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Analysis of Network Revenue Management under Uncertainty **

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Abstract: This paper investigates robust dynamic policies for network revenue management problems with uncertainty involved. We formulate such a problem in a setting of semi-definite programming and propose a heuristic procedure to find robust solutions. We also derive sufficient conditions for finding an approximation of the value function. Numerical experiments are included to illustrate the proposed approach.

Keywords: *Robust optimization; dynamic programming; revenue management; uncertainty.*

Mathematics Subject Classification (2000): 90B36, 90B15.

1 Introduction

This paper studies the revenue management problem on a given network in which all involved variables allow uncertainty. The optimal policy of the problem is investigated which determines what a quantity of resource should be offered at each different rate. The problem is important since we can find its broad applications, especially in airline network.

Revenue management is a technique concerned with a number of capacity constrained service industries such as airline, hotel, media, transposition, car rental, tourism and so on. Following the airline deregulation in 1970s, revenue management technique has

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obtained a great progress both in theory and methodology. This progress further pushes the development of service industries.

No matter the difference in the definitions on revenue management, most researchers agree that the primary goal of revenue management is to maximize the revenues in the industry. They also agree that price is a major control tool to achieve this goal among various mechanisms. In the literature of revenue management, Littlewood [16] first proposed a marginal seat revenue principle and applied it to a single leg problem with two fare classes. Belobaba [1] developed a stochastic seat inventory control model to solve the multi-fare-class single leg problem. His model generalized the marginal seat revenue concept to the expected marginal seat revenue principle. The multi-fare-class problem was also studied by Brumelle and McGill [4] and Robinson [19].

There have been various studies of pricing policies in the continuous-time revenue management framework. In a two-fare model that allows a single price change, Feng and Gallego [12] proposed an optimal threshold control policy in 1995. Later, Feng and Xiao [13] generalized their result by considering risk analysis. Using the dynamic programming approach, Liang [15] showed that a threshold control policy is optimal for a continuous-time dynamic yield management model. In 1999, Subramanian et al. [21] incorporated the overbooking control on a single-leg flight into a Markov decision process. In the same year, Chatwin [10] discussed a continuous-time airline-overbooking model with time-dependent fares and refunds. To capture the time dependency of demand, most of airline revenue management models need an assumption of nonhomogeneous demand intensities.

A natural extension of single-leg problem is the network revenue management. A major concept in the study of network revenue management, bid price control, was proposed by Simpson [20] in 1989 and further studied by Williamson [25] and Talluri and Van Ryzin [24].

Up to now, the models that we list above are all based on the certain environment. However, in real applications, circumstances are variable and the data we obtain is uncertain because of various complications. Developing a model to deal with the revenue management under uncertainty is an interesting problem. In 2000, Bertsimas and Popescu [3] proposed a dynamic programming model with demand uncertainty only. But the general procedure for the network revenue management under uncertainty is still an open problem.

We know that optimization technique is the base of research in revenue management. With the recent advances in conic and robust optimization theory [5, 6], we can see its various applications in industries such as mechanical structure design [7], VLSI circuit design [11], systems and control [8] and signal processing [17].

In the paper, we incorporate the robust optimization technique into network revenue management and establish a robust dynamic model to deal with uncertainty produced by demand uncertainty, data perturbation and variable errors. We transform the problem into a robust semi-definite programming and provide a heuristic procedure to obtain the optimal solution of the problem. Furthermore, we discuss the Hamilton-Jacobi-Bellman equation under uncertainty, which establishes a sufficient condition for an optimal solution existing.

This paper is organized as follows. In the next section, we will introduce the background of our problem and establish a robust dynamic model for the problem. In Section 3, we propose a method to determine the optimal policy and provide a heuristic algorithm. In section 4, we establish a sufficient optimal condition under uncertainty. In Section 5, we report some numerical results. Finally, in Section 6, we give our conclusion.

2 Dynamic Model under Uncertainty

2.1 Problem under Uncertainty

We are given an airline network which is composed of m legs providing n origin-destination itineraries. Let a_{ij} be the number of units on leg i used by itinerary j, which induces a $m \times n$ matrix $A = (a_{ij})$. The j-th column of A, denoted by A^j, is a multiple of the incidence vector for itinerary j. Here, we do not restrict that A is a 0-1 matrix, which means that group demand is permitted. $a_{ij} = 0$ implies that leg i is not a part of itinerary j.

The inventory state of the network is described by a vector $x = (x^1, x^2, \dots, x^m)^T$ of leg capacities. We have $0 \le x \le X$, where $X = (X^1, X^2, \dots, X^m)^T$ is the capacity limit vector of the system. If itinerary j is sold, the state of the network changes to $x - A^j$. To simplify our analysis, we do not consider such problems involving cancellations, no-shows or overbooking. In our problem, time is discrete. We assume a finite booking horizon of length T, with time line being partitioned sufficiently fine such that almost surely at most one request appears at each period.

Time is counted backward: time T is the beginning of the booking horizon and time 0 is the end of booking horizon. We assume that ticketing operation will stop when $t = 0$. At time $0 \le t \le T$, all booking events are denoted as a random vector $d(t) = (d_1(t), ..., d_n(t))$. $d_j(t) > 0$ for $1 \le j \le n$ indicates that a requirement for itinerary j occurs at time t while $d_j(t) = 0$ means no requirement for itinerary j at time t. The tickets for each itinerary $j = 1, 2, ..., n$ can be sold at h fares $p_j^r, r = 1, ..., h$. $p_j^r = 0$ implies that the fare p_j^r is unavailable for itinerary j and $p_j = 0$ the itinerary j is unavailable. Suppose that demands for different fares are independent of one another. Let $D_j^r(t)$, $1 \le r \le h$ and $1 \le j \le n$, be the demand flow for the r-th fare on the j-th itinerary, which is a nonhomogeneous Poisson process about time $0 \le t \le T$. The intensity of $D_j^r(t)$ is $\lambda_j^r(t)$, a deterministic function of time t.

We know that in the dynamic world, any unexpected sudden affair would bring a corresponding perturbation to circumstance. For example, weather condition such as fog or storm often cause airlines to adjust their flight schedule. Some flights may be cancelled and some additional flights may be added. Hence, any model, if it would like to simulate the reality, must consider the uncertainty within its parameters.

Let us denote by ΔA the perturbation to itinerary network and Δx the perturbation to the state of inventory. Then, the itinerary-leg matrix under uncertainty should be $A + \Delta A$. The capacity vector under uncertainty should be $x + \Delta x$. Here, Δx also can be observed as overbooking.

Airlines often offer a variety of fares in each fare class of itinerary and also pay varying commissions on these fares. In other industries such as advertising, broadcasting and hotel, the fare negotiation also cause the uncertainty in fares. Thus, we suppose that revenue from selling a ticket on itinerary $j \in \{1, ..., n\}$ at class $r \in \{1, ..., h\}$ is $p_j^r + \Delta p_j^r$, where Δp_j^r is the perturbation to fare p_j^r. In vector, $p^r = (p_1^r, \cdots, p_n^r)^T$ and $\Delta p^r = (\Delta p_1^r, \cdots, \Delta p_n^r)^T$.

In general, demand is concerned with the price. The uncertainty in fares would cause the requirement for itineraries at these fares is uncertain. Another reason for requirement uncertainty is the data error produced in our approaches for requirement forecast. Hence, we suppose that demand flow for class $r \in \{1, ..., h\}$ on itinerary $j \in \{1, ..., n\}$ at time t

is a nonhomogeneous Poisson process with intensity $\lambda_j^r(t) + \Delta\lambda_j^r(t)$, where $\Delta\lambda_j^r(t)$ is the perturbation to $\lambda_j^r(t)$.

Suppose there is an upper bound $\epsilon > 0$ such that for any r, $\|(\Delta x, \Delta A, \Delta p^r)\| \leq \epsilon$ holds for all perturbations, where $\|\cdot\|$ is the norm under Euclidian. Given the time-to-go, k, we call $S(k, x, A, p, \epsilon)$, or simply $S(k, x, \epsilon)$, the *current state* of system.

Our problem is that: Under the current state $S(k, x, \epsilon)$, should we accept or refuse the current request?

2.2 Dynamic Model

To answer the question proposed in the end of the last subsection, we establish a dynamic programming model in the subsection. Then, we solve the model to decide whether the current request is accepted or not.

Let $u_k = \{u_k^{r,j}\}$ denote the decision at time k, where

$$u_k^{r,j} = \begin{cases} 1, & \text{if } p_j^r \text{ is accepted at time } k; \\ 0, & \text{otherwise.} \end{cases}$$

In general, the decision u_k, accepting or refusing, is the function of time k, the capacity vector x and perturbation bound ϵ. Thus, $u_k = u_k(x, \epsilon)$. From our assumption, there is at most one request at each sufficiently small period, i.e., $\sum_{r,j} u_k^{r,j} \leq 1$. The feasible set for u_k at current state is defined as:

$$\begin{aligned} U_k(x, \epsilon) = \{u_k : \sum_{r,j} u_k^{r,j} \leq 1, u_k^{r,j} \in \{0,1\}, (A + \Delta A)u_k \leq (x + \Delta x), \\ \text{for all } \|(\Delta A, \Delta x)\| \leq \epsilon\}, \end{aligned} \tag{1}$$

where $(A + \Delta A)u_k := \sum_{r,j}(A^j + \Delta A^j)u_k^{r,j}$ and $\epsilon > 0$ is a given scalar.

Let $J_k(x + \Delta x)$ denote the maximum expected revenue at current system state $S(k, x, \epsilon)$. Then $J_k(x + \Delta x)$ should satisfy the Bellman equation [2]:

$$J_k(x + \Delta x) = \max_{u_k \in U_k(x,\epsilon)} E[(p + \Delta p)u_k + J_{k-1}((x + \Delta x) - (A + \Delta A)u_k)], \tag{2}$$

where $(p + \Delta p)u_k := \sum_{r,j}(p_j^r + \Delta p_j^r)u_k^{r,j}$, with the boundary conditions:

$$J_0(x + \Delta x) = 0, \quad \forall x, \ \Delta x. \tag{3}$$

We call $J_k(x + \Delta x)$ satisfying (2) and (3) the *value function* under a given state $S(k, x, \epsilon)$. Define the *minimum acceptable fare* (MAF) [12] for itinerary j under state $S(k, x, \epsilon)$ as follows:

$$G_j(x + \Delta x, k) = J_{k-1}(x + \Delta x) - J_{k-1}(x + \Delta x - (A^j + \Delta A^j)).$$

In view of [23], the request for class r on itinerary j at current state $S(k, x, \epsilon)$ is accepted if and only if

$$(p_j^r + \Delta p_j^r) - G_j(x + \Delta x, k) \geq 0 \text{ and } (A^j + \Delta A^j) \leq (x + \Delta x). \tag{4}$$

The intuition of formulation (4) is clear: Under uncertainty, we only accept a fare exceeding the MAF while we have sufficient remaining capacity.

3 Computation of MAF

In the section, we develop a robust linear programming model to obtain the approximation of value function under uncertainty.

Over the remaining period from k to 0, the expected accumulation demand for itinerary j at fare class r can be calculated as $D_j^r(k) + \Delta D_j^r(k) := \sum_{t=1}^{k}(\lambda_j^r(t) + \Delta\lambda_j^r(t))$. Similar as [25, 3], we consider following deterministic integer programming:

$$\begin{aligned} J_k(x+\Delta x) \quad &= \quad \max_{y^r} \min_{\|\Delta p^r\|} \sum_{r=1}^{h} (p^r + \Delta p^r)^T y^r \qquad (5) \\ s.t. \quad & (A+\Delta A)(\sum_{r=1}^{h} y^r) \le (x+\Delta x), \\ & 0 \le y^r \le (D^r(k) + \Delta D^r(k)), \quad \forall r, \\ & y^r \text{ integer vector}, \quad \forall r \\ & \text{for all } \|(\Delta x, \Delta A, \Delta p^r, \Delta D^r(k))\| \le \epsilon, \quad \forall r, \end{aligned}$$

where $y^r = (y_1^r, ..., y_n^r)^T$. This is a robust linear integer optimization problem. In formulation (5), variable y_j^r denotes the amount of accepted demands for itinerary j at fare class r over the remaining horizon. The first inequality in the constrain means that the total amount of accepted demands can not exceed the current capacity. The second inequality means that the amount of accepted demands for various itineraries at fare class r over the remaining horizon should be less then or equal to the expected accumulation demand for various itineraries at fare class r. We want to maximize the revenue under bounded perturbations.

Let $z^r = y^r - \Delta z^r$, where $\Delta z^r := \Delta D^r(k)$. Then, problem (5) can be transformed as a robust linear programming problem in following form:

$$\begin{aligned} L_k(x+\Delta x) \quad &= \quad \max_{\epsilon_0 \le z^r \le D^r(k)} \min_{\|\Delta z^r\| \le \epsilon_0} \sum_{r=1}^{h} (p^r + \Delta p^r)^T (z^r + \Delta z^r) \qquad (6) \\ s.t. \quad & (A+\Delta A)[\sum_{r=1}^{h} (z^r + \Delta z^r)] \le (x+\Delta x), \\ & \text{for all } \|(\Delta x, \Delta A, \Delta p^r)\| \le \epsilon, \ \forall r, \end{aligned}$$

where $\epsilon > 0$ and $\epsilon_0 > 0$ are perturbation bounds given by the problem we consider.

In formulation (6), the integral requirement on variables is relaxed and the perturbation on expected accumulation demand is transformed as that on variables. Thus, problem (6) is a relaxation of problem (5). We assume that both $\epsilon > 0$ and $\epsilon_0 > 0$ are small. The reason we make such an assumption is based on following analysis. First, $\|\Delta p^r\|$ denotes the price perturbation. This perturbation is small in general. Second, $\|\Delta A\|$ and $\|\Delta x\|$ denote variations of flight and capacity caused by some emergent affairs. Although these variations may be great in case of copping with the unexpected emergency, the unexpected accident affair happens at a low probability. It is unimaginable that we always treat routine affairs by the standard for emergency. From the view of long run, the average infection to flight and capacity should be small. Third, we hope to find an optimal solution with small perturbation to itself. Hence, (6) provides us a possible approximation of the allocation of inventory.

Problem (6) is a max-min problem with perturbations to all parameters and variables. To simplify the problem, we transform it to a semi-definite programming just with perturbation to variable via S-lemma in [26, 18].

We call a solution $\epsilon_0 \leq z^r \leq D^r(k), r = 1, ...h$ is *robust feasible* for problem (6) if

$$(A_i + \Delta A_i)[\sum_{r=1}^{h}(z^r + \Delta z^r)] \leq (x_i + \Delta x_i), \quad i = 1, ..., m$$

hold for all $\|(\Delta p^r, \Delta A, \Delta x,)\| \leq \epsilon,\ \|\Delta z^r\| \leq \epsilon_0,\ r = 1, ..., h$, where A_i is the $i-$th row of A. In view of [6], $\epsilon_0 \leq z^r \leq D^r(k), r = 1, ..., h$ is robust feasible if and only if for each i,

$$-A_i \sum_r z^r + x_i - \epsilon\sqrt{\|\sum_r z^r + \sum_r \Delta z^r\|^2 + 1} \geq 0 \tag{7}$$

holds for all $\|\Delta z^r\| \leq \epsilon_0$. The formulation of (7) can be reformulated as:

$$\begin{pmatrix} I & \sqrt{\epsilon}\begin{pmatrix} \sum_r z^r + \sum_r \Delta z^r \\ 1 \end{pmatrix} \\ \sqrt{\epsilon}\begin{pmatrix} (\sum_r z^r + \sum_r \Delta z^r)^T & 1 \end{pmatrix} & -A_i(\sum_r z^r + \sum_r \Delta z^r) + x_i \end{pmatrix} \succeq 0 \tag{8}$$

holds for each i and all $\|\Delta z^r\| \leq \epsilon_0$, where $A \succeq 0$ implies that A is a positive semi-definite matrix.

Now consider the objective function of problem (6). By introducing an additional variable $v \geq 0$ to be maximized, we obtain a new constraint: $\sum_{r=1}^{h}(p^r + \Delta p^r)^T(z^r + \Delta z^r) - v \geq 0$ for all $\|\Delta p^r\| \leq \epsilon, \|\Delta z^r\| \leq \epsilon_0, r = 1, ..., h$. Since both $(z^r + \Delta z^r) \geq 0$ and $(p^r + \Delta p^r) \geq 0$, this constraint is equivalent to $(p^r + \Delta p^r)^T(z^r + \Delta z^r) - v_r \geq 0$ for all $\|\Delta p^r\| \leq \epsilon, \|\Delta z^r\| \leq \epsilon_0, r = 1, ..., h$. Furthermore, each constraint is equivalent to

$$\begin{pmatrix} I & \sqrt{\epsilon}(z^r + \Delta z^r) \\ \sqrt{\epsilon}(z^r + \Delta z^r)^T & (p^r)^T(z^r + \Delta z^r) - v_r \end{pmatrix} \succeq 0, \quad \textit{for all } \|\Delta z^r\| \leq \epsilon_0. \tag{9}$$

In view of S-Lemma in [26, 18], we can obtain the conclusion: (8) holds if and only if there exists a $\mu_i \geq 0$ for each i such that

$$\begin{pmatrix} I & \sqrt{\epsilon}\begin{pmatrix} \sum_r z^r \\ 1 \end{pmatrix} & \sqrt{\epsilon}\begin{pmatrix} I \\ 0 \end{pmatrix} \\ \sqrt{\epsilon}\begin{pmatrix} (\sum_r z^r) \\ 1 \end{pmatrix}^T & -A_i(\sum_r z^r) + x_i & -\frac{1}{2}A_i \\ \sqrt{\epsilon}\begin{pmatrix} I \\ 0 \end{pmatrix}^T & -\frac{1}{2}A_i^T & 0 \end{pmatrix} - \mu_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \\ 0 & 0 & -I \end{pmatrix} \succeq 0. \tag{10}$$

Similarly, (9) holds if and only if there exists a $\mu_r \geq 0$ for each r such that

$$\begin{pmatrix} I & \sqrt{\epsilon}\sum_r z^r & \sqrt{\epsilon}I \\ \sqrt{\epsilon}(\sum_r z^r)^T & (p^r)^T z^r - v_r & -\frac{1}{2}p^r \\ \sqrt{\epsilon}I & -\frac{1}{2}(p^r)^T & 0 \end{pmatrix} - \mu_r \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \\ 0 & 0 & -I \end{pmatrix} \succeq 0. \tag{11}$$

Combining (7)–(11), we can obtain following theorem.

Theorem 3.1 *The robust linear programming (6) is equivalent to the following problem:*

$$\begin{array}{ll} \text{maximize} & \sum_{r=1}^{h} v_r, \\ \textit{subject to} & (10),(11) \text{ and } \epsilon_0 \leq z^r \leq D^r(k), \forall r. \end{array} \tag{12}$$

Problem (12) is a typical semi-definite programming. There are many effective methods [22, 14] to solve this problem.

Now we present a robust heuristic algorithm for network revenue management under uncertainty based on the robust optimization technique.

Robust Heuristic Algorithm (RHA):

At any current state $S(k, x, \epsilon)$, where ϵ is the perturbation bound for all parameters.

1. *For a request for itinerary j at class r, computer MAF via solving (12).*
2. *Sell to itinerary j if and only if its fare p_j^r exceeds its MAF, i.e., $p_j^r \geq G_j(x, k)$.*
3. *Back to step 1 for next request.*

4 Hamilton-Jacobi Equation

In this section, we further explore the property of value function and prove that Hamilton-Jacobi equation [9] still holds under uncertainty.

From (2) and (4), the value function can be expressed an inductive formulation as follows:

$$\begin{aligned} J_k(x+\Delta x) &= \max_{u_k \in U_k(x,\epsilon)} E[(p+\Delta p)u_k + J_{k-1}((x+\Delta x) - (A+\Delta A)u_k)] \\ &= J_{k-1}(x+\Delta x) + \max_{u_k \in U_k(x,\epsilon)} E[(p+\Delta p)u_k - G_j(x+\Delta x, k)u_k]^+ \\ &= J_{k-1}(x+\Delta x) + \sum_{r,j} (\lambda_j^r(k) + \Delta\lambda_j^r(k))[(p_j^r + \Delta p_j^r) - G_j(x+\Delta x, k)]^+, \end{aligned}$$

where $[\cdot]^+ := \max\{0, \cdot\}$. Let $\Delta J_k(x+\Delta) = J_{k-1}(x+\Delta x) - J_k(x+\Delta x)$. We obtain the difference equation as follows:

$$0 = \Delta J_k(x+\Delta x) + \sum_{r,j} (\lambda_j^r(k) + \Delta\lambda_j^r(k))[(p_j^r + \Delta p_j^r) - G_j(x+\Delta x, k)]^+.$$

We call $f(x)$ is an *ϵ-approximation* of $F(x)$ on X if there exists $\epsilon > 0$ and a constant α such that $\|f(x) - F(x)\| \leq \alpha\epsilon$ for all $x \in X$. In view of Hamilton-Jacobi equation [9], if take $\Delta J_k(x+\Delta x)$ as an ϵ-approximation of derivation, then we have following approximately sufficient optimality condition.

Theorem 4.1 *Suppose $\lambda_j^r(t)$ is continuous about $0 \leq t \leq T$. Partition $[0, T]$ into K sufficiently small intervals and arbitrarily take a point k from each small interval $T_k, k = 1, ..., K$. If for any given $\eta > 0$, there exist continuous function $J_t(x)$ such that $\Delta J_k(x+\Delta x)$ is an ϵ-approximation of $\frac{\partial J_t(x)}{\partial t}$ and satisfies*

$$\left|\sum_{r,j} (\lambda_j^r(k) + \Delta\lambda_j^r(k))[(p_j^r + \Delta p_j^r) - G_j(x+\Delta x, k)]^+ + \Delta J_k(x+\Delta x)\right| \leq \eta \tag{13}$$

and

$$J_0(x+\Delta x) = 0$$

for all $\|(\Delta x, \Delta A, \Delta p, \Delta\lambda)\| \le \epsilon$, *where* $\epsilon > 0$ *is a given parameter. Then,* $J_k(x+\Delta x)$ *is an* ϵ*-approximately value function.*

Proof Since $J_t(x)$ is continuous and T_k is sufficiently small, we only need to prove that $J_t(x)$ satisfies Hamilton-Jacobi equation [9].

From $|\Delta J_k(x+\Delta x) - \frac{\partial}{\partial t}J_t(x)| \le \epsilon$, the continuity of $\lambda_j^r(t)$ and $J_t(x)$ and (13), we have

$$|\sum_{r,j} \lambda_j^r(t)[(p_j^r - J_{t-1}(x) + J_{t-1}(x-A^j)]^+ + \frac{\partial}{\partial t}J_t(x)| \le \rho_1\eta + \rho_2\epsilon,$$

where ρ_1, ρ_2 are constants. Taking $\epsilon = \eta$ will finish the proof. □

Theorem 4.1 has an important meaning: The value function under uncertainty determined by (13) is the ϵ-approximation of value function in certainty.

5 Numerical Experiments

In the section, we will exhibit some numerical examples on the optimal booking control by the following example.

Example 5.1 *Consider the airline network whose leg-itinerary matrix is given as follows:*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The current state we consider is $S = (x, T, \epsilon)$, *where the capacity vector* $x = (600, 500, 280)^T$ *and* $T = 200$. *There are* $h = 2$ *fare classes for itinerary* $i = 1, 2, 3, 4$ *in the problem. The fares and their demand rates are tabulated in Table 5.1.*

We take $\mu_1 = 0.75, \mu_2 = 0.8$ in (12) and calculate by Robust Heuristic Algorithm the value function for each j at various ϵ. The results are presented in Figure 5.1 – Figure 5.4 as follows. The figures display the monotone evolution of the MAFs of disparate itineraries. The curves in the figures do not intersect with each other, which numerically depicts the corresponding monotone behaviors. This example shows the algorithm RHA is effective for a kind of robust revenue management problems.

Itinerary	1	2	3	4
p_i^1	400	300	560	320
p_i^2	350	260	400	280
λ_i^1	$50+25t$	$60+10t$	$30+15t$	$25+12t$
λ_i^2	$40+5t$	$60+10t$	$50+10t$	$20+11t$

Table 5.1: The data for Example 5.1.

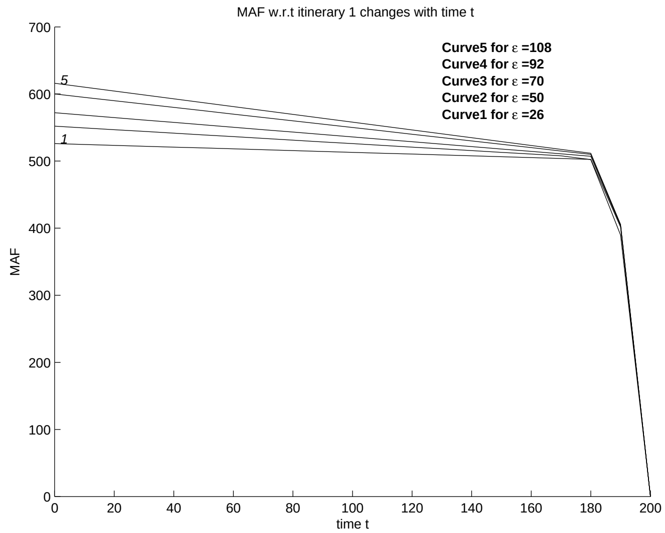

Figure 5.1: Picture of MAF for itinerary 1 changes with t for various ϵ.

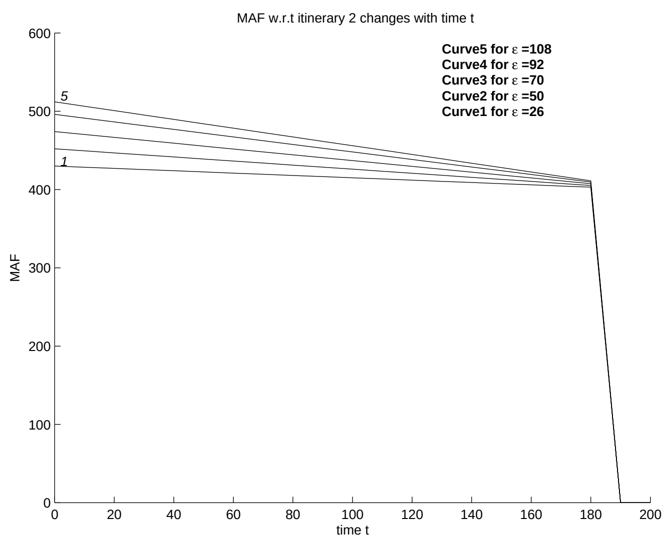

Figure 5.2: Picture of MAF for itinerary 2 changes with t for various ϵ.

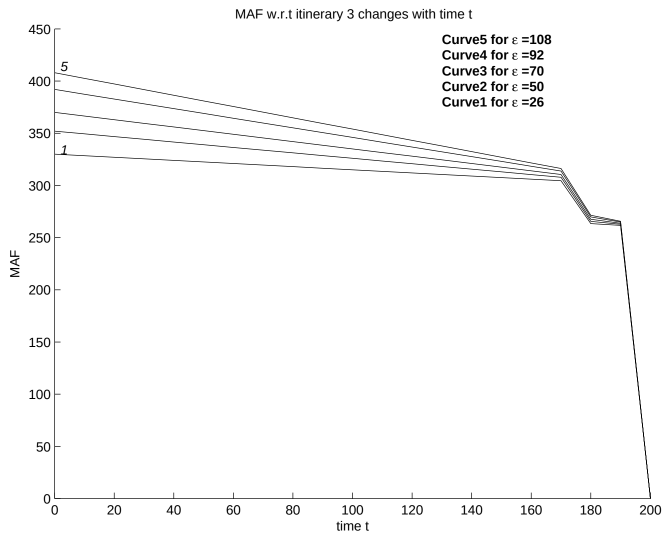

Figure 5.3: Picture of MAF for itinerary 3 changes with t for various ϵ.

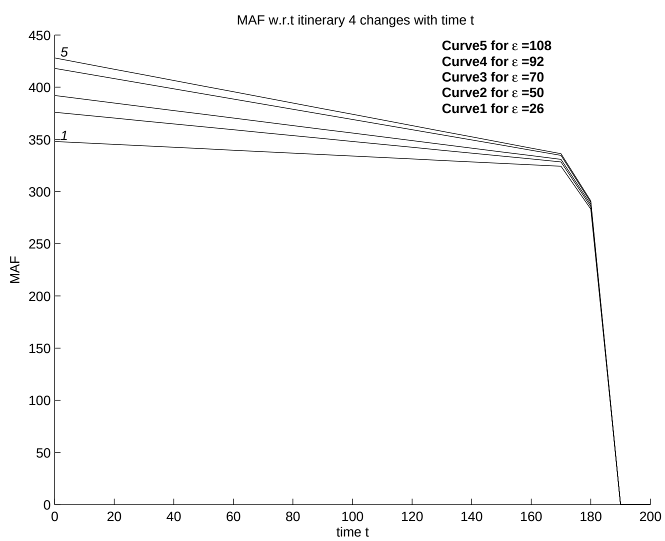

Figure 5.4: Picture of MAF for itinerary 4 changes with t for various ϵ.

6 Conclusion

This paper studies the network revenue management under uncertainty. A robust dynamic model for the problem is established and a heuristic is provided to find the robust solutions. Some numerical results are given to show that the algorithm is efficient. From the figures, we can observe that MAF is monotone of time for small ϵ. We estimate that MAF is also monotone of remaining capacity x for small ϵ.

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