



Transformation Synthesis for Euler-Lagrange Systems

M. Mabrouk¹, S. Ammar^{2*} and J.-C. Vivalda³

¹ *Faculté des sciences de Gabès – Cité Riadh – Zirig 6072 Gabès – Tunisie*

² *ISIT.Com de Hammam Sousse – Route principale Numéro 1 – 4011
Hammam Sousse – 4002*

³ *Inria-Lorraine et LMAM (UMR 7122) – Université de Metz
Ile du Saulcy – 57045 Metz Cedex 01 – France*

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Abstract: The transformation of Euler-Lagrange systems, with the variable of position as output, in order to solve some interesting problem as the design of observer is considered in this paper. First, we will provide a necessary and sufficient condition, which ensures the transformation of such system into some structure affine in the velocities, as well as a method to compute this transformation. For a particular family of Euler-lagrange systems with two degree of freedom we will present a change of coordinates which makes the dynamics triangular with respect to the velocities and a globally asymptotically converging observer is provided. To illustrate the approach, it is applied to the Cart-pendulum system.

Keywords: *Euler-Lagrange systems; state transformation; affine forms; cart-pendulum.*

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* Corresponding author: ammar_3021@yahoo.fr

1 Introduction

Euler-Lagrange systems with n generalized configuration coordinates $q = (q_1, \dots, q_n)^T$ are described by equations of the form

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q, v)v + V(q) &= \tau, \end{aligned} \tag{1}$$

where $M(q)$ denotes the inertia matrix, while $C(q, v)v$, with $v = \dot{q} = (\dot{q}_1, \dots, \dot{q}_n)^T$ the generalized velocities, denotes the centrifugal and Coriolis forces, $V(q)$ consists of the gravity terms and τ is the vector of input torques. This celebrated family of systems has been the subject of an important literature over half a century, because the equations of many physical devices belong to this family (see [18], [20], [17], [4] and references therein). When these systems are fully-actuated, they are globally feedback linearizable. But feedback linearization can be performed only when all the variables are measured. Unfortunately in practice, very often the variables of velocity cannot be measured. Therefore, the global output feedback stabilization of these systems with $y = q$ as output is challenging from a practical point of view. But, from a theoretical point of view, it is one of the most difficult problems in the field of nonlinear control: indeed, the matrix $C(q, v)v$ is a nonaffine function of the unmeasured part of the state v : this fact precludes from applying most of the classical techniques; for instance, the methods of [16], [15] and [14]. For more explanations on the obstacles due to the presence of terms which are nonaffine with respect to the unmeasured variables, see the introduction of [10].

Recently, in [2], an elegant alternative for one-degree-of-freedom systems was reported. The author presented a reduced order observer which converge exponentially. This observer is based upon a global nonlinear change of coordinates which makes the system affine in the unmeasured part of the state. This is crucial to define a very simple controller to solve the problem of tracking trajectory. So a very natural question arises: which conditions ensure that an Euler-Lagrange systems (1) can be transformed, with the help of a change of coordinates, into some structure affine in the unmeasured part of the state.

This question has been addressed in [2] and [17]. However the questions of existence and computation of the required solution were not answered. In the present paper, we address these question: we show that this problem can be brought back to the resolution of a set of partial differential equation for which an explicit solution is given.

The paper is organized as follows. In Section 4, we present first necessary and sufficient condition which gives to system (1) some structure affine in the unmeasured part of the state. Next we introduce triangular forms. A method of construction of observers is proposed. Section 7 contains concluding remarks.

2 Preliminary

In this section we briefly review some results and terminology from Euler-Lagrange dynamics that will be useful in the sequel. The interested reader should consult [12], [13] and [18] for a more detailed discussion.

The dynamics of equations (1) has the following properties [21]:

Property 2.1 The matrix $M(q) = (M_{ij})_{1 \leq i, j \leq n}$ is symmetric positive definite for all q .

Property 2.2 The inertia and centripetal-Coriolis matrices satisfy the following relationship

$$\frac{dM(q)}{dt} = C^T(q, v) + C(q, v), \tag{2}$$

where T denotes the transposition and $\frac{dM}{dt}$ is a shorthand for $\sum_{i=1}^n v_i \frac{\partial M}{\partial q_i}$.

It is also well known [18], that the (j, k) -entry of the matrix $C(q, v)$ is given by

$$C_{jk}(q, v) = \sum_{i=1}^n C_{ijk}(q)v_i, \tag{3}$$

where

$$C_{ijk}(q) = \frac{1}{2} \left(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j} \right) \tag{4}$$

are the so called Christoffel symbols of the first kind.

Equality (3) shows that we can write the matrix $C(q, v)$ as

$$C(q, v) = \sum_{i=1}^n v_i C_i(q), \tag{5}$$

where the entries of matrix C_i are the $C_{ijk}(q)$'s; these matrices satisfy the relation $C_i + C_i^T = \frac{\partial M}{\partial q_i}$.

Now, we state the following theorem which is proved in [1] and will be used in the next section.

Theorem 2.1 Let x_1, \dots, x_m denote the coordinates of a point $x \in R^m$ and y_1, \dots, y_n the coordinates of a point $y \in R^n$. Let M^1, \dots, M^m be smooth functions

$$M^i : R^m \rightarrow R^{n \times n} \tag{6}$$

such that

$$\frac{\partial M^i}{\partial x_k} - \frac{\partial M^k}{\partial x_i} + M^i M^k - M^k M^i = 0. \tag{7}$$

Consider the set of partial differential equations

$$\frac{\partial y(x)}{\partial x_i} = M^i(x)y(x), \quad 1 \leq i \leq m. \tag{8}$$

Given a point $(x^0, y^0) \in R^m \times R^n$, there exist a neighborhood U of x^0 and a unique smooth function $y(x)$ which satisfies (8) and is such that $y(x^0) = y^0$.

Throughout the paper,

- $M_n(R)$ denotes the set of n -square real matrices;
- $GL_m(R)$ denotes the set of n -square real invertible matrices;
- for $S \in M_n(R)$ symmetric positive definite $S^{1/2}$ denotes the square root of S .

3 Problem statement

We consider the family of Euler-Lagrange systems described by equations (1) where the output is $q = (q_1, \dots, q_n)^T \in R^n$, and the input is $\tau = (\tau_1, \dots, \tau_n)^T \in R^n$. The unmeasured part of the state is $v = (\dot{q}_1, \dots, \dot{q}_n)^T$.

As pointed out in the introduction the difficulty to stabilize or to construct observers for system (1) mainly stems from the fact that Coriolis and centrifugal forces vector in (1), have a quadratic growth in the generalized velocities v , which are not measured. The global change of coordinates introduced in [2] for one-degree-of freedom (i.e. $n = 1$) systems overcomes this problem by rewriting the dynamics with functions which are linear in the unmeasured velocities. As it is discussed in [2], the design procedure might be extended to the case of systems with more degrees of freedom, as soon as the same kind of change of coordinates can be found, that is to say if we can select an invertible matrix $T(q)$ such that

$$\frac{dT(q)}{dt} = T(q)M^{-1}(q)C(q, v). \quad (9)$$

Remark 3.1 We can notice that a more general condition which allows us to rewrite system (1) with an unmeasured part which is linear is the existence of a nonsingular matrix T such that

$$\frac{dT(q)}{dt}v = T(q)M^{-1}(q)C(q, v)v. \quad (10)$$

The following example shows that condition (10) is weaker than condition (9). Consider the following inertia matrix $M(q)$

$$M(q) = \begin{pmatrix} e^{-q_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the Christoffel symbols of the first kind [18], matrix C is given by

$$C(q, v) = \frac{1}{2}e^{-q_2} \begin{pmatrix} -v_2 & -v_1 \\ v_1 & 0 \end{pmatrix}$$

and an easy calculation shows that the matrix

$$T(q) = \begin{pmatrix} e^{-q_2} & 0 \\ \frac{1}{2}q_1 e^{-q_2} & 1 \end{pmatrix} \quad (11)$$

satisfies equation (10), but not (9). In fact, equation (9) does not admit any solution (as we will see later).

Necessary geometric conditions, so that (9) admits a solution are given in [6], furthermore necessary conditions in terms of Riemannian curvature are given in [18].

The main contribution of the paper is to give an algebraic necessary and sufficient condition in terms of the matrix of centrifugal and Coriolis forces, so that (9) admits a solution, and make the relation between it and Riemannian curvature as in [18].

4 Main results

4.1 Equation $\frac{dT(q)}{dt} = T(q)M^{-1}(q)C(q, v)$

This subsection is composed of two parts. In the first part, we propose a necessary and sufficient condition which ensures the existence of a solution of equation (9) as well as

methods to compute it (Lemma 4.4).

In the second part, we explain the relation between (9) and Riemannian curvature.

4.2 Necessary and sufficient conditions

Theorem 4.1 *Consider the nonlinear system (1); equation (9) admits a solution if and only if*

$$\frac{\partial C_i}{\partial q_j} - \frac{\partial C_j}{\partial q_i} = C_j^T M^{-1} C_i - C_i^T M^{-1} C_j, \tag{12}$$

for all $1 \leq i, j \leq n$ where the matrices C_i are defined by relation (5).

To establish Theorem 4.1, we need to prove the following preliminary lemma.

Lemma 4.1 *Let $M_1(q), \dots, M_n(q)$ be matrices in $M_m(R)$ depending smoothly on q and consider the set of partial differential equations*

$$\frac{\partial T}{\partial q_i}(q) = T(q)M_i(q), \quad \forall i = 1, \dots, n. \tag{13}$$

Given any matrix $T_0 \in GL_m(R)$ and $q_0 \in R^n$, there exists an unique smooth matrix $T(q)$ which satisfies (13) and is such that $T(q_0) = T_0$ if and only if the functions $M_1(q), \dots, M_n(q)$ satisfy the conditions

$$\forall i < j \leq n; \quad M_j M_i - M_i M_j = \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j}. \tag{14}$$

Proof *Necessity* Let $T(q)$ be a solution of equations (13), then from the property

$$\frac{\partial^2 T(q)}{\partial q_i \partial q_j} = \frac{\partial^2 T(q)}{\partial q_j \partial q_i} \tag{15}$$

one has

$$\frac{\partial(T(q) M_j(q))}{\partial q_i} = \frac{\partial(T(q) M_i(q))}{\partial q_j}. \tag{16}$$

Expanding the derivatives on both sides we obtain

$$T(q) \left(M_i(q)M_j(q) + \frac{\partial M_j(q)}{\partial q_i} \right) = T(q) \left(M_j(q)M_i(q) + \frac{\partial M_i(q)}{\partial q_j} \right) \tag{17}$$

which, due to the fact that $T(q)$ is invertible (since $T(q_0) \in GL_m(R)$), yields the condition (14).

Sufficiency The proof of this part of the demonstration can be easily derived from Theorem 2.3 as follows. Let $T_0 \in GL_m(R)$ and denote by $(\Gamma_0^1, \dots, \Gamma_0^n)$, Γ_0^i the columns of matrix T_0^{-1} . Conditions (14) ensure the existence of a family of functions Γ^k such that for all k we have

$$\frac{\partial \Gamma^k}{\partial q_i} = -M_i \Gamma^k, \quad \Gamma^k(q_0) = \Gamma_0^k. \tag{18}$$

The matrix Γ with columns $\Gamma^1, \dots, \Gamma^n$ satisfies the equality:

$$\frac{\partial \Gamma}{\partial x_i} = -M_i \Gamma, \quad \Gamma(q_0) = T_0^{-1}. \tag{19}$$

Since $\Gamma(q_0) = T_0^{-1}$ which is non singular, we conclude that there exists a neighborhood U of q_0 such that Γ is non singular as a solution of (13), then for $T(q)$ we take the matrix $\Gamma^{-1}(q)$. \square

The above proof gives a condition of existence, but not a method allowing the construction of the solution; however, the control implementation needs the knowledge of a matrix $T(q)$.

In the sequel we will give another proof of the sufficient part of Lemma 4.4, based on a reasoning by induction which provides an explicit solution of (9). Moreover this solution is defined on the whole domain of definition of the matrices M_i and not only locally.

Alternative proof of the sufficient part of Lemma 4.4. By induction on n we show that if (14) holds, then we have the following property denoted by $\mathcal{P}(n)$.

For all $m \geq 1$, there exists an invertible matrix $T(q) \in \text{GL}_m(R)$ such that equations (13) holds.

For $n = 1$: Equation (13) becomes

$$\frac{\partial T(q_1)}{\partial q_1} = T'(q_1) = T(q_1)M(q_1)$$

which admits solutions defined on the whole domain of definition of $M_1 \in \text{M}_m(R)$ and so $\mathcal{P}(1)$ is true.

Assume that $\mathcal{P}(n)$ is true and let $M_1, \dots, M_{n+1} \in \text{M}_m(R)$ be such that

$$M_j M_i - M_i M_j = \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j} \quad \text{for } i, j = 1, \dots, n+1. \quad (21)$$

The induction hypothesis implies that there exists an invertible matrix $T_{q_{n+1}} = T_{q_{n+1}}(q_1, q_2, \dots, q_n)$ such that

$$\frac{\partial T_{q_{n+1}}}{\partial q_i} = T_{q_{n+1}} M_i, \quad i = 1, \dots, n.$$

We will show that there exists a solution of the form $T = \Psi_1(q_{n+1})T_{q_{n+1}}$. First, observe that

$$\frac{\partial T}{\partial q_i} = \Psi_1(q_{n+1}) \frac{\partial T_{q_{n+1}}}{\partial q_i} = \Psi_1(q_{n+1}) T_{q_{n+1}} M_i = T M_i,$$

for $i = 1, \dots, n$. Moreover T satisfies the $(n+1)$ -th equation if and only if

$$\frac{d\Psi_1}{dq_{n+1}} T_{q_{n+1}} + \Psi_1 \frac{\partial T_{q_{n+1}}}{\partial q_{n+1}} = \Psi_1 T_{q_{n+1}} M_{n+1} \quad (22)$$

which is equivalent to

$$\frac{d\Psi_1}{dq_{n+1}} = \Psi_1 \left(T_{q_{n+1}} M_{n+1} - \frac{\partial T_{q_{n+1}}}{\partial q_{n+1}} T_{q_{n+1}}^{-1} \right). \quad (23)$$

This equation with unknown function Ψ_1 depending only on q_{n+1} admits a solution if and only if the term $\left(T_{q_{n+1}} M_{n+1} - \frac{\partial T_{q_{n+1}}}{\partial q_{n+1}} T_{q_{n+1}}^{-1} \right)$ does not depend on q_1, \dots, q_n . Now, taking into account that matrix $T_{q_{n+1}}$ satisfies equations (13), a straightforward

calculation leads to the following expression for the derivative of this term in respect of q_i :

$$T_{q_{n+1}} \left(M_i M_{n+1} + \frac{\partial M_{n+1}}{\partial q_i} - \frac{\partial M_i}{\partial q_{n+1}} - M_{n+1} M_i \right) T_{q_{n+1}}^{-1}$$

which is zero because matrices M_i satisfy equalities (21). \square

Proof of the main result.

Since we have

$$\frac{dT(q)}{dt} = \sum_{i=1}^n \frac{\partial T(q)}{\partial q_i} v_i$$

and from the decomposition of the matrix $C(q, v)$ (see equality (5)), equation (9) is equivalent to the set of equations

$$\frac{\partial T(q)}{\partial q_i} = T(q)M_i(q) \quad i = 1, \dots, n, \tag{24}$$

where $M_i(q) = M^{-1}(q)C_i(q)$. According to Lemma 4.4, we deduce that a solution of (9) exists if and only if

$$M_j(q)M_i(q) - M_i(q)M_j(q) = \frac{\partial M_j(q)}{\partial q_i} - \frac{\partial M_i(q)}{\partial q_j}.$$

Now,

$$\begin{aligned} \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j} &= -M^{-1}(C_i + C_i^T)M^{-1}C_j + M^{-1}\frac{\partial C_j}{\partial q_i} - M^{-1}\frac{\partial C_i}{\partial q_j} + M^{-1}(C_j + C_j^T)M^{-1}C_i \\ &= M_jM_i - M_iM_j + M^{-1} \left(\frac{\partial C_j}{\partial q_i} - \frac{\partial C_i}{\partial q_j} - C_i^T M^{-1}C_j + C_j^T M^{-1}C_i \right). \end{aligned}$$

It follows that a necessary and sufficient condition for the existence of a solution $T(q)$ of equation (9) is given by

$$\frac{\partial C_i}{\partial q_j} - \frac{\partial C_j}{\partial q_i} = C_j^T M^{-1}C_i - C_i^T M^{-1}C_j.$$

this concludes the proof. \square

The preceding theorem gives an algebraic characterization of a family of Euler-Lagrange systems which can be transformed, with the help of a change of coordinates into some structure, affine in the unmeasured part of the state $v = \dot{q}$. The following one gives another characterization for the existence of a solution of equation (9).

Theorem 4.2 *Consider an Euler-Lagrange system (1). The following conditions are equivalent.*

1. *There exists a matrix $T(q)$ such that (9) holds.*
2. *There exists a matrix $N(q)$ such that $M(q) = N^T(q)N(q)$ and $N^T(q)\frac{dN(q)}{dt} = C(q, v)$.*
3. *There exists a function $\Theta(q) R^n \rightarrow R^n$ and $N(q)$ nonsingular such that $M(q) = N^T(q)N(q)$ and the Jacobian matrix of Θ is equal to $N(q)$.*

Proof $1 \Rightarrow 2$

Suppose that (9) admits a solution; the computation of $\frac{d\overline{M}}{dt}$, where $\overline{M} = (T^T)^{-1}MT^{-1}$, gives

$$\begin{aligned} \frac{d\overline{M}}{dt} &= -(T^T)^{-1}(C^T M^{-1}T^T)(T^T)^{-1}M(T^T)^{-1} - (T^T)^{-1}MT^{-1}(TM^{-1}C)T^{-1} \\ &\quad + (T^T)^{-1}\frac{dM}{dt}T^{-1} = -(T^T)^{-1}(C^T + C - \frac{dM}{dt})T^{-1} = 0 \end{aligned}$$

because $C^T + C = \frac{dM}{dt}$. So, $\overline{M} = (T^T)^{-1}MT^{-1}$ is a constant symmetric positive definite matrix. Letting $N = \overline{M}^{\frac{1}{2}}T$, one can check easily that $M(q) = N^T(q)N(q)$ and $N(q)^T \frac{d}{dt}N(q) = C(q, v)$.

$2 \Rightarrow 1$ Suppose that conditions (2) are satisfied then $N(q)$ is nonsingular and an easy computation shows that this matrix is a solution of equation (9).

$2 \Rightarrow 3$ Let us denote the columns of matrix N by N^i ; $N(q)$ is the Jacobian matrix of a function Θ if and only if

$$\frac{\partial N^i}{\partial q_j} = \frac{\partial N^j}{\partial q_i}. \quad (25)$$

Now the equality $N^T \frac{dN}{dt} = C$ is equivalent to

$$\frac{\partial N^i}{\partial q_j} = N^T C_j^i, \quad i, j = 1, \dots, n,$$

where C_j^i denotes the i -th column of C_j . But from formula (3), we know that $C_i^j = C_j^i$; this proves formula (25).

$3 \Rightarrow 2$ Denoting by N_{ij} the entries of matrix $N(q)$, conditions (3) imply that

$$\frac{\partial N_{ij}}{\partial q_k} = \frac{\partial N_{ik}}{\partial q_j}$$

for all triple (i, j, k) . From (4) and taking into account that $M(q) = N(q)^T N(q)$, we have

$$\begin{aligned} 2C_{ijk} &= \frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j} = \sum_{s=1}^n \left(\frac{\partial N_{sj}}{\partial q_i} N_{sk} + N_{sj} \frac{\partial N_{sk}}{\partial q_i} \right) \\ &\quad + \sum_{s=1}^n \left(\frac{\partial N_{sj}}{\partial q_k} N_{si} + N_{sj} \frac{\partial N_{si}}{\partial q_k} \right) - \sum_{s=1}^n \left(\frac{\partial N_{si}}{\partial q_j} N_{sk} + N_{si} \frac{\partial N_{sk}}{\partial q_j} \right) \\ &= \sum_{s=1}^n \left(N_{sj} \frac{\partial N_{sk}}{\partial q_i} + N_{sj} \frac{\partial N_{si}}{\partial q_k} \right) = 2 \left(N^T \frac{\partial N}{\partial q_i} \right)_{jk} \end{aligned}$$

so we have

$$C_i = N^T \frac{\partial N}{\partial q_i}$$

which is equivalent to

$$C(q) = N^T(q) \frac{dN(q)}{dt}.$$

□

4.3 The Riemmanian curvature

Now suppose that the conditions of theorem 4.5 are fulfilled, then there exists a function $\Theta: R^n \rightarrow R^n$ such that, denoting by $N(q)$ the Jacobian matrix of Θ ,

$$M(q) = N^T(q)N(q). \tag{26}$$

In terms of the new variables $Q = \Theta(q)$, $V = N(q)v$ the Lagrangian dynamics equations (1) can be shown to reduce to

$$\dot{Q} = V, \tag{27}$$

$$\dot{V} = \dot{N}v + N\dot{v} = (N^T)^{-1}(\tau - V(q)). \tag{28}$$

Thus a double integrator model in terms of Q is achieved by the much simpler inner loop feedback control law

$$\tau - V(q) = N^T(q)\nu. \tag{29}$$

The point is that, in the new coordinates, the computation of the Coriolis and centrifugal terms in the inner loop is avoided. However, a necessary and sufficient condition for existence of the factorization (26) is that the Riemannian curvature of the metric defined by the robot inertia matrix be zero [5, 18]. More precisely we have the following theorem which summarizes our result and the result of papers [5, 18].

Theorem 4.3 *Consider an Euler-Lagrange system (1). The following conditions are equivalent:*

1. *There exists a matrix $T(q)$ such that (9) holds.*
2. *The Riemmanian Curvature Tensor defined by*

$$R_{ijkl} = \frac{\partial^2 M_{ik}(q)}{\partial q_i \partial q_j} + \frac{\partial^2 M_{jl}(q)}{\partial q_k \partial q_i} - \frac{\partial^2 M_{il}(q)}{\partial q_k \partial q_j} - \frac{\partial^2 M_{jk}(q)}{\partial q_i \partial q_i} + \frac{1}{2} \sum_{r,s=1}^n M_{r,s}^{-1}(q)[C_{rjl}C_{sik} - C_{ril}C_{sjk}] \tag{30}$$

are identically zero, where $M_{r,s}^{-1}(q)$ are the components of the inverse $M^{-1}(q)$ of the inertia matrix $M(q)$ and C_{rjl} are the Christoffel symbols of the first kind defined by (3).

4.4 Example: The cart pendulum system

As an example, we will consider the inverted pendulum. The Euler-Lagrange equations write:

$$\begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta &= \tau_1, \\ ml\ddot{x} \cos \theta + ml^2\ddot{\theta} - mlg \sin \theta &= 0, \end{aligned} \tag{31}$$

where M and x denote the mass and the position of the cart (which is moving horizontally), m , l and θ denote the mass, the length and the angular derivation from the upward vertical position of the pendulum which is pivoting around a point fixed on the cart. We denote the state vector $(x, \theta, \dot{x}, \dot{\theta})^T$ as $(q_1, q_2, v_1, v_2)^T$. The output is $y = (q_1, q_2)^T$.

The inertia matrix is

$$M(q) = \begin{pmatrix} a_1 & a_2 \cos q_2 \\ a_2 \cos q_2 & a_3 \end{pmatrix}$$

with $a_1 = M + m$, $a_2 = ml$ and $a_3 = ml^2$.

Using the Christoffel symbols, we obtain

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & -a_2 \sin(q_2) \\ 0 & 0 \end{pmatrix}.$$

One can check easily that condition (14) are verified so according to Theorem 4.3, equation (9) admits a solution that we will make explicit by using the method explained in the proof of Lemma 4.4.

First, if we denote by T_{q_2} the 2-dimensional identity matrix, T_{q_2} is obviously a solution of the differential equation

$$\frac{dT_{q_2}}{dq_1}(q_1) = T_{q_2} M^{-1} C_1.$$

so we can find a solution of equations (9) under the form $\Psi(q_2)$, a 2-dimensional square matrix solution of the equation

$$\frac{d\Psi(q_2)}{dq_2} = \Psi(q_2) M^{-1} C_2. \quad (32)$$

An easy calculations shows that the solution of equation (32) with initial condition $\Psi(0) =$ the identity matrix is

$$\Psi(q_2) = \begin{pmatrix} 1 & \frac{a_2 \beta(0) \cos(q_2) - a_2 \beta(q_2)}{a_1 \beta(0)} \\ 0 & \frac{\beta(q_2)}{\beta(0)} \end{pmatrix}, \quad (33)$$

where $\beta(q_2) = \sqrt{a_1 a_3 - a_2^2 \cos^2(q_2)}$.

Moreover the diffeomorphism $\Theta = (\Theta_1, \Theta_2)^T$ defined by

$$\Theta_1 = q_1 + \int_0^{q_2} \frac{a_2 \beta(0) \cos(s) - a_2 \beta(s)}{a_1 \beta(0)} ds,$$

$$\Theta_2 = \int_0^{q_2} \frac{\beta(s)}{\beta(0)} ds,$$

is such that Jacobian $(\Theta) = \Psi(q_2)$.

The following change of coordinates:

$$\begin{aligned}\Theta_1 &= q_1 + \int_0^{q_2} \frac{a_2\beta(0)\cos(s) - a_2\beta(s)}{a_1\beta(0)} ds, \\ \Theta_2 &= \int_0^{q_2} \frac{\beta(s)}{\beta(0)} ds, \\ z_1 &= v_1 + \frac{a_2\beta(0)\cos(q_2) - a_2\beta(q_2)}{a_1\beta(0)} v_2, \\ z_2 &= \frac{\beta(q_2)}{\beta(0)} v_2\end{aligned}$$

transforms the dynamics of the Cart-Pendulum into a double integrator

$$\dot{\Theta} = p, \quad (34)$$

$$\dot{z} = T(q)M^{-1}(q)(\tau - V(q)) = u. \quad (35)$$

where $\tau = (\tau_1, 0)^T$ and $V(q) = (0, -mgl\sin q_2)^T$.

Clearly this system is linear in the unmeasured part of the state and an exponentially converging observer can be constructed.

5 Discussion about equation (10)

Let us consider the problem of finding T such that (10) is satisfied.

Observe first that the matrix $M^{-1}C(q, v)v$ is quadratic in v with coefficients depending only on q , i.e. there exists R_i such that

$$M^{-1}(q)C(q, v)v = \sum_{i=1}^n v_i R_i v. \quad (36)$$

The matrices R_i are not uniquely determined.

In the case of one degree of freedom, a solution of (10) always exists [2]. In the case of higher order system, equation (9) can admit no solution while equation (10) admits one: see the example of Remark 3.1. In this example, observe that there is no function $\Theta(q)$ such that the Jacobian matrix of Θ is equal to $T(q)$ and thus the Riemmanian curvature Tensor are not identically zero.

The following theorem gives a necessary and sufficient condition for equation (10) has a solution.

Theorem 5.1 *Consider the Euler-Lagrange system (1), equation (10) admits a solution if and only if there exist matrices R_i satisfying equality (36) and such that*

$$R_j R_i - R_i R_j = \frac{\partial R_j}{\partial q_i} - \frac{\partial R_i}{\partial q_j} \quad (37)$$

for $i, j = 1, \dots, n$.

Proof Let us suppose that there exist a family of matrices $R_k(q)$, $k = 1 \dots, n$, such that equality (36) holds. Condition (37) ensures the existence of an invertible matrix $T(q)$ such that

$$\frac{\partial T}{\partial q} = TR_k, \quad (38)$$

moreover a method of construction of solution is given by Lemma 4.4. So we get

$$\dot{T}(q)v = \sum_{i=1}^n v_i \frac{\partial T}{\partial q_k} v = \sum_{i=1}^n v_i TR_i v = T(q)M^{-1}(q)C(q, v)v.$$

For the necessary part assume that there exists a matrix $T(q)$ such that (10) and (36) hold. It follows that

$$\sum_{k=1}^n v_k T^{-1}(q) \frac{\partial T}{\partial q_k} v = M^{-1}C(q, v)v.$$

Let $R_k = T^{-1}(q) \frac{\partial T}{\partial q_k}$, it follows that

$$\begin{aligned} \frac{\partial R_j}{\partial q_i} - \frac{\partial R_i}{\partial q_j} &= \frac{\partial}{\partial q_i} \left(T^{-1} \frac{\partial T}{\partial q_j} \right) - \frac{\partial}{\partial q_j} \left(T^{-1} \frac{\partial T}{\partial q_i} \right) \\ &= -T^{-1} \frac{\partial T}{\partial q_i} T^{-1} \frac{\partial T}{\partial q_j} + T^{-1} \frac{\partial^2 T}{\partial q_i \partial q_j} + T^{-1} \frac{\partial T}{\partial q_j} T^{-1} \frac{\partial T}{\partial q_i} - T^{-1} \frac{\partial^2 T}{\partial q_j \partial q_i} \\ &= R_j R_i - R_i R_j, \end{aligned}$$

which proves the result. \square

6 Triangular form for a particular family of Euler-Lagrange systems

It is now well-known that under certain conditions, we can carry out the transformation of a system, by a diffeomorphism into a state affine system in the velocity and carry out the synthesis of an observer. In the same way, we know that the necessary and sufficient conditions under which a system is transformable are very restrictive.

For that, we propose the triangular form in the unmeasured part of the state $\dot{q} = v$ for the analysis of observability. We will consider a particular family of Euler-lagrange systems, and we show that it can be transformed into some triangular structure for which an almost exponentially converging observer is given.

6.1 A family of Euler-Lagrange systems

In this section, we restrict ourselves to a particular family of Euler-Lagrange systems. We consider systems having two degrees of freedom and which satisfy the following properties.

Property 6.1 The inertia matrix depend only on the variable q_2 , this allows us to introduce the following notations:

$$M(q_2) = \begin{pmatrix} M_{11}(q_2) & M_{12}(q_2) \\ M_{12}(q_2) & M_{22}(q_2) \end{pmatrix}.$$

Property 6.2 There exist three positive constants m_1 , m_2 and K such that for all q

$$m_1 I_2 \leq M(q) \leq m_2 I_2, \tag{39}$$

$$\|C(q, v)\| \leq K \|v\|, \tag{40}$$

where I_2 denotes the 2-dimensional identity matrix.

Property 6.3 The function $\tau_1 - V_1$ is bounded in norm.

These properties are satisfied by many Euler-Lagrange systems with two degree of freedom: e.g. the cart-pole system [6, 19], the manipulator system [9]. In the problem under consideration, matrices C_1 and C_2 write

$$C_1(q_2) = \begin{pmatrix} 0 & \frac{1}{2}M'_{11}(q_2) \\ -\frac{1}{2}M'_{11}(q_2) & 0 \end{pmatrix}, \quad C_2(q_2) = \begin{pmatrix} \frac{1}{2}M'_{11}(q_2) & M'_{12}(q_2) \\ 0 & \frac{1}{2}M'_{22}(q_2) \end{pmatrix}$$

(the ' denotes the derivative). So, according to Theorem 4.3, equation (9) admits a solution iff

$$\frac{\partial C_1}{\partial q_2} - \frac{\partial C_2}{\partial q_1} = C_2^T M^{-1} C_1 - C_1^T M^{-1} C_2,$$

which is equivalent to

$$M''_{11}(q_2) = \frac{M'_{11}(q_2)\Delta'(q_2)}{2\Delta(q_2)}, \tag{41}$$

where $\Delta = M_{11}(q_2)M_{22}(q_2) - M_{12}(q_2)^2$, which is positive since matrix M is positive definite.

An example of systems satisfying equation (41) is, for instance, the cart-pendulum system [6] and the tora system [20]. But, other systems such that the manipulator or the two links manipulator do not satisfy this conditions. In spite of this, we will show that, this class of systems can be turned with the help of an appropriate change of coordinates into some triangular form near to feedforward form.

More precisely, we have the following result.

Proposition 6.1 *Under properties 2.1–6.3, the map*

$$\Phi: (q_1, v_1, q_2, v_2) \rightarrow (x_1, x_2, x_3, x_4)$$

defined by

$$\begin{aligned} x_1 &= q_1 + \int_0^{q_2} \frac{M_{12}(s)}{M_{11}(s)} ds, \\ x_2 &= M_{11}(q_2)v_1 + M_{12}(q_2)v_2, \\ x_3 &= q_2, \\ x_4 &= \alpha(q_2)v_2, \end{aligned}$$

where

$$\alpha(q_2) = \sqrt{\frac{\Delta(q_2)}{M_{11}(q_2)}},$$

defines a global change of coordinates which transforms system (1) into

$$\begin{aligned}
\dot{x}_1 &= \frac{x_2}{M_{11}(x_3)}, \\
\dot{x}_2 &= u_1, \\
\dot{x}_3 &= \frac{x_4}{\alpha(x_3)}, \\
\dot{x}_4 &= \frac{1}{\alpha(x_3)} \left(\frac{M'_{11}(x_3)}{2M_{11}^2(x_3)} x_2^2 + u_2 \right), \\
Y &= (x_1, x_3)^T.
\end{aligned} \tag{42}$$

where $u_1 = \tau_1 - V_1$ and $u_2 = \tau_2 - V_2 - \frac{M_{12}}{M_{11}}u_1$.

Proof The proposed transformation is obviously one-to-one and onto, moreover its jacobian matrix is equal to

$$\begin{pmatrix}
1 & 0 & \frac{M_{12}}{M_{11}} & 0 \\
0 & M_{11} & M'_{11}v_1 + M'_{12}v_2 & M_{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & \alpha'v_2 & \alpha
\end{pmatrix}$$

and we can see that this transformation is a global diffeomorphism.

On the other hand equations for \dot{x}_1 and \dot{x}_3 are obvious. One can determine the expression of \dot{x}_2 as follows; from

$$M(q_2)\dot{v} = -C(q_2, v)v + \tau - V$$

we have

$$M_{11}(q_2)\dot{v}_1 + M_{12}(q_2)\dot{v}_2 = -M'_{11}(q_2)v_1v_2 - M'_{12}(q_2)v_2^2$$

and so

$$\dot{x}_2 = M_{11}(q_2)\dot{v}_1 + M_{12}(q_2)\dot{v}_2 + M'_{11}(q_2)v_1v_2 + M'_{12}(q_2)v_2^2 = \tau_1 - V_1 = u_1.$$

We will now compute the expression of \dot{x}_4 . From

$$\dot{v} = -M(q_2)^{-1}(C(q_2, v) + \tau - V)$$

we have

$$\begin{aligned}
\Delta(q_2)\dot{v}_2 &= \frac{1}{2}M_{11}M'_{11}v_1^2 + M_{12}M'_{11}v_1v_2 + (M_{12}M'_{12} - \frac{1}{2}M_{11}M'_{22})v_2^2 \\
&\quad - M_{12}(\tau_1 - V_1) + M_{11}(\tau_2 - V_2) \\
&= \frac{M'_{11}}{2M_{11}}(M_{11}^2v_1^2 + 2M_{12}M_{11}v_1v_2 + M_{12}^2v_2^2) \\
&\quad + \frac{2M_{11}M_{12}M'_{12} - M_{11}^2M'_{12} - M_{12}^2M'_{11}}{2M_{11}}v_2^2 - M_{12}(\tau_1 - V_1) + M_{11}(\tau_2 - V_2) \\
&= \frac{M'_{11}}{2M_{11}}x_2^2 + \frac{2M_{11}M_{12}M'_{12} - M_{11}^2M'_{12} - M_{12}^2M'_{11}}{2M_{11}\alpha^2}x_4^2 - M_{12}(\tau_1 - V_1) \\
&\quad + M_{11}(\tau_2 - V_2).
\end{aligned}$$

Now

$$\dot{x}_4 = \alpha'(q_2) v_2^2 + \alpha(q_2) \dot{v}_2$$

and, taking into account that

$$\alpha' = \frac{1}{2\alpha} \frac{M_{11}^2 M_{22}' - 2M_{11} M_{12} M_{12}' + M_{12}^2 M_{11}'}{M_{11}^2},$$

we get the formula stated in the above proposition. \square

6.2 Construction of observers

Let us point out some particular interests of the system exhibited in Proposition 6.5. System (42) is triangular with respect to the unmeasured part of the state. The difficulty in designing an observer for the above system lies in the presence of the nonlinearity $\frac{M_{11}'(x_3)}{2M_{11}^2(x_3)} x_2^2$, which depends on the unmeasured part of the state x_2 . Moreover, due to the presence of term x_2^2 in the dynamics of x_4 , hypothesis $[H2']$ of paper [3] is not satisfied. This fact precludes from applying the techniques of [3] to construct an observer.

However, from the \dot{x}_1, \dot{x}_2 -equations in (42) we can see that the unmeasured state x_4 not appears in the derivative \dot{x}_1, \dot{x}_2 .

Therefore, we can obtain the information about x_2^2 from the x_1, x_2 -subsystem.

Consider x_1, x_2 -subsystem constituted by the two first equations of system (42)

$$\begin{aligned} \dot{x}_1 &= \frac{x_2}{M_{11}(x_3)}, \\ \dot{x}_2 &= u_1, \\ Y_1 &= x_1. \end{aligned} \tag{43}$$

This subsystem does not depend on x_4 . Moreover it is linear with respect to the unmeasured variable x_2 . In fact it can be considered as a linear system with a time-varying coefficient $\frac{1}{M(x_3)}$. Consequently, one can easily determine a globally exponentially converging observer. More precisely we have,

Proposition 6.2 *The auxiliary dynamical system*

$$\begin{aligned} \dot{\hat{x}}_1 &= \frac{1}{M_{11}(x_3)} (\hat{x}_2 + k_1(\hat{x}_1 - x_1)), \\ \dot{\hat{x}}_2 &= \frac{1}{M_{11}(x_3)} k_2(\hat{x}_1 - x_1) + u_1, \end{aligned} \tag{44}$$

is a globally exponentially converging observer for system (43), provided that the parameters k_1 and k_2 are negative.

Proof Let $(\varepsilon_1, \varepsilon_2) = (\hat{x}_1 - x_1, \hat{x}_2 - x_2)$. The error equation is

$$\begin{aligned} \dot{\varepsilon}_1 &= \frac{1}{M_{11}(x_3)} (\varepsilon_2 - k_1 \varepsilon_1), \\ \dot{\varepsilon}_2 &= \frac{1}{M_{11}(x_3)} k_2 \varepsilon_1, \end{aligned} \tag{45}$$

or, in more compact form,

$$\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \frac{1}{M_{11}(x_3)} \begin{pmatrix} k_1 & 1 \\ k_2 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

The matrix in the system above is Hurwitz since k_1 and k_2 are negative. Moreover the inequality (39) in Assumption 6.3 implies the existence of two constants κ_1 and κ_2 such that

$$0 < \kappa_1 \leq M_{11}(x_3) \leq \kappa_2.$$

Consequently, we can find a positive definite quadratic Lyapunov function $V(\varepsilon_1, \varepsilon_2)$ whose derivative along the trajectories of system (45) satisfies

$$\dot{V} = -\frac{1}{M_{11}(x_3)} W(\varepsilon_1, \varepsilon_2) \leq -\frac{1}{\kappa_2} W(\varepsilon_1, \varepsilon_2), \quad (46)$$

where $W(\varepsilon_1, \varepsilon_2)$ is a quadratic positive definite function. this implies that the system (44) is an exponential observer for the system (43). \square

We are ready to give an observer for the system (42).

Proposition 6.3 *Consider the following auxiliary dynamical system:*

$$\begin{aligned} \dot{\hat{x}}_3 &= \frac{\hat{x}_4}{\alpha(x_4)} + k_3(\hat{x}_1 - x_1) + \frac{k_4}{\alpha(x_3)}(\hat{x}_3 - x_3), \\ \dot{\hat{x}}_4 &= \frac{1}{\alpha(x_3)} \left(\frac{M'_{11}(x_3)}{2M_{11}^2(x_3)} \hat{x}_2^2 + u_2 + k_6(\hat{x}_3 - x_3) \right) + k_5(\hat{x}_1 - x_1), \\ \dot{\hat{x}}_1 &= \frac{1}{M_{11}(x_3)} (\hat{x}_2 + k_1(\hat{x}_1 - x_1)), \\ \hat{x}_2 &= \frac{1}{M_{11}(x_3)} k_2(\hat{x}_1 - x_1) + u_1, \end{aligned} \quad (47)$$

where the parameters k_1, k_2, k_4 and k_6 are chosen negative. Under the Assumptions 6.2–6.4, system (47) is a globally converging observer for system (42).

Proof Let $\varepsilon_3 = \hat{x}_3 - x_3$ and $\varepsilon_4 = \hat{x}_4 - x_4$. The error equation writes:

$$\begin{aligned} \dot{\varepsilon}_3 &= \frac{\varepsilon_4}{\alpha(x_4)} + k_3\varepsilon_1 + \frac{k_4}{\alpha(x_3)}\varepsilon_3, \\ \dot{\varepsilon}_4 &= \frac{1}{\alpha(x_3)} \left(\frac{M'_{11}(x_3)}{2M_{11}^2(x_3)} (\hat{x}_2^2 - x_2^2) + k_6\varepsilon_3 \right) + k_5\varepsilon_1, \\ \dot{\varepsilon}_1 &= \frac{1}{M_{11}(x_3)} (\varepsilon_2 - k_1\varepsilon_1), \\ \dot{\varepsilon}_2 &= \frac{1}{M_{11}(x_3)} k_2\varepsilon_1, \end{aligned} \quad (48)$$

From inequalities (39), (40) in property 6.3 and the positive definiteness of the inertia matrix, we can show easily that there exist α_1, α_2 and $c > 0$ such that

$$\alpha_1 \leq \alpha(x_3) \leq \alpha_2, \quad \left| \frac{m'_{11}(x_3)}{2k_2 m_{11}(x_3)^2} \right| \leq c.$$

Moreover since k_4 and k_6 are negative, one can determine a positive definite quadratic function $Q(\varepsilon_3, \varepsilon_4)$ such that it's derivative along the trajectories of (48) satisfies

$$\begin{aligned} \dot{Q} &\leq -\varepsilon_3^2 - \varepsilon_4^2 + c(|\varepsilon_3| + |\varepsilon_4|)(|\varepsilon_1| + \widehat{x}_2^2 - x_2^2) \\ &\leq -\varepsilon_3^2 - \varepsilon_4^2 + c(|\varepsilon_3| + |\varepsilon_4|)(|\varepsilon_1| + |\varepsilon_2||\varepsilon_2 + 2x_2|) \\ &\leq -\frac{1}{2}\varepsilon_3^2 - \frac{1}{2}\varepsilon_4^2 + 2c^2(|\varepsilon_1| + |\varepsilon_2||\varepsilon_2 + 2x_2|)^2. \end{aligned} \tag{49}$$

Now, property 6.4 ensures that \dot{x}_2 is bounded and (46) holds. It follows that there exist three constants a, k, β such that for all $t \geq 0$,

$$\begin{aligned} |\varepsilon_1(t)| &\leq k(|\varepsilon_1(0)| + |\varepsilon_2(0)|)e^{-\beta t}, \\ |\varepsilon_2(t)| &\leq k(|\varepsilon_1(0)| + |\varepsilon_2(0)|)e^{-\beta t}, \\ |x_2(t)| &\leq |x_2(0)| + at. \end{aligned} \tag{50}$$

It follows readily that there exists two constants K_1, K_2 which depends on $\varepsilon_1(0), \varepsilon_2(0)$ and $x_2(0)$ such that

$$\dot{Q} \leq -K_1 Q(\varepsilon_3, \varepsilon_4) + K_2 e^{-\frac{\beta}{2}t}, \tag{51}$$

which implies

$$Q(\varepsilon_3(t), \varepsilon_4(t)) \leq -K_1 \int_0^t Q(\varepsilon_3(s), \varepsilon_4(s)) ds + K_3 + Q(\varepsilon_3(0), \varepsilon_4(0)) \tag{52}$$

with $K_3 > 0$. It follows from Gronwall's Lemma that

$$Q(\varepsilon_3(t), \varepsilon_4(t)) \leq (K_3 + Q(\varepsilon_3(0), \varepsilon_4(0))) e^{-K_1 t}. \tag{53}$$

This concludes the proof. \square

6.3 Example

Consider the two-link manipulator studied in [4, 11]. The equations of motion are given by

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q, v)v + V(q) &= \tau, \end{aligned} \tag{54}$$

with $q = (q_1, q_2)^T, \tau = (\tau_1, \tau_2)^T$,

$$\begin{aligned} M(q) &= \begin{pmatrix} p_1 + 2p_3 \cos q_2 & p_2 + p_3 \cos q_2 \\ p_2 + p_3 \cos q_2 & p_2 \end{pmatrix}, \\ C(q, v) &= \begin{pmatrix} -v_2 p_3 \sin q_2 & -(v_1 + v_2) p_3 \sin q_2 \\ v_1 p_3 \sin q_2 & 0 \end{pmatrix}, \end{aligned}$$

$V(q) = 0$ and $p_1 = 3.473, p_2 = 0.193, p_3 = 0.242$.

Easy calculations show that

$$M''_{11}(q_2) - \frac{M'_{11}(q_2)\Delta'(q_2)}{2\Delta(q_2)} = 2p_3 \left(-1 - \frac{2p_3^2 \sin^2 q_2}{-p_1 p_2 + p_2^2 + p_3^2 \cos^2 q_2} \right) \cos q_2$$

which is non zero. It yields that equation (9) does not admits any solution. But one can check readily that this fully-actuated system satisfies Assumptions 2.1–6.3. Thanks to Proposition 6.5 the change of coordinates

$$\begin{aligned}x_1 &= q_1 + \frac{p_2 + p_3 \cos(s)}{p_1 + 2p_3 \cos(s)} ds = q_1 + \frac{p_1 - 2p_2}{\sqrt{4p_3^2 - p_1^2}} \operatorname{arctanh}\left(\frac{2p_3 - p_1 \tan(\frac{q_2}{2})}{\sqrt{4p_3^2 - p_1^2}}\right), \\x_2 &= (p_1 + 2p_3 \cos q_2)v_1 + (p_2 + p_3 \cos q_2) v_2, \\x_3 &= q_2, \\x_4 &= \alpha(q_2)v_2\end{aligned}$$

with

$$\alpha(q_2) = \sqrt{\frac{p_1 p_2 - p_2^2 - p_3^2 \cos^2(q_2)}{p_1 + 2p_3 \cos(q_2)}}$$

transforms (54) into

$$\begin{aligned}\dot{x}_1 &= \frac{x_2}{p_1 + 2p_3 \cos q_2}, \\ \dot{x}_2 &= u_1, \\ \dot{x}_3 &= \frac{x_4}{\alpha(x_3)}, \\ \dot{x}_4 &= \frac{1}{\alpha(x_3)} \left(\frac{-p_3 \sin x_3}{2(p_1 + 2p_3 \cos x_3)^2} x_2^2 + u_2 \right), \\ Y &= (x_1, x_3)^T,\end{aligned}\tag{55}$$

where

$$u_1 = (p_1 + 2p_3 \cos q_2)\tau_1, \quad u_2 = \tau_2 - \frac{p_2 + p_3 \cos q_2}{p_1 + 2p_3 \cos q_2} \tau_1.$$

According to Proposition 6.8, the following system

$$\begin{aligned}\dot{\hat{x}}_1 &= \frac{\hat{x}_2}{p_1 + 2p_3 \cos q_2} + \frac{k_1}{p_1 + 2p_3 \cos q_2}(\hat{x}_1 - x_1), \\ \dot{\hat{x}}_2 &= \tau_1 + \frac{k_2}{p_1 + 2p_3 \cos q_2}(\hat{x}_2 - x_1), \\ \dot{\hat{x}}_3 &= \frac{\hat{x}_4}{\alpha(x_3)} + k_3(\hat{x}_1 - x_1) + k_4(\hat{x}_3 - x_3), \\ \dot{\hat{x}}_4 &= \frac{1}{\alpha(x_3)} \left(\frac{-p_3 \sin x_3}{2(p_1 + 2p_3 \cos x_3)^2} \hat{x}_2^2 + u_2 \right) + k_5(\hat{x}_1 - x_1) + \frac{k_6}{\alpha(x_3)}(\hat{x}_3 - x_3),\end{aligned}$$

is a global observer for (55) when the k_i , $i = 1, 2, 4, 6$, are negative.

7 Conclusion

A necessary and a sufficient condition for determining a state change of coordinate which transform an Euler-Lagrange system into an affine system in the unmeasured part of state was given. Obviously in the case of one degree of freedom, a solution always exists. A case of higher order system, is for instance, that of the cart-pendulum system [10], the tora system [20] and the overhead crane [7]. We conjecture the result several others

problems in nonlinear control. Whereas, we know that these necessary and sufficient conditions so that a system is transformable are very restrictive. For that, we proposed the triangular forms in the unmeasured part of the state $\dot{q} = v$ for the analysis of observability. We have considered a particular family of Euler-lagrange systems, and we show that it can be transformed into some triangular structure for which an almost exponentially converging observer is given. Thanks to this triangular forms, a globally converging observer presented so called “two-link manipulator” system. Moreover the rate of convergence can be chosen arbitrary. Note also that our approach applies to the “cart-pendulum” system and an exponentially converging observers with an arbitrary rate of convergence can be constructed.

References

- [1] Isidori, A. *Non Linear Control Systems*. Second edition, Springer-Verlag, 1989.
- [2] Besançon, G. Global output feedback tracking control for a class of lagrangian systems. *Automatica* **36** (2000) 1915–1921.
- [3] Besançon, G. and Hammouri, H. On observer design for interconnected systems. *Math. Systems, Estimation and Control* **8**(4) (1998).
- [4] Besançon, G., Battilotti, S. and Lanari, L. On output feedback tracking control with disturbance attenuation for Euler-Lagrange systems. *37th CDC Conference*, Tampa, 1998, 3139–3143.
- [5] Bedrossian, N.S. Linearizing coordinate transformations and Riemann curvature. *Proc. of the 31th CDC*. Tucson, 1992, Vol. 1, 80–85.
- [6] Besançon, G., Battilotti, S. and Lanari, L. State transformation and global output feedback disturbance attenuation for a class of mechanical systems. *IEEE Mediterranean control conference*. Alghero, Sardinia, Italy, 1998, 561–566.
- [7] D’Andrea-Novel, B. and Lévine, J. Modeling and nonlinear control of an overhead crane. In: *Robust Control of Linear and Nonlinear Systems* (Eds: M.A. Kashoek, J.H. van Schuppen, and A.C.M. Rand), MTNS’89, vol. II, Birkäuser, Boston, 1990, 523–529.
- [8] Wan, C.J., Bernstein, D.S. and Coppola, V.T. Global stabilization of the oscillating eccentric rotor. *Nonlinear dynamics* **10** (1996) 49–62.
- [9] Angeli, D. Input to state stability of PD controlled robotic systems. *Automatica* **35** (1999) 1285–1290.
- [10] Mazenc, F. and Vivalda, J.C. Global asymptotic output feedback stabilization of feedforward systems. *European J. of Control* **8** (2002) 519–530.
- [11] Zhang, F., Dawson, D.M., Queiroz, M.S. de and Dixon, W. *Global Adaptive Output Feedback Tracking Control Of Robot Manipulators*. CDC, 1997.
- [12] Greenwood, D.T. *Principles of Dynamics*. Englewood Cliffs, Prentice-hall, New Jersey, 1988.
- [13] Goldstein, H. *Classical mechanics*. Reading, Massachusetts, Addison-Wesely, 1980.
- [14] Pomet, J-B., Hirschorn, R.M. and Cebuhar, W.A. Dynamic output feedback regulation for a class of nonlinear systems. *Math. Control Signals And Systems* **6** (1993) 106–124.
- [15] Gauthier, J.P. and Kupka, I. Observability and observers for non-linear systems. *Control And Optimisation* **32** (1994) 975–999.
- [16] Praly, L. and Jiang, Z.P. Stabilization by output feedback for systems with ISS inverse dynamics. *Syst. Cont. Lett.* **21** (1993) 19–33.

- [17] Loria, A. and Pantely, E. A separation principle for a class of Euler-Lagrange systems. In: *Lecture Notes in Information Sciences, N244, New directions in nonlinear observer design*. Springer, Berlin, 1999, 229–247.
- [18] Spong, M. and Vidyasagar. *Robot Dynamics and Control*. Wiley & Sons, New York, 1989.
- [19] Mazenc, F. and Praly, L. Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE Trans. Aut. Contr.* **41** (11) (1996) 1559–1578.
- [20] Jiang, Z.P. and Kanellakopoulos, I. Global Output-Feedback Tracking for a Benchmark Nonlinear System. *IEEE Trans. Aut. Contr.* **45** (2000) 1023–1027.
- [21] Johanson, R. Adaptive control of robot manipulators. *IEEE Trans. RA-6* (1999) 1230–1233.