight Dynamics*, 2001.

[7] Volpe, R.A. *Real and Artificial Forces in the Control of Manipulators: Theory and Experiments*. Ph.D. Thesis, Dept. of Physics, Carnegie Mellon University, September, 1990.

[8] Zienkiewicz, O.C. and Taylor, R.L. *The finite element method*. McGraw-Hill, New York, 1994.