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Numerical Search of Bounded Relative Satellite Motion

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Abstract: Relative motion between two or more satellites has been studied for a long time, as the works of W.H. Clohessy and R.S. Wiltshire, dated 1960, or the studies of J. Tschauner, dated 1967, can testify. Not only these early works are milestones for the relative motion modelling, as they provide linear models whose accuracy in terms of motion prediction is granted in the simplified assumption of pure Keplerian motion, but they are also powerful tools to gain insight into the complex dynamical properties of this type of motion. These models supply conditions on the initial relative position and velocity that allow the relative orbits to be periodic, that is closed orbits. When perturbations, such as Earth oblateness and air drag effects, or even the simple nonlinearities of the keplerian gravitational attraction are taken into account in the model, an analytical solution appears more and more complicated to be derived, if not impossible. Simple relations on the initial conditions leading to periodic orbits, such as those that are well known when considering Hill-Clohessy-Wiltshire (HCW) equations, are not to be expected without introducing some simplifications. In these cases a numerical approach could still be able to locate the exact conditions that result in a minimum drift per orbit. This work investigates the possibility of using a global optimization technique to locate the initial conditions resulting into minimal drift per orbit. Before using this approach in the nonlinear problem, the methodology is tested on Hill's and Tschauner-Hempel's models, where an analytical solution is well known. The global optimizer is essentially a genetic algorithm that considers the initial relative velocities between the satellites as the chromosomes of the individuals of the population, the initial relative position is considered as given. This not only reduces the number of variables the GA has to optimize, but it also allows to search for closed relative orbits of a predefined dimension. Results show that the methodology is returning the analytical results with a satisfactorily precision and that is able to locate bounded motion also when nonlinearities become important.

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Nomenclature

LVLH = Local Horizontal Local Vertical
x, y, z = relative position in LVLH frame
$\dot{x}, \dot{y}, \dot{z}$ = relative velocity in LVLH frame
subscript i= values at the initial time
subscript f= values at the final time
f = fitness function
ω_0 = angular velocity of the circular orbit
a = semi-major axis
e = eccentricity
i = inclination
Ω= RAAN
ω = argument of perigee
n = mean motion.

1 Introduction

Many efforts have been made in the last years on modelling and controlling satellites relative dynamic. In [7] conditions for relative orbits invariant with respect to the J2 perturbation are given in terms of mean orbital elements. In literature several linear models of relative dynamics including the second harmonic of the gravitational field, eccentricity and the air drag can be found (see [4, 6, 8]) but nor the analytical solution neither the initial conditions for periodic relative orbits are obtainable in most of these cases. The use of evolutionary/genetic approaches in the aerospace research, especially in mission analysis and design phase, is quite recent [5]. The difficulties encountered when using genetic algorithm in this field stand in the strong dependence that convergence speed shows upon the choice of the fitness function, the mutation and crossover probabilities, the population size and the number of generations. There is not a rigorous mathematical rule to choose these parameters in the best possible way, many times convergence can be achieved only after trial and error adjustment of the parameters with respect to the particular problem. For these reasons many think that global optimization using stochastic algorithms is more art than science. The benefits of these techniques are, though, huge. Stochastical global optimizers may approach many problems, otherwise unsolvable. A review on the use of global optimization techniques in problems related to Mission Analysis and System Design may be found in some recent studies funded under the European Space Agency ARIADNA scheme (see [2]).

2 Genetic Algorithms

Genetic Algorithms (GA) are stochastic global search methods that are based on the principle of natural selection and evolution of the species. These kinds of algorithms result to be effective for optimization problems containing different local optima with discontinuous parts between them. In these cases the calculus-based methods can converge to a local optimum rather then to the desired one. In the present paper a genetic algorithm is applied to minimize the drift per orbit in the simple case of two different linearized keplerian dynamic models. The first model of dynamics (HCW) considers a reference orbit without eccentricity (Hill-Clohessey-Wiltshire equations, [10]), the second model (TH) takes into account the eccentricity (Tschauner-Hempel equations [8] and [11]). After showing the convergence of the method to the well-known analytical conditions that exist for these simple dynamics, the algorithm is run with the nonlinear keplerian relative motion equations and considering an eccentricity value of $e = 0.3$. The relative motion is shown in the LVLH frame. The objective function chosen and maximized by the genetic algorithm has the following form

$$f(x, y, z) = -\sqrt{(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2} \tag{1}$$

representing the error in relative position between the initial conditions and those obtained at the end of the integration. The integration is performed over one orbital period in the linearized cases. For the nonlinear model five periods have been used to make the algorithm converge in a satisfying manner.

If a non linear dynamic, that takes into account all the possible perturbative effects, is considered, there is no clear argument that tells us information on the convexity of the objective function. On the other hand, in the Hill and T.-H. models, the objective function is expected to be convex in the $[e, \dot{y}, f]$ space. The software used for the numerical search is the online PIKAIA freely available tool (see [1]). PIKAIA uses a decimal alphabet made of 10 simple integers (0 through 9) for encoding the chromosome $(\dot{x}, \dot{y}, \dot{z})$. The mutation and crossover characteristic are the default PIKAIA's ones (see [1]).

3 Validation of the GA

3.1 Using the genetic algorithm to find analytical Hill's solutions

Considering the simple HCW equations to describe the relative dynamic between two orbiting objects we have that the analytical condition,

$$\frac{\dot{y_0}}{x_0} = -2\omega_0 \tag{2}$$

on the relative initial conditions, assures a periodic motion. If we now perform some numerical simulations considering a circular orbit with semimajor axis of 7000 km for which the above relation returns a ratio of $\frac{\dot{y_0}}{x_0} = -2.156E - 3s^{-1}$ we get the results contained in Table 3.1. The optimizer is able to converge to the global minimum that, in this case, is also the only minimum of the problem.

3.2 Using genetic algorithm to find analytical Tschauner-Hempel's solutions

As soon as we consider also the effect of the eccentricity on the relative satellite motion, the linear equations become with time periodic coefficients (Tschauner, 1967, [8]). In this

Individuals	Generations	$\frac{\dot{y}_0}{x_0}$ pikaia	Fitness function
20	50	-2.154459E-3	-2.721005E-2
20	100	-2.154459E-3	-2.721005E-2
50	100	-2.155201E-3	-1.425242E-2
100	100	-2.155892E-3	-2.164824E-3
100	500	-2.155892E-3	-2.164824E-3
100	1000	-2.155892E-3	-2.164824E-3

Table 3.1: Convergence of the GA increasing generations and population size.

case the relations between the initial conditions in order to obtain a periodic orbit are dependent on the true anomaly of the reference orbit and are quite complicated (Inalhan et al., 2002, [3]). When the true anomaly is zero, though, it is possible to write a simple analytical relation shown in Equation (3) (Inalhan et al., 2002, [3]):

$$\frac{\dot{y}_0}{x_0} = -\frac{n(2+e)}{(1-e)^{\frac{3}{2}}(1+e)^{\frac{1}{2}}} \,. \tag{3}$$

To perform the numerical simulations 100 individuals and 100 generations have been used. Higher values do not improve the quality of the solution.

Eccentricity	$\frac{\dot{y}_0}{x_0}$ eq.(1)	$\frac{\dot{y}_0}{x_0}$ pikaia	Percentage difference	Fitness function
$e = 0$	-2	-1.999999	-0.00005%	-3.4e-10
$e = 0.1$	-1.9091	-1.90899	-0.0058%	-5.2e-8
$e = 0.2$	-1.8333	-1.83339	0.00491%	-6.9e-8
$e = 0.3$	-1.7692	-1.76920	-0.000056%	-2.9e-8
$e = 0.4$	-1.7143	-1.714199	-0.005892%	-8.9e-7
$e = 0.5$	-1.6667	-1.666599	-0.006060%	-1.2e-6
$e = 0.6$	-1.625	-1.624998	-0.000123%	-3.3e-10
$e = 0.7$	-1.58823	-1.58819	-0.002519%	-4.9e-6
$e = 0.8$	-1.55556	-1.555599	0.002507%	-6.4e-5
$e = 0.9$	-1.526315	-1.526399	0.005503%	-1.6e-5

Table 3.2: GA restitution of the analytical TH conditions for different eccentricities.

The results in Table 3.2 are plotted in Figure 3.1.

As it was expected, the behaviour of the fitness function indicates an infinite number of minima but located in agreement to the Equation (3) (the real fitness has opposite sign with respect to the one here reported as the minima shown in Figure 3.2 are optimal maximum for the GA).

4 Searching Closed Relative Orbits for the Nonlinear Dynamic

When applying the above presented methodology to find the formations with minimal drift per orbit in a non linear dynamic case, problems arises in the propagation scheme that introduces difficulties in the calculation time and in the accuracy of the solution.

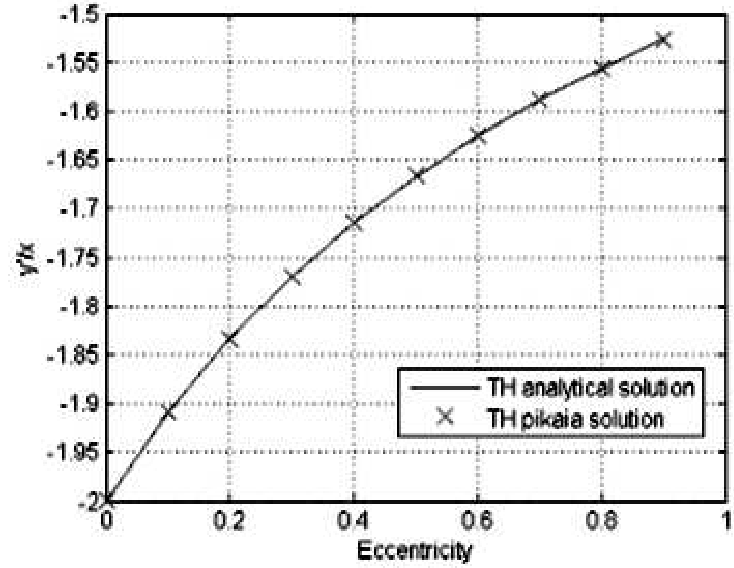

Figure 3.1: TH analytical solution vs. TH solution with GA.

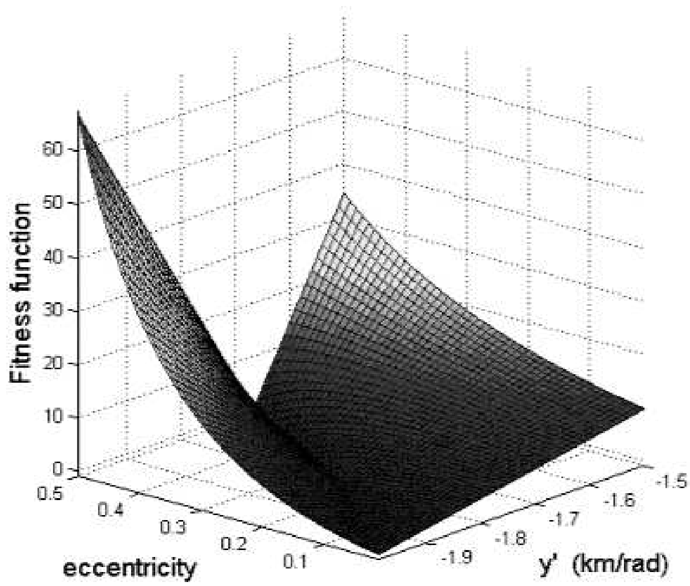

Figure 3.2: Fitness value vs. eccentricity and $\dot{y}$.

Subtracting directly the Cartesian coordinates of the two satellites can easily degrade the quality of the relative position obtained, as it subtract two very close values. In [9] an approach based on a geometric method (called unit sphere projection) is proposed. Integrating the relative dynamic in terms of orbital elements (for the Keplerian case just the true anomaly has to be used, see [9]) and subsequently translate the differences in terms $\delta x, \delta y, \delta z$ is numerically more accurate and the computation time is dramatically reduced. This approach has been used here. In Table 4.1 the used genetic parameters are reported.

Crossover probability	Mutation rate		
	initial	minimum	maximum
0.85	0.005	0.0005	0.25

Table 4.1: Genetic parameters for nonlinear approach.

After numerous trials the number of generations has been set to 500 with a population of 100 individuals and simulations have been performed for different relative orbit sizes. The initial dimension of these orbits increase from a 2 km relative position on the three axes to a 500 km one.

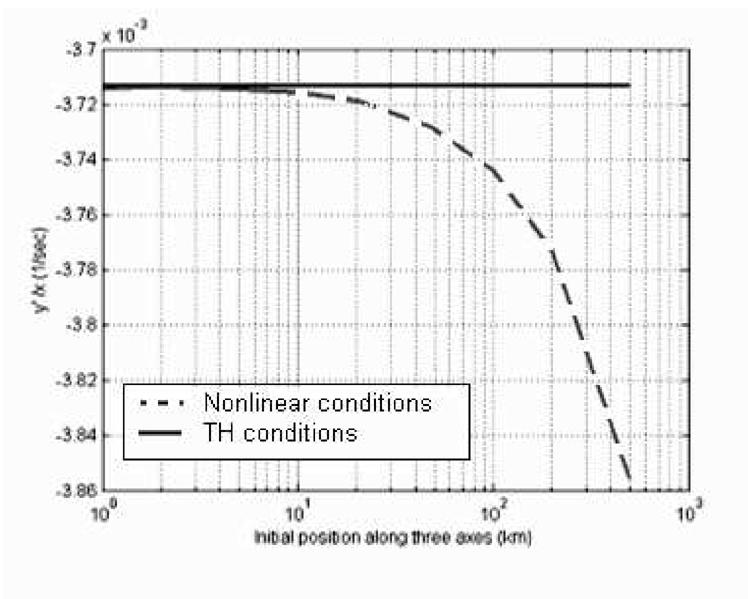

Figure 4.1: $\frac{\dot{y}_0}{x_0}$ ratio compared for the linearized and the nonlinear models (logarithmic scale on x axis).

Comparing the analytical relation of TH with the $\frac{\dot{y}_0}{x_0}$ rate obtained trough the GA it is easy to notice how the linear condition loses its capability to produce bounded orbits as the formation dimension increase. Figure 4.1 shows the comparison between the results given by Equation (3) and the one given by the genetic algorithm. As the size increases the ratio $\frac{\dot{y}_0}{x_0}$, as obtained from our numerical simulations, decreases drastically.

As expected the relation given in Equation (3) allows to obtain closed orbit only for small formations. The GA approach is therefore suitable to get the condition of periodic motion in these cases. In Figure 4.2 and Figure 4.3 a plot of ten orbits is shown as obtained propagating the initial conditions given by Equation (3) and by the GA and for two different formations of different sizes: a small one (2 km) ad a larger one (200 km).

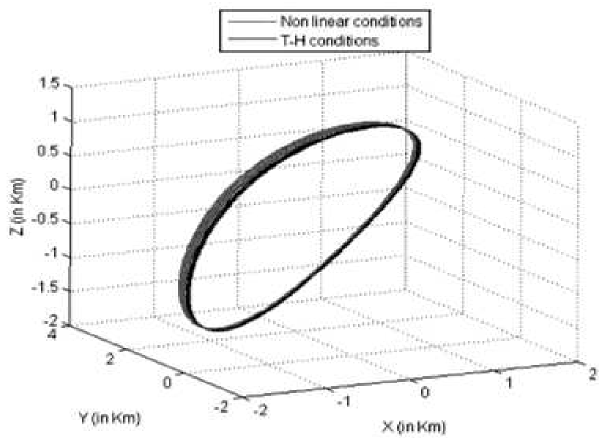

Figure 4.2: 10 orbits (TH vs. nonlinear) for low size (2 km).

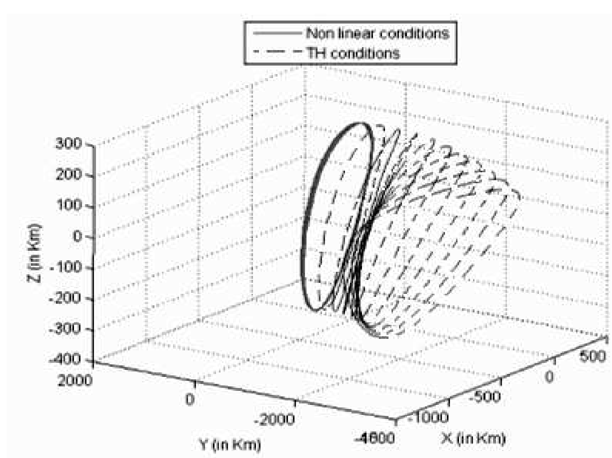

Figure 4.3: 110 orbits (TH vs. nonlinear) for high size (200 km).

As an additional proof of the GA convergence to a satisfying solution the orbital parameters of the two spacecrafts are calculated and compared in Table 4.2.

The only parameter that clearly maintains its value unchanged (considering the numerical errors) is the semi-major axis a. This result coincides with the only constrain to close a relative orbit in a Keplerian motion: the equality of the semi-major axis. In this

Vehicle N1		Vehicle N2		% difference
a	7000 km	a	7000.124 km	0.0018 %
e	0.3	e	0.2928	2.4 %
i	35	i	35.012	0.034%
Ω	35	Ω	33.99	2.9%
ω	35	ω	1.405	95.99 %

Table 4.2: Orbital parameters comparison.

way the two orbits have the same orbital period T and obviously the relative position is repeated every T seconds.

5 Conclusion

The GA strategy here used resulted to be a valid instrument to analyze the behavior of the nonlinear relative dynamics between two satellites in Keplerian orbit. After having re-obtained the Hill's and T.-H.'s solutions for bounded trajectories to check the validity of the algorithm, the GA has been run for the complete mathematical model of relative motion in Keplerian orbit. Considering the numerical approach and the limitations in terms of accuracy for the solutions, the matching period condition have been obtained for closing the relative orbit. The initial velocities generated with the genetic calculation match the analytic relation for T.-H. demonstrating the validity of the linear approach for low dimensions orbits. Increasing size results in a obliged switching to the conditions obtained numerically. Future developments of this new approach to the formation flying problem include the analysis of J2 and drag effects. The present paper represents an introduction and a validation work for the authors whose aim is to apply and study the possibilities given by the genetic algorithm to the most complete as possible model of the relative dynamics of satellites.

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