Nonlinear Dynamics and Systems Theory, **7** (2) (2007) 217–224

Stability Results for Large-Scale Difference Systems via Matrix-Valued Liapunov Functions

A.A. Martynyuk* and V.I. Slyn'ko

Stability of Processes Department,
S.P. Timoshenko Institute of Mechanics, Nesterov Str., 3, 03057, Kyiv-57, Ukraine

Received: April 10, 2005; Revised: March 18, 2007

Abstract: New results concerned with the Liapunov stability of composite or interconnected systems, described by linear difference equations are established. These results involve a matrix-valued Liapunov function. Furthemore, using a new approach for constructing Liapunov functions we obtain some results related to uniform asymptotic stability and compare our results with some know results which were obtained via vector Liapunov functions. The examples illustrating the efficiency of the proposed approach are given.

Keywords: *Large scale difference system; matrix-valued Liapunov function; uniform asymptotic stability.*

Mathematics Subject Classification (2000): 39A10, 93D05, 93D20, 93D30.

1 Introduction and Main Results

The aim of this paper is to study stability in the sense of Liapunov of a linear large-scale system of difference equations in the form

$$x_i(\tau+1) = A_{ii}x_i(\tau) + \sum_{j=1,\, j\neq i}^{m} A_{ij}(\tau)x_j(\tau), \quad i = 1,2,\dots,m, \tag{1}$$

where $x = (x_1^{\mathrm T},\dots,x_m^{\mathrm T})^{\mathrm T}$, $\tau \in N_\tau^+ = \{\tau_0 + k,\ k = 0,1,\dots\}$ $\tau_0 > 0$, $x_i \in R^{n_i}$, $x \in R^n$, $n = \sum\limits_{i=1}^{m} n_i$, A_{ii}, $i = 1,\dots,m$, are constant matrices of appropriate dimensions, $A_{ij}(\tau)$, $i,j = 1,\dots,m$, $i \neq j$, are determined on the set N_τ^+.

* Corresponding author: anmart@stability.kiev.ua

The transformation of initial systems to the form (1) is made by means of mathematical decomposition for the preassigned order of independent subsystems or in terms of some physical speculations formed in the description of real physical system by a system of difference equations.

For system (1) we construct the matrix-valued function $U(\tau, x)$ (for the details see [5]). The diagonal elements $v_{ii}(x_i)$ are taken as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1, 2, \dots, m, \tag{2}$$

where P_{ii} are symmetric positive definite matrices. We assume that at least one of the matrices A_{ij} or A_{ji} is not equal to constant and takes the corresponding non-diagonal elements $v_{ij}(\tau, x_i, x_j)$ as the bilinear form

$$v_{ij}(\tau, x_i, x_j) = v_{ji}(\tau, x_i, x_j) = x_i^{\mathrm{T}} P_{ij}(\tau) x_j, \quad i, j = 1, 2, \dots, m, \quad i \neq j, \tag{3}$$

where the matrix $P_{ij}(\tau)$ satisfies difference equation

$$\begin{aligned} P_{ij}(\tau+1) - P_{ij}(\tau) + A_{ii}^{\mathrm{T}} P_{ij}(\tau+1) A_{jj} - P_{ij}(\tau+1) \\ = -\frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) - \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj}. \end{aligned} \tag{4}$$

Equation (4) can be solved in the explicit form. Consider two cases.

Case 1. Assume that the matrices A_{ii} and A_{jj} are such that

$$q = \max_{k,l} |\lambda_k(A_{ii})\, \lambda_l(A_{jj})| < 1.$$

We consider the linear operators

$$F_{ij} \ \ R^{n_i \times n_j} \to R^{n_i \times n_j}, \quad F_{ij} X = A_{ii}^{\mathrm{T}} X A_{jj}.$$

and present equation (4) as

$$P_{ij}(\tau) = -F_{ij} P_{ij}(\tau+1) + \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj}. \tag{5}$$

Using the method of mathematical induction it is easy to show that

$$\begin{aligned} P_{ij}(\tau) = F_{ij}^{\nu} P_{ij}(\tau+\nu) + \sum_{k=0}^{\nu} F_{ij}^{k} \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau+k) \\ + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau+k) P_{jj} A_{jj}. \end{aligned} \tag{6}$$

for any positive integer ν. It is shown (see [1]) that the eigenvalues of the operators F_{ij} are $\lambda_k(A_{ii})\, \lambda_l(A_{jj})$, therefore the norm of the operator F_{ij}^{ν} admits the estimate

$$\|F_{ij}^{\nu}\| = \left\| \frac{1}{2\pi i} \int\limits_{|z|=\frac{1+q}{2}} z^{\nu} R_z(F_{ij})\, dz \right\| \leq \frac{c}{2\pi} \int\limits_{|z|=\frac{1+q}{2}} |z|^{\nu}\, dl = c\Big(\frac{1+q}{2}\Big)^{\nu},$$

where $c = \max\limits_{|z|=\frac{1+q}{2}} \|R_z(F_{ij})\|$, $R_z(F_{ij})$ is a resolvent of the operator F_{ij}. Taking into account $\frac{1+q}{2} < 1$, we get $\|F_{ij}^{\nu}\| \to 0$ as $\nu \to \infty$.

Further we are interested only in bounded solutions of equation (4). Passing to the limit in (6) as $\nu \to \infty$, we get

$$P_{ij}(\tau) = \sum_{k=\tau}^{\infty} F_{ij}^{-\tau+k} \left\{ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(k) P_{jj} A_{jj} \right\}. \tag{7}$$

Further it is assumed that the series in the right-side part of (7) converges.

Case 2. Assume that

$$q = \max_{k,l} |\lambda_k(A_{ii})\, \lambda_l(A_{jj})| \geq 1.$$

It is easy to notice that the operator F_{ij} is non-degenerated. We present equation (4) as

$$P_{ij}(\tau+1) = F_{ij}^{-1} P_{ij}(\tau) - F_{ij}^{-1} \left\{ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj} \right\}. \tag{8}$$

Using the method of mathematical induction it is easy to show that

$$\begin{aligned} P_{ij}(\tau) &= F_{ij}^{-\tau+\tau_0} P_{ij}(\tau_0) \\ &\quad - \sum_{k=0}^{\tau-\tau_0-1} F_{ij}^{-\tau+\tau_0+k} \left[\frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau_0+k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau_0+k) P_{jj} A_{jj} \right]. \end{aligned}$$

Setting $P_{ij}(\tau_0) = 0$ we find partial solution of equation (4) in the form

$$P_{ij}(\tau) = -\sum_{k=0}^{\tau-\tau_0-1} F_{ij}^{-\tau+\tau_0+k} \left[\frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau_0+k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau_0+k) P_{jj} A_{jj} \right]. \tag{9}$$

Assuming that the matrices $P_{ij}(\tau)$ are bounded for all $\tau \geq \tau^*$ we introduce designations

$$\begin{aligned} \bar{c}_{ii} &= \lambda_M(P_{ii}), \quad \bar{c}_{ij} = \sup_{\tau \geq \tau^*} \|P_{ij}(\tau)\|, \\ \underline{c}_{ii} &= \lambda_m(P_{ii}), \quad \underline{c}_{ij} = -\sup_{\tau \geq \tau^*} \|P_{ij}(\tau)\|. \end{aligned}$$

In view of the results from [2, 4] the estimates for the elements matrix-valued function $U(\tau, x)$ are

$$\begin{gathered} \underline{c}_{ii} \|x_i\|^2 \leq v_{ii}(x_i) \leq \bar{c}_{ii} \|x_i\|^2, \quad i = 1, 2, \dots, m, \\ \underline{c}_{ij} \|x_i\| \, \|x_j\| \leq v_{ij}(\tau, x_i, x_j) \leq \bar{c}_{ij} \|x_i\| \, \|x_j\|, \quad i, j = 1, 2, \dots, m, \quad i \neq j. \end{gathered}$$

Therefore for scalar function $v(\tau, x, \eta) = \eta^{\mathrm{T}} U(\tau, x) \eta$, $\eta \in R_+^m$, $\eta > 0$, the bilateral inequality

$$w^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H w \leq v(\tau, x, \eta) \leq w^{\mathrm{T}} H^{\mathrm{T}} \overline{C} H w, \tag{10}$$

is satisfied, where

$$\begin{aligned} \overline{C} &= [\bar{c}_{ij}]_{i,j=1}^m, \qquad \underline{C} = [\underline{c}_{ij}]_{i,j=1}^m, \\ H &= \operatorname{diag}(\eta_1, \dots, \eta_m), \qquad w = (\|x_1\|, \dots, \|x_m\|)^{\mathrm{T}}. \end{aligned}$$

For the first difference of function $v(\tau,x,\eta)$ along solutions of system (1) in view of (4) one can get the estimate

$$\Delta v(\tau,x,\eta)\Big|_{(1)} \le w^{\mathrm{T}} S(\tau) w, \tag{11}$$

where $w = (\|x_1\|,\dots,\|x_m\|)^{\mathrm{T}}$, $S(\tau) = [\sigma_{ij}(\tau)]_{i,j=1}^m$. The elements of matrix $S(\tau)$ have the following structure

$$\begin{aligned}
\sigma_{ii}(\tau) = & -\lambda_m(G_{ii})\eta_i^2 + \sum_{j=1,\, j\ne i}^{m} A_{ji}\|^2 \|P_{jj}\|\eta_j^2 \\
& + \sum_{k,j=1,\, k\ne j}^{m} \lambda_M(A_{ki}^{\mathrm{T}} P_{kj} A_{ji} + A_{ji}^{\mathrm{T}} P_{jk}^{\mathrm{T}} A_{ki})\eta_k\eta_j, \\
\sigma_{ij}(\tau) = & \sum_{k=1,\, k\ne j,\, k\ne i}^{m} \eta_i^2 \|A_{kj}\|\,\|P_{kk}\|\,\|A_{ki}\| \\
& + \sum_{k,l=1,\, k\ne i,\, k\ne j,\, l\ne j}^{m} \|A_{ki}\|\,\|P_{kl}\|\,\|A_{lj}\|\eta_j\eta_l, \quad i\ne j,
\end{aligned}$$

where $G_{ii} = -(A_{ii}^{\mathrm{T}} P_{ii} A_{ii} - P_{ii})$, $\|\cdot\|$ is a spectral norm of the corresponding matrix. Using the function $U(\tau,x)$, estimate (10) of the scalar function $v(\tau,x,\eta)$ and estimate (11) of the first difference of this function along solutions of system (1) we formulate sufficient conditions of stability and uniform asymptotic stability of the equilibrium state $x=0$ of system (1).

Theorem 1.1 *Let system of equations (1) be such that*

(1) *matrices $\overline{C}$ and $\underline{C}$ in estimate (10) are positive definite;*

(2) *there exist negative semidefinite (negative definite) matrix $\overline{S}$ such that*

$$\frac{1}{2}\big[S(\tau) + S^{\mathrm{T}}(\tau)\big] \le \overline{S} \quad \textit{for all} \quad \tau \ge \tau_0.$$

Then the equilibrium state $x=0$ of system (1) is uniformly stable (uniformly asymptotically stable).

Proof Condition (2) of Theorem 1.1 ensures the existence of $\tau_1 \in N_{\tau_0}^+$ such that for all $\tau \ge \tau_1$ for matrix $S(\tau)$ the generalized Silvester conditions are satisfied. So, for function $v(\tau,x,\eta) = \eta^{\mathrm{T}} U(\tau,x)\eta$ for all $\tau \ge \widetilde{\tau} = \max\{\tau_1,\tau^*\}$ all conditions of Theorem 16.3 from Hahn [3] are satisfied. Thus, the equilibrium state $x=0$ is stable (uniformly asymptotically stable) with respect to $N_{\widetilde{\tau}}^+$. Taking into account continuity of solutions $x(\tau,\tau_0,x_0)$ of system (1) in x_0 and discreteness of the set $N_{\tau_0}^+$ one can conclude on stability (uniform asymptotic stability) of the equilibrium state of system (1).

2 Examples

Consider the system

$$\begin{aligned}
x(\tau+1) &= \rho_1 x(\tau) + \alpha A(\omega,\tau) y(\tau), \\
y(\tau+1) &= \rho_2 y(\tau) + \beta A^{\mathrm{T}}(\omega,\tau) x(\tau),
\end{aligned} \tag{12}$$

where $x, y \in R^2$, and $\alpha, \beta, \rho_1, \rho_2 \in R$, $\omega \in [0, 2\pi)$,

$$A(\omega, \tau) = \begin{pmatrix} \cos\omega\tau & \sin\omega\tau \\ -\sin\omega\tau & \cos\omega\tau \end{pmatrix}, \qquad \tau \in N_0^+.$$

Moreover, we designate $q = \rho_1\rho_2$. Applying the approach proposed in Section 1 for system (12) we construct an auxiliary function

$$v(\tau, x, y) = x^{\mathrm{T}}x + y^{\mathrm{T}}y + 2x^{\mathrm{T}}P(\tau)y, \tag{13}$$

where

$$P(\tau) = \begin{cases} \dfrac{\alpha\rho_1 + \beta\rho_2}{1 - 2q\cos\omega + q^2} A(\omega, \tau - 1)(A(\omega, 1) - qI), & \text{if} \quad |q| \le 1; \\ -\dfrac{\alpha\rho_1 + \beta\rho_2}{1 - 2q\cos\omega + q^2}\Big[qA(\omega, \tau - 1) - A(\omega, \tau) - \\ \qquad q^{-\tau+1}A^{\mathrm{T}}(1) + q^{-\tau}I\Big], & \text{if} \quad |q| > 1, \end{cases}$$

and I is an identify matrix of dimension 2. Theorem 1.1 allows us to establish sufficient stability conditions of system (12) in the form of a system of inequalities

$$\begin{gathered} |\alpha\rho_1 + \beta\rho_2| < \sqrt{1 - 2q\cos\omega + q^2}; \\ \sigma_{11} < 0, \quad \sigma_{11}\sigma_{22} - \sigma_{12}^2 > 0, \end{gathered} \tag{14}$$

where

$$\sigma_{11} = \rho_1^2 - 1 - \frac{2\rho_1\beta(\alpha\rho_1 + \beta\rho_2)(q - \cos\omega)}{1 - 2q\cos\omega + q^2} + \beta^2,$$

$$\sigma_{22} = \rho_2^2 - 1 - \frac{2\rho_2\alpha(\alpha\rho_1 + \beta\rho_2)(q - \cos\omega)}{1 - 2q\cos\omega + q^2} + \alpha^2,$$

and

$$\sigma_{21} = \sigma_{12} = |\alpha\beta|\frac{|\alpha\rho_1 + \beta\rho_2|}{\sqrt{1 - 2q\cos\omega + q^2}}.$$

It this case the equilibrium state $x = y = 0$ of system (12) is uniformly asymptotically stable, and the constructed function (13) is the Liapunov function.

In order to compare the obtained stability conditions with the conditions obtained in terms of vector Liapunov function we employ the results from [6]. Construct the vector function $V(x, y) = (v_1(x), v_2(y))^{\mathrm{T}}$ with the components $v_1(x) = x^{\mathrm{T}}x$ and $v_2(y) = y^{\mathrm{T}}y$. Applying Theorem 3.3.14 from [6] we present sufficient conditions of uniform asymptotic stability of system (12) in the form of the system of inequalities

$$\begin{gathered} \rho_1^2 + \beta^2 - 1 < 0, \\ (\rho_1^2 + \beta^2 - 1)(\rho_2^2 + \alpha^2 - 1) - 4|\alpha\beta||\rho_1\rho_2| > 0. \end{gathered} \tag{15}$$

To compare conditions (15) and (14) obtained in terms of Theorem 1.1 we consider a system of difference equations

$$\begin{aligned} x(\tau + 1) &= 0.95\,x + \alpha A\left(\frac{\pi}{3}, \tau\right)y, \\ y(\tau + 1) &= -0.95\,y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right)x \end{aligned} \tag{16}$$

and construct in the space of parameters (α, β) the domains of stability of the equilibrium space $x = y = 0$ of system (16). Figures 2.1 and 2.2 show that the domain constructed in terms of conditions (14) is wider than the domain constructed in terms of conditions (15).

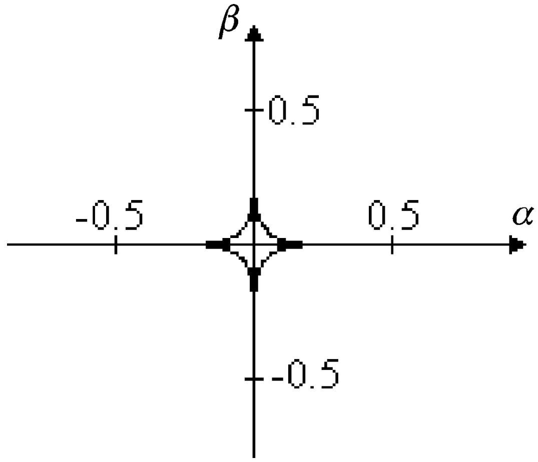

Figure 2.1: The domain of stability of (16) in the parameter space via Liapunov's vector function.

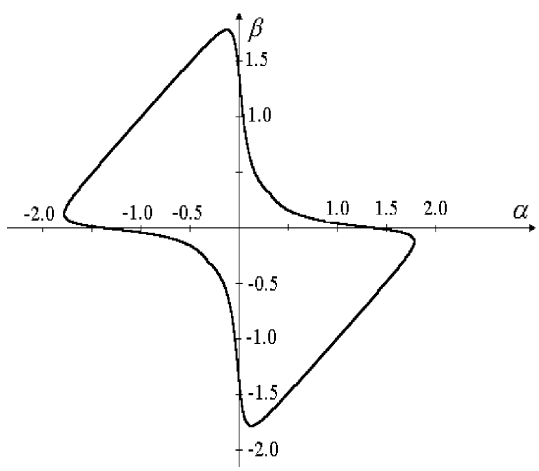

Figure 2.2: The domain of stability of (16) in the parameter space via Liapunov's matrix-valued function.

Note that for the system

$$
\begin{aligned}
x(\tau+1) &= 1.2\,x + \alpha A\left(\frac{\pi}{3}, \tau\right) y, \\
y(\tau+1) &= -0.8\,y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right) x
\end{aligned}
\tag{17}
$$

it is impossible to apply the vector function, because subsystem $x(\tau+1) = 1.2\,x$ is not exponentially stable. Nevertheless conditions (14) allow us to construct for system (16) in the space (α, β) a domain of stability shown in Figure 2.3.

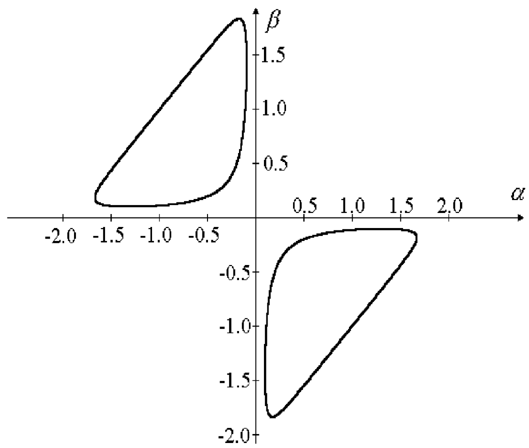

Figure 2.3: The domain of stability of (17) in the parameter space.

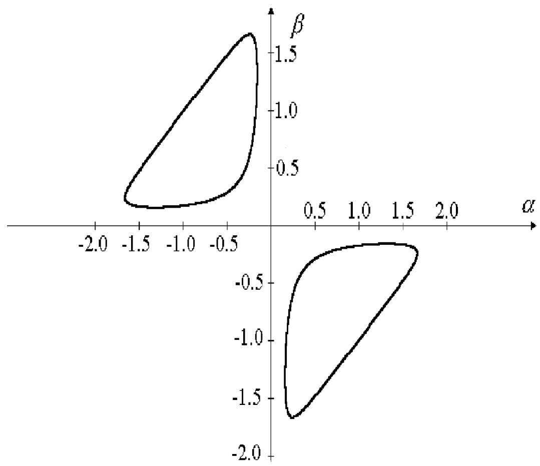

Figure 2.4: The domain of stability of (18) with exponentially unstable subsystem.

The system

$$\begin{aligned} x(\tau+1) &= 1.05\,x + \alpha A\left(\frac{\pi}{3}, \tau\right) y, \\ y(\tau+1) &= -1.05\,y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right) x \end{aligned} \tag{18}$$

has exponentially unstable subsystems. However in this case as well conditions (14) allow us to construct for system (18) a domain of stability in the space of parameters shown in Figure 2.4.

3 Concluding Remarks

Generalized Liapunov function method for a class of large-scale difference systems (1) were developed. In particular, stability and uniform asymptotic stability theorems were presented. The efficiency of the proposed approach was demonstrated by two examples. An important aspect of the new results is that they account an estimation stability domain of parameters of the systems. In connection with the developed theory, there remain many open problems. Some of these include the following: to established guides for choosing "best" vector η in the scalar function $v(\tau, x, \eta)$; to apply the developed theory to specific problems of uncertain systems. Because in general, one is not only interested in stability of systems (1), but also in trajectory bounds, it is desirable to investigate the behavior of systems (1) with respect to sub-sets of the state space.

References

[1] Daletskii, Yu.L. and Krene, M.G. *Stability of Solutions of Differential Equations in Banach Space*. Nauka, Moscow, 1970. [Russian]

[2] Djordjevic, M.Z. *Zur Stabilitat Nichtlinearer Gekoppelter Systeme mit der Matrix-Ljapunov-Methode*. Diss. ETH 7690, Zurich, 1984.

[3] Hahn, W. *Stability of Motion*. Springer-Verlag, Berlin, 1967.

[4] Krasovskii, N.N. *Certain Problems of the Theory of Stability of Motion*. Fizmatgiz, Moscow, 1959. [Russian]

[5] Martynyuk, A.A. *Qualitative Methods in Nonlinear Dynamics. Novel Approaches to Liapunov's Matrix Functions*. Marcel Dekker, New York, 2003.

[6] Michel, A.N. and Miller, R.K. *Qualitative Analysis of Large Scale Dynamical Systems*. Academic Press, New York, 1977.