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Robustly Global Exponential Stability of Time-varying Linear Impulsive Systems with Uncertainty [†]

Bin Liu [1*] and Fangyi Liu [2]

[1] *Department of Control Science and Engineering*
Huazhong University of Science and Technology, Wuhan, 430074, China;
Department of Information and Computation Sciences
Hunan University of Technology, Zhuzhou, 412008, China
[2] *Special Class for the Gifted Young*
University of Science and Technology of China, Hefei, Anhui 230026, China

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Abstract: This paper studies linear impulsive systems with varying time-delay and uncertainty. By using the method of the variation of constants formula for impulsive system, robustly global exponential stability criteria are established in terms of fairly simple algebraic conditions. Estimate of the decay rate of the solutions of such systems are also derived. Some examples are given to illustrate the main results.

Keywords: *Uncertainty; linear impulsive system; interval matrix; robustly global exponential stability; decay rate.*

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1 Introduction

Many real world systems display both continuous and discrete characteristics. For example, evolutionary processes such as biological neural networks, bursting rhythm models in pathology, optimal control models in economics, frequency-modulated signal processing systems, and flying object motions, etc., are characterized by abrupt changes of states at certain time instants. Those sudden and sharp changes are often of very short duration and are thus assumed to occur instantaneously in the form of impulses. Such impulses may be represented by discrete maps. Systems undergoing abrupt changes may not be

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[*] Corresponding author: oliverliu78@163.com

well described by using purely continuous or purely discrete models. However, they can be appropriately modelled by impulsive systems. It is now recognized that the theory of impulsive systems provides a natural framework for mathematical modelling of many such real world phenomena. Significant progress has been made in the theory of impulsive systems in recent years, see [1-8, 15] and references therein. Meantime, the robust stability problems for discrete systems have also been studied in recent literatures, see [15-18] and references therein. However, the corresponding theory for impulsive systems with uncertainty has not been fully developed. Recently, some robust asymptotic stability results for impulsive systems and impulsive hybrid systems with uncertainty have been established in [9-14]. In this paper, by using the variation of constants formula for impulsive systems, we shall establish some criteria on robustly global exponential stability and provide some estimate on the decay rate for time-varying linear impulsive systems with uncertainty.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we establish robustly exponential stability for time-varying linear impulsive systems with uncertainty. In Section 4, some examples are also worked out to demonstrate the main results.

2 Preliminaries

Let R^n denote the n-dimensional real vector space and $||A||$ be the norm of a matrix A induced by the Euclidean norm, i.e., $||A|| = [\lambda_{\max}(A^T A)]^{\frac{1}{2}}$. Let N denote the set of positive integers, i.e., $N = \{1, 2, \cdots\}$, and $R^+ = [0, +\infty)$. Let $PC[R^+, R]$ denote the class of piecewise continuous functions from R^+ to R, with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \cdots$. Let $\lambda_i(X), i = 1, 2, \cdots, n$, be all the eigenvalues of the matrix X and $\lambda_{\max}(X)$ (respectively, $\lambda_{\min}(X)$) the maximum (respectively, minimum) eigenvalue of the matrix X.

Consider the following time-varying linear impulsive system with uncertainty

$$\begin{cases} \dot{y}(t) = A(t)y(t) + \tilde{A}(t)y(t), & t \in (t_{k-1}, t_k], \\ \Delta y(t) = C_k y(t^-) + \tilde{C}_k y(t^-), & t = t_k, k \in N, \end{cases} \tag{1}$$

and its nominal system

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \in (t_{k-1}, t_k], \\ \Delta x(t) = x(t^+) - x(t^-) = C_k x(t^-), & t = t_k, k \in N, \end{cases} \tag{2}$$

under the following assumptions:

(A_1) The sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots$, with $\lim_{k\to\infty} t_k = \infty$.

(A_2) $A(t) = (a_{ij}(t))$ is an $n \times n$ matrix, and $a_{ij} \in PC[R^+, R], i, j = 1, 2, \cdots, n$.

(A_3) $\tilde{A}(t) = (\tilde{a}_{ij}(t))$ is a disturbance matrix of $A(t)$ with $\tilde{a}_{ij} \in PC[R^+, R], i, j = 1, 2, \cdots, n$.

(A_4) For every $k \in N$, C_k and its disturbance matrix $\tilde{C}_k$ are $n \times n$ matrices.

(A_5) Every solution of (1) (respectively, (2)) exists globally and uniquely on R^+ and is continuous except at $t_k, k \in N$, at which it is left-hand continuous, i.e., $y(t_k) = y(t_k^-)$. Let $y(t) = y(t, t_0, y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$.

Let Ω_1 be the set of all disturbance matrices $\tilde{A}(t)$ satisfying (A_3) such that, for any $t \in R^+$, $\|\tilde{A}(t)\| \le K_1$, where K_1 is some positive constant. Furthermore, let Ω_2 be the set of all disturbance matrices $\tilde{C}_k, k \in N$, satisfying (A_4) such that, for any $k \in N$, $\|\tilde{C}_k\| \le K_2$, where K_2 is an appropriate positive constant.

Definition 2.1 System (1) is said to be robustly global exponential stable with decay rate $\alpha > 0$ if, for any initial condition $y(t_0^+) = y_0$ and for every disturbance matrices $\tilde{A}(t) \in \Omega_1, \tilde{C}_k \in \Omega_2$, $k \in N$, the trivial solution of system (1) is globally exponentially stable with decay rate $\alpha > 0$, i.e., there exist two positive numbers $\alpha > 0$, and $K \geq 1$, such that

$$\|y(t)\| \leq K\|y_0\|e^{-\alpha(t-t_0)}, \quad t \geq t_0. \tag{3}$$

The aim of this paper is to establish the robustly exponential stability criteria for the time-varying linear impulsive system with uncertainty. The following preliminaries are adopted from [1].

Let $\Phi_k(t,s)$ be the fundamental matrix solution (Cauchy Matrix) (see [1]) of the linear system

$$\dot{x}(t) = A(t)x, \qquad t_{k-1} < t < t_k. \tag{4}$$

Then the solution $x(t)$ to system (2), which satisfies the initial condition $x(t_0^+) = x_0$, can be written in the form

$$x(t) = W(t, t_0^+)x_0, \quad t \geq t_0, \tag{5}$$

where $W(t,s)$ is the fundamental matrix solution (Cauchy Matrix) of the linear system (2) with $W(t,t) = I$ given by (see [1])

$$W(t,s) = \begin{cases} \Phi_k(t,s), \qquad \text{for} \quad t,s \in (t_{k-1}, t_k]; \\ \Phi_{k+1}(t,t_k)(I + C_k)\Phi_k(t_k, s), \qquad \text{for} \quad t_{k-1} < s \leq t_k < t \leq t_{k+1}; \\ \Phi_{k+1}(t,t_k)\Pi_{j=k}^{i+1}(I + C_j)\Phi_j(t_j, t_{j+1}) \cdot (I + C_i)\Phi_i(t_i, s), \\ \qquad\qquad\qquad\qquad \text{for} \quad t_{i-1} < s \leq t_i < t_k < t \leq t_{k+1}. \end{cases} \tag{6}$$

Lemma 2.1 [1] *Assume that* (A_1) *holds. Suppose that* $m \in PC^1[R^+, R], p \in C[R^+, R^+]$ *and* $m(t)$ *is left-continuous at* $t_k, k = 1, 2, \cdots$. *If for* $k = 1, 2, \cdots$,

$$m(t) \leq C + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0 < t_k < t} \beta_k m(t_k), \quad t \geq t_0, \tag{7}$$

where $\beta_k \geq 0$, *and* C *are constants, then*

$$m(t) \leq C\Pi_{t_0 < t_k < t}(1 + \beta_k)e^{\int_{t_0}^{t} p(s)ds}, \quad t \geq t_0. \tag{8}$$

3 Main Results

In this section, we shall establish the robust exponential stability criteria for system (1).

Theorem 3.1 *Suppose Assumptions* (A_1)-(A_5) *hold. Then, the system (2) is exponentially stable with decay rate* $\alpha > 0$ *if and only if there exists a constant* $M \geq 1$ *such that*

$$\|W(t, s^+)\| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq t_0. \tag{9}$$

Proof *Sufficiency.* Suppose (9) holds. Then, by (5), we get

$$\|x(t)\| \leq \|W(t, t_0^+)\|\|x_0\| \leq M\|x_0\|e^{-\alpha(t-t_0)}, \quad \text{for all} \quad t \geq t_0, \quad x_0 \in R^n. \tag{10}$$

Hence, the system (2) is globally exponentially stable with decay rate α.

Necessity. If the system (2) is globally exponentially stable with decay rate $\alpha > 0$, then there exists a positive constant $M \geq 1$ such that $\|x(t)\| \leq M\|x_0\|e^{-\alpha(t-t_0)}$ holds. Thus, for any $x_0 \neq 0$, we get

$$Me^{-\alpha(t-t_0)} \geq \frac{\|x(t)\|}{\|x_0\|} = \frac{\|W(t,t_0^+)x_0\|}{\|x_0\|}, \tag{11}$$

which implies that

$$\|W(t,t_0^+)\| = \sup_{x_0\neq 0}\left\{\frac{\|W(t,t_0^+)x_0\|}{\|x_0\|}\right\} \leq Me^{-\alpha(t-t_0)} \tag{12}$$

and hence, by the properties of Cauchy matrix, we have

$$\begin{aligned}\|W(t,s^+)\| &= \|W(t,t_0^+)W^{-1}(s,t_0^+)\| \leq \|W(t,t_0^+)\|\|W^{-1}(s,t_0^+)\| \\ &\leq Me^{-\alpha(t-t_0)}\cdot\{Me^{-\alpha(s-t_0)}\}^{-1} = e^{-\alpha(t-s)} \leq Me^{-\alpha(t-s)}.\end{aligned} \tag{13}$$

The proof is complete. □

Theorem 3.2 *Assume that Assumption (A_1)-(A_5) hold. Furthermore, suppose that the following conditions hold.*
(1) For any $k \in N$, the system (4) is exponentially stable with decay rate $\alpha > 0$, i.e., there exists a constant $M \geq 1$ such that

$$\|\Phi_k(t,s)\| \leq Me^{-\alpha(t-s)}, \quad t_{k-1} < s \leq t \leq t_k, k \in N. \tag{14}$$

(2) There exist constants $\gamma > 0$, $M_1 \geq 0$, with $0 < \gamma < \min\{\frac{\alpha}{M}, K_1\}$ such that

$$\int_{t_0}^{t}\|\tilde{A}(s)\|ds \leq \gamma(t-t_0) + M_1, \quad t \geq t_0. \tag{15}$$

(3) There exists a constant β with $0 < \beta < \alpha - M\gamma$ such that

$$\sup_{\tilde{C}_i\in\Omega_2}\left\{\sum_{i=0}^{k}\ln\left(M\|I + C_i + \tilde{C}_i\|\right)\right\} \leq \beta(t_k - t_0), \quad \text{for all} \quad k \in N. \tag{16}$$

Then system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta > 0$.

Proof Let $y(t) = y(t,t_0,y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$. For $t \in (t_{k-1}, t_k], k \in N$, by the variation of constants formula, we get

$$\begin{aligned}y(t) &= W(t,t_{k-1}^+)y(t_{k-1}^+) + \int_{t_{k-1}}^{t}W(t,s)\tilde{A}(s)y(s)ds \\ &= \Phi_k(t,t_{k-1}^+)y(t_{k-1}^+) + \int_{t_{k-1}}^{t}\Phi_k(t,s)\tilde{A}(s)y(s)ds\end{aligned} \tag{17}$$

Thus, by (14) and (17), for $t \in (t_{k-1}, t_k], k \in N$, we obtain

$$\begin{aligned}\|y(t)\| &\leq \|\Phi_k(t,t_{k-1}^+)\|\|y(t_{k-1}^+)\| + \int_{t_{k-1}}^{t}\|\Phi_k(t,s)\|\|\tilde{A}(s)\|\|y(s)\|ds \\ &\leq Me^{-\alpha(t-t_{k-1})}\|y(t_{k-1}^+)\| + M\int_{t_{k-1}}^{t}e^{-\alpha(t-s)}\|\tilde{A}(s)\|\|y(s)\|ds.\end{aligned} \tag{18}$$

This implies

$$\|y(t)\|e^{\alpha t} \leq Me^{\alpha t_{k-1}}\|y(t_{k-1}^+)\| + M\int_{t_{k-1}}^t e^{\alpha s}\|\tilde{A}(s)\|\|y(s)\|ds,\ t \in (t_{k-1}, t_k], k \in N. \quad (19)$$

By Gronwall-Bellman inequality, we have

$$\|y(t)\|e^{\alpha t} \leq Me^{\alpha t_{k-1}}\|y(t_{k-1}^+)\|e^{M\int_{t_{k-1}}^t \|\tilde{A}(s)\|ds},\ t \in (t_{k-1}, t_k], k \in N. \quad (20)$$

Hence, by (20), for $t \in (t_{k-1}, t_k], k \in N$, we get

$$\begin{aligned}\|y(t)\| &\leq Me^{-\alpha(t-t_{k-1})}\|y(t_{k-1}^+)\|e^{M\int_{t_{k-1}}^t \|\tilde{A}(s)\|ds} \\ &= Me^{-\alpha(t-t_{k-1})}e^{M\int_{t_{k-1}}^t \|\tilde{A}(s)\|ds}\|I + C_{k-1} + \tilde{C}_k\|\|y(t_{k-1})\|. \quad (21)\end{aligned}$$

Specially, we have

$$\|y(t_k)\| \leq Me^{-\alpha(t_k-t_{k-1})}e^{M\int_{t_{k-1}}^{t_k} \|\tilde{A}(s)\|ds}\|I + C_{k-1} + \tilde{C}_k\|\|y(t_{k-1})\|. \quad (22)$$

Thus, by (21)-(22) and conditions (2)-(3), for $t \in (t_{k-1}, t_k], k \in N$, it follows that

$$\begin{aligned}\|y(t)\| &\leq \left(\Pi_{i=1}^{k-1} M\|I + C_i + \tilde{C}_i\|\right)e^{-\alpha(t-t_0)+M\int_{t_0}^t \|\tilde{A}(s)\|ds}\|y_0\| \\ &= e^{-\alpha(t-t_0)+M\int_{t_0}^t \|\tilde{A}(s)\|ds + \sum_{i=1}^{k-1}\ln M\|I+C_i+\tilde{C}_i\|}\|y_0\| \\ &\leq e^{-\alpha(t-t_0)+M\gamma(t-t_0)+MM_1+\beta(t_{k-1}-t_0)}\|y_0\| \\ &\leq e^{MM_1}e^{-(\alpha-M\gamma-\beta)(t-t_0)}\|y_0\|. \quad (23)\end{aligned}$$

Hence, the system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta$. The proof is complete. □

Theorem 3.3 *Assume that Assumptions (A_1)-(A_5) hold and system (2) is exponentially stable with decay rate $\alpha > 0$, i.e., (9) holds. Furthermore, suppose that the condition (2) of Theorem 3.2 holds and the following condition is satisfied.*
(1) There exists a constant β with $0 < \beta < \alpha - M\gamma$ such that*

$$\sup_{\tilde{C}_i \in \Omega_2}\left\{\sum_{i=0}^k \ln\left(1 + M\|\tilde{C}_i\|\right)\right\} \leq \beta(t_k - t_0), \quad \textit{for all} \quad k \in N. \quad (24)$$

Then system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta > 0$.

Proof Let $y(t) = y(t, t_0, y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$. For $t \in (t_{k-1}, t_k], k \in N$, by the variation of constants formula for impulsive system (Theorem 2.5.1 in [1]), we get

$$y(t) = W(t, t_0^+)y(t_0^+) + \int_{t_0}^t W(t, s)\tilde{A}(s)y(s)ds + \sum_{i=1}^{k-1} W(t, t_i^+)\tilde{C}_i y(t_i). \quad (25)$$

Thus, by (9) and (25), for $t \in (t_{k-1}, t_k], k \in N$, we obtain

$$\begin{aligned}\|y(t)\| &\le \|W(t,t_0^+)\|\|y(t_0^+)\| + \int_{t_0}^{t} \|W(t,s)\|\|\tilde{A}(s)\|\|y(s)\|ds \\ &\quad + \sum_{i=1}^{k-1} \|W(t,t_i^+)\|\|\tilde{C}_i\|\|y(t_i)\| \le Me^{-\alpha(t-t_0)}\|y_0\| \\ &\quad + M\int_{t_0}^{t} e^{-\alpha(t-s)}\|\tilde{A}(s)\|\|y(s)\|ds + M\sum_{i=1}^{k-1} e^{-\alpha(t-t_i)}\|\tilde{C}_i\|\|y(t_i)\|. \end{aligned} \tag{26}$$

This implies that for $t \in (t_{k-1}, t_k], k \in N$,

$$\|y(t)\|e^{\alpha t} \le Me^{\alpha t_0}\|y_0\| + M\int_{t_0}^{t} e^{\alpha s}\|\tilde{A}(s)\|\|y(s)\|ds + M\sum_{i=1}^{k-1} e^{\alpha t_i}\|\tilde{C}_i\|\|y(t_i)\|. \tag{27}$$

By Lemma 2.1, we have

$$\|y(t)\|e^{\alpha t} \le Me^{\alpha t_0}\Pi_{i=1}^{k-1}(1+M\|\tilde{C}_i\|) \cdot e^{M\int_{t_0}^{t}\|\tilde{A}(s)\|ds}\|y_0\|,\ t \in (t_{k-1}, t_k], k \in N. \tag{28}$$

Hence, by (28), for $t \in (t_{k-1}, t_k], k \in N$, we get

$$\begin{aligned}\|y(t)\| &\le Me^{-\alpha(t-t_0)}\Pi_{i=1}^{k-1}(1+M\|\tilde{C}_i\|) \cdot e^{M\int_{t_0}^{t}\|\tilde{A}(s)\|ds}\|y(t_0)\| \\ &\le Me^{-\alpha(t-t_0)+\sum_{i=1}^{k-1}\ln(1+M\|\tilde{C}_i\|)+M\gamma(t-t_0)+MM_1}\|y_0\| \\ &\le Me^{MM_1}e^{-(\alpha-\beta-M\gamma)(t-t_0)}\|y_0\| \end{aligned} \tag{29}$$

Hence, the system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta$. The proof is complete. □

In the following, we specialize the results obtained above to a class of interval linear impulsive systems (see [13-14]). Interval linear impulsive systems can be described as:

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t), & t \in (t_{k-1}, t_k], \\ \Delta x(t) = \tilde{C}_k x(t), & t = t_k, k \in N, \end{cases} \tag{30}$$

where $\tilde{A}, \tilde{C_k} \in R^{n\times n}$ are interval matrices satisfying

$$\tilde{A} \in N[A^{(1)}, A^{(2)}] = \{\tilde{A} = (\tilde{a}_{ij})_{n\times n} : {a_{ij}}^{(1)} \le \tilde{a}_{ij} \le {a_{ij}}^{(2)}\},$$

and

$$\tilde{C}_k \in N[C_k^{(1)}, C_k^{(2)}] = \{\tilde{C}_k = (\tilde{c}_{ij_k})_{n\times n} : {c_{ij_k}}^{(1)} \le \tilde{c}_{ij_k} \le {c_{ij_k}}^{(2)}\}.$$

By [13], an interval matrix $\tilde{X} \in N[X^{(1)}, X^{(2)}]$ can be described as:

$$\tilde{X} = X + E_X\Sigma_X F_X, \tag{31}$$

where $X = \frac{1}{2}(X^{(1)} + X^{(2)})$, $H = (h_{ij})_{n\times n} = \frac{1}{2}(X^{(2)} - X^{(1)})$,
$\Sigma_X \in \Sigma^* = \Big\{\Sigma \in R^{n^2\times n^2} : \Sigma = \mathrm{diag}\{\varepsilon_{11}, \cdots, \varepsilon_{n^2n^2}\}, |\varepsilon_{ij}| \le 1; i,j = 1,2,\cdots,n.\Big\}$,
$E_X E_X^T = \mathrm{diag}\Big\{\sum_{j=1}^n h_{1j}, \sum_{j=1}^n h_{2j}, \cdots, \sum_{j=1}^n h_{nj}\Big\} \in R^{n\times n}$,
$F_X^T F_X = \mathrm{diag}\Big\{\sum_{j=1}^n h_{j1}, \sum_{j=1}^n h_{j2}, \cdots, \sum_{j=1}^n h_{jn}\Big\} \in R^{n\times n}$.

By (31), we denote $\tilde{A} = A + E_A \Sigma_A F_A$, and $\tilde{C}_k = C_k + E_{C_k} \Sigma_{C_k} F_{C_k}, k \in N$.

Let J_A be the Jordan matrix of A and $PAP^{-1} = J_A$ for some $n \times n$ nonsingular matrix P. Denote $M_A(P) = \|P\| \|P^{-1}\|$. Clearly, $M_A(P) \geq 1$.

Corollary 3.1 *Assume that the following conditions hold.*
(1) A is a Hurwitz matrix.
(2) Let $\alpha = -\max_{1 \leq i \leq n} \big\{(Re(\lambda_i(A))\big\}$. Then

$$\|E_A\| \|F_A\| < \frac{\alpha}{M_A(P)}. \tag{32}$$

(3) There exists a constant β with $0 < \beta < \alpha - M_A(P) \|E_A\| \|F_A\|$ such that

$$\sum_{i=0}^{k} \ln \big(M_A(P) \|I + C_i\| + M_A(P) \|E_{C_i}\| \|F_{C_i}\|\big) \leq \beta (t_k - t_0), \quad \text{for all} \quad k \in N. \tag{33}$$

Then system (30) is robustly exponentially stable and at least with decay rate: $\alpha - M_A(P) \|E_A\| \cdot \|F_A\| - \beta$.

Proof Obviously, for the linear system (30), we have

$$\Phi_k(t, s^+) = e^{A(t-s)}, \quad t_{k-1} < s \leq t \leq t_k, k \in N. \tag{34}$$

Since A is a Hurwitz matrix, we get $\max_{1 \leq i \leq n} \big\{(Re(\lambda_i(A))\big\} < 0$ and

$$\|\Phi_k(t, s^+)\| = \|e^{A(t-s)}\| \leq M_A(P) \cdot \|e^{PAP^{-1}(t-s)}\| \leq M_A(P) e^{-\alpha(t-s)}, \tag{35}$$

where $\alpha = -\max_{1 \leq i \leq n} \big\{(Re(\lambda_i(A))\big\} > 0$.

The rest of the proof follows as a direct consequence of Theorem 3.2 with $\gamma = \|E_A\| \|F_A\|$, and the inequality

$$\ln \big(M_A(P)(\|I + C_i + E_{C_i} \Sigma_{C_i} F_{C_i}\|)\big) \leq \ln \big(M_A(P) \|I + C_i\| + M_A(P) \|E_{C_i}\| \|F_{C_i}\|\big), \quad i \in N. \tag{36}$$

The proof is thus complete. □

Corollary 3.2 *For system (2), if $A(t) = A$, where A is a constant matrix, and (2) is exponentially stable with decay rate $\alpha > 0$, i.e., (9) holds. Furthermore, suppose that the following conditions hold.*
(1)

$$\|E_A\| \|F_A\| < \frac{\alpha}{M}. \tag{37}$$

(2) There exists a constant β with $0 < \beta < \alpha - M\gamma$ such that

$$\sum_{i=0}^{k} \ln \big(1 + M \|E_{C_i}\| \|F_{C_i}\|\big) \leq \beta (t_k - t_0), \quad \text{for all} \quad k \in N. \tag{38}$$

Then system (30) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta > 0$.

Proof By Theorem 3.3, it is easy to show that the results of the corollary are valid. The details are omitted. □

4 Examples

In this Section, we shall consider two examples to illustrate the results obtained in Section 3.

Example 4.1 Consider system (1) in the form of system (30), where $t_0 = 0, t_k = k, k \in N$, and

$$A^{(1)} = \begin{pmatrix} -3.5 & -0.5 \\ 0 & -2.8 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -2.5 & 0.5 \\ 0 & -1.2 \end{pmatrix},$$

$$C_k^{(1)} = \begin{pmatrix} -2.5 & -0.4 \\ 0 & -2.6 \end{pmatrix}, \quad C_k^{(2)} = \begin{pmatrix} -1.5 & 0.4 \\ 0.2 & -1.4 \end{pmatrix}.$$

Obviously, $A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}, C_k = \begin{pmatrix} -2 & 0 \\ 0.1 & -2 \end{pmatrix}, k \in N$.
Let $P = I$. Then,

$$\alpha = -\max_{1 \le i \le n} \left\{ (Re(\lambda_i(A)) \right\} = 2 > 0, M_A(P) = 1, \|E_A\| = 1, \|F_A\| = 1.1402,$$

$$\|E_{C_k}\| = 0.9487, \|F_{C_k}\| = 1, k \in N.$$

Let $\beta = 0.6931$. Then, we obtain

$$\beta + M_A(P)\|E_A\|\|F_A\| = 1.8333 < 2 = \alpha,$$

$$\sum_{i=0}^{k} \ln \left(M_A(P)\|I + C_i\| + M_A(P)\|E_{C_i}\|\|F_{C_i}\| \right) = 0.6931 \cdot k \le \beta(t_k - t_0).$$

Hence, by Corollary 3.1, we conclude that the system is robustly global exponential stable and at least with decay rate 0.1667.

Example 4.2 Consider system (1), where $t_0 = 0, t_k = k, k \in N$, and

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}, \tilde{A}(t) = \begin{pmatrix} \tilde{a}_{11}(t) & 0 \\ 0 & \tilde{a}_{22}(t) \end{pmatrix}, C_k = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \tilde{C}_k = \begin{pmatrix} \tilde{c}_{k_{11}} & 0 \\ 0 & \tilde{c}_{k_{22}} \end{pmatrix}.$$

The uncertainty entries satisfy:

$$|\tilde{a}_{11}(t)| \le |\sin t|, |\tilde{a}_{22}(t)| \le |\cos t|, |\tilde{c}_{k_{11}}| \le 1 + \frac{1}{(1+k)^2}, |\tilde{c}_{k_{22}}| \le 1 + \frac{1}{(1+k)^2}, k \in N.$$

Then, we obtain

$$\|W(t, s^+)\| \le e^{-2(t-s)}$$

and

$$\int_{t_0}^{t} \|\tilde{A}(s)\| ds = \int_0^t \|\tilde{A}(s)\| ds \le t,$$

and hence, $\alpha = -2, M = 1, \gamma = 1, M_1 = 0$.
Moreover,

$$\sum_{i=0}^{k} \ln \left(1 + M\|\tilde{C}_i\| \right) \le \ln(2.25) \cdot k = 0.8109 \cdot (t_k - t_0).$$

Thus, by letting $\beta = 0.8109$, we obtain $0 < \beta + \gamma M < \alpha$.

Hence, by Theorem 3.3, we conclude that the system is robustly global exponential stable and at least with decay rate 0.1891.

5 Conclusions

In this paper, by employing the variation of constants formula for impulsive system, we have established some global exponential stability criteria for time-varying linear impulsive system with uncertainties. We have also obtained estimates for decay rates. The criteria obtained are verifiable via solving algebraic inequalities in Matlab environment. Some examples have been worked out to demonstrate the main results.

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