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NONLINEAR DYNAMICS & SYSTEMS THEORY

**Volume 7, No. 2, 2007**

# Nonlinear Dynamics and Systems Theory

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**PERSONAGE IN SCIENCE**

# Alexander Mikhaylovich Liapunov

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*The sixth of June, 2007 is the 150-th birthday anniversary of the outstanding Russian mathematician and mechanical scientist, Academician Liapunov. Taking into account great significance of Liapunov's works for modern development of nonlinear dynamics and systems theory, the Editorial Board of the journal is publishing a sketch of his life, a brief survey of main directions of his scientific activity and a list of his works published to date.*

## 1 Biographical sketch

Alexander Mikhaylovich Liapunov was born on the 6th of June, 1857 in the family of the prominent astronomer Mikhail Vasilievich Liapunov in the town of Yaroslavl.

The grandfather of Alexander Mikhaylovich, Vasiliy Alexandrovich Liapunov, served for the Kazan University. The elder son of Vasiliy Alexandrovich was the grandfather of Academician Krylov and his younger daughter Ekaterina was the wife of R.M. Sechenov, the brother of the prominent Russian physiologist I.M.Sechenov. Their daughter Nataliya Rafailovna Sechenova was the cousin of Alexander Mikhaylovich. She became his wife in 1886.

In the family of Mikhail Vasilievich and Sofiya Alexandrovna Shipilova there were seven children born but only three sons survived: Alexander (1857–1918), Sergey (1859–1924), and Boris (1864–1942). Four other children died in infancy.

In 1863 Alexander Mikhaylovich's father resigned and settled in the family estate of his parents but soon he moved to the family estate of his wife Sofiya Alexandrovna in the village of Bolobonovo, Simbirskaya province.

Sofiya Alexandrovna was the first to raise the Liapunov's sons until they reached the age of 7 and then Mikhail Alexandrovich took over. He trained his children on his own, applying deliberately developed technique to practicing fast calculation and stirring the

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interest to geography by encouraging the children's games with the elements of travelling around the world.

After the death of Mikhail Vasilievich in 1868 Alexander Mikhaylovich was brought up in the family of his uncle R.M.Sechenov. In 1870 together with his mother Alexander Mikhaylovich moved to Nizhniy Novgorod, where he was admitted to the third class of Nizhegorodskaya gymnasium. At gymnasium he was a bright student and read a lot of books on Russian and European literature, history and natural science. He finished gymnasium with a golden medal in 1876 and entered the Natural Science Department of Physico-Mathematical Faculty of the St Petersburg University where Professor Mendeleyev delivered his lectures, but soon changed it for the Mathematical Department. At that time professors of the Mathematical Department were P.L. Chebyshev, A.N. Korkin and E.I. Zolotariov. In 1880 Alexander Mikhaylovich finished his education and joined the Chair of Mechanics of the University to be prepared for professor rank, as Professor Bobylev had proposed.

In 1882 Alexander Mikhaylovich passed the master's examinations and got down to his master's thesis. P.L. Chebyshev proposed him to investigate the loss of ellipsoidal equilibrium forms of rotating fluid. He was to find out if in this case they would turn into other forms of equilibrium that under slight increase of angular velocity would be little different from ellipsoids. This problem proved to be quite difficult, but its statement involved the other one, namely, the problem on stability of ellipsoidal equilibrium forms. Alexander Mikhaylovich solved this problem in 1885 and defended his master's thesis "On stability of ellipsoidal equilibrium forms of rotating fluid". His opponents were Professors Bobylev and Budayev.

In August, 1885 Alexander Mihaylovich moved to Kharkov on invitation of the Kharkov University to deliver lectures on mechanics at the University and the Technological Institute.

In January, 1886 he went to St Petersburg to get married to Natalia Rafailovna Sechenova. Natalia Rafailovna was a highly educated woman, very sophisticated, she had a profound knowledge of Slavonic philology and was good at painting.

In June–July, 1886 Alexander Mikhaylovich and his family went on a trip to Germany, Switzerland, Austria and also to Serbia for his wife doing research in philology.

Since 1888 Alexander Mikhaylovich began publication of his works on motion stability of mechanical systems with finite number of degrees of freedom. In 1892 the Kharkov Mathematical Society published Liaponov's work "A general problem on stability of motion". This work was defended by Alexander Mikhaylovich as the doctor thesis at Moscow University in 1892. His opponents were Professors Zhukovskii and Mlodzeevskii. Soon A.M. Liapunov was assigned as ordinary professor. In December, 1900 A.M. Liapunov was elected a corresponding member of the St Petersburg Academy of Sciences and in October, 1901 he became an ordinary academician of the Academy.

In spring, 1902 A.M. Liapunov returned to St Petersburg to take up exclusively the scientific work. At the St Petersburg University he headed the Chair of Applied Mathematics Department, this position being vacant since P.L. Chebyshev's death in 1884.

At this period of his scientific activity Alexander Mikhaylovich turned back to the problem on equilibrium figures of rotating fluid. In 1908 A.M. Liapunov took part in the work of the 4th International Mathematical Congress in Rome. In November, 1907 Alexander Mikhaylovich was elected a member of the Palermo Mathematical Society and in September, 1908 — a member of the Academy of Sciences dei Lincei in Rome. Since

1909 Alexander Mikhaylovich was involved in publication of the collected works by Euler who once had commented with humour that it would take quite a bit of time for the Academy of Sciences to publish his papers after his death.

In June, 1917 on doctors' request Alexander Mikhaylovich took his wife away from starving St Petersburg to settle in Odessa where at that time his brother Boris Mikhaylovich lived and worked. In spring, 1918 Natalia Rafailovna suffered from a severe cold which caused an acute attack of a pulmonary tuberculosis. In the end of summer her state became critical.

By that time the wave of revolutionary transformation had reached the Liapunovs' family estate. The house was destroyed and the unique library was burnt. The "Ghost of communism" had strayed to the Russian empire from the West and warmed itself by bonfires made of libraries of intellectuals which were not been in demand of revolutionary crowd.

On the 31st of October, 1918 Natalia Rafailovna died. On the same day Alexander Mikhaylovich was brought to Professor Sapezhko's surgical clinics with a gun-shot wound of his head. Three days later on the 3rd of November, 1918 Alexander Mikhaylovich passed away. His ante-mortem note expressed his last will to be buried in his wife's grave and it was executed.

So was the tragic end of the life of a mathematical genius of the 19th century who under other circumstances would have done a lot of good for his country and world science.

## 2 Main Directions of Scientific Activity

Being the closest disciple of P.L. Chebyshev Alexander Mikhaylovich upheld the best traditions of the St Petersburg mathematical school founded by Chebyshev. Hence, the fundamental importance of problems, accuracy of statement and strictness of solutions are the characteristics of Liapunov's research. Now we shall briefly outline the main directions of his scientific activity.

### *2.1 Stability of equilibrium and motion of mechanical systems with a finite number of degrees of freedom*

The problem on stability of equilibrium and motion which is traced back to the ancient times had remained unsolved until A.M. Liapunov undertook his research in this direction. The strict definition of stability was given by Liapunov in 1892 and was the completion of his intensive work during 1889–1892. The notion "stability by Liapunov" accepted nowadays defines the stability of solutions with respect to perturbations of initial data on infinite time interval. The accurate formulation of the notion of stability was of great importance for further searching for the criteria of equilibrium stability and/or motion of mechanical systems.

A.M. Liapunov considers differential equations of perturbed motion of a general type to discover two general methods for analyzing their solutions. The first method is based on integration of the equations considered by means of special form series. The second is based on application of a certain auxiliary function whose properties together with the properties of its total time derivative along solutions of the system under study allow one to draw a conclusion about the system dynamic behavior. Along with these two methods A.M. Liapunov introduces a new concept of a function characteristic number to apply it to analyzing the stability of solutions of linear systems of differential equations with variable coefficients. A.M. Liapunov completely solved the problem on the first approximation stability and investigated the problem on solutions stability in certain critical cases.

### *2.2 Equilibrium figures of uniformly rotating fluid*

A.M. Liapunov dedicated the last 15 years of his life to this field of research to obtain results of utmost importance. No strict theory existed before Liapunov. His precursors, including Newton, Makloren, Jacobi, Darvin, and Laplace failed to develop a faultless theory, the convergence matters being involved. It was Liapunov, who succeeded. In his work dated by 1903 he and established the existence of figures of equilibrium close to a sphere in the case of heterogeneous fluid slowly rotating around its axis. In a series of works dated by 1905–1914 he studied a more complex problem on existence of equilibrium figures close to known ellipsoidal figures in the case of homogeneous fluid. The subsequent works published in 1915–1917 investigated the problems on equilibrium figures of weakly heterogeneous fluids close to the Macloren or Jacobi ellipsoids. Moreover it was proved that any Macloren or Jacobi ellipsoid different from the bifurcation one generated a series of new equilibrium figures of almost the same shape as the initial ellipsoid, the new figures being also similar to the initial ellipsoid in heterogeneity of density and angular velocity of rotation. To solve the problem considered A.M. Liapunov applied various means of mathematical analysis required for obtaining the result.

### *2.3 Stability of equilibrium figures of rotating fluid*

Works of Liouville and Riemann preceded A.M. Lipunov's research in this field. The first work of Alexander Mikhaylovich on this problem was his master's thesis. Of principle importance was the formulation of definition of equilibrium stability of rotating fluid. Having done this, Alexander Mikhaylovich reduced the problem considered to purely mathematical problem on minimum of a certain expression representing a potential energy of fluid. Analyzing the expression obtained, Liapunov established stability conditions of Macloren and Jacobi ellipsoids as well as instability of pear-shaped figures. In so doing the erroneous Darvin's result on stability of pear-shaped figures was corrected.

As far as the viscous fluid is concerned A.M. Liapunov noticed the following "According to this principle (principle of energy minimum), if the fluid considered is viscous, then the equilibrium figure will be stable or unstable depending on the complete energy corresponding to this figure having minimum or not having minimum provided invariability of momentum with respect to center of gravity. Although this principle has never been proved satisfactorily, there is a good reason to believe it to be valid." This principle for the ideal fluid was proved by Liapunov yet in 1884.

### *2.4 Equations of mathematical physics*

The results obtained by Liapunov in this area of research were of great value both for substantiation of the methods of mathematical physics (Green, Neyman, Roben methods) and for Alexander Mikhaylovich to gain a foothold in the international mathematical community. While investigating properties of simple and double layer potentials the fundamentals of the potential theory and harmonic function theory were first established. In 1899 he found sufficient condition for existence and equality of limiting values of double layer normal derivative. The results obtained caused a sensation since they either refuted or justified the methods of mathematical physics the versions of which belonged to the famous authors such as Poincare, Shvarts and others.

### *2.5 Probability theory*

In this field A.M. Liapunov published two papers in Russian and two in French in 1900 and 1901 correspondingly. These works deal with the problem on probability limit of a sum of infinitely growing number of magnitudes, dependent on random causes, being in the given limits. To prove the limit theorem for this case A.M. Liapunov developed a new method referred to as the method of characteristic functions. It is one of the basic methods in the modern probability theory.

### *2.6 Lecture courses on theoretical mechanics*

During Kharkov period of his activity (1885–1902) A.M. Liapunov prepared the courses of lectures on theoretical mechanics which he delivered at the University and the Technological Institute. On the occasion of the 125-th birthday of A.M. Liapunov these lectures were issued by "Naukova Dumka" Publisher in 1982. The book begins with the sketches about life and activity of A.M. Liapunov and about his lectures on theoretical mechanics written by Boris Liapunov and A.N. Krylov. The first section of the book contains the course of theoretical mechanics which was delivered by Liapunov at the Technological Institute. The second, third and fourth sections contain the university course of lectures on analytical mechanics including "bases of deformable bodies and hydrostatics" and "attraction theory".

These lectures, as Academician Steklov pointed out, had been written by Liapunov himself and were a valuable contribution to theoretical mechanics.

## 3 A.M. Liapunov's credo

It is generally acknowledged that in many branches of mathematics and mechanics A.M. Liapunov established a certain level of consideration accuracy and proof strictness that raised the mathematical sciences to the state of the art and made them classical.

The characteristic feature of A.M.Liaponov's creative work is his interest to the most difficult problems of mechanics arised due to the development of science and whose solution was of great importance for applications. In all his scientific work Alexander Mikhaylovich adheared to the rule:

> "It is not appropriate to make use of ambiguous means in solving a certain problem no matter if it refers to mechanics or physics but is definitely stated from the mathematical point of view. It becomes a pure analysis problem and must be dealt with as it is."

In the end of our survey of scientific and pedagogical activity of A.M. Liapunov we note the permanent effect of his works on further development of mathematics and mechanics in the 20-th century. In all cases when for a real process or a natural phenomenon there was constructed a mathematical model in the form of differential or other type equation or system of such equations the application of Liapunov methods provides the possibility to carry out dynamic analysis of the phenomenon considered no matter whether this phenomenon occurs in biology or astrodynamics.

Entirely devoted to science, a person of perfect integrity Alexander Mikhaylovich actively supported democratic transformations in Russia, freedom of press and opposed reactionaries' impact on secondary and high school. V.A.Steklov wrote that "his spiritual values matched each other so well and nobly that Russia can do be proud of her son".

## List of the published works of A.M. Liapunov*


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* Many scientific works by Liapunov did not published up to day and are keeping in the archive of Russian Academy of Sciences

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# Observation for the descriptor systems with disturbances

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**Abstract:** In this paper, the observation problem for the descriptor systems with disturbances is studied. It is assumed that the disturbances and their first order derivatives are bounded, where the upper and lower bounds are unknown. First, the formulated descriptor system is decomposed into a dynamical system and an algebraic equation. The dynamical system is the relation among a part of the descriptor state, the input-output and the disturbance. The algebraic equation is the relation between the descriptor state variable and the disturbance. Second, the disturbances and one part of the descriptor state are estimated based on the obtained dynamical system. Finally, the other part of the descriptor state is estimated based on the obtained algebraic equation. Examples are presented to illustrate the proposed method.

**Keywords:** *Descriptor system, disturbance observer, state observer.*

**Mathematics Subject Classification (2000):** 93B07, 93B30, 93C15, 93C35.

## 1 Introduction

In the last years, considerable attention has been focused on the control synthesis problems of linear descriptor systems. Structures of such control systems were first studied in the frequency domain by Rosenbrock using matrix pencil theory [13]. Later, controllability, observability and feedback control problems have been investigated by many researchers [2, 5, 7, 8, 9, 15, 16, 19, 20]. However, little effort has been made to develop a theory of observers for descriptor systems. Based on singular-value decomposition,

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El-Tohami et al. have proposed the reduced-order observer for a class of descriptor systems satisfying a simple rank condition [9]. By using the concept of a matrix generalized inverse, a new method is given for constructing a minimal reduced-order observer under a certain observability condition on the constructed observer [14]. Then, a Luenberger-type observer is formulated by using the descriptor standard form [12]. It should be noted that these results are restricted to linear time-invariant descriptor systems with known parameters and without any additional uncertainties.

Recently, the problem of constructing a state observer for the input unknown systems has received some attention. Construction of a state variable observer is a very difficult task for the dynamical system with disturbances, not to say descriptor system with disturbances. For the dynamical systems with disturbances, one typical method is the disturbance decoupled observer by using an elegant geometric approach [3, 18]. Then, this method is applied to disturbance decoupling problems for descriptor systems [11, 10, 1]. However, the results are very complicated and far from complete. The index and stability of the resulting combined systems and the numerical computation of the desired observer have not been considered. As a matter of fact, these geometric solution methods are not suited for numerical computations. The need for reliable numerical method was pointed out in [18]. Later, the computation of the desired disturbance decoupling observer is effectively considered in [6] by using the orthogonal matrix transformation, where the descriptor systems under consideration must be regular and of index at most one.

For the input unknown dynamical systems, another typical effective method about the construction of the state observer is the VSS-type one [17]. However, this approach can only cope with the minimum phase dynamical systems with relative degree one, and the upper and lower bounds of the disturbances are required. It should be noted that this method cannot be applied to the state observation problem for input unknown descriptor systems.

In this paper, the observation problem for the descriptor systems with disturbances is studied by using a totally different approach, where both the descriptor state and the disturbances are estimated. The requirement that the descriptor system must be of index at most one is not needed. It is assumed that the disturbances and their first order derivatives are bounded in the open loop. However, the upper and lower bounds are unknown. The formulated descriptor system is decomposed into a dynamical system and an algebraic equation. The dynamical system is the relation among a part of the descriptor state, the input-output and the disturbance. The algebraic equation is the relation between the descriptor state variable and the disturbance. Based on the obtained dynamical system, the disturbances are first estimated, where the nonlinear method proposed by the authors in [4] for single disturbance single output (SDSO) systems is applied; then, one part of the descriptor state is estimated. Finally, the other part of the descriptor state is calculated based on the obtained algebraic equation.

This paper is organized as follows. Section 2 gives the problem formulation. In Section 3, the disturbance and the state variable are estimated for a special case, the dynamical system case, of the formulated descriptor system. In Section 4, the observation for the general descriptor system with disturbances is studied. In Section 5, design examples and computer simulation results are presented to illustrate the proposed method. Section 6 concludes this paper.

## 2 Problem formulation

Let us consider the following uncertain system

$$\begin{cases} E\dot{x}(t) = Fx(t) + Gu(t) + Kv(t), \\ y(t) = Hx(t) + Bu(t) + Dv(t), \end{cases} \tag{2.1}$$

where $u(t) \in R^q$, $y(t) \in R^r$ and $x(t) \in R^n$ are the input, output and the unknown descriptor state variable, respectively; $v(t) \in R^p$ represents the disturbance, which may include modeling errors, noise, higher order terms in linearization or just an unknown input to the system; $E \in R^{n\times n}$ is a known matrix which may not be nonsingular; $F \in R^{n\times n}$, $G \in R^{n\times q}$, $K \in R^{n\times p}$, $H \in R^{r\times n}$, $B \in R^{r\times q}$ and $D \in R^{r\times p}$ are known matrices.

About the system (2.1), the following assumptions are made.

**Assumption 1** rank $\begin{bmatrix} E \\ H \end{bmatrix} = n$, rank $\begin{bmatrix} F - cE \\ H \end{bmatrix} =$ n for all $c \in C$ , where $C$ denotes the complex plane.

**Assumption 2** For any $c \in C$ satisfying Re(c) $\geq 0$ , $\begin{bmatrix} F - cE & K \\ H & D \end{bmatrix}$ is of full rank, i.e. the system (2.1) is in "minimum phase" with respect to the relation between the disturbance and the output.

**Assumption 3** The signals $u(t)$, $y(t)$ and $v(t)$ are bounded. However, the upper bound of $\|v(t)\|_2$ is unknown.

**Assumption 4** The disturbance $v(t)$ is continuous and piecewise differentiable. Furthermore, the derivative (at the undifferentiable points, we mean the right- and left-hand derivatives) is bounded.

**Assumption 5** $r \geq p$ , i.e. the number of the outputs is not smaller than that of the disturbances.

**Remark 2.1** When the disturbance $v(t)$ is absent, Assumption 1 means that the system (2.1) is observable [12].

The purpose of this paper is to estimate the uncertain signal $v(t)$ and the descriptor state variable $x(t)$ by using the input-output information even though the matrix $E$ may not be nonsingular.

In the following, we assume $B = 0$. Otherwise, we regard the signal $y(t) - Bu(t)$ as $y(t)$.

First, the observation problem is discussed for the case that $E$ is nonsingular. Then, the observation problem is studied for the general descriptor system.

## 3 Observation for the system when E is nonsingular

Without loss of generality, we assume $E = I$. Otherwise, we pre-multiply the first equation of (2.1) with $E^{-1}$.

### 3.1 Observation for the system when D is of full rank

If $D$ is of full rank, then the difference between the observed state $\hat{x}(t)$ and the genuine state $x(t)$ can be designed to decay to zero exponentially, and the disturbance can be asymptotically observed, where Assumption 4 about the disturbance $v(t)$ is not needed.

**Theorem 3.1** *If $D$ is of full rank, then the state observer of the system (2.1) with $E = I$ can be constructed as*

$$\begin{cases} \dot{\hat{x}}(t) = \big(F - KD_1^{-1}\Omega_1 H\big)\hat{x}(t) + Gu(t) + KD_1^{-1}\Omega_1 y(t) + \bar{L}\big(\Omega_2 y(t) - \hat{y}(t)\big), \ \hat{x}(t_0) = 0, \\ \hat{y}(t) = \Omega_2 H\hat{x}(t), \end{cases} \tag{3.1}$$

*where $\hat{x}(t)$ is the estimated state, $\bar{L}$ is chosen such that $F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H$ is a stable matrix, $\Omega = \begin{bmatrix}\Omega_1 \\ \Omega_2\end{bmatrix}$ is a $r \times r$ nonsingular matrix such that*

$$\Omega D = \begin{bmatrix}\Omega_1 \\ \Omega_2\end{bmatrix} D = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \tag{3.2}$$

*in which $D_1$ is a $p \times p$ nonsingular matrix. Furthermore, the disturbance $v(t)$ can be observed by*

$$\hat{v}(t) = D_1^{-1}\Omega_1 y(t) - D_1^{-1}\Omega_1 H\hat{x}(t), \tag{3.3}$$

*where $\hat{x}(t)$ is the estimated state generated in (3.1). For the estimated state and the disturbance, we have*

$$x(t) - \hat{x}(t) \to 0, \ v(t) - \hat{v}(t) \to 0 \tag{3.4}$$

*as $t \to \infty$.*

***Proof*** Equation (2.1) gives

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) + Kv(t), \\ \Omega_1 y(t) = \Omega_1 Hx(t) + D_1 v(t), \\ \Omega_2 y(t) = \Omega_2 Hx(t). \end{cases} \tag{3.5}$$

From the second equation in (3.5), we have

$$v(t) = D_1^{-1}\Omega_1 y(t) - D_1^{-1}\Omega_1 Hx(t). \tag{3.6}$$

By substituting (3.6) into the first equation in (3.5), equation (3.5) yields

$$\begin{cases} \dot{x}(t) = \big(F - KD_1^{-1}\Omega_1 H\big)x(t) + Gu(t) + KD_1^{-1}\Omega_1 y(t), \\ \Omega_2 y(t) = \Omega_2 Hx(t). \end{cases} \tag{3.7}$$

Since

$$\begin{bmatrix} I & -KD_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Omega_1 \\ 0 & \Omega_2 \end{bmatrix} \begin{bmatrix} F - cI & K \\ H & D \end{bmatrix} = \begin{bmatrix} F - KD_1^{-1}\Omega_1 H - cI & 0 \\ \Omega_1 H & D_1 \\ \Omega_2 H & 0 \end{bmatrix}, \tag{3.8}$$

from Assumption 2, it can be seen that $\begin{bmatrix} F - KD_1^{-1}\Omega_1 H - cI \\ \Omega_2 H \end{bmatrix}$ is of full rank for all $c \in C$ satisfying Re(c) $\geq 0$ by observing that $D_1 \in R^{p\times p}$ is a nonsingular matrix. Thus, the system (3.7) is detectable, i.e. the matrix $\bar{L}$ exists such that $F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H$ is a stable matrix. If the observer is constructed as in (3.1), it yields

$$\frac{d}{dt}(x(t) - \hat{x}(t)) = \big(F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H\big)(x(t) - \hat{x}(t)). \tag{3.9}$$

It can be easily seen that $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. From (3.6), it can be concluded that (3.3) is an observer of the disturbance $v(t)$ and $v(t) - \hat{v}(t) \to 0$ as $t \to \infty$.

### 3.2 Observation for the system when D is not of full rank

#### 3.2.1 Some preliminaries

Let s denote the differential operator. Then, equation (2.1) can be written as

$$\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -Gu(t) \\ y(t) \end{bmatrix}. \tag{3.10}$$

Now, pre-multiplying (3.10) by $\left(adj\Big(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\Big)\right)\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T$ yields

$$k(s)\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \left(adj\Big(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\Big)\right)\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T\begin{bmatrix} -Gu(t) \\ y(t) \end{bmatrix}, \tag{3.11}$$

where $k(s)$ is defined as

$$k(s) = det\left(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\right) = k_0 s^{q_0} + \cdots + k_{q_0}, \quad k \neq 0. \tag{3.12}$$

By Assumption 2, it can be easily known that $k(s)$ is a Hurwitz polynomial.

By observing the calculation methods of the adjoint of a matrix and the multiplication of the matrices, equation (3.11) can be expressed as

$$\begin{cases} s^{l_{11}}(\beta_{11}y(t)) = \Phi_{11}(s)y(t) + \Psi_{11}(s)u(t) + k(s)x_1(t), \\ \qquad\vdots \\ s^{l_{1n}}(\beta_{1n}y(t)) = \Phi_{1n}(s)y(t) + \Psi_{1n}(s)u(t) + k(s)x_n(t), \\ s^{l_{21}}(\beta_{21}y(t)) = \Phi_{21}(s)y(t) + \Psi_{21}(s)u(t) + k(s)v_1(t), \\ \qquad\vdots \\ s^{l_{2p}}(\beta_{2p}y(t)) = \Phi_{1p}(s)y(t) + \Psi_{2p}(s)u(t) + k(s)v_p(t), \end{cases} \tag{3.13}$$

where $\beta_{ji} \neq 0$ are row vectors whose entries are constants, $\Phi_{ji}(s)$ are row vectors whose entries are at most $(l_{ji} - 1) - th$ order polynomials of $s$, $\Psi_{ji}(s)$ are row vectors whose entries are at most $(l_{ji} - 1) - th$ order polynomials of $s$.

Because $F,\ G,\ K,\ H$ and $D$ are known matrices, $\beta_{ji}$ , $\Phi_{ji}(s)$ , $\Psi_{ji}(s)$ and $k(s)$ can be calculated.

**Remark 3.1** If $r = p$, i.e. the number of the outputs equals to that of the disturbances, then we can simply pre-multiply the both sides of (3.10) by $\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}$.

### 3.2.2 Observation of the disturbances

About the disturbance $v(t) = [v_1(t) \ \cdots \ v_p(t)]^T$, from (3.13), we have

$$\begin{cases} s^{l_{21}}(\beta_{21}y(t)) = \Phi_{21}(s)y(t) + \Psi_{21}(s)u(t) + k(s)v_1(t), \\ \qquad\qquad \vdots \\ s^{l_{2p}}(\beta_{2p}y(t)) = \Phi_{2p}(s)y(t) + \Psi_{2p}(s)u(t) + k(s)v_p(t). \end{cases} \tag{3.14}$$

For the $i-th$ equation in (3.14), $l_{2i} - q_0$ can be regarded as the "relative degree" with respect to the relation between the disturbance $v_i(t)$ and the "output" $\beta_{2i}y(t)$. It is easy to see that $l_{2i} \geq q_0$, otherwise, equation (3.13) contradicts with the original differential equation (2.1).

We start with equation (3.14) to estimate the disturbances.

For simplicity, let

$$\eta_i = l_{2i} - q_0. \tag{3.15}$$

To estimate the disturbances, the discussion is divided into the following two cases.

**Case 1:** $l_{2i} = q_0$

In this case, from (3.14), it gives

$$v_i(t) = \frac{s^{l_{2i}}}{k(s)}(\beta_{2i}y(t)) - \frac{\Phi_{2i}(s)}{k(s)}y(t) - \frac{\Psi_{2i}(s)}{k(s)}u(t), \tag{3.16}$$

i.e. the disturbance $v_i(t)$ can be expressed by the outputs and the filters of the inputs and outputs, where only the input and output information is employed. Thus,

$$w_{i,0} \triangleq \frac{s^{l_{2i}}}{k(s)}(\beta_{2i}y(t)) - \frac{\Phi_{2i}(s)}{k(s)}y(t) - \frac{\Psi_{2i}(s)}{k(s)}u(t) \tag{3.17}$$

can be regarded as the estimate of $v_i(t)$.

**Remark 3.2** For a complex constant $\Gamma \in C$ satisfying $\mathrm{Re}(\Gamma) > 0$, $\frac{1}{s+\Gamma}y(t)$ is defined as the solution of the following differential equation

$$\dot{\xi}(t) + \Gamma\xi(t) = y(t), \ \xi(t_0) = 0, \tag{3.18}$$

where $t_0$ is the starting time. Thus, the filters in (3.17) and the upcoming ones can be analogously defined.

**Case 2:** $l_{2i} > q_0$

Introduce a monic $l_i - th$ order Hurwitz polynomial

$$g_i(s) = \frac{1}{k_0}k(s) \cdot (s + \lambda)^{\eta_i}, \tag{3.19}$$

where $\lambda$ is a positive constant. Then, the $i-th$ equation in (3.14) can be rewritten as

$$\dot{z}_i(t) + \lambda z_i(t) = L_i(y(t), u(t)) + \frac{k_0}{(s+\lambda)^{\eta_i - 1}}v_i(t), \tag{3.20}$$

where $z_i(t)$ and $L_i(y(t),u(t))$ are respectively defined as

$$z_i(t) = \beta_{2i} y(t), \tag{3.21}$$

$$L_i(y(t),u(t)) = (s+\lambda)\left\{\frac{g_i(s)-s^{l_i}}{g_i(s)}\{\beta_{2i}y(t)\} + \frac{\Phi_{2i}(s)}{g_i(s)}y(t) + \frac{\Psi_{2i}(s)}{g_i(s)}u(t)\right\}. \tag{3.22}$$

**Remark 3.3** It should be pointed out that $z_i(t)$ and $L_i(y(t),u(t))$ are computable signals.

Since $v_i(t)$ are bounded signals, it can be seen that, for a positive constant $\lambda$, signals $\left|\frac{1}{(s+\lambda)^{j_i}}v_i(t)\right|$ are also bounded for any positive integer $j_i$.

The next theorem gives a method to estimate $\frac{1}{(s+\lambda)^{\eta_i-j_i}}v_i(t)$, where the upper bounds of $\left|\frac{1}{(s+\lambda)^{\eta_i-j_i}}v_i(t)\right|$ are adaptively updated.

**Theorem 3.2** *Construct the following differential equations*

$$\dot{\hat{z}}_i(t) + \lambda\hat{z}_i(t) = L_i(y(t),u(t)) + k_0 w_{i,1}(t),\ \ \hat{z}_i(t_0) = z_i(t_0), \tag{3.23}$$

$$\dot{\hat{w}}_{i,\mu_i-1}(t) + \lambda\hat{w}_{i,\mu_i-1}(t) = w_{i,\mu_i}(t),\ \ \hat{w}_{i,\mu_i-1}(t_0) = 0, \tag{3.24}$$

*where $\hat{z}_i(t)$ and $\hat{w}_{i,\mu_i-1}(t)$ $(1<\mu_i\le\eta_i)$ are the variables which can be obtained by respectively solving (3.23) and (3.24); $w_{i,1}(t)$ and $w_{i,\mu_i}(t)$ are the inputs described respectively by*

$$w_{i,1}(t) = \hat{\omega}_{i,1}(t)\frac{k_0\{z_i(t)-\hat{z}_i(t)\}}{\left|k_0\{z_i(t)-\hat{z}_i(t)\}\right| + \delta_{i,1}} \tag{3.25}$$

*and*

$$w_{i,\mu_i}(t) = \hat{\omega}_{i,\mu_i}(t)\frac{w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)}{\left|w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)\right| + \delta_{i,\mu_i}},\ \ (1<\mu_i\le\eta_i) \tag{3.26}$$

*$\delta_{i,j_i} > 0$ $(i=1,\cdots,p;\ j_i=1,\cdots,\eta_i)$ are design parameters which are usually chosen to be very small; $\hat{\omega}_{i,\mu_i}(t)$ $(1\le\mu_i\le\eta_i)$ are updated by the following adaptive algorithms*

$$\dot{\hat{\omega}}_{i,1}(t) = \begin{cases} 2\alpha_{i,1}|z_i(t)-\hat{z}_i(t)| & if\ |k_0\{z_i(t)-\hat{z}_i(t)\}| > \delta_{i,1} \\ 0 & otherwise \end{cases}, \tag{3.27}$$

$$\dot{\hat{\omega}}_{i,\mu_i}(t) = \begin{cases} 2\alpha_{i,\mu_i}|w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)| & if\ |w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)| > \delta_{i,\mu_i} \\ 0 & otherwise \end{cases} \tag{3.28}$$

*for $1<\mu\le\eta_i$, $\hat{\omega}_{i,\mu_i}(t_0)$ can be chosen as any positive constants, $\alpha_{i,\mu_i}$ are positive constants for $i=1,\cdots,p$, $1\le\mu_i\le\eta_i$. It can be concluded that $w_{i,\mu_i}(t)$ are the corresponding approximate estimates of $\frac{1}{(s+\lambda)^{\eta_i-\mu_i}}v_i(t)$ for $1\le\mu_i\le\eta_i$ as $t$ is large* *enough, i.e. there exist $T_{i,\mu_i} \geq t_0$ and functions $\epsilon_{i,\mu_i}(\nu_1, \cdots, \nu_{\mu_i}) > 0$ with the property $\lim\limits_{\sum_{j=1}^{\mu_i} |\nu_i| \to 0} \epsilon_{i,\mu_i}(\nu, \cdots, \nu_{\mu_i}) = 0$ such that*

$$\left| \frac{1}{(s+\lambda)^{\eta_i - \mu_i}} v_i(t) - w_{i,\mu_i(t)} \right| < \epsilon_{i,\mu_i}(\delta_{i,1}, \cdots, \delta_{i,\mu_i}) \tag{3.29}$$

*for all $t \geq T_{i,\mu_i}$*

***Proof*** This theorem can be proved by a similar procedure as in [4], where Assumptions 3 and 4 are employed.

**Remark 3.4** The design parameters $\delta_{i,j_i} > 0$ $(1 \leq j_i \leq \eta_i)$ and $\lambda > 0$ determine the estimating precision and the estimating speed. The parameters $\alpha_{i,j_i} > 0$ should be chosen large enough to adjust the estimated upper bounds $\hat{\omega}_{i,j_i}(t)$ rapidly for $1 \leq j_i \leq \eta_i$. The estimation error for the disturbances can be designed to be arbitrarily small by choosing the design parameters. The influence of the measurement noises in the output can be similarly discussed as in [4].

**Remark 3.5** For $i \neq j$, it can be seen that the estimation of $v_i(t)$ is independent of the estimation of $v_j(t)$.

### 3.2.3 Observation of the state

About the state $x(t) = [x_1(t) \ \cdots \ x_n(t)]^T$, from (3.13), we have

$$\begin{cases} s^{l_{11}}(\beta_{11} y(t)) = \Phi_{11}(s) y(t) + \Psi_{11}(s) u(t) + k(s) x_1(t), \\ \qquad \vdots \\ s^{l_{1n}}(\beta_{1n} y(t)) = \Phi_{1n}(s) y(t) + \Psi_{1n}(s) u(t) + k(s) x_n(t). \end{cases} \tag{3.30}$$

To estimate the state, the discussion is divided into the following two cases.

**Case 1:** $l_{1i} \leq q_0$

In this case, from (3.30), it gives

$$x_i(t) = \frac{s^{l_{1i}}}{k(s)}(\beta_{1i} y(t)) - \frac{\Phi_{1i}(s)}{k(s)} y(t) - \frac{\Psi_{1i}(s)}{k(s)} u(t), \tag{3.31}$$

i.e. the partial state $x_i(t)$ can be expressed by the outputs and the filters of the inputs and outputs, where only the input and output information is employed. Thus,

$$\hat{x}_i(t) \triangleq \frac{s^{l_{1i}}}{k(s)}(\beta_{1i} y(t)) - \frac{\Phi_{1i}(s)}{k(s)} y(t) - \frac{\Psi_{1i}(s)}{k(s)} u(t) \tag{3.32}$$

can be regarded as the estimate of $x_i(t)$.

**Remark 3.6** If $l_{1i} \leq q_0$ for all $i = 1, \cdots, n$, then there is no steady error between the estimated state and the genuine state $x(t)$.

**Case 2:** $l_{2i} > q_0$

In this case, the partial state $x_i(t)$ can be similarly estimated by the method proposed for estimating the disturbances in Section 3.2.2 if the partial state $x_i(t)$ is bounded.

However, the computation for all such partial states $x_i(t)$ satisfying $l_{2i} > q_0$ will become very complicated, and the partial state $x_i(t)$ may not be bounded.

One simple method of estimating the partial state in this case is to construct a Luenberger-type state observer for the full state $x(t)$ by using the estimates of the disturbances obtained in Section 3.2.2, and then extract the partial states $x_i(t)$ satisfying $l_{2i} > q_0$. We have the following theorem to approximately construct the full state observer. The estimation error is controlled by the design parameters.

**Theorem 3.3** *The state observer of the system (2.1) with $E = I$ can be considered as*

$$\begin{cases} \dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + Kw(t) + L(y(t) - \hat{y}(t)), \ \hat{x}(t_0) = 0, \\ \hat{y}(t) = H\hat{x}(t) + Dw(t), \end{cases} \tag{3.33}$$

*where $\hat{x}(t)$ is the estimated state, $w(t) = [w_{1,\eta_1} \ \cdots \ w_{p,\eta_p}]^T$ is the estimate of the disturbance $v(t)$ obtained in Section 3.2.2, the design matrix $L$ is chosen such that the matrix $F - LH$ is stable. Then, there exists a function $\epsilon(\nu_{i,j_i}\big| i \in S;\ j_i = 1, \cdots, \eta_i) > 0$ with the property $\lim\limits_{\sum_{i=1}^{p}\sum_{j_i=1}^{\eta_i}|\nu_{i,j_i}|\to 0} \epsilon(\nu_{i,j_i}\big| i \in S;\ j_i = 1, \cdots, \eta_i) \to 0$ such that*

$$\|x(t) - \hat{x}(t)\|_2 \le \epsilon(\delta_{i,j_i}\big| i \in S;\ j = 1, \cdots, \eta_i), \tag{3.34}$$

*as $t \to \infty$, where $S$ is the subset of $\{1, \cdots, p\}$ satisfying the condition: if $i \in S$, then $\eta_i > 0$.*

***Proof*** It can be seen from Assumption 1 that there exists a matrix $L$ such that $F - LH$ is stable. From (2.1) and (3.33), it gives

$$\dot{e}(t) = (F - LH)e(t) - (K + LD)\{v(t) - w(t)\}, \tag{3.35}$$

where $e(t)$ is defined as $e(t) = x(t) - \hat{x}(t)$. As $w(t)$ is the estimate of $v(t)$, by employing Theorem 3.1 and the stability of matrix $F - LH$, the result can be easily proved.

### 3.3 The numerical observation algorithm for the case that E is nonsingular

Suppose $E = I$. Otherwise, pre-multiply the first equation of (2.1) with $E^{-1}$.

**S1** If $D$ is of full rank, then the disturbance $v(t)$ and the state $x(t)$ are asymptotically identified by Theorem 3.1. Otherwise, go to S2.

**S2** Derive the system (3.13) based directly on (2.1).

**S3** Identify the disturbance $v_i(t)$ by (3.17) or Theorem 3.2.

**S4** Identify the state $x_i(t)$ by using (3.32) or extracting from the constructed Luenberger-type state observer formulated in Theorem 3.3.

## 4 Observation for the general descriptor system

### 4.1 Some preparations

Suppose the matrix $E$ is of rank $l$ ($l < n$). Since $E$ is known, we can find nonsingular matrices $P, Q \in R^{n\times n}$ such that

$$PEQ^{-1} = \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.1}$$

Thus, by taking the transformation

$$\bar{x}(t) = Qx(t), \tag{4.2}$$

the system (2.1) can be rewritten as

$$\begin{cases} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \dot{\bar{x}}(t) = PFQ^{-1}\bar{x}(t) + PGu(t) + PKv(t), \\ y(t) = HQ^{-1}\bar{x}(t) + Dv(t). \end{cases} \tag{4.3}$$

**Lemma 4.1** *For the system (4.3), we have*

$$\operatorname{rank} \begin{bmatrix} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = n, \tag{4.4}$$

$$\operatorname{rank} \begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = n \ for\ all\ c \in C, \tag{4.5}$$

*and* $\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} & PK \\ HQ^{-1} & D \end{bmatrix}$ *is of full rank for any* $c \in C$ *satisfying* $Re(c) \geq 0$.

***Proof*** The lemma can be easily proved by observing the following facts

$$\begin{bmatrix} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E \\ H \end{bmatrix} Q^{-1}, \tag{4.6}$$

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F - cE \\ H \end{bmatrix} Q^{-1}, \tag{4.7}$$

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} & PK \\ HQ^{-1} & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F - cE & K \\ H & D \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}, \tag{4.8}$$

From now on, we will start with the system (4.3) to estimate the disturbance and the state. Now rewrite the system (4.3) as

$$\begin{cases} \dot{\bar{x}}_1(t) = F_{11}\bar{x}_1(t) + F_{12}\bar{x}_2(t) + G_1u(t) + K_1v(t), \\ 0 = F_{21}\bar{x}_1(t) + F_{22}\bar{x}_2(t) + G_2u(t) + K_2v(t), \\ y(t) = H_1\bar{x}_1(t) + H_2\bar{x}_2(t) + Dv(t), \end{cases} \tag{4.9}$$

where

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},\ PFQ^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},\ PG = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},\ HQ^{-1} = [H_1\ H_2],\ PK = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{4.10}$$

**Lemma 4.2** *$H_2 \in R^{r\times(n-l)}$ is of full rank and $r$ satisfies $r \geq n-l$.*

***Proof*** From Lemma 4.1 and the assumptions, the lemma is obvious.

Since the matrix $H_2 \in R^{r\times(n-l)}$ is of full rank, there exists a nonsingular matrix $O = \begin{bmatrix} O_1 \\ O_2 \end{bmatrix} \in R^{r\times r}$ such that

$$OH_2 = \begin{bmatrix} H_{21} \\ 0 \end{bmatrix}, \tag{4.11}$$

where $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix.

Therefore, by pre-multiplying the third equation in (4.9) with $O$, equation (4.9) yields

$$\begin{cases} \dot{\bar{x}}_1(t) = F_{11}\bar{x}_1(t) + F_{12}\bar{x}_2(t) + G_1u(t) + K_1v(t), \\ 0 = F_{21}\bar{x}_1(t) + F_{22}\bar{x}_2(t) + G_2u(t) + K_2v(t), \\ O_1y(t) = O_1H_1\bar{x}_1(t) + H_{21}\bar{x}_2(t) + O_1Dv(t), \\ O_2y(t) = O_2H_1\bar{x}_1(t) + O_2Dv(t). \end{cases} \tag{4.12}$$

By the third equation in (4.12), $\bar{x}_2(t)$ can be expressed as

$$\bar{x}_2(t) = H_{21}^{-1}O_1\big(y(t) - H_1\bar{x}_1(t) - Dv(t)\big). \tag{4.13}$$

By substituting (4.13) into the first two equations in (4.12), equation (4.12) yields

$$\begin{cases} \dot{\bar{x}}_1(t) = (F_{11} - F_{12}H_{21}^{-1}O_1H_1)\bar{x}_1(t) + F_{12}H_{21}^{-1}O_1y(t) + G_1u(t) + (K_1 - F_{12}H_{21}^{-1}O_1D)v(t), \\ \begin{bmatrix} F_{22}H_{21}^{-1}O_1 \\ O_2 \end{bmatrix} y(t) = \begin{bmatrix} F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2D \end{bmatrix} v(t). \end{cases} \tag{4.14}$$

Now, for simplicity, we rewrite the system (4.14) in the following compact form

$$\begin{cases} \dot{\bar{x}}_1(t) = \bar{F}\bar{x}_1(t) + \bar{u}(t) + \bar{K}v(t), \\ \bar{y}(t) = \bar{H}\bar{x}_1(t) + \bar{D}v(t), \end{cases} \tag{4.15}$$

where the matrices $\bar{F},\ \bar{K},\ \bar{H},\ \bar{D}$ are defined as

$$\bar{F} = F_{11} - F_{12}H_{21}^{-1}O_1H_1,\ \bar{K} = K_1 - F_{12}H_{21}^{-1}O_1D, \tag{4.16}$$

$$\bar{H} = \begin{bmatrix} F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{bmatrix},\ \bar{D} = \begin{bmatrix} K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2D \end{bmatrix}, \tag{4.17}$$

$\bar{u}(t)$ and $\bar{y}(t)$ are represented by

$$\bar{u}(t) = F_{12}H_{21}^{-1}O_1y(t) + G_1u(t),\ \bar{y}(t) = \begin{bmatrix} F_{22}H_{21}^{-1}O_1y(t) - G_2u(t) \\ O_2y(t) \end{bmatrix}. \tag{4.18}$$

**Remark 4.1** The matrices $\bar{F} \in R^{l\times l},\ \bar{K} \in R^{l\times p},\ \bar{H} \in R^{r\times l},\ \bar{D} \in R^{r\times p}$ are available because they can be computed out by using the known matrices $E,\ F,\ G,\ K,\ H$, and $D$.

**Remark 4.2** $\bar{u}(t) \in R^l$ and $\bar{y}(t) \in R^r$ are available signals. Since $r \geq p$, the number of the outputs of the system (4.15) is not smaller than that of the disturbances.

**Lemma 4.3** *The system (4.15) is observable in the absence of the disturbance $v(t)$.*

***Proof*** Since

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = \left[\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc} F_{11} - cI & F_{12} \\ F_{21} & F_{22} \\ \hdashline O_1 H_1 & H_{21} \\ O_2 H_1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc:cc} I & 0 & F_{12}H_{21}^{-1} & 0 \\ 0 & I & F_{22}H_{21}^{-1} & 0 \\ \hdashline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI & 0 \\ F_{21} - F_{22}H_{21}^{-1}O_1H_1 & 0 \\ \hdashline O_1H_1 & H_{21} \\ O_2H_1 & 0 \end{array}\right], \tag{4.19}$$

we obtain by Lemma 4.1 that $\left[\begin{array}{c} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI \\ \hdashline F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{array}\right]$, i.e. $\begin{bmatrix} \bar{F} - cI \\ \bar{H} \end{bmatrix}$, is of full rank for all $c \in C$ by using the fact that $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix. Thus, the observability of the system (4.15) is verified.

**Lemma 4.4** *The system (4.15) is in minimum phase with respect to the relation between the disturbance $v(t)$ and the "output" $\bar{y}(t)$.*

***Proof*** Since

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} & PK \\ HQ^{-1} & D \end{bmatrix} = \left[\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc:c} F_{11} - cI & F_{12} & K_1 \\ F_{21} & F_{22} & K_2 \\ \hdashline O_1 H_1 & H_{21} & O_1 D \\ O_2 H_1 & 0 & O_2 D \end{array}\right]$$

$$= \left[\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc:cc} I & 0 & F_{12}H_{21}^{-1} & 0 \\ 0 & I & F_{22}H_{21}^{-1} & 0 \\ \hdashline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \times$$

$$\left[\begin{array}{cc:c} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI & 0 & K_1 - F_{12}H_{21}^{-1}O_1D \\ F_{21} - F_{22}H_{21}^{-1}O_1H_1 & 0 & K_2 - F_{22}H_{21}^{-1}O_1D \\ \hdashline O_1H_1 & H_{21} & O_1D \\ O_2H_1 & 0 & O_2D \end{array}\right], \tag{4.20}$$

we obtain from Lemma 4.1 that $\left[\begin{array}{c:c} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI & K_1 - F_{12}H_{21}^{-1}O_1D \\ \hdashline F_{21} - F_{22}H_{21}^{-1}O_1H_1 & K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2H_1 & O_2D \end{array}\right]$, i.e. $\begin{bmatrix} \bar{F} - cI & \bar{K} \\ \bar{H} & \bar{D} \end{bmatrix}$, is of full rank for any $c \in C$ satisfying $\text{Re}(c) \geq 0$ by observing that $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix. Thus, the lemma is proved.

From Lemmas 4.3 and 4.4, the system (4.15) can be used to estimate the disturbances and to observe the partial state $\bar{x}_1(t)$ by the algorithm proposed in Section 3.3. Furthermore, from (4.13), the partial state $\bar{x}_2(t)$ can be estimated as

$$\hat{\bar{x}}_2(t) = H_{21}^{-1}O_1(y(t) - H_1\hat{\bar{x}}_1(t) - Dw(t)), \tag{4.21}$$

where $\hat{\bar{x}}_1(t)$ is the estimate of the partial state $\bar{x}_1(t)$, $w(t)$ is the estimate of the disturbance $v(t)$. Therefore, the state $x(t)$ can be estimated by using the transformation

$$\hat{x}(t) = Q^{-1}\begin{bmatrix}\hat{\bar{x}}_1(t)\\ \hat{\bar{x}}_2(t)\end{bmatrix}. \tag{4.22}$$

### 4.2 The numerical observation algorithm for the general descriptor systems with disturbances

**Step1** If $E$ is nonsingular, then the algorithm is given in Section 3.3. Otherwise, go to step 2.

**Step2** Determine the nonsingular matrices $P$ and $Q$ satisfying (4.1), derive the system (4.9), and consider the state observer and the disturbance observer for the system (4.9).

**Step3** For the matrix $H_2$ in the system (4.9), determine the nonsingular matrix $O$ satisfying (4.11). The system (4.10) is rearranged as the dynamical system (4.15) and relation (4.13). For the dynamical system (4.15), the algorithm presented in Section 3.3 can be used to estimate the disturbance $v(t)$ and the partial state $\bar{x}_1(t)$.

**Step4** Construct the observer $\hat{\bar{x}}_2(t)$ for the partial descriptor state $\bar{x}_2(t)$ by (4.21).

**Step5** The descriptor state $x(t)$ is estimated by (4.22).

## 5 Design examples and simulation results

**Example 5.1** Consider the descriptor system

$$\begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 1 & -2 & 0\\ -2 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 2\end{bmatrix}, \tag{5.1}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}, \tag{5.2}$$

where the input $u(t)$ is assumed as zero, the disturbances are governed by

$$v_1(t) = \phi(t) + \psi(t),\ \ v_2(t) = 1 + \psi(t) \tag{5.3}$$

with $\phi(t) = \begin{cases} t & 0 \le t \le 3\\ 3 & t > 3\end{cases}$ and $\psi(t) = \begin{cases} t & 0 \le t \le 6\\ 4 & t > 6\end{cases}$.

It can be easily checked that the assumptions in Section 2 are all satisfied. Furthermore, it can be checked that $\deg(\det(sE-F)) = rank(E)$, i.e. this descriptor system is of index at most one.

In the following, the disturbance observer and the descriptor state observer will be formulated by following the algorithm summarized in Section 4.2.

**Step1** Since $E$ is singular with $rank(E) = 2$, go to step 2.

**Step2** The nonsingular matrices $P$ and $Q$ satisfying (4.1) are determined as

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \left[\begin{array}{c} \bar{x}_{11} \\ \bar{x}_{12} \\ \hline \bar{x}_{21} \end{array}\right] = Qx(t)$. Then, corresponding to (4.9), it yields

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} v(t), \\ 0 = [-1 \quad 2]\bar{x}_1(t) + \bar{x}_2(t) + [0 \quad -1]v(t), \\ y(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} v(t). \end{cases}$$

**Step3** For the matrix $H_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the nonsingular matrix $O$ satisfying (4.11) can be determined as

$$O = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Thus, corresponding to (4.12), it gives

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} v(t), \\ 0 = [-1 \ 2]\bar{x}_1(t) + \bar{x}_2(t) + [0 \ -1]v(t), \\ [1 \ 0]\, y(t) = [0 \ 0]\bar{x}_1(t) + \bar{x}_2(t) + [0 \ 1]v(t), \\ [-1 \ 1]\, y(t) = [0 \ 1]\bar{x}_1(t) + [0 \ 1]v(t). \end{cases} \tag{5.4}$$

Then, from the third equation in (5.4), $\bar{x}_2(t)$ can be expressed as

$$\bar{x}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]v(t). \tag{5.5}$$

By substituting the expression of $\bar{x}_2(t)$ into the other equations in (5.4), the equation corresponding to (4.15) is given by

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v(t), \\ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} y(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} v(t). \end{cases} \tag{5.6}$$

Now, based on (5.6), the disturbance $v(t)$ and the variable $\bar{x}_1(t)$ will be estimated by following the algorithm summarized in Section 3.3.

**S1** Since $\bar{D} = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$ is not of full rank, go to S2.

**S2** Rewrite (5.6) as

$$\left[\begin{array}{cc:cc} 2-s & 0 & 1 & 1 \\ 0 & -2-s & 0 & 1 \\ \hdashline -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} \bar{x}_1(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{cc} 2 & 0 \\ -1 & 0 \\ \hdashline -1 & 0 \\ -1 & 1 \end{array}\right] y(t). \tag{5.7}$$

By pre-multiplying the both sides of (5.7) with

$$adj \left[\begin{array}{cc:cc} 2-s & 0 & 1 & 1 \\ 0 & -2-s & 0 & 1 \\ \hdashline -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} 0 & 4 & s+3 & 2s+2 \\ 0 & 1 & 0 & -1 \\ -s-3 & 4s-7 & s^2+s-6 & 2s^2-s-2 \\ 0 & -1 & 0 & -s-2 \end{bmatrix}$$

the system corresponding to (3.13) is derived as

$$\begin{cases} s(3y_1 - 2y_2) + 9y_1 - 2y_2 = (s+3)\bar{x}_{11}, \\ y_2 = (s+3)\bar{x}_{12}, \\ s^2(3y_1 - 2y_2) = -s(6y_1 + y_2) + 9y_1 - 2y_2 + (s+3)v_1, \\ s(-y_1 + y_2) = 3y_1 - 2y_2 + (s+3)v_2. \end{cases} \tag{5.8}$$

**S3** Based on (5.8), the disturbance $v_2(t)$ can be simply estimated by

$$w_{2,0} \triangleq \frac{s}{s+3}(-y_1 + y_2) - \frac{1}{s+3}(3y_1 - 2y_2).$$

The disturbance $v_1(t)$ is estimated as follows.
Introduce the Hurwitz polynomial

$$g_1(s) = (s+3)(s+2),$$

where $\lambda$ is chosen as $\lambda = 2$.
Define $z_1 = 3y_1 - 2y_2$. Corresponding to (3.10), the third equation in (5.8) can be rewritten as

$$\dot{z}_1(t) + 2z_1(t) = \frac{s}{s+3}(9y_1 - 11y_2) + \frac{1}{s+3}(27y_1 - 14y_2) + v_1(t).$$

By Theorem 3.2, construct the following differential equation

$$\dot{\hat{z}}_1(t) + 2\hat{z}_1(t) = \frac{s}{s+3}(9y_1 - 11y_2) + \frac{1}{s+3}(27y_1 - 14y_2) + w_{1,1}(t),$$

$$\hat{z}_1(0) = z_1(0),$$

$$w_{1,1}(t) = \hat{\omega}_{1,1}(t)\frac{z_1(t) - \hat{z}_1(t)}{|z_1(t) - \hat{z}_1(t)| + \delta_{1,1}},$$

$\hat{\omega}_{1,1}(t)$ is updated by the following adaptive algorithms

$$\dot{\hat{\omega}}_{1,1}(t) = \begin{cases} 1200|z_1(t) - \hat{z}_1(t)| & \text{if} \quad |z_1(t) - \hat{z}_1(t)| > \delta_{1,1} \\ 0 & \text{otherwise} \end{cases}, \quad \dot{\hat{\omega}}_{1,1}(0) = 5\,.$$

Then, $w_{1,1}(t)$ can be regarded as an estimate of $v_1(t)$.

**S4** By the theory in Section 3.2.3, the variables $\bar{x}_{11}(t)$ and $\bar{x}_{12}(t)$ can be respectively estimated by $\hat{\bar{x}}_{11}(t)$ and $\hat{\bar{x}}_{12}(t)$ defined by

$$\hat{\bar{x}}_{11} = \frac{s}{s+3}(3y_1 - 2y_2) + \frac{1}{s+3}(9y_1 - 2y_2),$$

$$\hat{\bar{x}}_{12} = \frac{1}{s+3}y_2.$$

**Step4** Construct the observer $\hat{\bar{x}}_2(t)$ for the partial descriptor state $\bar{x}_2(t)$ by (5.5).

$$\hat{\bar{x}}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]\begin{bmatrix} w_{1,1}(t) \\ w_{2,0}(t) \end{bmatrix} = y_1(t) - w_{2,0}(t).$$

**Step5** The descriptor state $x(t)$ is estimated by

$$\hat{x}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_{11}(t) \\ \hat{\bar{x}}_{12}(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix}.$$

It can be seen that some steady error exists in the estimation of $v_1(t)$, and the error depends on the design parameter $\delta_{11}$. Furthermore, there are no steady errors existing in the estimation of the disturbance $v_2(t)$ and the descriptor state $x(t)$.

Computer simulation results show that the disturbance $v_2(t)$ and the descriptor state $x(t)$ can be perfectly identified. The figures are omitted. The estimation error of the disturbance $v_1(t)$ is shown in Figure 5.1, where the parameter $\delta_{11}$ is chosen as $\delta_{11} = 0.0001$.

It should be noted $v_1(t)$ is not differentiable at $t = 3$ and $t = 6$ and is not continuous at $t = 6$. Simulation results show that the disturbance observer works well at the continuous points and has a transient error at the discontinuous points. This is because that the proposed method is trying to identify the unknown signals by using a differentiable approach. It is considered that the new method can be applied to practical problems with piecewise differentiable disturbances. For the sake of strictness, the disturbances are assumed to be continuous and piecewise differentiable.

**Example 5.2** Consider the descriptor system

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \tag{5.9}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{5.10}$$

where the input $u(t)$ is assumed as zero, $v_1(t)$ and $v_2(t)$ are the disturbances.

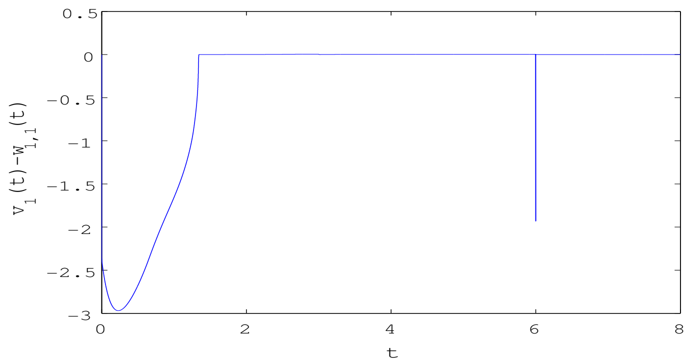


**Figure 5.1: The difference between the disturbance $v_1(t)$ and its estimate**

It can be easily checked that the assumptions in Section 2 are all satisfied. Furthermore, it can be checked that $\deg(\det(sE-F)) \neq rank(E)$, i.e. this descriptor system is not of index at most one.

Since $E$ is singular with $rank(E) = 2$, the nonsingular matrices $P$ and $Q$ satisfying (3.32) are determined as

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \left[\begin{array}{c} \bar{x}_{11} \\ \bar{x}_{12} \\ \hdashline \bar{x}_{21} \end{array}\right] = Qx(t)$. Then, by a computation similar to that in Example 5.1, the relations corresponding to (5.5) and (5.8) are derived as

$$\bar{x}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]v(t), \tag{5.11}$$

$$\begin{cases} s(-y_2) + 15y_1 - 4y_2 = (2s+3)\bar{x}_{11}, \\ 6y_1 - y_2 = (2s+3)\bar{x}_{12}, \\ s^2(-y_2) = -s(15y_1 - y_2) + 33y_1 - 10y_2 + (2s+3)v_1, \\ s(y_2) = 3y_1 - 2y_2 + (2s+3)v_2. \end{cases} \tag{5.12}$$

Similar to Example 5.1, the disturbances and the descriptor state can be identified. It can be seen that the proposed method can also deal with the descriptor systems which are not of index at most one.

## 6 Conclusions

In this paper, the observation problem for the descriptor systems with disturbances is studied. It is assumed that the disturbances and their first order derivatives are bounded in the open loop. However, the upper and lower bounds are unknown. The formulated descriptor system can be decomposed into a dynamical system and an algebraic equation. Based on this obtained dynamical system, first, the disturbances are estimated; then, one part of the descriptor state is observed. Finally, the other part of the descriptor state is estimated based on the obtained algebraic equation.

If $D$ (if $E$ is nonsingular; or $\bar{D}$ if $E$ is singular) is of full rank, then the estimation errors of the full state and all the disturbances decay to zero exponentially. For the cases $l_{ji} < q_0$, the estimation errors of the corresponding partial states and disturbances still remain, and they can be controlled to be as small as necessary by choosing the design parameters. For the cases $l_{ji} \geq q_0$, no steady errors exist in the estimates of the corresponding partial states and disturbances. After the disturbance and the descriptor state are estimated, the controller can be designed by referring to the results in [8, 16, 20].

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# Near-Time-Optimal Path Planning of a Rigid Machine with Multiple Axes

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**Abstract:** A path planning method which is nearly time-optimal is designed for computer numerical control machines which must handle sharp corners. The nominal geometrical trajectory is modified in a way that limitations of the drives' accelerations are taken into account, which will avoid acceleration discontinuities at the cornering point. The method uses two consecutive optimization procedures based on the theory of time-optimal control of single axes while maximizing the travel length of the fastest axis. Simulation results show that the method, which can be generalized to a machine with several axes, is quite effective.

**Keywords:** *Computer numerical control machines; path-planning; contour error.*

**Mathematics Subject Classification (2000):** 49K15, 93C05, 93C95.

## 1 Introduction

The need for increased productivity leads computer numerical control (CNC) machine tools to be faster, i.e. to reduce cycle time, while keeping a good contouring quality (i.e. keeping the tool path within prescribed bounds). Whereas the main goal of trajectory planning is to ensure the following of a nominal geometrical path, smooth modifications of the path can be used as pre-filtering functions which act as a feedforward controller for each individual axis. Afterwards, a feedback control algorithm will be designed which will allow to maintain the positioning accuracy while taking the dynamics into account. However, high speed machines are generally flexible and have to bear vibrations which

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are harmful for the mechanical parts and deteriorate the accuracy, which excludes discontinuities in the drive speed and/or the acceleration. The path planning is thus an important stage of control because it has to take into account speed, acceleration or jerk limitations, which is necessary to obtain good overall performances [2].

Time-optimal path planning of a machine tool with one single rigid axis usually results in a bang-bang scheme [5]. The problem is more complex when considering a machine with multiple axes, because geometrical constraints generate coupling terms between the trajectories of the different axes. When considering limitations in acceleration and jerk, the optimal trajectory of each of the individual axes cannot generally be left to an optimal bang-bang scheme because the geometrical trajectory would be outside the prescribed error bounds. For example, when a discontinuity in curvature occurs, the speed of the axes before and after the cornering point have to be adapted, and, thus, a discontinuity in acceleration will appear, which is not acceptable in practice. Indeed, this would excite vibratory modes – that could be neglected or compensated when the acceleration is smooth [2]. A first way to keep close to the nominal trajectory without bearing discontinuities in accelerations consists of letting the manipulator come at a full stop at the corner, and then accelerate again, or gradually reduce the speed to zero (e.g. introducing jerk limitations in the individual axes) [2], [1], [4]. One can also design a feedback controller which will manage in a way such that the contouring accuracy keeps acceptable, e.g. by cross-coupling controllers [8]. This is partially achieved by the look-ahead function that is built into CNC machines which will ensure that acceleration commands in the interpolated trajectory never exceed their allowed limitations, or by low-pass filtering of acceleration commands [9], [10], [11].

Sharp corners can also be traveled by modifying the toolpath and adjusting the feedrate, which is classical in robotics applications where interpolation is only needed,[3] and related references. It is possible to replace sharp corner with a smooth curve, which can be, for example, a circular arc [6] or an under or over-corner quintic spline [4]. However, very few indications exist how to perform this smoothing in a way that the traveling time keeps close to the optimum, while respecting the geometrical error bandwidth.

This paper proposes a method to obtain a near-time-optimal path planning for machines with several axes, considering speed and acceleration limitations. This optimal trajectory will be given as a geometrical path where the time is not directly given, and will be a function of the allowed contouring error. For the sake of simplicity, the algorithm will be presented for only two axes. The trajectory will be divided into 3 parts, the first one consist of a sequence where the path follows the nominal trajectory which will be a straight line. Then, the modified geometrical path leaves the nominal trajectory before corner crossing and will reach the new direction after the cornering point. The second sequence consists of a point-to-point motion between the leaving and reaching points. The motion will be designed in a way that it is time-optimal for each part of the path taken separately, and, in a second time, that the resulting geometrical path uses the fastest axis at full speed during the maximal time, while staying within the contouring error bandwidth.

## 2 Point to Point Time-Optimal Trajectory Planning for One-Axis Rigid Machines

The rigid machine is supposed to exhibit an ideal dynamics :

$$\dot{X} = ku,$$

where $X$ is the position, $u$ is the driving force. For the sake of simplicity, $k$ will be set to 1.

The limitations in speed and acceleration of the drive will be considered, i.e. there exist $A$, $U$ such that $|\dot{X}| = u \leq U$, $|\ddot{X}| = \dot{u} \leq A$.

The general solution is given for example in [5]. A particular solution is recalled hereafter when the constraints are met (trajectory with full speed and maximum acceleration):

$$\exists\,(t_1, t_2)\colon\ |\dot{X}(t_1)| = U, \quad |\ddot{X}(t_2)| = A.$$

A "point to point" trajectory starts from $X_0(t_0)$, $\dot{X}_0(t_0)$, where $\ddot{X}_0(t_0), \dots, X_0^{(n)}(t_0) = 0$ and reaches $X_f(t_f)$, $\dot{X}_f(t_f)$, where $\ddot{X}_f(t_f), \dots, X_f^{(n)}(t_f) = 0$.

Particular cases include "rest-to-rest" motion ($\dot{X}_0(t_0) = \dot{X}_f(t_f) = 0$), "starting stage" ($\dot{X}_0(t_0) = 0,\ \dot{X}_f(t_f) \neq 0$) and "stop stage" ($\dot{X}_0(t_0) \neq 0,\ \dot{X}_f(t_f) = 0$).

Time minimal control $t = t_f$ leads to maximize the speed along the trajectory which increases from 0 to $U$, which yields a piecewise-polynomial curve,i.e, for a rest-to-rest motion from $X_0(t_0 = 0) = 0$ to $X_f$:

$$\begin{aligned}
t \leq \frac{U}{A}, \quad \dot{X} = At, \quad X = At^2/2,\\
\frac{U}{A} \leq t \leq \frac{X_f}{U}, \quad \dot{X} = U, \quad X = Ut - \frac{U^2}{2A},\\
\frac{X_f}{U} \leq t \leq \frac{X_f}{U} + \frac{U}{A} = t_f, \quad X = X_f - A(t - t_f)^2/2.
\end{aligned}$$

## 3 Optimal Control of a 2-Axes Rigid Machine: Objectives And Constraints

The aim of time-optimal control is to minimize the final time $t_f$ for a motion between $(X_0, Y_0)$ and $(X_f, Y_f)$ (where $\dot{X}_0 = \dot{Y}_0 = \dot{X}_f = \dot{Y}_f = 0$), when spatial and drive constraints are taken into account. This is a far most difficult problem than in Section 1, because, even when the axes are not coupled dynamically, they are made dependent by the geometric constraints imposed by the trajectory. This is particularly crucial when a change in angle occurs, because the speed both axes have to change "simultaneously". Without any constraints on speed and acceleration, it would be only necessary to follow the nominal trajectory and adapt the driving forces at the cornering point. In fact, this is not possible because of drive speed and acceleration limitations, and, in practice, abrupt changes are not desirable because they would excite oscillating modes that are present in mechanical structures. For high-speed machining which are lighter and thus very flexible, these oscillations enforce, in industrial drives, to decrease the speed to zero (or nearly zero) at the crossing point, thus generating an important loss of time.

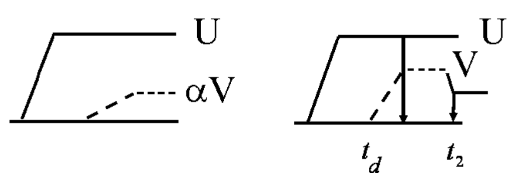


**Figure 3.1:** Corner crossing.

For the sake of simplicity, the case where one axis will move in the first straight line ($Y_0 = 0$), and the final direction has a slope $\alpha$ will be addressed. Both axes are supposed to exhibit rigid dynamics:

$$\dot{X} = u, \quad \dot{Y} = v.$$

Constraints are of three kinds:

– saturation on drive speed and acceleration,

– constraints on final states (zero derivative and acceleration), which are not considered in the present case;

– geometrical constraints;

The machine should follow the following contour:

$$|Y| \leq \varepsilon, \quad X \leq X_c, \quad |Y - \alpha(X - X_c)| \leq \varepsilon, \quad X \geq X_c.$$

As shown in Figure 3.1, is is difficult to stick to the nominal trajectory when drive constraints exist, when the speed is changed and does not decrease to zero. Moreover, the trajectory is supposed to be modified as follows: the motion stays on the nominal trajectory until the point $X_d$ (which is to be determined), and reaches the new direction at the point $X_a$ (also to be determined), while staying in the error bounds.

The following additional hypotheses are taken:

– Straight lines before and after corner crossing are long enough to reach maximum speed and acceleration.

– Limitations in speed and acceleration occur, i.e. $|\dot{X}| = |u| \leq U$, $|\ddot{X}| = |\dot{u}| \leq A$, $|\dot{Y}| = |v| \leq V$, $|\ddot{Y}| = |\dot{v}| \leq B$.

The methodology will be presented with an illustrative case, but can be generalized to multiple axes and additional configurations (e.g. maximum speed is not reached, etc...).

## 4 Near time-Optimal Control

### 4.1 Basic algorithm

The time optimal criterion can be written as follows

$$J = \int_0^{t_f} dt, \tag{1}$$

and can be separated into three parts

$$J = \int_0^{t_d} dt + \int_{t_d}^{t_a} dt + \int_{t_a}^{t_d} dt = J_1 + J_2 + J_3. \tag{2}$$

where $t_d$ is the time where the motion leaves the horizontal axis, $t_a$ is the time where the new direction is reached $t_f$ is the final time where the final position is reached.

The near-optimal trajectory planning consists of 3 steps.

Given the positions, $X_0$, $X_d$ $X_a$, $X_f$ the *first step* consists in minimizing the final time which leads to minimize the time of motion for each of the three parts

(a) minimize $t_d$ for fixed $X_0, X_d$ given initial conditions in $X_0$,

(b) minimize $t_f - t_a$ for fixed $X_f, X_a$ given final conditions in $X_f$,

(c) minimized $t_a - t_d$ for a motion from fixed $X_d$ to $X_a$ starting from initial conditions in $X_d$ given by the solution in (a) and final conditions $X_a$ given by (b).

In the case (a), the time optimal control is a "start" from $X_0(t_0)$ to $X_d$, where $\dot{X}_0 = 0$ which yields $\dot{X} = U$.

In the case (b), geometric constraints imply that $Y = \alpha(X - X_c)$. The maximum speed of the axis $Y$ is $\min(\alpha U, V)$ and the maximum speed of $X$ is $\min\left(\frac{V}{\alpha}, U\right)$. If the maximum speed is reached at the point $(X_a, Y_a)$, the minimum-time control $t_f - t_a$ is of the "stop" type.

The time-optimal control (c) will be a "point-to-point" strategy for both axes, where the speed in $Y$ increases from 0 to $\min(\alpha U, V)$ between $Y_d$ and $Y_a$ and the speed of axis $X$ stays equal to $U$ or decreases from $U$ to $\frac{V}{\alpha}$ between $X_d$ and $Y_a$.

In summary, the problem is simplified by solving three time-optimal control problems for one-axis machines where the solutions are given in Section 2. These solutions are parametrized by the points $X_d, X_a$. This is of course a near-optimal control because it is well known that the sum of optima is not necessarily the optimal solution. However, the solution is quite simple to obtain and can be expected to be close to the true optimal control.

The *second step* consists of minimizing $t_f$, by the optimization of the location of $X_d$ (and thus of $X_a$) which will consists of keeping the longest possible trajectory on the axis which exhibits the higher velocity. Two cases arise based on the comparative values of $\alpha U$ and $V$. The strategy will be different whether the axis X is faster or if the motion is faster along the slope.

In fact, one now tries to keep the maximum speed on the fastest axis, and thus try to adapt the trajectory and point $X_d$. Only the first case $\alpha U \leq V$ will be considered for illustration of the methodology, as the other case can be considered as "dual".

### 4.2 Illustrative example

Let us suppose that $\alpha U \leq V$ and $\alpha A \leq B$. In this case, the velocity of the axis $X$ is kept to $U$, from $t = 0$ to $t = t_f$ (while respecting acceleration constraints). Since the axis $Y$ is faster, it has to adapt and to be bounded.

Applying point (a), the axis $Y$ starts to move at time $t_d$, until $Y = Y_a$, the velocity of axis $Y$ will increase from 0 to $\alpha U$.

The near optimal control consists of minimizing $t_d$ and thus $X_d$ while respecting drive constraints and geometric constraints i.e. $|Y| \leq \varepsilon$, $X \leq X_c$, $|Y - \alpha(X - X_c)| \leq \varepsilon$, $X \geq X_c$.

The configuration (1) does not answer the problem correctly since $t_d$ is imposed by geometric considerations which does not leave any degree of freedom for optimization. Configuration (2) will allow to maximize the speed on the fastest (slope) axis: one supposed, for the sake of simplicity, that the maximum speed $V$ is reached by axis $Y$ and decreases again to reach the nominal speed.

On the $X$ axis, the motion will be:

$$X = Ut - \frac{U^2}{2A}, \quad t \geq \frac{U}{A}. \tag{3}$$

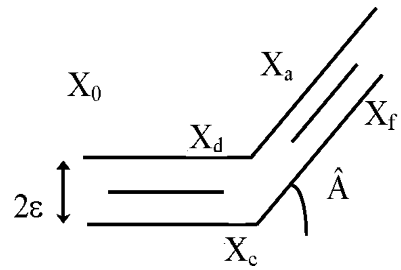


**Figure 4.1:** Speed profile for X and Y axes.

On the axis $Y$, time-optimal motion is:

$$Y = B(t-t_d)^2/2, \quad t - t_d \leq \frac{V}{B},$$
$$Y = V(t-t_d) - \frac{(V)^2}{2B}, \quad \text{if} \;\; t_1 \geq t - t_d \geq \frac{V}{B},$$
$$Y = V(t-t_1) - B(t-t_1)^2/2 + Y_1, \quad \text{where} \;\; t_2 \geq t - t_d \geq t_1,$$
$$Y = U\alpha(t-t_2) + Y_2, \quad \text{where} \;\; t - t_d \geq t_2$$

with $Y_1 = Y(t_1)$, $Y_2 = Y(t_2)$.

Continuity considerations (same derivative and position at breaking points) for axis Y yield:

$$U\alpha = V - B(t_2 - t_1), \quad Y_2 = V(t_2-t_1) - \frac{B(t_2-t_1)^2}{2} + Y_1, \quad Y_1 = V(t_1 - t_d) - \frac{(V)^2}{2B}.$$

Condition $Y = \alpha(X - X_c),\ t \geq t_2,$ yields $U\alpha(t-t_2) + Y_2 = \alpha\left(Ut - \frac{U^2}{2A} - X_c\right)$, i.e.

$$U\alpha t_2 - Y_2 = \alpha\left(\frac{U^2}{2A} + X_c\right).$$

All variables can be expressed as a function of one degree of freedom (i.e. $t_d$ or $t_2$ can be "freely" chosen), one obtains:

$$U\alpha = V - B(t_2 - t_1), \tag{4}$$
$$Y_2 = V(t_2-t_1) - B(t_2-t_1)^2/2 + Y_1, \tag{5}$$
$$Y_1 = V(t_1-t_d) - (V)^2/2B, \tag{6}$$
$$U\alpha t_2 - Y_2 = \alpha\left(\frac{U^2}{2A} + X_c\right), \tag{7}$$

where

$$t_1 = \frac{AU^2\alpha^2 - 2AUV\alpha + B\alpha U^2 + 2AB\alpha X_c - 2ABVt_d}{2AB(U\alpha - V)}.$$

One can eliminate the expression of time within the equations. Since $X - X_d = U(t-t_d)$ one obtains a piecewise polynomial curve:

$$Y = \frac{B}{2}\left(\frac{X - X_d}{U}\right)^2, \quad X \geq X_d, \quad Y \leq \frac{V^2}{2B}, \tag{8}$$

$$Y = V\frac{X - X_d}{U} - \frac{V^2}{2B}, \quad \frac{V^2}{2B} \le Y \le Y_1 \tag{9}$$

as $X - X_1 = U(t - t_1)$,

$$Y = V\frac{X - X_1}{U} - \frac{B}{2}\left(\frac{X - X_1}{U}\right)^2 + Y_1, \quad Y_1 \le Y \le Y_2, \tag{10}$$

$$Y = \alpha(X - X_c), \quad Y_2 \le Y, \tag{11}$$

where $X_d$, $Y_1$, $Y_2$ are given above.

This solution is particularly interesting because it is a geometric path which is supposed to yield the time-optimal controller. One can use it as a geometrical reference trajectory which the axes should follow using feedback control to eliminate the effect of disturbances.

Now, one should determine $t_d$ (or $X_d$) such that $|Y| \le \varepsilon,\ X \le X_c,\ |Y - \alpha(X - X_c)| \le \varepsilon,\ X \ge X_c$. Since corner crossing $X_c$ can be met in the piecewise parts (8,9,10), the resulting constraints will be different, and several cases can be considered.

Lets now write the constraints on the trajectory as a function of $t_d$:

The profile in Figure 4.1 can exist if the nominal speed is reached after the full acceleration stage:

$$t_d + V/B < t_1 \tag{12}$$

and the leaving time should occur after the acceleration step has been completed:

$$\begin{gathered} \frac{U}{A} \le t_d, \\ t_d + \frac{V}{B} \le \frac{AU^2\alpha^2 - 2AUV\alpha + B\alpha U^2 + 2AB\alpha X_c - 2ABVt_d}{2AB(U\alpha - V)}, \end{gathered} \tag{13}$$

i.e.

$$\frac{AU^2\alpha^2 - 4AUV\alpha + B\alpha U^2 + 2AB\alpha X_c + 2V^2A}{2ABU\alpha} \le t_d.$$

Suppose that the cornering point $X_c$ is met during the motion (8). This implies that the leaving point lies before the corner and $t_d \le t_c = \dfrac{X_c}{U} + \dfrac{U}{2A}$, which can be turned, eliminating the time, $\dfrac{B}{2}\left(\dfrac{X_c - X_d}{U}\right)^2 \le \dfrac{V^2}{2B}$ and thus

$$\frac{X_c}{U} - \frac{V}{B} + \frac{U}{2A} \le t_d \le \frac{X_c}{U} + \frac{U}{2A}\,. \tag{14}$$

Once the path has left the nominal trajectory, it should stay nevertheless between prescribed error bounds, e.g. for the section described by equation (8), when the trajectory stays ahead of the corner: $\dfrac{B}{2}\left(\dfrac{X_c - X_d}{U}\right)^2 < \varepsilon$ and thus

$$\frac{X_c}{U} + \frac{U}{2A} - U\sqrt{\frac{2\varepsilon}{B}} < t_d. \tag{15}$$

Since when $X \ge X_c,\ Y \le \frac{V^2}{2B}$ one must have for equation (8), when the path travels the corner and the trajectory should be close to the new direction $-\varepsilon < Y - \alpha(X - X_c) < \varepsilon$, and, replacing:

$$-\varepsilon < \frac{B}{2}\left(\frac{X - X_d}{U}\right)^2 - \alpha(X - X_c) < \varepsilon. \tag{16}$$

The maximum of this function is given for $X = X_d + \frac{\alpha U}{B}^2$ which yields:

$$-\frac{\varepsilon}{\alpha U} + \frac{\alpha U}{2B} \leq \frac{X_c}{U} + \frac{U}{2A} - t_d \leq \frac{\varepsilon}{\alpha U} + \frac{\alpha U}{2B},$$

i.e.

$$\frac{X_c}{U} + \frac{U}{2A} - \frac{\varepsilon}{\alpha U} - \frac{\alpha U}{2B} \leq t_d \leq \frac{X_c}{U} + \frac{U}{2A} + \frac{\varepsilon}{\alpha U} - \frac{\alpha U}{2B}\,. \tag{17}$$

When, in a second time, the trajectory is given by equation (9), it should also stay within prescribed error bounds

$$-\varepsilon \leq V\frac{X - X_d}{U} - \frac{V^2}{2B} - \alpha(X - X_c) \leq \varepsilon,$$

where

$$\frac{V^2}{2B} \leq V\frac{X - X_d}{U} - \frac{V^2}{2B} \leq Y_1.$$

Since the function is increasing, one has only to verify, that $-\varepsilon \leq Y_1 - \alpha(X_1 - X_c) \leq \varepsilon$, where $Y_1 = V(t_1 - t_d) - (V)^2/2B$ which yields:

$$-\varepsilon \leq V(t_1 - t_d) - \frac{V^2}{2B} - \alpha\left(Ut_1 - \frac{U^2}{2A} - X_c\right) \leq \varepsilon$$

and one obtains

$$\begin{aligned} -\varepsilon \leq V\frac{X - X_1}{U} - \frac{B}{2}\left(\frac{X - X_1}{U}\right)^2 + Y_1 - \alpha(X - X_c) \leq \varepsilon, \\ \text{if } \ Y_1 \leq V\frac{X - X_1}{U} - \frac{B}{2}\left(\frac{X - X_1}{U}\right)^2 + Y_1 \leq Y_2. \end{aligned} \tag{18}$$

Last, when the trajectory is described by equation (10), one has to check that

$$-\varepsilon \leq V\frac{X_2 - X_1}{U} - \frac{B}{2}\left(\frac{X_2 - X_1}{U}\right)^2 + Y_1 - \alpha(X_2 - X_c)) \leq \varepsilon$$

which leads to $-\varepsilon \leq U\alpha t_2 - \alpha\left(\frac{U^2}{2A} + X_c\right) - \alpha\left(Ut_2 + \frac{U^2}{2A} - X_c\right) \leq \varepsilon$, i.e.

$$\alpha\left(\frac{U^2}{A}\right) \leq \varepsilon. \tag{19}$$

In summary, one obtains easily a set of inequality constraints (13)–(19) which should in a first time be all compatible in a way such that the profile (2) in Figure 4.1 is really feasible. This gives upper and lower bounds on the value of $t_d$, and, since the motion on the fastest axis (the vertical one) should be preferred, the value of $t_d$ will be the minimum one.

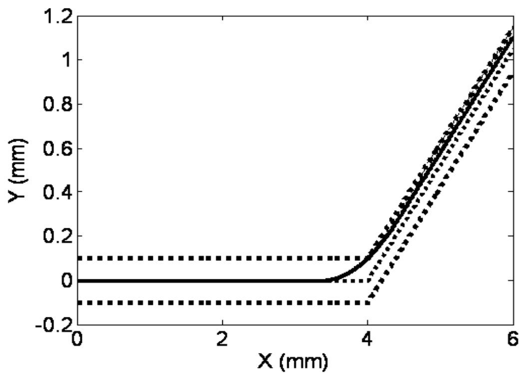


**Figure 4.2:** Near-time-optimal trajectory.

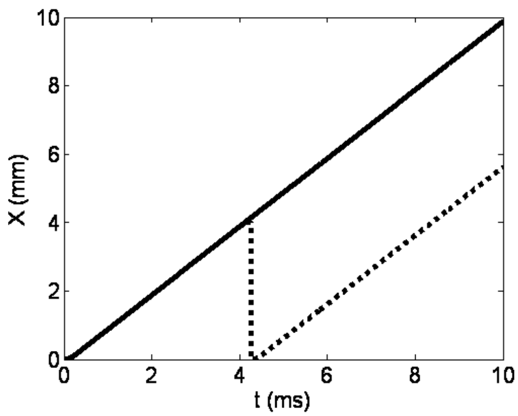


**Figure 4.3:** Near-optimal versus full stop at corner trajectory.

**Example 4.1** Taking numerical values as

$$U = 1, \quad A = 4, \quad B = 0.4, \quad V = 1, \quad X_c = 4, \quad \varepsilon = 0.2, \quad \alpha = 0.8.$$

The maximum constraints are given by (13) and (18) which yields $3.25 \leq t_d \leq 3.37$.

The optimal path is given in Figure 4.2. The "nominal" path (a stop of the axis $X$ at point $X_c$, and a "start" from point $X_c$ of axes $X$ and $Y$, considering speed and acceleration limitations) is also represented in Figure 4.2. One sees that the result is an "under corner" trajectory smoothing [4]. In Figure 4.3, the time history of axis X is represented; for the nominal trajectory following, ones sees that a stop is needed for $X = X_c$. Modified trajectory (solid), nominal trajectory and error bounds (dotted) X position as a function of time, near-optimal trajectory in solid. Classical (with full stop and restart at the corner) dotted. In the case of the modified trajectory, the axis $X$ stays at full speed. The saved time exceeds that which would have been saved by canceling the start and stop procedures, i.e. $\frac{2U}{A}$. The time for which the modified trajectory reaches $X = 8$ equals 8.14 s compared to 12.38 s for the traditional algorithm. One also can verify that the modified trajectory does not reach the upper breaking point $(X = X_c, Y = \varepsilon)$

since, in this case, other limitations (in speed, acceleration, or geometrical) would not be respected. This demonstrates that the optimum path planning does not reduce to taking the chord.

## 5 Conclusion

A near-time-optimal path planning method for traveling sharp corners has been designed for a machine with multiple axes. Its main originality consists of modifying, on purpose, the geometric path in order to smooth the nominal trajectory and to respect the drives' capabilities in term of acceleration and speed. The time-dependent trajectory is bang-bang when traveling straight lines and is a point-to-point optimal trajectory between the two points where the trajectory deviates from the geometrical discontinuity. The second step of the algorithm consists of maximizing the travelling time of the fastest axis, by moving forward or backward the point where the modified trajectory leaves the nominal path, while staying within the prescribed contouring accuracy.

This method proves to be quite effective and can be generalized to a machine with more than two axes. In a next work, this algorithm will be tested on a real-time cartesian machine tool.

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# $\mathcal{H}_\infty$ Filtering for Uncertain Bilinear Stochastic Systems[†]

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**Abstract:** This paper is concerned with the problem of $\mathcal{H}_\infty$ filtering for continuous-time uncertain stochastic systems. The model under consideration contains both state-dependent stochastic noises and deterministic parameter uncertainties residing in a polytope. According to the online availability of the information on the uncertain parameters, we propose two approaches, namely robust stochastic $\mathcal{H}_\infty$ filtering and parameter-dependent stochastic $\mathcal{H}_\infty$ filtering. Both approaches solve the filtering problems based on a modified (improved) bounded real lemma for continuous-time stochastic systems, which decouples the product terms between the Lyapunov matrix and systems matrices and enables us to exploit parameter-dependent stability idea in the filter designs. Sufficient conditions for the existence of admissible robust stochastic $\mathcal{H}_\infty$ filters and parameter-dependent stochastic $\mathcal{H}_\infty$ filters are obtained in terms of linear matrix inequalities, upon which the filter designs are cast into convex optimization problems. Since the filter designs make full use of the parameter-dependent stability idea, the obtained results are less conservative than the existing one in the quadratic framework. A numerical example is provided to illustrate the effectiveness and advantage of the filter design methods proposed in this paper.

**Keywords:** *Linear matrix inequality; $\mathcal{H}_\infty$ filtering; parameter uncertainty; robust filtering; stochastic systems.*

**Mathematics Subject Classification (2000):** 93E11.

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## 1 Introduction

During the past decades, stochastic modeling has come to play an important role in many branches of science such as biology, economics and engineering applications. Therefore, much attention has been drawn to systems with stochastic perturbations from researchers working in related areas. By stochastic systems, we generally refer to systems whose parameter uncertainties are modeled as white noise processes. These parameter uncertainties are usually due to some stochastic environment, and thus it is a natural way to represent them in the model by stochastic parameters fluctuating around some deterministic nominal values. This kind of systems has been called systems with random parametric excitation [2], stochastic bilinear systems [18] and linear stochastic systems with multiplicative noise [15, 31]. Analysis and synthesis of stochastic systems have been investigated extensively and many fundamental results for deterministic systems have been extended to stochastic cases. To mention a few, the analysis of asymptotic behavior can be found in [19, 21, 24]; the optimal control problems were reported in [15, 31]; and recently with the development of $H_\infty$ control theory, the robust control and filtering results have also been extended to stochastic systems through Riccati-like approaches as well as by means of linear matrix inequality (LMI) [3, 4, 9, 16, 29, 33].

On the other hand, for the purpose of analysis and synthesis, estimating the state variables of a dynamic model is important in helping to improve our knowledge about the system concerned [1]. Hence, state estimation has long been an important and interesting problem in the control and signal processing area. Among the existing approaches for estimating the state variables of a linear system described by a state-space equation, arguably, the most popular and useful one is the celebrated Kalman filter [6, 7, 17] which has been applied to a wide range of problems (biology, economics, aerospace, and even population analysis etc. [23, 26]). Usually, it is supposed that a precisely known system model is available and that the dynamic and measurement equations are additively affected by white noise processes satisfying standard assumptions. In many practical situations, however, the availability of the *a priori* information about the external noise is unrealistic. In this case, the filtering problem is more involved and many researchers have made great efforts in proposing useful algorithms in different contexts (see, for instance, [11, 13, 27, 34, 35] and the references therein). Among these available filtering results, the $\mathcal{H}_\infty$ filtering approach provides both a guaranteed noise attenuation level and robustness against unmodeled dynamics. In the presence of both unknown statistics of the external noises and uncertain parameters in the system model, a common approach is to design robust $\mathcal{H}_\infty$ filters. The problem of robust $\mathcal{H}_\infty$ filtering consists on designing a linear stationary asymptotically stable filter that assures a prescribed $\mathcal{H}_\infty$ performance for the filtering error system, irrespective of modeling uncertainties. In general, two popular approaches used to solve the aforementioned filtering problem are Riccati equation approach [30] and linear matrix inequality (LMI) approach [22, 32, 33], and two kinds of parameter uncertainty have been widely used in the literature: norm-bounded uncertainty and polytopic uncertainty. In solving the robust $\mathcal{H}_\infty$ filtering problem, most of the reported results are based on quadratic Lyapunov functions, which have been largely used for robust analysis and synthesis in the past decades. Although being able to ensure stability for systems with arbitrarily fast time-varying parameters, methods based on quadratic stability can produce conservative results since the same parameter-independent Lyapunov function must be used for the entire uncertainty domain. One recognized way to overcome this conservativeness is to consider a parameter-dependent Lyapunov function. An example of a less conservative stability condition based on parameter-dependent Lyapunov functions can be found in [8].

Recently, the problem of robust $\mathcal{H}_\infty$ filtering for uncertain stochastic systems has been investigated in [14] by using LMI technique. It is worth mentioning that the filter designs are based on the quadratic stability notion, which requires a common Lyapunov function for the entire uncertainty domain, and thus much overdesign has been introduced in the derivation process. In this paper, we revisit the problem solved in [14], and present two approaches to solve the $\mathcal{H}_\infty$ filtering problem for continuous-time stochastic systems with parameter uncertainties residing in a polytope. One approach is concerned with the robust stochastic $\mathcal{H}_\infty$ filter design, where stationary constant filters are designed to ensure the filtering error system to be asymptotically stable and has a guaranteed $\mathcal{H}_\infty$ performance for the entire uncertainty domain. The other approach designs parameter-dependent filters whose system matrices are dependent on the available information of the uncertain parameters. Both approaches solve the filtering problems based on a modified (improved) bounded real lemma for continuous-time stochastic systems, which decouples the product terms between the Lyapunov matrix and systems matrices and enables us to exploit parameter-dependent stability idea in the filter designs. Sufficient conditions for the existence of admissible robust stochastic $\mathcal{H}_\infty$ filters and parameter-dependent stochastic $\mathcal{H}_\infty$ filters are obtained in terms of LMIs, upon which the filter designs are cast into convex optimization problems. Since the filter designs make full use of the parameter-dependent stability idea, the obtained results are less conservative than the existing one in the quadratic framework. A numerical example is provided to illustrate the effectiveness and advantage of the filter design methods proposed in this paper.

The remainder of this paper is organized as follows. The problem of $\mathcal{H}_\infty$ filtering for uncertain continuous-time stochastic systems is formulated in Section 2. Sections 3 and 4 present results for parameter-dependent and robust stochastic $\mathcal{H}_\infty$ filtering problems respectively. An illustrative example is provided to show the effectiveness and advantages of the proposed filter designs in Section 5. Finally, some concluding remarks are given in Section 6.

*Notations:* The notations used throughout the paper are fairly standard. The superscript "$T$" stands for matrix transposition; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$ and the notation $P > 0$ means that $P$ is real symmetric and positive definite. $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$; the notation $|\cdot|$ refers to the Euclidean vector norm and $\|\cdot\|_2$ stands for the usual $L_2[0,\infty)$ norm. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and diag$\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In addition, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. the filtration contains all $\mathcal{P}$-null sets and is right continuous) and $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to the probability measure $\mathcal{P}$.

## 2 Problem Description

Consider a mean-square stable system $\mathcal{S}$ with state-dependent noise:

$$\begin{aligned} \mathcal{S}: \quad dx(t) &= [A(\lambda)x(t) + B(\lambda)w(t)]\,dt + E(\lambda)x(t)d\beta(t), \\ dy(t) &= [C(\lambda)x(t) + D(\lambda)w(t)]\,dt + F(\lambda)x(t)d\zeta(t), \\ z(t) &= L(\lambda)x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^m$ is the measured output; $z(t) \in \mathbb{R}^p$ is the signal to be estimated; $w(t) \in \mathbb{R}^q$ is the disturbance input which belongs to $L_2[0,\infty)$. The variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy

$$\begin{aligned}
\mathbb{E}\{d\beta(t)\} &= 0, \quad \mathbb{E}\left\{d\beta(t)^2\right\} = dt, \\
\mathbb{E}\{d\zeta(t)\} &= 0, \quad \mathbb{E}\left\{d\zeta(t)^2\right\} = dt, \\
\mathbb{E}\{d\beta(t)d\zeta(t)\} &= \alpha dt, \quad |\alpha| < 1,
\end{aligned}$$

$A(\lambda)$, $B(\lambda)$, $E(\lambda)$, $C(\lambda)$, $D(\lambda)$, $F(\lambda)$ and $L(\lambda)$ are appropriately dimensioned matrices. It is assumed that

$$\Omega(\lambda) \triangleq (A(\lambda), B(\lambda), E(\lambda), C(\lambda), D(\lambda), F(\lambda), L(\lambda)) \in \mathcal{R},$$

where $\mathcal{R}$ is a given convex bounded polyhedral domain described by $s$ vertices:

$$\mathcal{R} \triangleq \left\{\Omega(\lambda): \ \Omega(\lambda) = \sum_{i=1}^{s} \lambda_i \Omega_i; \ \sum_{i=1}^{s} \lambda_i = 1, \ \lambda_i \geq 0\right\}$$

and $\Omega_i \triangleq (A_i, B_i, E_i, C_i, D_i, F_i, L_i)$ denotes the vertex of the polytope.

Since the signal $z(t)$ cannot be measured directly, our purpose in this paper is to estimate $z(t)$ via the available measurement $y(t)$, such that the estimation error is small in the $\mathcal{H}_\infty$ sense with respect to the energy bounded noise $w(t)$.

According to practical situations, we make two different assumptions on the uncertain parameter $\lambda$.

**Assumption 1** *The uncertain parameter $\lambda$ is unknown, and cannot be measured online.*

**Assumption 2** *The uncertain parameter $\lambda$ does not depend explicitly on the time variable but can be measured online. The uncertain parameter $\lambda$ can vary slowly due to changes in temperature, wind, pressure, humidity, atmosphere, or operating points [20].*

For Assumption 1, since the uncertain parameter $\lambda$ cannot be measured online, a natural way to deal with the filtering problem is to consider a robust filter of the following form (whose filter matrices are not dependent on the parameter $\lambda$):

$$\begin{aligned}
\mathcal{F}_R: \quad dx_F(t) &= A_F x_F(t)dt + B_F dy(t), \quad x_F(0) = 0, \\
z_F(t) &= C_F x_F(t).
\end{aligned} \tag{2}$$

In some situations, however, the uncertain parameter $\lambda$ does not depend explicitly on the time variable but can be measured online. In such cases (Assumption 2), it may be desirable to utilize the available information on parameter $\lambda$ to reduce the conservatism of the robust filter designs. That is, to design a parameter-dependent filter of the following form (whose filter matrices are explicitly dependent on the parameter $\lambda$):

$$\begin{aligned}
\mathcal{F}_P: \quad dx_F(t) &= A_F(\lambda) x_F(t)dt + B_F(\lambda) dy(t), \quad x_F(0) = 0, \\
z_F(t) &= C_F(\lambda) x_F(t).
\end{aligned} \tag{3}$$

Throughout the paper, the estimation error is denoted by $e(t) \triangleq z(t) - z_F(t)$. We define, for a given scalar $\gamma > 0$, the following performance index:

$$\mathcal{J} \triangleq \|e\|_E^2 - \gamma^2 \|w\|_2^2,$$

where

$$\|e\|_E^2 \triangleq \mathbb{E}\left\{\int_0^\infty |e(t)|^2\,dt\right\}.$$

In the following sections, we will present LMI-based approaches to solve the above two stochastic filtering problems. We first present results on the parameter-dependent stochastic $\mathcal{H}_\infty$ filtering problem, and then solve the robust stochastic $\mathcal{H}_\infty$ filtering problem.

## 3 Parameter-Dependent Stochastic $\mathcal{H}_\infty$ Filtering

In the parameter-dependent stochastic $\mathcal{H}_\infty$ filtering problem, by augmenting the model of $\mathcal{S}$ to include the states of the filter $\mathcal{F}_P$, we obtain the filtering error system $\mathcal{E}_P$:

$$\begin{aligned} \mathcal{E}_P: \quad & d\xi(t) = \left[\bar{A}(\lambda)\xi\,(t) + \bar{B}(\lambda)w(t)\right]dt + \bar{E}(\lambda)\xi\,(t)\,d\beta(t) + \bar{F}(\lambda)\xi\,(t)\,d\zeta(t), \\ & e(t) = \bar{C}(\lambda)\xi(t), \end{aligned} \tag{4}$$

where $\xi(t) = \left[x^{\mathrm{T}}(t),\ x_F^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ and

$$\begin{aligned} \bar{A}(\lambda) &= \begin{bmatrix} A(\lambda) & 0 \\ B_F(\lambda)C(\lambda) & A_F(\lambda) \end{bmatrix}, \quad \bar{B}(\lambda) = \begin{bmatrix} B(\lambda) \\ B_F(\lambda)D(\lambda) \end{bmatrix}, \\ \bar{E}(\lambda) &= \begin{bmatrix} E(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{F}(\lambda) = \begin{bmatrix} 0 & 0 \\ B_F(\lambda)F(\lambda) & 0 \end{bmatrix}, \\ \bar{C}(\lambda) &= \left[L(\lambda), \quad -C_F(\lambda)\right]. \end{aligned} \tag{5}$$

Then, the parameter-dependent stochastic $\mathcal{H}_\infty$ filtering problem to be addressed in this section can be expressed as follows.

**Problem PDSHinfF (Parameter-dependent Stochastic $\mathcal{H}_\infty$ Filtering)**: Given system $\mathcal{S}$ in (1), determine the parameter-dependent matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ of the filter $\mathcal{F}_P$ in (3), such that the filtering error system $\mathcal{E}_P$ in (4) is mean-square asymptotically stable and $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$. Filters satisfying the above conditions are called parameter-dependent stochastic $\mathcal{H}_\infty$ filters.

### 3.1 Preliminaries

To solve Problem PDSHinfF, we need the following lemma (see, for instance, Lemma 1 in [14]).

**Lemma 3.1** *Suppose system $\mathcal{S}$ in (1) and filter $\mathcal{F}_P$ in (3) are given, the filtering error system $\mathcal{E}_P$ in (4) is mean-square asymptotically stable with $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$ under zero initial conditions if and only if there exists a matrix function $Q(\lambda) > 0$ satisfying*

$$\begin{aligned} &\bar{A}^{\mathrm{T}}(\lambda)Q(\lambda) + Q(\lambda)\bar{A}(\lambda) + \bar{C}^{\mathrm{T}}(\lambda)\bar{C}(\lambda) + \gamma^{-2}Q(\lambda)\bar{B}(\lambda)\bar{B}^{\mathrm{T}}(\lambda)Q(\lambda) + \bar{E}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{E}(\lambda) \\ &\quad + \bar{F}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{F}(\lambda) + \alpha\bar{E}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{F}(\lambda) + \alpha\bar{F}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{E}(\lambda) < 0 \end{aligned} \tag{6}$$

The above lemma characterizes the $\mathcal{H}_\infty$ performance for continuous-time stochastic systems by using matrix inequality. Denoting $\bar{\alpha} \triangleq \sqrt{1-\alpha^2}$, by Schur complement [5], condition (6) in Lemma 3.1 can be transformed into

$$\begin{bmatrix} -Q(\lambda) & 0 & \bar{\alpha}Q(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & -Q(\lambda) & Q(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right) & 0 & 0 \\ * & * & \bar{A}^{\mathrm{T}}(\lambda)Q(\lambda)+Q(\lambda)\bar{A}(\lambda) & Q(\lambda)\bar{B}(\lambda) & \bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0. \tag{7}$$

(7) is an LMI formulation of the $\mathcal{H}_\infty$ performance presented in Lemma 3.1 for continuous-time stochastic systems. A robust stochastic $\mathcal{H}_\infty$ filtering result has been presented in [14] based on the performance condition (7). Due to the existence of product terms between the Lyapunov matrix $Q(\lambda)$ and system matrices, the robust filtering result in [14] is obtained by imposing $Q(\lambda) \equiv Q$, which leads to a filtering result within the quadratic framework. In the following, we will present an improved version of (7) by decoupling the product terms between the Lyapunov matrix $Q(\lambda)$ and system matrices, which will be used in our filter designs.

**Proposition 3.1** *Suppose system $\mathcal{S}$ in (1) and filter $\mathcal{F}_P$ in (3) are given, the filtering error system $\mathcal{E}_P$ in (4) is mean-square asymptotically stable with $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$ under zero initial conditions if and only if for a sufficiently small scalar $\epsilon > 0$, there exist matrix functions $Q(\lambda) > 0$ and $W(\lambda)$ satisfying*

$$\begin{bmatrix} \Upsilon & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}W^{\mathrm{T}}(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & \Upsilon & 0 & \sqrt{\epsilon}W^{\mathrm{T}}(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right) & 0 & 0 \\ * & * & \Upsilon & W^{\mathrm{T}}(\lambda)\left(I+\epsilon\bar{A}(\lambda)\right) & \sqrt{\epsilon}W^{\mathrm{T}}(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & -Q(\lambda) & 0 & \sqrt{\epsilon}\bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{8}$$

*where*

$$\Upsilon \triangleq Q(\lambda) - W^{\mathrm{T}}(\lambda) - W(\lambda).$$

***Proof*** We first show that (8) is equivalent to

$$\begin{bmatrix} -Q(\lambda) & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}Q(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & -Q(\lambda) & 0 & \sqrt{\epsilon}Q(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right) & 0 & 0 \\ * & * & -Q(\lambda) & Q(\lambda)\left(I+\epsilon\bar{A}(\lambda)\right) & \sqrt{\epsilon}Q(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & -Q(\lambda) & 0 & \sqrt{\epsilon}\bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0. \tag{9}$$

The equivalence between (8) and (9) can be proved as follows. On one hand, if there exists a matrix function $Q(\lambda) > 0$ satisfying (9), (8) is readily established by choosing $W^{\mathrm{T}}(\lambda) = W(\lambda) = Q(\lambda)$. On the other hand, if there exist matrix functions $Q(\lambda) > 0$ and $W(\lambda)$ satisfying (8), we can easily see that $W(\lambda)$ is nonsingular. In addition, we have $(Q(\lambda) - W(\lambda))^{\mathrm{T}} Q^{-1}(\lambda)(Q(\lambda) - W(\lambda)) \geq 0$, which implies that $\Gamma \triangleq -W^{\mathrm{T}}(\lambda)Q^{-1}(\lambda)W(\lambda) \leq Q(\lambda) - W^{\mathrm{T}}(\lambda) - W(\lambda)$. Therefore we can conclude from

(8) that

$$\begin{bmatrix} \Gamma & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}W^{\mathrm{T}}(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & \Gamma & 0 & \sqrt{\epsilon}W^{\mathrm{T}}(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right) & 0 & 0 \\ * & * & \Gamma & W^{\mathrm{T}}(\lambda)\left(I+\epsilon\bar{A}(\lambda)\right) & \sqrt{\epsilon}W^{\mathrm{T}}(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & -Q(\lambda) & 0 & \sqrt{\epsilon}\bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0. \tag{10}$$

Performing a congruence transformation to (10) by $\operatorname{diag}\{W^{-1}(\lambda)Q(\lambda), W^{-1}(\lambda)Q(\lambda), W^{-1}(\lambda)Q(\lambda), I, I, I\}$ yields (9).

Now, performing a congruence transformation to (9) by $\operatorname{diag}\left\{I, I, I, \epsilon^{-1/2}I, I, I\right\}$, we obtain

$$\begin{bmatrix} -Q(\lambda) & 0 & 0 & \bar{\alpha}Q(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & -Q(\lambda) & 0 & Q(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right) & 0 & 0 \\ * & * & -Q(\lambda) & Q(\lambda)\left(\epsilon^{-1/2}I+\sqrt{\epsilon}\bar{A}(\lambda)\right) & \sqrt{\epsilon}Q(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & -\epsilon^{-1}Q(\lambda) & 0 & \bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{11}$$

by Schur complement, (11) is equivalent to

$$\begin{bmatrix} -Q(\lambda) & 0 & \bar{\alpha}Q(\lambda)\bar{E}(\lambda) & 0 & 0 \\ * & -Q(\lambda) & Q(\lambda)(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)) & 0 & 0 \\ * & * & \begin{matrix} Q(\lambda)\bar{A}(\lambda)+\bar{A}^{\mathrm{T}}(\lambda)Q(\lambda)+ \\ \epsilon\bar{A}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{A}(\lambda) \end{matrix} & \begin{matrix} Q(\lambda)\bar{B}(\lambda)+ \\ \epsilon\bar{A}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{B}(\lambda) \end{matrix} & \bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & -\gamma^2 I+\epsilon\bar{B}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{12}$$

which is further equivalent to

$$\begin{bmatrix} \tilde{\Upsilon} & Q(\lambda)\bar{B}(\lambda) \\ * & -\gamma^2 I \end{bmatrix} + \epsilon \begin{bmatrix} \bar{A}^{\mathrm{T}}(\lambda) \\ \bar{B}^{\mathrm{T}}(\lambda) \end{bmatrix} Q(\lambda) \begin{bmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \end{bmatrix} < 0, \tag{13}$$

where

$$\begin{aligned} \tilde{\Upsilon} \triangleq\ & Q(\lambda)\bar{A}(\lambda)+\bar{A}^{\mathrm{T}}(\lambda)Q(\lambda)+\bar{C}^{\mathrm{T}}(\lambda)\bar{C}(\lambda)+\bar{\alpha}^2\bar{E}^{\mathrm{T}}(\lambda)Q(\lambda)\bar{E}(\lambda) \\ & +\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right)^{\mathrm{T}} Q(\lambda)\left(\alpha\bar{E}(\lambda)+\bar{F}(\lambda)\right). \end{aligned}$$

Since $Q(\lambda) > 0$ and $\epsilon$ is sufficiently small positive, (13) is in fact equivalent to (6), and the proof is completed. □

The advantage of Proposition 3.1 lies in the fact that by introducing the slack (in the sense that no structural restriction is imposed) matrix function $W(\lambda)$ and a sufficient small positive constant $\epsilon$, (8) does not contain product terms between the Lyapunov matrix $Q(\lambda)$ and system matrices. This decoupling property has been proved to be an advantage for polytopic uncertain systems concerning reducing conservativeness [25]. In the following (sub)sections, we will develop parameter-dependent and robust stochastic $\mathcal{H}_\infty$ filters based on Proposition 3.1.

It is noted that if the filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ are given, (8) is a linear matrix inequality over the matrix variables $Q(\lambda)$ and $W(\lambda)$ for fixed $\lambda$. However, since our purpose is to determine the filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$, condition (8) is actually a nonlinear matrix inequality. In addition, to test the feasibility of these conditions is an infinite-dimensional problem in terms of the uncertain parameter $\lambda$. Our main objective hereafter is to transform (8) into finite-dimensional LMI condition.

### 3.2 Main Results

Our result depends on the following proposition.

**Proposition 3.2** *Given system $\mathcal{S}$ in (1). For a sufficiently small scalar $\epsilon > 0$, there exist matrix functions $Q(\lambda) > 0$ and $W(\lambda)$ satisfying (8) if and only if there exist matrices $\bar{Q}(\lambda) \triangleq \begin{bmatrix} \bar{Q}_1(\lambda) & \bar{Q}_2(\lambda) \\ * & \bar{Q}_3(\lambda) \end{bmatrix} > 0$, $R(\lambda)$, $S(\lambda)$, $T(\lambda)$, $\bar{A}_F(\lambda)$, $\bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$ satisfying*

$$\Psi(\lambda) \triangleq \begin{bmatrix} \Pi_1 & \Pi_2 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}R^{\mathrm{T}}(\lambda)E(\lambda) & 0 & 0 & 0 \\ * & \Pi_3 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}S^{\mathrm{T}}(\lambda)E(\lambda) & 0 & 0 & 0 \\ * & * & \Pi_1 & \Pi_2 & 0 & 0 & \Pi_4 & 0 & 0 & 0 \\ * & * & * & \Pi_3 & 0 & 0 & \Pi_5 & 0 & 0 & 0 \\ * & * & * & * & \Pi_1 & \Pi_2 & \Pi_6 & T(\lambda)+\epsilon\bar{A}_F(\lambda) & \Pi_8 & 0 \\ * & * & * & * & * & \Pi_3 & \Pi_7 & T(\lambda)+\epsilon\bar{A}_F(\lambda) & \Pi_9 & 0 \\ * & * & * & * & * & * & -\bar{Q}_1(\lambda) & -\bar{Q}_2(\lambda) & 0 & \sqrt{\epsilon}L^{\mathrm{T}}(\lambda) \\ * & * & * & * & * & * & * & -\bar{Q}_3(\lambda) & 0 & -\sqrt{\epsilon}\bar{C}_F^{\mathrm{T}}(\lambda) \\ * & * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{14}$$

*where*

$$\Pi_1 = \bar{Q}_1(\lambda) - R^{\mathrm{T}}(\lambda) - R(\lambda), \quad \Pi_2 = \bar{Q}_2(\lambda) - T(\lambda) - S(\lambda),$$
$$\Pi_3 = \bar{Q}_3(\lambda) - T(\lambda) - T^{\mathrm{T}}(\lambda), \quad \Pi_4 = \sqrt{\epsilon}\alpha R^{\mathrm{T}}(\lambda)E(\lambda) + \bar{B}_F(\lambda)F(\lambda),$$
$$\Pi_5 = \sqrt{\epsilon}\alpha S^{\mathrm{T}}(\lambda)E(\lambda) + \bar{B}_F(\lambda)F(\lambda), \quad \Pi_6 = R^{\mathrm{T}}(\lambda) + \epsilon R^{\mathrm{T}}(\lambda)A(\lambda) + \epsilon\bar{B}_F(\lambda)C(\lambda),$$
$$\Pi_7 = S^{\mathrm{T}}(\lambda) + \epsilon S^{\mathrm{T}}(\lambda)A(\lambda) + \epsilon\bar{B}_F(\lambda)C(\lambda), \quad \Pi_8 = \sqrt{\epsilon}R^{\mathrm{T}}(\lambda)B(\lambda) + \sqrt{\epsilon}\bar{B}_F(\lambda)D(\lambda),$$
$$\Pi_9 = \sqrt{\epsilon}S^{\mathrm{T}}(\lambda)B(\lambda) + \sqrt{\epsilon}\bar{B}_F(\lambda)D(\lambda).$$

*Moreover, under the above condition, the matrix functions for an admissible parameter-dependent stochastic $\mathcal{H}_\infty$ filter $\mathcal{F}_P$ in the form of (3) are given by*

$$\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} T^{-1}(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix}. \tag{15}$$

***Proof*** *Necessity.* Given a sufficiently small scalar $\epsilon > 0$, suppose there exist filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ and matrices $Q(\lambda) > 0$ and $W(\lambda)$ satisfying (8). Let the matrix functions $Q(\lambda)$ and $W(\lambda)$ be partitioned as

$$Q(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ Q_2^{\mathrm{T}}(\lambda) & Q_3(\lambda) \end{bmatrix}, \quad W(\lambda) = \begin{bmatrix} W_1(\lambda) & W_2(\lambda) \\ W_4(\lambda) & W_3(\lambda) \end{bmatrix}. \tag{16}$$

By invoking a small perturbation if necessary, we can assume that $W_4(\lambda)$ and $W_3(\lambda)$ are nonsingular. Define the following invertible matrix functions

$$J(\lambda) = \begin{bmatrix} I & 0 \\ 0 & W_3^{-1}(\lambda)W_4(\lambda) \end{bmatrix}, \quad K(\lambda) = \mathrm{diag}\{J(\lambda),\ J(\lambda),\ J(\lambda),\ J(\lambda),\ I,\ I\} \tag{17}$$

and define

$$\bar{Q}(\lambda) = \begin{bmatrix} \bar{Q}_1(\lambda) & \bar{Q}_2(\lambda) \\ * & \bar{Q}_3(\lambda) \end{bmatrix} = J^{\mathrm{T}}(\lambda)Q(\lambda)J(\lambda). \tag{18}$$

Then, performing a congruence transformation to (8) by $K(\lambda)$ together with the consideration of (5) yields

$$\begin{bmatrix} \bar{Q}(\lambda)-\Psi_1-\Psi_1^{\mathrm{T}} & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}\Psi_5 & 0 & 0 \\ * & \bar{Q}(\lambda)-\Psi_1-\Psi_1^{\mathrm{T}} & 0 & \sqrt{\epsilon}(\alpha\Psi_5+\Psi_6) & 0 & 0 \\ * & * & \bar{Q}(\lambda)-\Psi_1-\Psi_1^{\mathrm{T}} & \Psi_1^{\mathrm{T}}+\epsilon\Psi_3 & \sqrt{\epsilon}\Psi_2 & 0 \\ * & * & * & -\bar{Q}(\lambda) & 0 & \sqrt{\epsilon}\Psi_4 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{19}$$

where

$$\begin{aligned}
\Psi_1 &= \begin{bmatrix} W_1(\lambda) & W_2(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \\ W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)W_4(\lambda) & W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)W_4(\lambda) \end{bmatrix}, \\
\Psi_2 &= \begin{bmatrix} W_1^{\mathrm{T}}(\lambda)B(\lambda)+W_4^{\mathrm{T}}(\lambda)B_F(\lambda)D(\lambda) \\ W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)W_2^{\mathrm{T}}(\lambda)B(\lambda)+W_4^{\mathrm{T}}(\lambda)B_F(\lambda)D(\lambda) \end{bmatrix}, \\
\Psi_3 &= \begin{bmatrix} W_1^{\mathrm{T}}(\lambda)A(\lambda)+W_4^{\mathrm{T}}(\lambda)B_F(\lambda)C(\lambda) & W_4^{\mathrm{T}}(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \\ \begin{gathered} W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)W_2^{\mathrm{T}}(\lambda)A(\lambda)+ \\ W_4^{\mathrm{T}}(\lambda)B_F(\lambda)C(\lambda) \end{gathered} & W_4^{\mathrm{T}}(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \end{bmatrix}, \\
\Psi_4 &= \begin{bmatrix} L^{\mathrm{T}}(\lambda) \\ -W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)C_F^{\mathrm{T}}(\lambda) \end{bmatrix}, \\
\Psi_5 &= \begin{bmatrix} W_1^{\mathrm{T}}(\lambda)E(\lambda) & 0 \\ W_4^{\mathrm{T}}(\lambda)W_3^{-T}(\lambda)W_2^{\mathrm{T}}(\lambda)E(\lambda) & 0 \end{bmatrix}, \\
\Psi_6 &= \begin{bmatrix} W_4^{\mathrm{T}}(\lambda)B_F(\lambda)F(\lambda) & 0 \\ W_4^{\mathrm{T}}(\lambda)B_F(\lambda)F(\lambda) & 0 \end{bmatrix}.
\end{aligned}$$

By defining

$$R(\lambda) = W_1(\lambda), \tag{20}$$

$$S(\lambda) = W_2(\lambda)W_3^{-1}(\lambda)W_4(\lambda), \tag{21}$$

$$T(\lambda) = W_4^{\mathrm{T}}(\lambda)W_3^{-1}(\lambda)W_4(\lambda), \tag{22}$$

$$\begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} W_4^{\mathrm{T}}(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} \begin{bmatrix} W_3^{-1}(\lambda)W_4(\lambda) & 0 \\ 0 & I \end{bmatrix}, \tag{23}$$

(19) is equivalent to (14), and the necessity is proved.

*Sufficiency.* Suppose for a sufficiently small scalar $\epsilon > 0$, there exist matrix functions $\bar{Q}(\lambda) > 0$, $R(\lambda)$, $S(\lambda)$, $T(\lambda)$, $\bar{A}_F(\lambda)$, $\bar{B}_F(\lambda)$ , and $\bar{C}_F(\lambda)$ satisfying (14), we will prove that there must exist filter matrices ($A_F(\lambda)$, $B_F(\lambda)$, $C_F(\lambda)$) and matrices $Q(\lambda) > 0$ and $W(\lambda)$ satisfying ( 8).

First (14) implies $T(\lambda) + T^{\mathrm{T}}(\lambda) - \bar{Q}_3(\lambda) > 0$, then we know that $T(\lambda)$ is nonsingular due to $\bar{Q}_3(\lambda) > 0$. Thus one can always find square and nonsingular matrix functions $W_3(\lambda)$ and $W_4(\lambda)$ satisfying (22). Now introduce the matrix functions $J(\lambda)$, $K(\lambda)$ as defined in (17) and

$$W(\lambda) = \begin{bmatrix} R(\lambda) & S(\lambda)W_4^{-1}(\lambda)W_3(\lambda) \\ W_3(\lambda) & W_4(\lambda) \end{bmatrix},$$

$$Q(\lambda) = J^{-T}(\lambda)\bar{Q}(\lambda)J^{-1}(\lambda),$$

$$\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} W_4^{-T}(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix} \begin{bmatrix} W_4^{-1}(\lambda)W_3(\lambda) & 0 \\ 0 & I \end{bmatrix}. \tag{24}$$

Then, we have $Q(\lambda) > 0$. Now, by some algebraic matrix manipulations, it can be established that (14) is equivalent to

$$\begin{bmatrix} \widetilde{\Phi} & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}J^{\mathrm{T}}(\lambda)W^{\mathrm{T}}(\lambda)\bar{E}(\lambda)J(\lambda) & 0 & 0 \\ * & \widetilde{\Phi} & 0 & \begin{array}{c}\sqrt{\epsilon}J^{\mathrm{T}}(\lambda)W^{\mathrm{T}}(\lambda)\times \\ (\alpha\bar{E}(\lambda)+\bar{F}(\lambda))J(\lambda)\end{array} & 0 & 0 \\ * & * & \widetilde{\Phi} & \begin{array}{c}J^{\mathrm{T}}(\lambda)W^{\mathrm{T}}(\lambda)\times \\ (I+\epsilon\bar{A}(\lambda))J(\lambda)\end{array} & \sqrt{\epsilon}J^{\mathrm{T}}(\lambda)W^{\mathrm{T}}(\lambda)\bar{B}(\lambda) & 0 \\ * & * & * & -J^{\mathrm{T}}(\lambda)Q(\lambda)J(\lambda) & 0 & \sqrt{\epsilon}J^{\mathrm{T}}(\lambda)\bar{C}^{\mathrm{T}}(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{25}$$

where $\widetilde{\Phi} = J^{\mathrm{T}}(\lambda)\Upsilon J(\lambda)$. Now, performing a congruence transformation to (25) by $K^{-1}(\lambda)$ yields (8), and the sufficiency proof is completed.

*Proof of Second Part.* If the condition in Proposition 3.2 has a set of feasible solutions $\{\bar{Q}(\lambda),\ R(\lambda),\ S(\lambda),\ T(\lambda),\ \bar{A}_F(\lambda),\ \bar{B}_F(\lambda),\ \bar{C}_F(\lambda)\}$, from the above proof we know that the filter with a state-space realization $\left(\bar{A}(\lambda), \bar{B}(\lambda), \bar{C}(\lambda)\right)$ defined in (24) guarantees the filtering error system $\mathcal{E}_P$ in (4) to be mean-square asymptotically stable with $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$. Now denote the operator from $y(t)$ to $z_F(t)$ by $\mathcal{T}_{z_F y}(\lambda) = (A_F(\lambda),\ B_F(\lambda), C_F(\lambda))$, then we have $\mathcal{T}_{z_F y}(\lambda)$ is equivalent to $\mathcal{G}_{z_F y}(\lambda)$ under a similarity transformation, where

$$\mathcal{G}_{z_F y}(\lambda) = \left(W_4^{-1}(\lambda)W_3(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda),\ W_4^{-1}(\lambda)W_3(\lambda)B_F(\lambda), C_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda)\right).$$

By substituting the matrices with (24) and by considering the relationship (22), we have

$$\mathcal{G}_{z_F y}(\lambda) = \left(T^{-1}(\lambda)\bar{A}_F(\lambda),\ T^{-1}(\lambda)\bar{B}_F(\lambda), \bar{C}_F(\lambda)\right).$$

Therefore, an admissible filter can be given by (15), and the proof is completed. □

Proposition 3.2 is a preliminary result for solving the parameter-dependent $\mathcal{H}_\infty$ filtering problem. It casts the nonlinear matrix inequality in Lemma 3.1 into an LMI condition by using linearization procedures, upon which desired filters can be constructed by using the obtained matrix functions $\bar{Q}(\lambda)$, $R(\lambda)$, $S(\lambda)$, $T(\lambda)$, $\bar{A}_F(\lambda)$, $\bar{B}_F(\lambda)$ , and $\bar{C}_F(\lambda)$. However, this LMI condition still cannot be implemented due to it infinite-dimensional nature in the parameter $\lambda$. Our purpose hereafter is to transform the infinite-dimensional condition in Proposition 3.2 into finite-dimensional condition that depends only on the vertex matrices of the polytope $\mathcal{R}$. Then, we have the main filtering result in the following theorem.

**Theorem 3.1 (Parameter-Dependent Stochastic $H_\infty$ Filtering)** *Given system $\mathcal{S}$ in (1), an admissible parameter-dependent stochastic $\mathcal{H}_\infty$ filter in the form of $\mathcal{F}_P$*

*in (3) exists if for a sufficiently small scalar* $\epsilon > 0$*, there exist matrices* $R_i$*,* $S_i$*,* $T_i$*,* $\bar{A}_{Fi}$*,* $\bar{B}_{Fi}$*,* $\bar{C}_{Fi}$ *and* $\bar{Q}_i = \begin{bmatrix} \bar{Q}_{1i} & \bar{Q}_{2i} \\ * & \bar{Q}_{3i} \end{bmatrix} > 0$*, satisfying*

$$\Psi_{ii} < 0, \quad i = 1, \dots, s, \tag{26}$$

$$\Psi_{ij} + \Psi_{ji} \le 0, \quad 1 \le i < j \le s, \tag{27}$$

*where*

$$\Psi_{ij} = \begin{bmatrix} \Phi_1 & \Phi_2 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}R_i^{\mathrm{T}}E_j & 0 & 0 & 0 \\ * & \Phi_3 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}S_i^{\mathrm{T}}E_j & 0 & 0 & 0 \\ * & * & \Phi_1 & \Phi_2 & 0 & 0 & \begin{matrix}\sqrt{\epsilon}\alpha R_i^{\mathrm{T}}E_j + \\ \bar{B}_{Fi}F_j\end{matrix} & 0 & 0 & 0 \\ * & * & * & \Phi_3 & 0 & 0 & \begin{matrix}\sqrt{\epsilon}\alpha S_i^{\mathrm{T}}E_j + \\ \bar{B}_{Fi}F_j\end{matrix} & 0 & 0 & 0 \\ * & * & * & * & \Phi_1 & \Phi_2 & \Phi_4 & T_i + \epsilon\bar{A}_{Fi} & \begin{matrix}\sqrt{\epsilon}R_i^{\mathrm{T}}B_j + \\ \sqrt{\epsilon}\bar{B}_{Fi}D_j\end{matrix} & 0 \\ * & * & * & * & * & \Phi_3 & \Phi_5 & T_i + \epsilon\bar{A}_{Fi} & \begin{matrix}\sqrt{\epsilon}S_i^{\mathrm{T}}B_j + \\ \sqrt{\epsilon}\bar{B}_{Fi}D_j\end{matrix} & 0 \\ * & * & * & * & * & * & -\bar{Q}_{1i} & -\bar{Q}_{2i} & 0 & \sqrt{\epsilon}L_j^{\mathrm{T}} \\ * & * & * & * & * & * & * & -\bar{Q}_{3i} & 0 & -\sqrt{\epsilon}\bar{C}_{Fi}^{\mathrm{T}} \\ * & * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\begin{aligned} &\Phi_1 = \bar{Q}_{1i} - R_i^{\mathrm{T}} - R_i, \quad \Phi_2 = \bar{Q}_{2i} - T_i - S_i, \quad \Phi_3 = \bar{Q}_{3i} - T_i - T_i^{\mathrm{T}}, \\ &\Phi_4 = R_i^{\mathrm{T}} + \epsilon R_i^{\mathrm{T}}A_j + \epsilon\bar{B}_{Fi}C_j, \quad \Phi_5 = S_i^{\mathrm{T}} + \epsilon S_i^{\mathrm{T}}A_j + \epsilon\bar{B}_{Fi}C_j. \end{aligned} \tag{28}$$

*Moreover, under the above conditions, the matrix functions for an admissible parameter-dependent stochastic* $\mathcal{H}_\infty$ *filter* $\mathcal{F}_P$ *in the form of (3) are given by*

$$\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} \left(\sum\limits_{i=1}^{s}\lambda_i T_i\right)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \sum\limits_{i=1}^{s}\lambda_i\bar{A}_{Fi} & \sum\limits_{i=1}^{s}\lambda_i\bar{B}_{Fi} \\ \sum\limits_{i=1}^{s}\lambda_i\bar{C}_{Fi} & 0 \end{bmatrix}. \tag{29}$$

***Proof*** From Propositions 3.1 and 3.2, an admissible parameter-dependent stochastic $\mathcal{H}_\infty$ filter $\mathcal{F}_P$ in the form of (3) exists if there exist matrix functions $\bar{Q}(\lambda) > 0$, $R(\lambda)$, $S(\lambda)$, $T(\lambda)$, $\bar{A}_F(\lambda)$, $\bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$ satisfying (14). Now assume the above matrix functions to be of the following form

$$\begin{aligned} \bar{Q}(\lambda) &= \sum_{i=1}^{s}\lambda_i\bar{Q}_i = \sum_{i=1}^{s}\lambda_i\begin{bmatrix} \bar{Q}_{1i} & \bar{Q}_{2i} \\ * & \bar{Q}_{3i} \end{bmatrix}, \\ R(\lambda) &= \sum_{i=1}^{s}\lambda_i R_i, \quad S(\lambda) = \sum_{i=1}^{s}\lambda_i S_i, \quad T(\lambda) = \sum_{i=1}^{s}\lambda_i T_i, \\ \bar{A}_F(\lambda) &= \sum_{i=1}^{s}\lambda_i\bar{A}_{Fi}, \quad \bar{B}_F(\lambda) = \sum_{i=1}^{s}\lambda_i\bar{B}_{Fi}, \quad \bar{C}_F(\lambda) = \sum_{i=1}^{s}\lambda_i\bar{C}_{Fi}. \end{aligned} \tag{30}$$

With (30) it is not difficult to rewrite $\Psi(\lambda)$ in (14) as

$$\Psi(\lambda) = \sum_{j=1}^{s}\sum_{i=1}^{s}\lambda_i\lambda_j\Psi_{ij} = \sum_{i=1}^{s}\lambda_i^2\Psi_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j\left(\Psi_{ij}+\Psi_{ji}\right), \tag{31}$$

where $\Psi_{ij}$ is defined in (28). Then, (26) and (27) guarantee $\Psi(\lambda) < 0$, and the first part of the proof is completed.

By substituting the matrices defined in (30) into (15), we readily obtain (29) and the proof is completed. □

**Remark 3.1** The idea behind Theorem 3.1 is to use convex combinations of vertex matrices in the form of (30) to substitute the matrix functions in Proposition 3.2. With the introduction of these matrices, the infinite-dimensional nonlinear matrix inequality condition in Proposition 3.2 is cast into finite-dimensional LMI condition, which depends only on the vertex matrices of the polytope $\mathcal{R}$, and therefore can be readily checked by using standard numerical software [10].

**Remark 3.2** Note that the condition in Theorem 3.1 is an LMI not only over the matrix variables, but also over the scalar $\gamma$. This implies that the scalar $\gamma$ can be included as an optimization variable to obtain a reduction of the attenuation level bound. Then the minimum (in terms of the feasibility of Theorem 3.1) attenuation level of $\mathcal{H}_\infty$ filters can be readily found by solving the following convex optimization problem:

Minimize $\gamma$ subject to (26) and (27) for sufficiently small $\epsilon > 0$.

## 4 Robust Stochastic $\mathcal{H}_\infty$ Filtering

In the robust stochastic $\mathcal{H}_\infty$ filtering problem, by augmenting the model of $\mathcal{S}$ to include the states of the filter $\mathcal{F}_R$, we obtain the filtering error system $\mathcal{E}_R$:

$$\begin{aligned}\mathcal{E}_R: \quad & d\xi(t) = \left[\bar{A}(\lambda)\xi(t) + \bar{B}(\lambda)w(t)\right]dt + \bar{E}(\lambda)\xi(t)\,d\beta(t) + \bar{F}(\lambda)\xi(t)\,d\zeta(t), \\ & e(t) = \bar{C}(\lambda)\xi(t),\end{aligned} \tag{32}$$

where

$$\begin{aligned}\bar{A}(\lambda) &= \begin{bmatrix} A(\lambda) & 0 \\ B_F C(\lambda) & A_F \end{bmatrix}, \quad \bar{B}(\lambda) = \begin{bmatrix} B(\lambda) \\ B_F D(\lambda) \end{bmatrix}, \quad \bar{E}(\lambda) = \begin{bmatrix} E(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{F}(\lambda) &= \begin{bmatrix} 0 & 0 \\ B_F F(\lambda) & 0 \end{bmatrix}, \quad \bar{C}(\lambda) = \begin{bmatrix} L(\lambda) & -C_F \end{bmatrix}.\end{aligned} \tag{33}$$

Then, the robust stochastic $\mathcal{H}_\infty$ filtering problem to be addressed in this section can be expressed as follows:

**Problem RSHinfF (Robust Stochastic $\mathcal{H}_\infty$ Filtering)**: Given system $\mathcal{S}$ in (1), determine the matrices $(A_F, B_F, C_F)$ of the filter $\mathcal{F}_R$ in (2), such that the filtering error system $\mathcal{E}_R$ in (32) is mean-square asymptotically stable and $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$ under zero initial conditions . Filters satisfying the above conditions are called robust stochastic $\mathcal{H}_\infty$ filters.

In the following, we will solve the robust stochastic $\mathcal{H}_\infty$ filtering problem. First according to Proposition 3.1, when system $\mathcal{S}$ in (1) and filter $\mathcal{F}_R$ in (2) are given, the filtering error system $\mathcal{E}_R$ in (32) is mean-square asymptotically stable with $\mathcal{J} < 0$ for all nonzero $w(t) \in L_2[0,\infty)$ under zero initial conditions if and only if for a sufficiently small scalar $\epsilon > 0$, there exist matrix functions $Q(\lambda) > 0$ and $W(\lambda)$ satisfying (8). It is worth noting that if we solve the robust filter design problem by following the idea in previous references [12, 28], we need to set the general-structured matrix $W(\lambda) \equiv W$ for the entire uncertainty domain. To reduce the conservativeness while keeping the filter synthesis problem tractable simultaneously, here we assume $W(\lambda)$ takes the following structure:

$$W(\lambda) = \begin{bmatrix} W_1(\lambda) & W_2(\lambda) \\ W_4 & W_3 \end{bmatrix}.$$

Then, by following similar lines as in the proof of Proposition 3.2, we have the following proposition.

**Proposition 4.1** *Given system $\mathcal{S}$ in (1), an admissible robust stochastic $\mathcal{H}_\infty$ filter in the form of $\mathcal{F}_R$ in (2) exists if for a sufficiently small scalar $\epsilon > 0$, there exist matrices $\bar{Q}(\lambda) \triangleq \begin{bmatrix} \bar{Q}_1(\lambda) & \bar{Q}_2(\lambda) \\ * & \bar{Q}_3(\lambda) \end{bmatrix} > 0$, $R(\lambda)$, $S(\lambda)$, $T$, $\bar{A}_F$, $\bar{B}_F$, and $\bar{C}_F$ satisfying*

$$\Delta(\lambda) \triangleq \begin{bmatrix} \bar{\Pi}_1 & \bar{\Pi}_2 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}R^{\mathrm{T}}(\lambda)E(\lambda) & 0 & 0 & 0 \\ * & \bar{\Pi}_3 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}S^{\mathrm{T}}(\lambda)E(\lambda) & 0 & 0 & 0 \\ * & * & \bar{\Pi}_1 & \bar{\Pi}_2 & 0 & 0 & \bar{\Pi}_4 & 0 & 0 & 0 \\ * & * & * & \bar{\Pi}_3 & 0 & 0 & \bar{\Pi}_5 & 0 & 0 & 0 \\ * & * & * & * & \bar{\Pi}_1 & \bar{\Pi}_2 & \bar{\Pi}_6 & T(\lambda)+\epsilon\bar{A}_F & \bar{\Pi}_8 & 0 \\ * & * & * & * & * & \bar{\Pi}_3 & \bar{\Pi}_7 & T(\lambda)+\epsilon\bar{A}_F & \bar{\Pi}_9 & 0 \\ * & * & * & * & * & * & -\bar{Q}_1(\lambda) & -\bar{Q}_2(\lambda) & 0 & \sqrt{\epsilon}L^{\mathrm{T}}(\lambda) \\ * & * & * & * & * & * & * & -\bar{Q}_3(\lambda) & 0 & -\sqrt{\epsilon}\bar{C}_F^{\mathrm{T}} \\ * & * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{34}$$

*where*

$$\bar{\Pi}_1 = \bar{Q}_1(\lambda) - R^{\mathrm{T}}(\lambda) - R(\lambda), \quad \bar{\Pi}_2 = \bar{Q}_2(\lambda) - T - S(\lambda), \quad \bar{\Pi}_3 = \bar{Q}_3(\lambda) - T - T^{\mathrm{T}},$$
$$\bar{\Pi}_4 = \sqrt{\epsilon}\alpha R^{\mathrm{T}}(\lambda)E(\lambda) + \bar{B}_F F(\lambda), \quad \bar{\Pi}_5 = \sqrt{\epsilon}\alpha S^{\mathrm{T}}(\lambda)E(\lambda) + \bar{B}_F F(\lambda),$$
$$\bar{\Pi}_6 = R^{\mathrm{T}}(\lambda) + \epsilon R^{\mathrm{T}}(\lambda)A(\lambda) + \epsilon\bar{B}_F C(\lambda), \quad \bar{\Pi}_7 = S^{\mathrm{T}}(\lambda) + \epsilon S^{\mathrm{T}}(\lambda)A(\lambda) + \epsilon\bar{B}_F C(\lambda),$$
$$\bar{\Pi}_8 = \sqrt{\epsilon}R^{\mathrm{T}}(\lambda)B(\lambda) + \sqrt{\epsilon}\bar{B}_F D(\lambda), \quad \bar{\Pi}_9 = \sqrt{\epsilon}S^{\mathrm{T}}(\lambda)B(\lambda) + \sqrt{\epsilon}\bar{B}_F D(\lambda).$$

*Moreover, under the above condition, the matrices for an admissible robust stochastic $\mathcal{H}_\infty$ filter are given by*

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F & \bar{B}_F \\ \bar{C}_F & 0 \end{bmatrix}. \tag{35}$$

Based on Proposition 4.1, we readily have the main robust filtering result.

**Theorem 4.1 (Robust Stochastic $\mathcal{H}_\infty$ Filtering)** *Given system $\mathcal{S}$ in (1), an admissible robust stochastic $\mathcal{H}_\infty$ filter in the form of $\mathcal{F}_R$ in (2) exists if for a sufficiently small scalar $\epsilon > 0$, there exist matrices $\bar{Q}_i \triangleq \begin{bmatrix} \bar{Q}_{1i} & \bar{Q}_{2i} \\ * & \bar{Q}_{3i} \end{bmatrix} > 0$, $R_i$, $S_i$, $T$, $\bar{A}_F$, $\bar{B}_F$,*

$\bar{C}_F$ *satisfying*

$$\Delta_{ii} < 0, \quad i = 1, \ldots, s, \tag{36}$$

$$\Delta_{ij} + \Delta_{ji} \leq 0, \quad 1 \leq i < j \leq s, \tag{37}$$

*where*

$$\Delta_{ij} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}R_i^{\mathrm{T}}E_j & 0 & 0 & 0 \\ * & \Lambda_3 & 0 & 0 & 0 & 0 & \sqrt{\epsilon}\bar{\alpha}S_i^{\mathrm{T}}E_j & 0 & 0 & 0 \\ * & * & \Lambda_1 & \Lambda_2 & 0 & 0 & \sqrt{\epsilon}\alpha R_i^{\mathrm{T}}E_j + \bar{B}_F F_j & 0 & 0 & 0 \\ * & * & * & \Lambda_3 & 0 & 0 & \sqrt{\epsilon}\alpha S_i^{\mathrm{T}}E_j + \bar{B}_F F_j & 0 & 0 & 0 \\ * & * & * & * & \Lambda_1 & \Lambda_2 & \Lambda_4 & T + \epsilon\bar{A}_F & \begin{array}{c}\sqrt{\epsilon}R_i^{\mathrm{T}}B_j + \\ \sqrt{\epsilon}\bar{B}_F D_j\end{array} & 0 \\ * & * & * & * & * & \Lambda_3 & \Lambda_5 & T + \epsilon\bar{A}_F & \begin{array}{c}\sqrt{\epsilon}S_i^{\mathrm{T}}B_j + \\ \sqrt{\epsilon}\bar{B}_F D_j\end{array} & 0 \\ * & * & * & * & * & * & -\bar{Q}_{1i} & -\bar{Q}_{2i} & 0 & \sqrt{\epsilon}L_j^{\mathrm{T}} \\ * & * & * & * & * & * & * & -\bar{Q}_{3i} & 0 & -\sqrt{\epsilon}\bar{C}_F^{\mathrm{T}} \\ * & * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix},$$

$$\begin{aligned} &\Lambda_1 = \bar{Q}_{1i} - R_i^{\mathrm{T}} - R_i, \quad \Lambda_2 = \bar{Q}_{2i} - T - S_i, \quad \Lambda_3 = \bar{Q}_{3i} - T - T^{\mathrm{T}}, \\ &\Lambda_4 = R_i^{\mathrm{T}} + \epsilon R_i^{\mathrm{T}}A_j + \epsilon\bar{B}_F C_j, \quad \Lambda_5 = S_i^{\mathrm{T}} + \epsilon S_i^{\mathrm{T}}A_j + \epsilon\bar{B}_F C_j. \end{aligned} \tag{38}$$

*Moreover, under the above conditions, the matrices for an admissible robust stochastic $\mathcal{H}_\infty$ filter in the form of (2) are given by (35).*

The theorem can be proved by following similar lines as in the proof of Theorem 3.1 and thus omitted.

With Theorem 4.1, the minimum (in terms of the feasibility of Theorem 4.1) attenuation level of robust stochastic $\mathcal{H}_\infty$ filters can be readily found by solving the following convex optimization problem:

$$\text{Minimize } \gamma \text{ subject to (36) and (37) for sufficiently small } \epsilon > 0.$$

## 5 Illustrative Example

Consider the following numerical example:

$$\begin{aligned} dx(t) &= \left\{ \begin{bmatrix} -0.6 & 4+a \\ -4 & -0.6 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1.5 & 0 \end{bmatrix} w(t) \right\} dt + \begin{bmatrix} -0.4 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} x(t) d\beta(t), \\ y(t) &= \left\{ \begin{bmatrix} 0 & -1.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t) \right\} + \begin{bmatrix} 0.3 & 0.4 + 0.1a \end{bmatrix} x(t) d\beta(t), \\ z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{aligned} \tag{39}$$

where $a$ represents an uncertain parameter satisfying $|a| \leq \bar{a}$. This uncertain system can be modeled with a two-vertex polytope.

First assume $\bar{a} = 0.5$, we solve the filtering problem for this system by several approaches described as follows:

1. By Theorem 4.1, the obtained minimum $\mathcal{H}_\infty$ performance of robust stochastic filters is $\gamma = 1.6988$ for ($\epsilon = 0.001$), and the associated matrices for filter $\mathcal{F}_R$ in (2) are given by

$$A_F = \begin{bmatrix} -7.2213 & 6.8684 \\ -5.4021 & -0.1494 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0.0024 \\ -0.0066 \end{bmatrix}, \quad C_F = \begin{bmatrix} 0.0000 & -1.0000 \end{bmatrix}.$$

The actual calculated $\mathcal{H}_\infty$ performance of the filtering error system for different $a$ by connecting the above filter to the original system is depicted in Figure 5.1. From this figures, we can see that the $\mathcal{H}_\infty$ performances for the entire uncertainty domain are below the prescribed value $\gamma = 1.6988$.

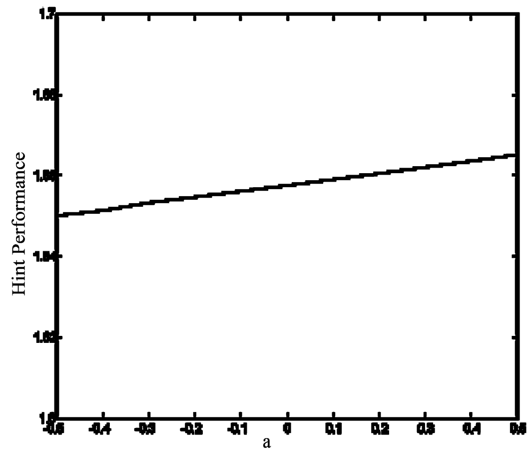


**Figure 5.1:** $\mathcal{H}_\infty$ performance of robust stochastic filter for entire uncertainty domain.

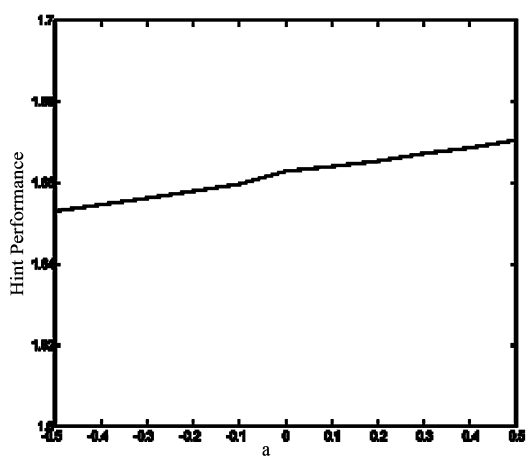


**Figure 5.2:** $\mathcal{H}_\infty$ performance of parameter-dependent stochastic filter for entire uncertainty domain.

2. By Theorem 3.1, the obtained minimum $\mathcal{H}_\infty$ performance of parameter-dependent stochastic filters is $\gamma = 1.6900$ for ($\epsilon = 0.001$), and the associated matrices needed for the calculation of (29) are given by

$$T_1 = \begin{bmatrix} 0.6035 & -0.1113 \\ -0.1113 & 0.5390 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.6251 & -0.1128 \\ -0.1129 & 0.7156 \end{bmatrix},$$
$$\bar{A}_{F1} = \begin{bmatrix} -0.1019 & 2.1424 \\ -2.0922 & -0.7071 \end{bmatrix}, \quad \bar{A}_{F2} = \begin{bmatrix} -0.0988 & 2.8630 \\ -2.7663 & -0.7701 \end{bmatrix},$$
$$\bar{B}_{F1} = \begin{bmatrix} 0.0009 \\ -0.00331 \end{bmatrix}, \quad \bar{B}_{F2} = \begin{bmatrix} 0.0031 \\ -0.0054 \end{bmatrix},$$
$$\bar{C}_{F1} = \begin{bmatrix} 0.0001 & -1.0000 \end{bmatrix}, \quad \bar{C}_{F2} = \begin{bmatrix} 0.0001 & -1.0003 \end{bmatrix}.$$

The actual calculated $\mathcal{H}_\infty$ performance of the filtering error system for different $a$ by connecting the above filter to the original system is depicted in Figure 5.2. It can be seen that the $\mathcal{H}_\infty$ performances for the entire uncertainty domain are below the prescribed value $\gamma = 1.6900$.

3. By Corollary 1 of [14], the obtained minimum $\mathcal{H}_\infty$ performance of robust stochastic filters is $\gamma = 2.0472$, and the associated matrices for filter $\mathcal{F}_R$ in (2) are given

$$A_F = \begin{bmatrix} -0.1735 & 4.0691 \\ -4.0141 & -1.9794 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0.0874 \\ -1.4642 \end{bmatrix}, \quad C_F = \begin{bmatrix} 0.0000 & 1.0000 \end{bmatrix}.$$

The above calculated results show that for this example, the robust filtering result in the quadratic framework [14] is conservative than the approaches presented in this paper. In addition, since the parameter-dependent stochastic filter design makes use of information of the uncertain parameter, it is reasonable to obtain less conservative filter designs than the robust filtering approach.

Finally, Table 5.1 presents a comparison of minimum $\mathcal{H}_\infty$ performance obtained by using Theorem 4.1, Theorem 3.1 and Corollary 1 of [14] for different cases. This table shows again the reduced conservativeness of the filtering approaches proposed in this paper. Notably for $1.0 \leq \bar{a} \leq 4$ where Corollary 1 of [14] fails to find feasible solutions, the parameter-dependent and robust approach presented here are still able to provide desired filters.

| | $\bar{a} = 0.5$ | $\bar{a} = 0.8$ | $\bar{a} = 1.0$ | $\bar{a} = 3$ | $\bar{a} = 4$ |
|---|---|---|---|---|---|
| Minumum $\gamma$ by Theorem 4.1 | 1.6900 | 1.7102 | 1.7280 | 2.5071 | 21.5990 |
| Minumum $\gamma$ by Theorem 3.1 | 1.6988 | 1.7189 | 1.7399 | 2.5293 | 22.5724 |
| Minumum $\gamma$ by [14] | 2.0472 | 6.0166 | infeasible | infeasible | infeasible |

**Table 5.1:** Minimum $\mathcal{H}_\infty$ performance for different cases.

## 6 Conclusions

The problem of $\mathcal{H}_\infty$ filtering for continuous-time stochastic systems with parameter uncertainties residing in a polytope has been investigated in this paper. Two approaches, namely robust stochastic $\mathcal{H}_\infty$ filtering and parameter-dependent stochastic $\mathcal{H}_\infty$ filtering, have been proposed according to the online availability of the information on the uncertain parameters. Sufficient conditions are derived based on an improved bounded real lemma for stochastic systems and formulated in terms of linear matrix inequalities, upon which desired filters can be obtained by solving convex optimization problems. Since the filter designs make full use of the parameter-dependent stability idea, the obtained results are less conservative than the existing one in the quadratic framework, which has been illustrated via a numerical example.

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# Estimations of Solutions Convergence of Hybrid Systems Consisting of Linear Equations with Delay

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**Abstract:** The logic-dynamical hybrid system given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay is investigated in the paper. The estimations of disturbances of such system are obtained. We consider the cases of stable and unstable subsystems. Besides the estimations of solutions of hybrid system given by a set of scalar subsystems are obtained.

**Keywords:** *Hybrid system; differential-difference equation; Lyapunov-Krasovsky functional; stable system*

**Mathematics Subject Classification (2000):** 34K20, 34K06, 34O20

## 1 Introduction

Nowadays the disturbances in hybrid systems dynamic is an actual research problem [2,9]. Since in different branches such as medicine, ecology, construction of control systems, the state at a given moment in time essentially depends on the previous history, more adequate instrument for researching the dynamic of separate subsystems is formed by equations with delay [4-6].

Let the logic-dynamical system be given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau), \quad i = \overline{1,n}, \quad x(t) \in R^n, \quad t \geq 0. \tag{1}$$

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Each of these subsystems describes the dynamics on a fixed finite time interval $t_{i-1} \le t < t_i$, $i = \overline{1, N}$, $t_0 = 0$. Subsystems can be stable or unstable. We suppose, the initial disturbance is in $\delta$-vicinity of the origin. It is required to estimate the size of the deviation of solutions $x(t)$ of the logic-dynamical system (1) from the origin at the final moment $t = t_N$. We consider finite time intervals, and at switching times coordinates have no discontinuity, i.e.

$$\lim_{s \to +0} x(t_i - s) = \lim_{s \to +0} x(t_i + s), \quad i = \overline{1,\, N-1}, \tag{2}$$

and on separate time intervals the subsystems are systems of linear differential-difference equations such that, by virtue of a continuity, all solutions which start from $\delta$-vicinity do not leave $\varepsilon(\delta)$-vicinity. On the contrary, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that $|x(t_N)| < \varepsilon$, if $\|x(0)\|_\tau < \delta(\varepsilon)$. In the paper the mentioned values are calculated. Special attention is given to the case of unstable subsystems. Here and further the following vector and matrix norms are used

$$|A| = \left\{\lambda_{\max}(A^T A\right\}^{1/2},$$
$$|x(t)| = \left\{\sum_{i=1}^{n} x_i^2(t)\right\}^{1/2},$$
$$\|x(t)\|_\tau = \max_{-\tau \le s \le 0} \left\{|x(s+t)|\right\},$$
$$\|x(t)\|_{\tau,\beta} = \left\{\int_{-\tau}^{0} e^{\beta s} \left|x(t+s)\right|^2 ds\right\}^{1/2},$$

$\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ are the largest and smallest eigenvalues of the corresponding symmetric, positive definite matrices.

For the derivation of estimations the method of Lyapunov-Krasovsky functionals [7–9] is used.

Research of such type of logic-dynamic systems has been carried out earlier. In [10] the logic-dynamical system consisting of linear differential equations subsystems was examined. The method of quadratic Lyapunov functions was used. The Lyapunov's functions were built as non-autonomous quadratic forms $V(x,t) = x^T H(t) x$, $H(t) = e^{-tA^T} e^{-tA}$ by using a first integral. This kind of Lyapunov function allows to derive the most exact estimations of solutions, as level surfaces $V_i(x,t) = \alpha_i$, $i = \overline{1,\, N-1}$ of Lyapunov functions $V_i(x,t)$, $i = \overline{1,\, N-1}$, completely consisting of integral curves. However, the construction of such functions is connected with the presence of a matrix exponential $e^{tA}$, i.e. with the presence of a fundamental matrix of solutions. That is a strong condition.

In [11] it has been proposed to use autonomous Lyapunov functions with symmetric, positive definite matrices $H_i$, $i = \overline{1,\, N-1}$ which are calculated using a solution of the matrix Lyapunov equations

$$A_i^T H_i + H_i A_i = C_i$$

for $i = \overline{1,\, N-1}$. However this requires the asymptotic stability of matrices $A_i$, $i = \overline{1,\, N-1}$ . Finally, in [12] estimations of disturbances of logic-dynamical system (1) without the requirement of asymptotic stability of matrices $A_i$, $i = \overline{1,\, N-1}$ has been obtained.

## 2 Estimations of solutions of stable subsystems

We'll first obtain some auxiliary results. We investigate the behavior of the solution $x(t)$ of a linear stationary subsystem with delay

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{3}$$

determined on an interval $t_0 \leq t \leq t_1$. For obtaining an estimation of solutions we use a functional of the form

$$V[x(t),t] = e^{\gamma t}\left\{x^T(t)Hx(t) + \int\limits_{-\tau}^{0} e^{\beta s}x^T(t+s)Gx(t+s)ds\right\}. \tag{4}$$

Let's denote

$$\begin{gathered}\varphi_{11}(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \varphi_{12}(G,H) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)},\\ \varphi_{21}(G,H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)}, \quad \varphi_{22}(G) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} \quad ,\\ S[G,H] = \begin{bmatrix} -A^TH - HA - G & -HB \\ -B^TH & G \end{bmatrix}.\end{gathered} \tag{5}$$

The following statement holds.

**Theorem 2.1** *Let there exist positive definite matrices $G$ and $H$ for which the matrix $S[G,H]$ is also positive definite. Then the system (3) is asymptotic stable and for its solutions $x(t)$ it follows the top exponential estimations of convergence hold:*

$$|x(t)| \leq \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0, \tag{6}$$

*and*

$$\|x(t)\|_{\tau,\beta} \leq \left[\sqrt{\varphi_{21}\,(G,H)}\,|x(0)| + \sqrt{\varphi_{22}(G)}\,\|x(0)\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0 \tag{7}$$

*for*

$$\varsigma(\beta,\gamma) = \min\left\{\frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)},\ \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right]\right\}. \tag{8}$$

*The value $\beta \geq 0$ can be arbitrary for*

$$\lambda_{\min}(S[G,H]) \geq \lambda_{\max}(G).$$

*And*

$$\beta \leq \frac{1}{\tau}\ln\left\{\frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}(S[G,H])}\right\},$$

*if*

$$\lambda_{\min}(S[G,H]) < \lambda_{\max}(H).$$

*The value $\gamma$ satisfies a condition $\gamma \leq \beta$.*

***Proof*** For the proof we use the Lyapunov-Krasovsky functional of the form (4) with positive definite matrices $G$ and $H$. It satisfies the following bilateral estimations:

$$\begin{aligned} e^{\gamma t}\left\{\lambda_{\min}(H)|x(t)|^2+\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} &\le V[x(t),t] \\ &\le e^{\gamma t}\left\{\lambda_{\max}(H)|x(t)|^2+\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} \end{aligned} \tag{9}$$

We find an estimation for its derivative in force of system (3). We make a substitution $t+s=\xi$. Then the functional transforms to

$$V[x(t),t]=e^{\gamma t}\left\{x^T(t)Hx(t)+\int\limits_{t-\tau}^{t}e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)\,d\xi\right\}. \tag{10}$$

We calculate a full derivative of the transformed functional (10) along solutions $x(t)$ of system (3). We obtain

$$\begin{aligned} \frac{d}{dt}V[x(t),t]=\gamma e^{\gamma t}&\left\{x^T(t)Hx(t)+\int\limits_{t-\tau}^{t}e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\} \\ &+e^{\gamma t}\big\{\left[Ax(t)+Bx(t-\tau)\right]^T Hx(t)+x^T(t)H\left[Ax(t)+Bx(t-\tau)\right] \\ &+x^T(t)Gx(t)-e^{-\beta\tau}x^T(t-\tau)Gx(t-\tau)\big\} \\ &-e^{\gamma t}\left\{\beta\int\limits_{t-\tau}^{t}e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\}. \end{aligned}$$

We transform the obtained expression as follows:

$$\begin{aligned} \frac{d}{dt}V[x(t),t]=-e^{\gamma t}&\left\{(\beta-\gamma)\int\limits_{t-\tau}^{t}e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\} \\ &-e^{\gamma t}\left(x^T(t),x^T(t-\tau)\right)\begin{bmatrix}-A^TH-HA-G & -HB\\ -B^TH & G\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix} \\ &+\gamma e^{\gamma t}x^T(t)Hx(t)+e^{\gamma t}\big(1-e^{-\beta\tau}\big)x^T(t-\tau)Gx(t-\tau). \end{aligned} \tag{11}$$

We suppose, as follows from the conditions of Theorem 1, there are positive definite matrices $G$ and $H$ for which the matrix $S[G,H]$ is also positive definite and $\beta\ge\gamma\ge 0$. Then we obtain

$$\begin{aligned} \frac{d}{dt}V[x(t),t]\le -e^{\gamma t}&\lambda_{\min}\big(S[G,H]\big)\big(|x(t)|^2+|x(t-\tau)|^2\big) \\ &+e^{\gamma t}\gamma\lambda_{\max}(H)|x(t)|^2+e^{\gamma t}\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\left|x(t-\tau)\right|^2 \\ &-e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}. \end{aligned}$$

Let's transform the obtained expression as follows

$$\begin{aligned} \frac{d}{dt}V[x(t),t]\le -e^{\gamma t}&\{\lambda_{\min}(S[G,H])-\gamma\lambda_{\max}(H)\}|x(t)|^2 \\ &-e^{\gamma t}\left\{\lambda_{\min}(S[G,H])-\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\right\}\left|x(t-\tau)\right|^2 \\ &-e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}. \end{aligned} \tag{12}$$

If the parameters of system and functional are

$$\lambda_{\min}(S[G,H]) \geq \lambda_{\max}(G)$$

then from inequality (12) it follows, that

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\left\{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right\}|x(t)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta} \tag{13}$$

for any $\beta \geq 0$. If

$$\lambda_{\min}(S[G,H]) < \lambda_{\max}(G),$$

then inequality (13) will be used for

$$0 \leq \beta < \frac{1}{\tau}\ln\left[\frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}(S[G,H])}\right].$$

We transform the right part of the inequality of quadratic forms (9) as

$$-e^{\gamma t}\lambda_{\max}(H)|x(t)|^2 - e^{\gamma t}\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta} \leq -V[x(t),t]\,. \tag{14}$$

Let's consider two cases.

1. Let's transform the inequality (14) as

$$-e^{\gamma t}|x(t)|^2 \leq -\frac{1}{\lambda_{\max}(H)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\left\|x(t)\right\|^2_{\tau,\beta}$$

and we substitute it in the first part of the inequalities (13). We obtain

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}V\left[x(t,t)\right]$$
$$- e^{\gamma t}\left\{(\beta-\gamma)\lambda_{\min}(G) - \left[\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\right\}\left\|x(t)\right\|^2_{\tau,\beta}.$$

If the parameters are

$$(\beta-\gamma)\lambda_{\min}(G) \geq \left[\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\,, \tag{15}$$

then

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}V[x(t),t].$$

Solving the obtained differential inequality, we get

$$V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}, \quad t \geq 0. \tag{16}$$

From here

$$\zeta = \alpha + \gamma = \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)}\,.$$

2. We transform inequality (14) to the following form

$$-e^{\gamma t}\left\|x(t)\right\|^2_{\tau,\beta} \leq -\frac{1}{\lambda_{\max}(G)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}|x(t)|^2$$

and again we substitute it in the second part of the inequalities (13). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V[x(t),t]$$
$$- e^{\gamma t}\left\{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}\right\}|x(t)|^2.$$

And if parameters are such that

$$\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} > 0, \tag{17}$$

then

$$\frac{d}{dt}V\left[x(t)\right] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V\left[x(t)\right].$$

Having integrated the obtained expression, we get

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = (\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \quad t \ge 0. \tag{18}$$

We get

$$\zeta = \alpha+\gamma = \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right].$$

For obtaining the required result we return to bilateral estimations of Lyapunov–Krasovsky functional (9). Using expressions (16), (18), we write down

$$e^{\gamma t}\left\{\lambda_{\min}(H)|x(t)|^2 + \lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} \le V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}$$
$$\le e^{-\alpha t}\left\{\lambda_{\max}(H)\left|x(0)\right|^2 + \lambda_{\max}(G)\left\|x(0)\right\|^2_{\tau,\beta}\right\}.$$

It is possible to obtain two estimations. First, we get

$$|x(t)|^2 \le \left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}\left|x(0)\right|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)}\left\|x(0)\right\|^2_{\tau,\beta}\right]e^{-(\alpha+\gamma)t}.$$

And, using denotations $\varphi_{11}(H)$, $\varphi_{12}(G,H)$, we obtain

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\left|x(0)\right| + \sqrt{\varphi_{12}(G,H)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t \ge 0.$$

Further it is possible to write down

$$\|x(t)\|^2_{\tau,\beta} \le \left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(G)}\left|x(0)\right|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}\left\|x(0)\right\|^2_{\tau,\beta}\right]e^{-(\alpha+\gamma)t}.$$

And, using designations $\varphi_{21}(G,H), \varphi_{22}(G)$, we obtain an inequality

$$\|x(t)\|_{\tau,\beta} \le \left[\sqrt{\varphi_{21}\left(G,H\right)}\left|x(0)\right| + \sqrt{\varphi_{22}(G)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \ge 0.$$

As follows from consideration of both cases we have

$$\varsigma = \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} \quad \text{for} \quad \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] \ge \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} \tag{19}$$

$$\varsigma = \frac{\beta\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] \quad \text{for} \quad \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] < \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}H)}. \tag{20}$$

Uniting these expressions, we obtain the statement of Theorem 2.1. □

## 3 Estimations of solutions of unstable subsystems

We consider a case where it is not possible to find matrices $G$ and $H$ for which the matrix $S[G,H]$ is positive definite. Let's denote

$$S[G,H,\gamma] = \begin{bmatrix} -A^T H - HA - \gamma H - G & -HB \\ -B^T H & G \end{bmatrix}. \tag{21}$$

Obviously, due to the choice of a scalar value $\gamma < 0$ the matrix $S[G,H,\gamma]$ can be made positive definite.

**Lemma 3.1** *Let the matrices $G$, $H$ be positive definite and let the following inequality hold*

$$\gamma < \frac{\lambda_{\min}\left[-A^T H - HA - G - HBG^{-1}B^T H\right]}{\lambda_{\max}(H)}. \tag{22}$$

*Then the matrix $S[G,H,\gamma]$ is also positive definite.*

***Proof*** We introduce a vector $z^T(t,\tau) = \left(x^T(t), x^T(t-\tau)\right)$. The condition of positive definiteness of matrix $S[G,H,\gamma]$ is equivalent to positiveness of the minimal eigenvalue

$$\lambda_{\min}\left[S\left(G,H\right)\right] = \min_{|z|=1}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} > 0,$$

or to the condition

$$\min_{x(t-\tau)}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} > 0$$

at an arbitrary $x(t) \in R^n$. In braces the quadratic form is written down

$$\begin{aligned} z^T(t,\tau)S[G,H,\gamma]z(t,\tau) &= x^T(t)\left[-A^T H - HA - \gamma H - G\right]x(t) \\ &\quad - x^T(t)HBx(t-\tau) - x^T(t-\tau)B^T Hx(t) + x^T(t-\tau)Gx(t-\tau). \end{aligned}$$

The necessary and sufficient condition for a minimum on a variable $x(t-\tau)$ is equality to zero of a partial derivative on $x(t-\tau)$ and positive definiteness of a matrix $G$, i.e.

$$\frac{\partial}{\partial x(t-\tau)}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} = 0.$$

Calculating the derivative, we get

$$-B^T Hx(t) + Gx(t-\tau) = 0.$$

As the matrix $G$ is positive definite, it is non special. From this it follows that $x(t-\tau) = G^{-1}B^T Hx(t)$. We calculate the value of the quadratic form in the obtained point $x(t-\tau)$

$$z^T(t,\tau)S[G,H,\gamma]z\left(t,\tau\right) = x^T(t)\left[-A^T H - HA - \gamma H - G - HBG^{-1}B^T H\right]x(t).$$

From this we obtain that the matrix $S[G,H,\gamma]$ is positive definite, if there are positive definite matrices $G$ and

$$Q[G,H,\gamma] = -A^T H - HA - \gamma H - G - HBG^{-1}B^T H.$$

This expression is used for

$$\lambda_{\min}\left(Q[G,H,\gamma]\right) > \lambda_{\min}\left[-A^T H - HA - G - HBG^{-1}B^T H\right] - \gamma\lambda_{\max}(H) > 0.$$

From this we obtain inequality (22), i.e. the statement of the Lemma. □

Using the proved Lemma, we obtain the following statement.

**Theorem 3.1** *Let there not be any positive definite matrices $G$, $H$ for which the matrix $S[G,H]$ is also positive definite. If the value $\gamma$ is chosen according to an inequality (22) and $\beta \geq \gamma$ then for the solutions $x(t)$ of system (3) there are truly top exponential estimations of convergence (6), (7)*

$$|x(t)| \leq \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_{\tau,\beta}\right] \exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0,$$

$$\|x(t)\|_{\tau,\beta} \leq \left[\sqrt{\varphi_{21}(G,H)}\,|x(0)| + \sqrt{\varphi_{22}(G)}\,\|x(0)\|_{\tau,\beta}\right] \exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0,$$

*and*

$$\varsigma(\beta,\gamma) = \min\left\{\frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} + \gamma,\ \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right]\right\}. \tag{23}$$

*The value $\beta$ can be arbitrary if*

$$\lambda_{\min}\left(S[G,H,\gamma]\right) \geq \lambda_{\max}(G)$$

*and*

$$\beta \leq \frac{1}{\tau}\ln\left\{\frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}\left(S[G,H,\gamma]\right)}\right\}$$

*if*

$$\lambda_{\min}\left(S[G,H,\gamma]\right) < \lambda_{\max}(H).$$

***Proof*** For the proof of the statements of Theorem 3.1 again we use a Lyapunov–Krasovsky functional of the form (4) with positive definite matrices $G$ and $H$. We write the full derivative of the functional (10) along solutions $x(t)$ of system (3) as

$$\begin{aligned}\frac{d}{dt}V[x(t),t] = &-e^{\gamma t}\left\{(\beta-\gamma)\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)\,d\xi\right\}\\ &- e^{\gamma t}(x^T(t), x^T(t-\tau))\begin{bmatrix}-A^TH - HA - \gamma H - G & -HB\\ -B^TH & G\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix}\\ &+ e^{\gamma t}(1-e^{-\beta\tau})x^T(t-\tau)Gx(t-\tau).\end{aligned}$$

Let the matrix $S[G,H]$ described in (4), be nonpositive definite. Then, as follows from the Lemma, if $\gamma$ satisfies conditions (22), then the matrix $S[G,H,\gamma]$ will be positive definite and the following inequality holds

$$\begin{aligned}\frac{d}{dt}V[x(t),t] \leq &-e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)\left(|x(t)|^2 + |x(t-\tau)|^2\right)\\ &+ e^{\gamma t}\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\,|x(t-\tau)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\,\|x(t)\|^2_{\tau,\beta}.\end{aligned}$$

Let's transform the obtained expression as follows

$$\begin{aligned}\frac{d}{dt}V[x(t),t] \leq &-e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)|x(t)|^2\\ &- e^{\gamma t}\left\{\lambda_{\min}\left(S[G,H,\gamma]\right) - \left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\right\}|x(t-\tau)|^2\\ &- e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\,\|x(t)\|^2_{\tau,\beta}.\end{aligned} \tag{24}$$

If the parameters of system and functional are such that

$$\lambda_{\min}\left(S[G,H,\gamma]\right) \geq \lambda_{\max}(G),$$

then

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)|x(t)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta} \tag{25}$$

for arbitrary $\beta \geq 0$. If

$$\lambda_{\min}\left(S[G,H,\gamma]\right) < \lambda_{\max}(G),$$

then inequality (25) is used for

$$0 \leq \beta < \frac{1}{\tau}\ln\left[\frac{\lambda_{\max}(G)}{\lambda_{\max}(G)-\lambda_{\min}\left(S[G,H,\gamma]\right)}\right].$$

We transform the right part of inequality of quadratic forms (9) to the form of expression (14)

$$-e^{\gamma t}\lambda_{\max}(H)|x(t)|^2 - e^{\gamma t}\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta} \leq -V[x(t),t]$$

and we consider two cases.

1. Let's transform the right part of the inequality (14) as

$$-e^{\gamma t}|x(t)|^2 \leq -\frac{1}{\lambda_{\max}(H)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\left\|x(t)\right\|^2_{\tau,\beta}$$

and we substitute it in the first part of inequalities (25). We get

$$\begin{aligned}\frac{d}{dt}V[x(t),t] \leq &-\frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}V\left[x(t,t)\right]\\ &- e^{\gamma t}\left\{(\beta-\gamma)\lambda_{\min}(G) - \left[\lambda_{\min}\left(S[G,H],\gamma\right)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\right\}\left\|x(t)\right\|^2_{\tau,\beta}.\end{aligned} \tag{26}$$

If the parameters are such that

$$(\beta-\gamma)\lambda_{\min}(G) \geq \lambda_{\min}\left(S[G,H,\gamma]\right)\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}, \tag{27}$$

then

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}V[x(t),t].$$

Solving the obtained differential inequality, we get

$$V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}, \quad t \geq 0. \tag{28}$$

2. Further we transform inequality (14) as follows:

$$-e^{\gamma t}\left\|x(t)\right\|^2_{\tau,\beta} \leq -\frac{1}{\lambda_{\max}(G)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}|x(t)|^2,$$

and we also substitute it in the second part of inequality (27). We get

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V[x(t),t] \\ - e^{\gamma t}\left\{\lambda_{\min}\left(S[G,H,\gamma]\right) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}\right\}|x(t)|^2.$$

And if parameters are

$$\lambda_{\min}\left(S[G,H,\gamma]\right) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} > 0$$

then

$$\frac{d}{dt}V\left[x(t)\right] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V\left[x(t)\right].$$

Having integrated it, we obtain

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = (\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \quad t \ge 0. \tag{29}$$

Let's return to bilateral estimations of Lyapunov–Krasovsky functional (9). Using expressions (28), (29), we obtain

$$e^{\gamma t}\big\{\lambda_{\min}(H)|x(t)|^2 + \lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\big\} \le V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t} \\ \le e^{-\alpha t}\left\{\lambda_{\max}(H)\left|x(0)\right|^2 + \kappa_{\max}(G)\left\|x(0)\right\|^2_{\tau,\beta}\right\}.$$

From this we obtain

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\left|x(0)\right| + \sqrt{\varphi_{12}(G,H)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t \ge 0,$$

$$\left\|x(t)\right\|_{\tau,\beta} \le \left[\sqrt{\varphi_{21}\left(G,H\right)}\left|x(0)\right| + \sqrt{\varphi_{22}(G)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t \ge 0.$$

From the consideration of both cases we get the following expressions

$$\alpha+\gamma = \begin{cases} \dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)} + \gamma, & \text{for } \beta\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] \ge \\ & \qquad \dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}, \\ \dfrac{\beta\lambda_{\min}(G)}{\lambda_{\max}(G)}, & \text{for } \beta\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] < \\ & \qquad \dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}H)}. \end{cases}$$

Uniting these expressions, we obtain the statement of Theorem 3.1. □

**Remark 3.1** As for the value $\left\|x(t)\right\|^2_{\tau,\beta}$ the top estimations hold

$$\left\|x(t)\right\|^2_{\tau,\beta} = \int\limits_{-\tau}^{0} e^{\beta s}\left|x(t+s)\right|ds \le \max_{-\tau\le s\le 0}\left\{\left|x\left(t+s\right)\right|^2\right\}\int\limits_{-\tau}^{0} e^{\beta s}ds \\ \le \frac{1}{\beta}\left(1-e^{-\beta\tau}\right)\left\|x(t)\right\|^2_{\tau} \le \tau\left\|x(t)\right\|_{\tau}$$

where

$$\|x(t)\|_\tau = \max_{-\tau \le s \le 0} \{|x(t+s)|\},$$

then it is possible to transform the inequality (6) to the following

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_\tau\right] e^{-\frac{1}{2}\varsigma t}, \quad t \ge 0,$$

or, even,

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)} + \sqrt{\varphi_{12}(G,H)}\right] \|x(0)\|_\tau\, e^{-\frac{1}{2}\varsigma t}, \quad t \ge 0. \tag{30}$$

**Remark 3.2** As estimations of majorant type, they contain two free parameters $\beta$ and $\gamma$, and in the second theorem $\gamma$ can be negative. If put to the task of finding an "optimum estimation" for a given class of functionals it is possible to calculate the parameters $\beta$ and $\gamma$ precisely.

## 4 Estimations of solutions of scalar subsystems

Let's consider the scalar linear differential equation with constant delay

$$\dot{x}(t) = -ax(t) + bx(t-\tau), \quad a > 0, \quad 0 \le t \le t_1, \quad \tau > 0. \tag{31}$$

For the equation (31) the Lyapunov–Krasovsky functional (10) looks like

$$V[x(t),t] = e^{\gamma t}\left\{hx^2(t) + g\int_{-\tau}^{0} e^{\beta s}x^2(t+s)\,ds\right\}, \tag{32}$$

where $h > 0$, $g > 0$ are positive constants. We obtain estimations of the divergence of disturbances on a finite time interval. As $h > 0$, $g > 0$ are scalar values then

$$\lambda_{\min}(H) = \lambda_{\max}(H) = h, \quad \lambda_{\min}(G) = \lambda_{\max}(G) = g.$$

For the full derivative of functional (32) along solutions of the equation (31) the equality holds

$$\begin{aligned}\frac{d}{dt}V[x(t),t] &= \gamma e^{\gamma t}\left\{hx^2(t) + g\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)\,d\xi\right\}\\ &\quad + e^{\gamma t}\left\{2hx(t)\left[-ax(t) + bx(t-\tau)\right] + gx^2(t) - ge^{-\beta\tau}x^2(t-\tau)\right\}\\ &\quad - e^{\gamma t}\left\{\beta g\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\}.\end{aligned}$$

Let's transform it similarly to the form of (11)

$$\begin{aligned}\frac{d}{dt}V[x(t),t] &= -e^{\gamma t}\left\{(\beta-\gamma)g\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\}\\ &\quad - e^{\gamma t}\left(x(t), x(t-\tau)\right)\begin{bmatrix}2ah-g & -hb\\ -hb & g\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix}\\ &\quad + e^{\gamma t}\gamma hx^2(t) + e^{\gamma t}\left(1 - e^{-\beta\tau}\right)gx^2(t-\tau).\end{aligned} \tag{33}$$

### 4.1 Derivation of estimations of disturbances in the case of stable equation

Let's find $h > 0$, $g > 0$ from the condition of "maximal" positive definiteness of the matrix
$$S[g,h] = \begin{bmatrix} 2ah-g & -hb \\ -hb & g \end{bmatrix}.$$
If the parameters of equation (31) and the Lyapunov–Krasovsky functional (32) are
$$g\,(2ah-g) - h^2b^2 > 0,$$
as follows from Silvester criterion, the matrix $S[g,h]$ is positive definite. As $h > 0$, $g > 0$, then, taking into account uniformity, we denote $h = 1$ and we transform the inequality to
$$g\,(2a-g) - b^2 > 0.$$
Function $F(g) = g(2a-g) - b^2$ with respect to the variable $g$ represents a parabola with the branches directed downwards. And it reaches the extreme value at $g = a$. Thus "maximal positive definiteness" of matrixes $S[g,h]$ is reached at $g = a$. And the Lyapunov – Krasovsky functional (32) is chosen as
$$V[x(t),t] = e^{\gamma t}\left\{x^2(t) + a\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\}. \tag{34}$$
In this case a matrix $S[g,h]$ looks like
$$S[g,h] = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}. \tag{35}$$
Let's transform the expression for a full derivative (33) in view of $h = 1$, $g = a$ to the form similar to (12)
$$\begin{aligned}\frac{d}{dt}V[x(t),t] \le & -e^{\gamma t}\left\{\lambda_{\min}\left(S[g,h]\right) - \gamma\right\}|x(t)|^2 \\ & - e^{\gamma t}\left\{\lambda_{\min}\left(S[g,h]\right) - \left(1-e^{-\beta\tau}\right)a\right\}|x(t-\tau)|^2 \\ & - e^{\gamma t}(\beta-\gamma)a\left\|x(t)\right\|^2_{\tau,\beta}\end{aligned}$$
If
$$\lambda_{\min}\left(S[g,h]\right) = a - |b|, \quad \lambda_{\min}\left(S[g,h]\right) - \left(1-e^{-\beta\tau}\right)a = e^{-\beta\tau}a - |b|,$$
then
$$\beta < \frac{1}{\tau}\ln\frac{a}{|b|}. \tag{36}$$
Then for a full derivative the inequality such as (13) becomes
$$\frac{d}{dt}V[x(t),t] \le -e^{\gamma t}\left\{a - |b| - \gamma\right\}|x(t)|^2 - e^{\gamma t}(\beta-\gamma)a\left\|x(t)\right\|^2_{\tau,\beta}. \tag{37}$$
It is easy to see that for the functional (33) the following inequality holds:
$$-e^{\gamma t}|x(t)|^2 - e^{\gamma t}a\left\|x(t)\right\|^2_{\tau,\beta} \le -V[x(t),t]. \tag{38}$$

a) We transform (38) to

$$-e^{\gamma t}|x(t)|^2 \le -V[x(t),t] + e^{\gamma t} a \,\|x(t)\|^2_{\tau,\beta}\,. \tag{39}$$

Also we substitute it in the first part of (37). We obtain

$$\frac{d}{dt}V[x(t),t] \le -\left(a-|b|-\gamma\right)V[x(t),t] - e^{\gamma t}\left[(\beta-\gamma)a - \left(a-|b|-\gamma\right)a\right]\|x(t)\|^2_{\tau,\beta}\,.$$

And, if for the parameters $\beta > a-|b|$ holds then

$$\frac{d}{dt}V[x(t),t] \le -(a-|b|-\gamma)V[x(t),t].$$

And from this

$$V[x(t),t] \le V\left[x(0),0\right]e^{-(a-|b|-\gamma)t}, \quad t \ge 0. \tag{40}$$

b) We transform (38) to

$$e^{\gamma t}\,\|x(t)\|^2_{\tau,\beta} \le -\frac{1}{a}V[x(t),t] + e^{\gamma t}\frac{1}{a}|x(t)|^2. \tag{41}$$

Also we substitute it in the second part of (37). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t] + \left(\beta - a + |b|\right)\|x(t)\|^2_{\tau,\beta}\,.$$

And, if $\beta \le a-|b|$, then

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t].$$

We get

$$V[x(t),t] \le V\left[x(0),0\right]e^{-(\beta-\gamma)t}, \quad t \ge 0. \tag{42}$$

Uniting inequalities (40), (41), we obtain

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad t \ge 0 \tag{43}$$

if

$$\alpha = \begin{cases} a-|b|-\gamma & \text{for } \ \beta > a-|b|, \\ \beta-\gamma & \text{for } \ \beta \le a-|b|. \end{cases}$$

Let's transform the inequality (43) as

$$e^{\gamma t}|x(t)|^2 + e^{\gamma t}a\,\|x(t)\|^2_{\tau,\beta} \le \left[|x(0)|^2 + a\,\|x(0)\|^2_{\tau,\beta}\right]e^{-\alpha t}, \quad t \ge 0.$$

We get

$$|x(t)| \le \sqrt{|x(0)|^2 + a\,\|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}(\alpha+\gamma)t},$$

$$\|x(0)\|^{\le}_{\tau,\beta}\sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}(\alpha+\gamma)t}, \quad t \ge 0.$$

Let's denote

$$\varsigma = \min t\{a-|b|,\, \beta\}.$$

As the value $\beta$ is chosen according to (36), finally the following most exact estimation of convergence is obtained.

**Proposition 4.1** *Let the condition $a > |b|$ be satisfied. Then the equation (31) is asymptotically stable and for its solutions the exponential estimation of convergence is valid*

$$|x(t)| \leq \sqrt{|x(0)|^2 + a\,\|x(0)\|^2_{\tau,\beta}}\, e^{-\frac{1}{2}\varsigma t}, \quad \|x(0)\|^{\leq}_{\tau,\beta} \sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|^2_{\tau,\beta}}\, e^{-\frac{1}{2}\varsigma t}, \quad t \geq 0,$$

*for*

$$\varsigma = \min\left\{a - |b|,\ \frac{1}{\tau}\ln\frac{a}{|b|}\right\}.$$

## 4.2 Derivation of estimations of disturbances in the case of unstable equation

Let's transform the expression for a full functional (34) derivative to

$$\begin{aligned} \frac{d}{dt}V[x(t),t] = &-e^{\gamma t}\left\{(\beta-\gamma)g\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\} \\ &- e^{\gamma t}\left(x(t),x(t-\tau)\right)\begin{bmatrix} 2a-g-\gamma h & -hb \\ -hb & g\end{bmatrix}\begin{pmatrix} x(t) \\ x(t-\tau)\end{pmatrix} \\ &+ e^{\gamma t}\left(1-e^{-\beta\tau}\right)gx^2(t-\tau). \end{aligned} \tag{44}$$

Similarly to the first case, we denote $h = 1$, $g = a$. Then

$$S\left[g,h,\gamma\right] = \begin{bmatrix} a-\gamma & -b \\ -b & a\end{bmatrix}, \quad \lambda_{\min}\left(S\left[g,h,\gamma\right]\right) = a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2}. \tag{45}$$

Let's suppose, that $a < |b|$, i.e. the equation is unstable. Then if

$$\gamma < \frac{a^2 - b^2}{a}, \tag{46}$$

the matrix $S\left[g,h.\gamma\right]$ is positive definite, i.e. $\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) > 0$ and expression for a full functional (34) derivative can be written down as

$$\begin{aligned} \frac{d}{dt}V[x(t),t] \leq &-e^{\gamma t}\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)|x(t)|^2 \\ &- e^{\gamma t}\left\{\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - \left(1-e^{-\beta\tau}\right)a\right\}|x(t-\tau)|^2 - e^{\gamma t}(\beta-\gamma)a\,\|x(t)\|^2_{\tau,\beta}. \end{aligned}$$

As the value

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - a = -\frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} < 0$$

is always negative, then if

$$\beta < \frac{1}{\tau}\ln\frac{a}{\frac{1}{2}\gamma + \sqrt{b^2 + \frac{1}{4}\gamma^2}} \tag{47}$$

it yields

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)|x(t)|^2 - e^{\gamma t}(\beta-\gamma)a\,\|x(t)\|^2_{\tau,\beta}. \tag{48}$$

1) We substitute inequality (39) in the first part of (48). We obtain

$$\frac{d}{dt}V[x(t),t] \le -\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)V[x(t),t] + e^{\gamma t}\left\{a\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - (\beta-\gamma)a\right\}\|x(t)\|^2_{\tau,\beta}\,.$$

And, if inequality

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) < \beta - \gamma \tag{49}$$

holds, then

$$\frac{d}{dt}V[x(t),t] \le -\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)V[x(t),t]. \tag{50}$$

From this, we have

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2}, \quad t \ge 0. \tag{51}$$

2) We substitute an inequality (41) in the second part of (48). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t] + e^{\gamma t}\left\{-\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) + (\beta-\lambda)\right\}|x(t)|^2$$

and, if

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) \ge \beta - \gamma, \tag{52}$$

then

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t]. \tag{53}$$

We get

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \beta - \gamma, \quad t \ge 0. \tag{54}$$

Uniting expressions (51), (54) connected by conditions (49), (52) and having substituted instead of $\lambda_{\min}(S[g,h,\gamma])$ its value, we obtain

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad t \ge 0,$$

if

$$\alpha = \begin{cases} a - \dfrac{1}{2}\gamma - \sqrt{b^2 + \dfrac{1}{4}\gamma^2} & \text{for } \; a - \dfrac{1}{2}\gamma - \sqrt{b^2 + \dfrac{1}{4}\gamma^2} < \beta - \gamma, \\ \beta - \gamma, & \text{for } \; a - \dfrac{1}{2}\gamma - \sqrt{b^2 + \dfrac{1}{4}\gamma^2} \ge \beta - \gamma. \end{cases}$$

Let's denote $\alpha + \gamma = \varsigma$, and we obtain

$$\varsigma(\beta,\gamma) = \begin{cases} a + \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} & \text{for } \; a - \sqrt{b^2 + \frac{1}{4}\gamma^2} < \beta; \\ \beta & \text{for } \; a - \sqrt{b^2 + \frac{1}{4}\gamma^2} \ge \beta. \end{cases}$$

As the values $\beta$ and $\gamma$ satisfy the expressions

$$\beta < \frac{1}{2}\ln\frac{a}{\frac{1}{2}\gamma + \sqrt{b^2 + \frac{1}{4}\gamma^2}}, \quad \gamma < \frac{a^2 - b^2}{a},$$

the following result holds.

**Proposition 4.2** *Let the condition $a < |b|$ be satisfied. Then the equation (31) is unstable and for its solutions the following exponential estimation holds*

$$|x(t)| \leq \sqrt{|x(0)|^2 + a\,\|x(0)\|_{\tau,\beta}^2} e^{-\frac{1}{2}\varsigma t}, \quad \|x(0)\|_{\tau,\beta}^{\leq} \sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|_{\tau,\beta}^2} e^{-\frac{1}{2}\varsigma t}, \quad t \geq 0,$$

*for*

$$\varsigma = \frac{a^2 - b^2}{a}.$$

## 5 Estimations of solutions of hybrid systems

In the previous sections majorant estimations of solutions of stable and unstable subsystems were separately obtained. Now we shall consider whole hybrid system (1). On each of intervals $t_{i-1} \leq t < t_i$, $i = \overline{1,N}$ let's select Lyapunov–Krasovsky functional of the form (4) with positive definite matrices $H_i$, $G_i$, $i = \overline{1,N}$ . If there are positive definite matrices $H_i$, $G_i$, $i \in I$, such that matrices

$$S_i\left[G_i, H_i\right] = \begin{bmatrix} -A_i^T H_i - H_i A_i - G_i & -H_i B_i \\ -B_i^T H_i & G_i \end{bmatrix}, \quad i \in I$$

are positive definite, then we designate

$$N_i = \left[\sqrt{\varphi_{11}\left(H_i\right)} + \sqrt{\varphi_{12}\left(G_i, H_i\right)}\right] \exp\left\{\varsigma_i\left(\beta_i, \gamma_i\right)\tau\right\},$$

where the value $\beta_i > 0$ can be arbitrary at

$$\lambda_{\min}\left(S\left[G_i, H_i\right]\right) \geq \lambda_{\max}(G_i)$$

and

$$\beta_i \leq \frac{1}{\tau} \ln\left\{\frac{\lambda_{\max}(G_i)}{\lambda_{\max}(G_i) - \lambda_{\min}\left(S\left[G_i, H_i\right]\right)}\right\},$$

if $\lambda_{\min}\left(S\left[G_i, H_i\right]\right) < \lambda_{\max}\left(H_i\right)$. The value $\gamma$ satisfies the condition $\gamma \leq \beta$. If such matrices $H_i$, $G_i$ , $j \in J$ do not exist, then we assume

$$\gamma_j < \frac{\lambda_{\min}\left[-A_j^T H_j - H_j A_j - G_j - H_j B_j G_j^{-1} B_j^T H_j\right]}{\lambda_{\max}(H_j)},$$

and we denote

$$S\left[G_j, H_j, \gamma_j\right] = \begin{bmatrix} -A_j^T H_j - H_j A_j - \gamma_j H_j - G_j & -H_j B_j \\ -B_j^T H_j & G_j \end{bmatrix},$$

$$N_j = \left[\sqrt{\varphi_{11}\left(H_j\right)} + \sqrt{\varphi_{12}\left(G_j, H_j\right)}\right] \exp\left\{\varsigma_j\left(\beta_j, \gamma_j\right)\right\},$$

for

$$\varsigma_j\left(\beta_j, \gamma_j\right) = \min\left\{\frac{\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right)}{\lambda_{\max}(H_j)} + \gamma_j,\ \beta_j \frac{\lambda_{\min}(G_j)}{\lambda_{\max}(G_j)} + \gamma_j\left[1 - \frac{\lambda_{\min}(G_j)}{\lambda_{\max}(G_j)}\right]\right\}.$$

The value $\beta_j$ can be arbitrary at

$$\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right) \geq \lambda_{\max}(G_j)$$

and

$$\beta_j \le \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max}(G_j)}{\lambda_{\max}(G_j) - \lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right)} \right\},$$

if

$$\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right) < \lambda_{\max}\left(H_j\right).$$

**Theorem 5.1** *Let the initial state of the logic-dynamical hybrid system (1) satisfy the condition* $\|x(0)\|_\tau < \delta$*. Then at* $t = t_N$ *the following inequality holds*

$$\|x(t_N)\| \le \prod_{i=1}^{N} N_i \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \varsigma_i \left(t_i - t_{i-1}\right) \right\}.$$

***Proof*** Let's consider the first time interval $t_0 \le t \le t_1$, $t_0 = 0$. If there are positive definite matrices $G_1$, $H_1$, for which the matrix $S[G_1, H_1]$ is also positive definite, then as follows from expression (30) of Remark 1, the following inequality holds:

$$\|x(t_1)\| \le \left[ \sqrt{\varphi_1(H_1)} + \varphi\left(G_1, H_1\right) \right] \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - \tau)}.$$

If there are no such matrices, for arbitrary positive definite matrices $G_1$, $H_1$, there exists $\gamma_1$, for which the matrix $S[G_1, H_1, \gamma_1]$ is also positive definite. Again using expression (30) of Remark 1, we get

$$\|x(t_1)\| \le \left[ \sqrt{\varphi_1(H_1)} + \varphi\left(G_1, H_1\right) \right] \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - t_0)}.$$

And for the moment $t = t_1$

$$\|x(t_1)\|_\tau \le N_1 \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - t_0)}$$

holds. Let us consider the next interval $t_1 \le t \le t_2$. As for the second interval a similar estimate

$$\|x(t_2)\|_\tau \le N_2 \|x(t_1)\|_\tau \, e^{-\frac{1}{2}\varsigma_2 (t_2 - t_1)}$$

holds we obtain

$$\|x(t_2)\|_\tau \le N_1 N_2 \|x(t_0)\|_\tau \exp\left\{ -\frac{1}{2} \left[ \varsigma_1 \left(t_1 - t_0\right) + \varsigma_2 \left(t_2 - t_1\right) \right] \right\}.$$

Continuing the process further, for the moment $t = t_N$ we get

$$\|x(t_N)\| \le \prod_{i=1}^{N} N_i \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \varsigma_i \left(t_i - t_{i-1}\right) \right\},$$

which was required to prove. □

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# Robustly Global Exponential Stability of Time-varying Linear Impulsive Systems with Uncertainty [†]

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**Abstract:** This paper studies linear impulsive systems with varying time-delay and uncertainty. By using the method of the variation of constants formula for impulsive system, robustly global exponential stability criteria are established in terms of fairly simple algebraic conditions. Estimate of the decay rate of the solutions of such systems are also derived. Some examples are given to illustrate the main results.

**Keywords:** *Uncertainty; linear impulsive system; interval matrix; robustly global exponential stability; decay rate.*

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## 1 Introduction

Many real world systems display both continuous and discrete characteristics. For example, evolutionary processes such as biological neural networks, bursting rhythm models in pathology, optimal control models in economics, frequency-modulated signal processing systems, and flying object motions, etc., are characterized by abrupt changes of states at certain time instants. Those sudden and sharp changes are often of very short duration and are thus assumed to occur instantaneously in the form of impulses. Such impulses may be represented by discrete maps. Systems undergoing abrupt changes may not be

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well described by using purely continuous or purely discrete models. However, they can be appropriately modelled by impulsive systems. It is now recognized that the theory of impulsive systems provides a natural framework for mathematical modelling of many such real world phenomena. Significant progress has been made in the theory of impulsive systems in recent years, see [1-8, 15] and references therein. Meantime, the robust stability problems for discrete systems have also been studied in recent literatures, see [15-18] and references therein. However, the corresponding theory for impulsive systems with uncertainty has not been fully developed. Recently, some robust asymptotic stability results for impulsive systems and impulsive hybrid systems with uncertainty have been established in [9-14]. In this paper, by using the variation of constants formula for impulsive systems, we shall establish some criteria on robustly global exponential stability and provide some estimate on the decay rate for time-varying linear impulsive systems with uncertainty.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we establish robustly exponential stability for time-varying linear impulsive systems with uncertainty. In Section 4, some examples are also worked out to demonstrate the main results.

## 2 Preliminaries

Let $R^n$ denote the $n$-dimensional real vector space and $||A||$ be the norm of a matrix $A$ induced by the Euclidean norm, i.e., $||A|| = [\lambda_{\max}(A^TA)]^{\frac{1}{2}}$. Let $N$ denote the set of positive integers, i.e., $N = \{1, 2, \cdots\}$, and $R^+ = [0, +\infty)$. Let $PC[R^+, R]$ denote the class of piecewise continuous functions from $R^+$ to $R$, with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \cdots$. Let $\lambda_i(X), i = 1, 2, \cdots, n$, be all the eigenvalues of the matrix $X$ and $\lambda_{\max}(X)$ (respectively, $\lambda_{\min}(X)$) the maximum (respectively, minimum) eigenvalue of the matrix $X$.

Consider the following time-varying linear impulsive system with uncertainty

$$\begin{cases} \dot{y}(t) = A(t)y(t) + \tilde{A}(t)y(t), & t \in (t_{k-1}, t_k], \\ \Delta y(t) = C_k y(t^-) + \tilde{C}_k y(t^-), & t = t_k, k \in N, \end{cases} \tag{1}$$

and its nominal system

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t \in (t_{k-1}, t_k], \\ \Delta x(t) = x(t^+) - x(t^-) = C_k x(t^-), & t = t_k, k \in N, \end{cases} \tag{2}$$

under the following assumptions:

$(A_1)$ The sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots$, with $\lim_{k\to\infty} t_k = \infty$.

$(A_2)$ $A(t) = (a_{ij}(t))$ is an $n \times n$ matrix, and $a_{ij} \in PC[R^+, R], i, j = 1, 2, \cdots, n$.

$(A_3)$ $\tilde{A}(t) = (\tilde{a}_{ij}(t))$ is a disturbance matrix of $A(t)$ with $\tilde{a}_{ij} \in PC[R^+, R], i, j = 1, 2, \cdots, n$.

$(A_4)$ For every $k \in N$, $C_k$ and its disturbance matrix $\tilde{C}_k$ are $n \times n$ matrices.

$(A_5)$ Every solution of (1) (respectively, (2)) exists globally and uniquely on $R^+$ and is continuous except at $t_k, k \in N$, at which it is left-hand continuous, i.e., $y(t_k) = y(t_k^-)$. Let $y(t) = y(t, t_0, y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$.

Let $\Omega_1$ be the set of all disturbance matrices $\tilde{A}(t)$ satisfying $(A_3)$ such that, for any $t \in R^+$, $\|\tilde{A}(t)\| \le K_1$, where $K_1$ is some positive constant. Furthermore, let $\Omega_2$ be the set of all disturbance matrices $\tilde{C}_k, k \in N$, satisfying $(A_4)$ such that, for any $k \in N$, $\|\tilde{C}_k\| \le K_2$, where $K_2$ is an appropriate positive constant.

**Definition 2.1** System (1) is said to be robustly global exponential stable with decay rate $\alpha > 0$ if, for any initial condition $y(t_0^+) = y_0$ and for every disturbance matrices $\tilde{A}(t) \in \Omega_1, \tilde{C}_k \in \Omega_2$, $k \in N$, the trivial solution of system (1) is globally exponentially stable with decay rate $\alpha > 0$, i.e., there exist two positive numbers $\alpha > 0$, and $K \geq 1$, such that

$$\|y(t)\| \leq K\|y_0\|e^{-\alpha(t-t_0)}, \quad t \geq t_0. \tag{3}$$

The aim of this paper is to establish the robustly exponential stability criteria for the time-varying linear impulsive system with uncertainty. The following preliminaries are adopted from [1].

Let $\Phi_k(t,s)$ be the fundamental matrix solution (Cauchy Matrix) (see [1]) of the linear system

$$\dot{x}(t) = A(t)x, \qquad t_{k-1} < t < t_k. \tag{4}$$

Then the solution $x(t)$ to system (2), which satisfies the initial condition $x(t_0^+) = x_0$, can be written in the form

$$x(t) = W(t, t_0^+)x_0, \quad t \geq t_0, \tag{5}$$

where $W(t,s)$ is the fundamental matrix solution (Cauchy Matrix) of the linear system (2) with $W(t,t) = I$ given by (see [1])

$$W(t,s) = \begin{cases} \Phi_k(t,s), \qquad \text{for} \quad t,s \in (t_{k-1}, t_k]; \\ \Phi_{k+1}(t,t_k)(I + C_k)\Phi_k(t_k,s), \qquad \text{for} \quad t_{k-1} < s \leq t_k < t \leq t_{k+1}; \\ \Phi_{k+1}(t,t_k)\Pi_{j=k}^{i+1}(I + C_j)\Phi_j(t_j,t_{j+1}) \cdot (I + C_i)\Phi_i(t_i,s), \\ \qquad\qquad\qquad\qquad \text{for} \quad t_{i-1} < s \leq t_i < t_k < t \leq t_{k+1}. \end{cases} \tag{6}$$

**Lemma 2.1** [1] *Assume that $(A_1)$ holds. Suppose that $m \in PC^1[R^+, R], p \in C[R^+, R^+]$ and $m(t)$ is left-continuous at $t_k, k = 1, 2, \cdots$. If for $k = 1, 2, \cdots$,*

$$m(t) \leq C + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0 < t_k < t} \beta_k m(t_k), \quad t \geq t_0, \tag{7}$$

*where $\beta_k \geq 0$, and $C$ are constants, then*

$$m(t) \leq C\Pi_{t_0 < t_k < t}(1 + \beta_k)e^{\int_{t_0}^{t} p(s)ds}, \quad t \geq t_0. \tag{8}$$

## 3 Main Results

In this section, we shall establish the robust exponential stability criteria for system (1).

**Theorem 3.1** *Suppose Assumptions $(A_1)$-$(A_5)$ hold. Then, the system (2) is exponentially stable with decay rate $\alpha > 0$ if and only if there exists a constant $M \geq 1$ such that*

$$\|W(t,s^+)\| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq t_0. \tag{9}$$

***Proof*** *Sufficiency.* Suppose (9) holds. Then, by (5), we get

$$\|x(t)\| \leq \|W(t,t_0^+)\|\|x_0\| \leq M\|x_0\|e^{-\alpha(t-t_0)}, \quad \text{for all} \quad t \geq t_0, \quad x_0 \in R^n. \tag{10}$$

Hence, the system (2) is globally exponentially stable with decay rate $\alpha$.

*Necessity.* If the system (2) is globally exponentially stable with decay rate $\alpha > 0$, then there exists a positive constant $M \geq 1$ such that $\|x(t)\| \leq M\|x_0\|e^{-\alpha(t-t_0)}$ holds. Thus, for any $x_0 \neq 0$, we get

$$Me^{-\alpha(t-t_0)} \geq \frac{\|x(t)\|}{\|x_0\|} = \frac{\|W(t,t_0^+)x_0\|}{\|x_0\|}, \tag{11}$$

which implies that

$$\|W(t,t_0^+)\| = \sup_{x_0\neq 0}\left\{\frac{\|W(t,t_0^+)x_0\|}{\|x_0\|}\right\} \leq Me^{-\alpha(t-t_0)} \tag{12}$$

and hence, by the properties of Cauchy matrix, we have

$$\begin{aligned}\|W(t,s^+)\| &= \|W(t,t_0^+)W^{-1}(s,t_0^+)\| \leq \|W(t,t_0^+)\|\|W^{-1}(s,t_0^+)\| \\ &\leq Me^{-\alpha(t-t_0)} \cdot \{Me^{-\alpha(s-t_0)}\}^{-1} = e^{-\alpha(t-s)} \leq Me^{-\alpha(t-s)}.\end{aligned} \tag{13}$$

The proof is complete. □

**Theorem 3.2** *Assume that Assumption ($A_1$)-($A_5$) hold. Furthermore, suppose that the following conditions hold.*
*(1) For any $k \in N$, the system (4) is exponentially stable with decay rate $\alpha > 0$, i.e., there exists a constant $M \geq 1$ such that*

$$\|\Phi_k(t,s)\| \leq Me^{-\alpha(t-s)}, \quad t_{k-1} < s \leq t \leq t_k, k \in N. \tag{14}$$

*(2) There exist constants $\gamma > 0$, $M_1 \geq 0$, with $0 < \gamma < \min\{\frac{\alpha}{M}, K_1\}$ such that*

$$\int_{t_0}^{t} \|\tilde{A}(s)\|ds \leq \gamma(t-t_0) + M_1, \quad t \geq t_0. \tag{15}$$

*(3) There exists a constant $\beta$ with $0 < \beta < \alpha - M\gamma$ such that*

$$\sup_{\tilde{C}_i\in\Omega_2}\left\{\sum_{i=0}^{k}\ln\left(M\|I + C_i + \tilde{C}_i\|\right)\right\} \leq \beta(t_k - t_0), \quad \textit{for all} \quad k \in N. \tag{16}$$

*Then system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta > 0$.*

***Proof*** Let $y(t) = y(t,t_0,y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$. For $t \in (t_{k-1}, t_k], k \in N$, by the variation of constants formula, we get

$$\begin{aligned} y(t) &= W(t,t_{k-1}^+)y(t_{k-1}^+) + \int_{t_{k-1}}^{t} W(t,s)\tilde{A}(s)y(s)ds \\ &= \Phi_k(t,t_{k-1}^+)y(t_{k-1}^+) + \int_{t_{k-1}}^{t} \Phi_k(t,s)\tilde{A}(s)y(s)ds \end{aligned} \tag{17}$$

Thus, by (14) and (17), for $t \in (t_{k-1}, t_k], k \in N$, we obtain

$$\begin{aligned}\|y(t)\| &\leq \|\Phi_k(t,t_{k-1}^+)\|\|y(t_{k-1}^+)\| + \int_{t_{k-1}}^{t} \|\Phi_k(t,s)\|\|\tilde{A}(s)\|\|y(s)\|ds \\ &\leq Me^{-\alpha(t-t_{k-1})}\|y(t_{k-1}^+)\| + M\int_{t_{k-1}}^{t} e^{-\alpha(t-s)}\|\tilde{A}(s)\|\|y(s)\|ds.\end{aligned} \tag{18}$$

This implies

$$\|y(t)\|e^{\alpha t} \leq Me^{\alpha t_{k-1}}\|y(t_{k-1}^+)\| + M\int_{t_{k-1}}^{t} e^{\alpha s}\|\tilde{A}(s)\|\|y(s)\|ds,\ t\in(t_{k-1},t_k], k\in N. \quad (19)$$

By Gronwall-Bellman inequality, we have

$$\|y(t)\|e^{\alpha t} \leq Me^{\alpha t_{k-1}}\|y(t_{k-1}^+)\|e^{M\int_{t_{k-1}}^{t}\|\tilde{A}(s)\|ds},\ t\in(t_{k-1},t_k], k\in N. \quad (20)$$

Hence, by (20), for $t\in(t_{k-1},t_k], k\in N$, we get

$$\begin{aligned}\|y(t)\| &\leq Me^{-\alpha(t-t_{k-1})}\|y(t_{k-1}^+)\|e^{M\int_{t_{k-1}}^{t}\|\tilde{A}(s)\|ds}\\ &= Me^{-\alpha(t-t_{k-1})}e^{M\int_{t_{k-1}}^{t}\|\tilde{A}(s)\|ds}\|I+C_{k-1}+\tilde{C}_k\|\|y(t_{k-1})\|. \quad (21)\end{aligned}$$

Specially, we have

$$\|y(t_k)\| \leq Me^{-\alpha(t_k-t_{k-1})}e^{M\int_{t_{k-1}}^{t_k}\|\tilde{A}(s)\|ds}\|I+C_{k-1}+\tilde{C}_k\|\|y(t_{k-1})\|. \quad (22)$$

Thus, by (21)-(22) and conditions (2)-(3), for $t\in(t_{k-1},t_k], k\in N$, it follows that

$$\begin{aligned}\|y(t)\| &\leq \left(\Pi_{i=1}^{k-1}M\|I+C_i+\tilde{C}_i\|\right)e^{-\alpha(t-t_0)+M\int_{t_0}^{t}\|\tilde{A}(s)\|ds}\|y_0\|\\ &= e^{-\alpha(t-t_0)+M\int_{t_0}^{t}\|\tilde{A}(s)\|ds+\sum_{i=1}^{k-1}\ln M\|I+C_i+\tilde{C}_i\|}\|y_0\|\\ &\leq e^{-\alpha(t-t_0)+M\gamma(t-t_0)+MM_1+\beta(t_{k-1}-t_0)}\|y_0\|\\ &\leq e^{MM_1}e^{-(\alpha-M\gamma-\beta)(t-t_0)}\|y_0\|. \quad (23)\end{aligned}$$

Hence, the system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta$. The proof is complete. □

**Theorem 3.3** *Assume that Assumptions ($A_1$)-($A_5$) hold and system (2) is exponentially stable with decay rate $\alpha > 0$, i.e., (9) holds. Furthermore, suppose that the condition (2) of Theorem 3.2 holds and the following condition is satisfied.*
*(1*) There exists a constant $\beta$ with $0 < \beta < \alpha - M\gamma$ such that*

$$\sup_{\tilde{C}_i\in\Omega_2}\left\{\sum_{i=0}^{k}\ln\left(1+M\|\tilde{C}_i\|\right)\right\} \leq \beta(t_k-t_0), \quad \textit{for all} \quad k\in N. \quad (24)$$

*Then system (1) is robustly exponentially stable and at least with decay rate $\alpha-M\gamma-\beta > 0$.*

***Proof*** Let $y(t) = y(t,t_0,y_0)$ be the solution of system (1) with initial condition $y(t_0^+) = y_0$. For $t\in(t_{k-1},t_k], k\in N$, by the variation of constants formula for impulsive system (Theorem 2.5.1 in [1]), we get

$$y(t) = W(t,t_0^+)y(t_0^+) + \int_{t_0}^{t}W(t,s)\tilde{A}(s)y(s)ds + \sum_{i=1}^{k-1}W(t,t_i^+)\tilde{C}_i y(t_i). \quad (25)$$

Thus, by (9) and (25), for $t \in (t_{k-1}, t_k], k \in N$, we obtain

$$\begin{aligned}\|y(t)\| &\le \|W(t,t_0^+)\|\|y(t_0^+)\| + \int_{t_0}^t \|W(t,s)\|\|\tilde{A}(s)\|\|y(s)\|ds \\ &+ \sum_{i=1}^{k-1} \|W(t,t_i^+)\|\|\tilde{C}_i\|\|y(t_i)\| \le Me^{-\alpha(t-t_0)}\|y_0\| \\ &+M\int_{t_0}^t e^{-\alpha(t-s)}\|\tilde{A}(s)\|\|y(s)\|ds + M\sum_{i=1}^{k-1} e^{-\alpha(t-t_i)}\|\tilde{C}_i\|\|y(t_i)\|. \end{aligned} \tag{26}$$

This implies that for $t \in (t_{k-1}, t_k], k \in N$,

$$\|y(t)\|e^{\alpha t} \le Me^{\alpha t_0}\|y_0\| + M\int_{t_0}^t e^{\alpha s}\|\tilde{A}(s)\|\|y(s)\|ds + M\sum_{i=1}^{k-1} e^{\alpha t_i}\|\tilde{C}_i\|\|y(t_i)\|. \tag{27}$$

By Lemma 2.1, we have

$$\|y(t)\|e^{\alpha t} \le Me^{\alpha t_0}\Pi_{i=1}^{k-1}(1+M\|\tilde{C}_i\|)\cdot e^{M\int_{t_0}^t \|\tilde{A}(s)\|ds}\|y_0\|,\ t \in (t_{k-1}, t_k], k \in N. \tag{28}$$

Hence, by (28), for $t \in (t_{k-1}, t_k], k \in N$, we get

$$\begin{aligned}\|y(t)\| &\le Me^{-\alpha(t-t_0)}\Pi_{i=1}^{k-1}(1+M\|\tilde{C}_i\|)\cdot e^{M\int_{t_0}^t \|\tilde{A}(s)\|ds}\|y(t_0)\| \\ &\le Me^{-\alpha(t-t_0)+\sum_{i=1}^{k-1}\ln(1+M\|\tilde{C}_i\|)+M\gamma(t-t_0)+MM_1}\|y_0\| \\ &\le Me^{MM_1}e^{-(\alpha-\beta-M\gamma)(t-t_0)}\|y_0\| \end{aligned} \tag{29}$$

Hence, the system (1) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta$. The proof is complete. □

In the following, we specialize the results obtained above to a class of interval linear impulsive systems (see [13-14]). Interval linear impulsive systems can be described as:

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t), & t \in (t_{k-1}, t_k], \\ \Delta x(t) = \tilde{C}_k x(t), & t = t_k, k \in N, \end{cases} \tag{30}$$

where $\tilde{A}, \tilde{C_k} \in R^{n\times n}$ are interval matrices satisfying

$$\tilde{A} \in N[A^{(1)}, A^{(2)}] = \{\tilde{A} = (\tilde{a}_{ij})_{n\times n} : {a_{ij}}^{(1)} \le \tilde{a}_{ij} \le {a_{ij}}^{(2)}\},$$

and

$$\tilde{C}_k \in N[C_k^{(1)}, C_k^{(2)}] = \{\tilde{C}_k = (\tilde{c}_{ij_k})_{n\times n} : {c_{ij_k}}^{(1)} \le \tilde{c}_{ij_k} \le {c_{ij_k}}^{(2)}\}.$$

By [13], an interval matrix $\tilde{X} \in N[X^{(1)}, X^{(2)}]$ can be described as:

$$\tilde{X} = X + E_X\Sigma_X F_X, \tag{31}$$

where $X = \frac{1}{2}(X^{(1)} + X^{(2)})$, $H = (h_{ij})_{n\times n} = \frac{1}{2}(X^{(2)} - X^{(1)})$,
$\Sigma_X \in \Sigma^* = \Big\{\Sigma \in R^{n^2\times n^2} : \Sigma = \mathrm{diag}\{\varepsilon_{11}, \cdots, \varepsilon_{n^2n^2}\}, |\varepsilon_{ij}| \le 1; i, j = 1, 2, \cdots, n.\Big\}$,
$E_X E_X^T = \mathrm{diag}\Big\{\sum_{j=1}^n h_{1j}, \sum_{j=1}^n h_{2j}, \cdots, \sum_{j=1}^n h_{nj}\Big\} \in R^{n\times n}$,
$F_X^T F_X = \mathrm{diag}\Big\{\sum_{j=1}^n h_{j1}, \sum_{j=1}^n h_{j2}, \cdots, \sum_{j=1}^n h_{jn}\Big\} \in R^{n\times n}$.

By (31), we denote $\tilde{A} = A + E_A \Sigma_A F_A$, and $\tilde{C}_k = C_k + E_{C_k} \Sigma_{C_k} F_{C_k}, k \in N$.

Let $J_A$ be the Jordan matrix of $A$ and $PAP^{-1} = J_A$ for some $n \times n$ nonsingular matrix $P$. Denote $M_A(P) = \|P\| \|P^{-1}\|$. Clearly, $M_A(P) \geq 1$.

**Corollary 3.1** *Assume that the following conditions hold.*
*(1) $A$ is a Hurwitz matrix.*
*(2) Let $\alpha = -\max_{1 \leq i \leq n} \{(Re(\lambda_i(A))\}$. Then*

$$\|E_A\| \|F_A\| < \frac{\alpha}{M_A(P)}. \tag{32}$$

*(3) There exists a constant $\beta$ with $0 < \beta < \alpha - M_A(P) \|E_A\| \|F_A\|$ such that*

$$\sum_{i=0}^{k} \ln \left( M_A(P) \|I + C_i\| + M_A(P) \|E_{C_i}\| \|F_{C_i}\| \right) \leq \beta(t_k - t_0), \quad \text{for all} \quad k \in N. \tag{33}$$

*Then system (30) is robustly exponentially stable and at least with decay rate: $\alpha - M_A(P) \|E_A\| \cdot \|F_A\| - \beta$.*

***Proof*** Obviously, for the linear system (30), we have

$$\Phi_k(t, s^+) = e^{A(t-s)}, \quad t_{k-1} < s \leq t \leq t_k, k \in N. \tag{34}$$

Since $A$ is a Hurwitz matrix, we get $\max_{1 \leq i \leq n} \{(Re(\lambda_i(A))\} < 0$ and

$$\|\Phi_k(t, s^+)\| = \|e^{A(t-s)}\| \leq M_A(P) \cdot \|e^{PAP^{-1}(t-s)}\| \leq M_A(P) e^{-\alpha(t-s)}, \tag{35}$$

where $\alpha = -\max_{1 \leq i \leq n} \{(Re(\lambda_i(A))\} > 0$.

The rest of the proof follows as a direct consequence of Theorem 3.2 with $\gamma = \|E_A\| \|F_A\|$, and the inequality

$$\ln \left( M_A(P)(\|I + C_i + E_{C_i} \Sigma_{C_i} F_{C_i}\|) \right) \leq \ln \left( M_A(P) \|I + C_i\| + M_A(P) \|E_{C_i}\| \|F_{C_i}\| \right), \quad i \in N. \tag{36}$$

The proof is thus complete. □

**Corollary 3.2** *For system (2), if $A(t) = A$, where $A$ is a constant matrix, and (2) is exponentially stable with decay rate $\alpha > 0$, i.e., (9) holds. Furthermore, suppose that the following conditions hold.*
*(1)*

$$\|E_A\| \|F_A\| < \frac{\alpha}{M}. \tag{37}$$

*(2) There exists a constant $\beta$ with $0 < \beta < \alpha - M\gamma$ such that*

$$\sum_{i=0}^{k} \ln \left( 1 + M \|E_{C_i}\| \|F_{C_i}\| \right) \leq \beta(t_k - t_0), \quad \text{for all} \quad k \in N. \tag{38}$$

*Then system (30) is robustly exponentially stable and at least with decay rate $\alpha - M\gamma - \beta > 0$.*

***Proof*** By Theorem 3.3, it is easy to show that the results of the corollary are valid. The details are omitted. □

## 4 Examples

In this Section, we shall consider two examples to illustrate the results obtained in Section 3.

**Example 4.1** Consider system (1) in the form of system (30), where $t_0 = 0, t_k = k, k \in N$, and

$$A^{(1)} = \begin{pmatrix} -3.5 & -0.5 \\ 0 & -2.8 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} -2.5 & 0.5 \\ 0 & -1.2 \end{pmatrix},$$

$$C_k^{(1)} = \begin{pmatrix} -2.5 & -0.4 \\ 0 & -2.6 \end{pmatrix}, \quad C_k^{(2)} = \begin{pmatrix} -1.5 & 0.4 \\ 0.2 & -1.4 \end{pmatrix}.$$

Obviously, $A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}, C_k = \begin{pmatrix} -2 & 0 \\ 0.1 & -2 \end{pmatrix}, k \in N.$

Let $P = I$. Then,

$$\alpha = -\max_{1 \le i \le n} \left\{(Re(\lambda_i(A))\right\} = 2 > 0, M_A(P) = 1, \|E_A\| = 1, \|F_A\| = 1.1402,$$

$$\|E_{C_k}\| = 0.9487, \|F_{C_k}\| = 1, k \in N.$$

Let $\beta = 0.6931$. Then, we obtain

$$\beta + M_A(P)\|E_A\|\|F_A\| = 1.8333 < 2 = \alpha,$$

$$\sum_{i=0}^{k} \ln\left(M_A(P)\|I + C_i\| + M_A(P)\|E_{C_i}\|\|F_{C_i}\|\right) = 0.6931 \cdot k \le \beta(t_k - t_0).$$

Hence, by Corollary 3.1, we conclude that the system is robustly global exponential stable and at least with decay rate 0.1667.

**Example 4.2** Consider system (1), where $t_0 = 0, t_k = k, k \in N$, and

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}, \tilde{A}(t) = \begin{pmatrix} \tilde{a}_{11}(t) & 0 \\ 0 & \tilde{a}_{22}(t) \end{pmatrix}, C_k = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \tilde{C}_k = \begin{pmatrix} \tilde{c}_{k_{11}} & 0 \\ 0 & \tilde{c}_{k_{22}} \end{pmatrix}.$$

The uncertainty entries satisfy:

$$|\tilde{a}_{11}(t)| \le |\sin t|, |\tilde{a}_{22}(t)| \le |\cos t|, |\tilde{c}_{k_{11}}| \le 1 + \frac{1}{(1+k)^2}, |\tilde{c}_{k_{22}}| \le 1 + \frac{1}{(1+k)^2}, k \in N.$$

Then, we obtain

$$\|W(t, s^+)\| \le e^{-2(t-s)}$$

and

$$\int_{t_0}^{t} \|\tilde{A}(s)\| ds = \int_0^t \|\tilde{A}(s)\| ds \le t,$$

and hence, $\alpha = -2, M = 1, \gamma = 1, M_1 = 0$.

Moreover,

$$\sum_{i=0}^{k} \ln\left(1 + M\|\tilde{C}_i\|\right) \le \ln(2.25) \cdot k = 0.8109 \cdot (t_k - t_0).$$

Thus, by letting $\beta = 0.8109$, we obtain $0 < \beta + \gamma M < \alpha$.

Hence, by Theorem 3.3, we conclude that the system is robustly global exponential stable and at least with decay rate 0.1891.

## 5 Conclusions

In this paper, by employing the variation of constants formula for impulsive system, we have established some global exponential stability criteria for time-varying linear impulsive system with uncertainties. We have also obtained estimates for decay rates. The criteria obtained are verifiable via solving algebraic inequalities in Matlab environment. Some examples have been worked out to demonstrate the main results.

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# Transformation Synthesis for Euler-Lagrange Systems

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**Abstract:** The transformation of Euler-Lagrange systems, with the variable of position as output, in order to solve some interesting problem as the design of observer is considered in this paper. First, we will provide a necessary and sufficient condition, which ensures the transformation of such system into some structure affine in the velocities, as well as a method to compute this transformation. For a particular family of Euler-lagrange systems with two degree of freedom we will present a change of coordinates which makes the dynamics triangular with respect to the velocities and a globally asymptotically converging observer is provided. To illustrate the approach, it is applied to the Cart-pendulum system.

**Keywords:** *Euler-Lagrange systems; state transformation; affine forms; cart-pendulum.*

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## 1 Introduction

Euler-Lagrange systems with $n$ generalized configuration coordinates $q = (q_1, \dots, q_n)^{\mathrm{T}}$ are described by equations of the form

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q, v)v + V(q) &= \tau, \end{aligned} \tag{1}$$

where $M(q)$ denotes the inertia matrix, while $C(q, v)v$, with $v = \dot{q} = (\dot{q}_1, \dots, \dot{q}_n)^{\mathrm{T}}$ the generalized velocities, denotes the centrifugal and Coriolis forces, $V(q)$ consists of the gravity terms and $\tau$ is the vector of input torques. This celebrated family of systems has been the subject of an important literature over half a century, because the equations of many physical devices belong to this family (see [18], [20], [17], [4] and references therein). When these systems are fully-actuated, they are globally feedback linearizable. But feedback linearization can be performed only when all the variables are measured. Unfortunately in practice, very often the variables of velocity cannot be measured. Therefore, the global output feedback stabilization of these systems with $y = q$ as output is challenging from a practical point of view. But, from a theoretical point of view, it is one of the most difficult problems in the field of nonlinear control: indeed, the matrix $C(q, v)v$ is a nonaffine function of the unmeasured part of the state $v$: this fact precludes from applying most of the classical techniques; for instance, the methods of [16], [15] and [14]. For more explanations on the obstacles due to the presence of terms which are nonaffine with respect to the unmeasured variables, see the introduction of [10].

Recently, in [2], an elegant alternative for one-degree-of-freedom systems was reported. The author presented a reduced order observer which converge exponentially. This observer is based upon a global nonlinear change of coordinates which makes the system affine in the unmeasured part of the state. This is crucial to define a very simple controller to solve the problem of tracking trajectory. So a very natural question arises: which conditions ensure that an Euler-Lagrange systems (1) can be transformed, with the help of a change of coordinates, into some structure affine in the unmeasured part of the state.

This question has been addressed in [2] and [17]. However the questions of existence and computation of the required solution were not answered. In the present paper, we address these question: we show that this problem can be brought back to the resolution of a set of partial differential equation for which an explicit solution is given.

The paper is organized as follows. In Section 4, we present first necessary and sufficient condition which gives to system (1) some structure affine in the unmeasured part of the state. Next we introduce triangular forms. A method of construction of observers is proposed. Section 7 contains concluding remarks.

## 2 Preliminary

In this section we briefly review some results and terminology from Euler-Lagrange dynamics that will be useful in the sequel. The interested reader should consult [12], [13] and [18] for a more detailed discussion.

The dynamics of equations (1) has the following properties [21]:

**Property 2.1** The matrix $M(q) = (M_{ij})_{1 \le i,j \le n}$ is symmetric positive definite for all $q$.

**Property 2.2** The inertia and centripetal-Coriolis matrices satisfy the following relationship

$$\frac{dM(q)}{dt} = C^{\mathrm{T}}(q,v) + C(q,v), \tag{2}$$

where T denotes the transposition and $\dfrac{dM}{dt}$ is a shorthand for $\displaystyle\sum_{i=1}^{n} v_i \frac{\partial M}{\partial q_i}$.

It is also well known [18], that the $(j,k)$-entry of the matrix $C(q,v)$ is given by

$$C_{jk}(q,v) = \sum_{i=1}^{n} C_{ijk}(q) v_i, \tag{3}$$

where

$$C_{ijk}(q) = \frac{1}{2}\left(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j}\right) \tag{4}$$

are the so called Christoffel symbols of the first kind.

Equality (3) shows that we can write the matrix $C(q,v)$ as

$$C(q,v) = \sum_{i=1}^{n} v_i C_i(q), \tag{5}$$

where the entries of matrix $C_i$ are the $C_{ijk}(q)$'s; these matrices satisfy the relation $C_i + C_i^{\mathrm{T}} = \dfrac{\partial M}{\partial q_i}$.

Now, we state the following theorem which is proved in [1] and will be used in the next section.

**Theorem 2.1** *Let $x_1,\dots,x_m$ denote the coordinates of a point $x \in R^m$ and $y_1,\dots,y_n$ the coordinates of a point $y \in R^n$. Let $M^1,\dots,M^m$ be smooth functions*

$$M^i \;\; R^m \to R^{n\times n} \tag{6}$$

*such that*

$$\frac{\partial M^i}{\partial x_k} - \frac{\partial M^k}{\partial x_i} + M^i M^k - M^k M^i = 0. \tag{7}$$

*Consider the set of partial differential equations*

$$\frac{\partial y(x)}{\partial x_i} = M^i(x) y(x), \quad 1 \le i \le m. \tag{8}$$

*Given a point $(x^0, y^0) \in R^m \times R^n$, there exist a neighborhood $U$ of $x^0$ and a unique smooth function $y(x)$ which satisfies (8) and is such that $y(x^0) = y^0$.*

Throughout the paper,

- $\mathrm{M}_n(R)$ denotes the set of $n$-square real matrices;
- $\mathrm{GL}_m(R)$ denotes the set of $n$-square real invertible matrices;
- for $S \in \mathrm{M}_n(R)$ symmetric positive definite $S^{1/2}$ denotes the *square root* of $S$.

## 3 Problem statement

We consider the family of Euler-Lagrange systems described by equations (1) where the output is $q = (q_1, \ldots, q_n)^{\mathrm{T}} \in R^n$, and the input is $\tau = (\tau_1, \ldots, \tau_n)^{\mathrm{T}} \in R^n$. The unmeasured part of the state is $v = (\dot{q}_1, \ldots, \dot{q}_n)^{\mathrm{T}}$.

As pointed out in the introduction the difficulty to stabilize or to construct observers for system (1) mainly stems from the fact that Coriolis and centrifugal forces vector in (1), have a quadratic growth in the generalized velocities $v$, which are not measured. The global change of coordinates introduced in [2] for one-degree-of freedom (i.e. $n = 1$) systems overcomes this problem by rewriting the dynamics with functions which are linear in the unmeasured velocities. As it is discussed in [2], the design procedure might be extended to the case of systems with more degrees of freedom, as soon as the same kind of change of coordinates can be found, that is to say if we can select an invertible matrix $T(q)$ such that

$$\frac{d\,T(q)}{dt} = T(q)M^{-1}(q)C(q,v). \tag{9}$$

**Remark 3.1** We can notice that a more general condition which allows us to rewrite system (1) with an unmeasured part which is linear is the existence of a nonsingular matrix $T$ such that

$$\frac{d\,T(q)}{dt}v = T(q)M^{-1}(q)C(q,v)v. \tag{10}$$

The following example shows that condition (10) is weaker than condition (9). Consider the following inertia matrix $M(q)$

$$M(q) = \begin{pmatrix} e^{-q_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the Christoffel symbols of the first kind [18], matrix $C$ is given by

$$C(q,v) = \frac{1}{2}e^{-q_2}\begin{pmatrix} -v_2 & -v_1 \\ v_1 & 0 \end{pmatrix}$$

and an easy calculation shows that the matrix

$$T(q) = \begin{pmatrix} e^{-q_2} & 0 \\ \frac{1}{2}q_1 e^{-q_2} & 1 \end{pmatrix} \tag{11}$$

satisfies equation (10), but not (9). In fact, equation (9) does not admit any solution (as we will see later).

Necessary geometric conditions, so that (9) admits a solution are given in [6], furthermore necessary conditions in terms of Riemmanien curvature are given in [18].

The main contribution of the paper is to give an algebraic necessary and sufficient condition in terms of the matrix of centrifugal and Coriolis forces, so that (9) admits a solution, and make the relation between it and Riemannain curvature as in [18].

## 4 Main results

### 4.1 Equation $\frac{d\,T(q)}{dt} = T(q)M^{-1}(q)C(q,v)$

This subsection is composed of two parts. In the first part, we propose a necessary and sufficient condition which ensures the existence of a solution of equation (9) as well as methods to compute it (Lemma 4.4).

In the second part, we explain the relation between (9) and Riemannain curvature.

### 4.2 Necessary and sufficient conditions

**Theorem 4.1** *Consider the nonlinear system (1); equation (9) admits a solution if and only if*

$$\frac{\partial C_i}{\partial q_j} - \frac{\partial C_j}{\partial q_i} = C_j^{\mathrm{T}} M^{-1} C_i - C_i^{\mathrm{T}} M^{-1} C_j, \tag{12}$$

*for all* $1 \leq i, j \leq n$ *where the matrices* $C_i$ *are defined by relation (5).*

To establish Theorem 4.1, we need to prove the following preliminary lemma.

**Lemma 4.1** *Let* $M_1(q), \ldots, M_n(q)$ *be matrices in* $\mathrm{M}_m(R)$ *depending smoothly on* $q$ *and consider the set of partial differential equations*

$$\frac{\partial T}{\partial q_i}(q) = T(q) M_i(q), \quad \forall\, i = 1, \ldots, n. \tag{13}$$

*Given any matrix* $T_0 \in \mathrm{GL}_m(R)$ *and* $q_0 \in R^n$*, there exists an unique smooth matrix* $T(q)$ *which satisfies (13) and is such that* $T(q_0) = T_0$ *if and only if the functions* $M_1(q), \ldots, M_n(q)$ *satisfy the conditions*

$$\forall\, i < j \leq n; \quad M_j M_i - M_i M_j = \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j}. \tag{14}$$

***Proof*** *Necessity* Let $T(q)$ be a solution of equations (13), then from the property

$$\frac{\partial^2 T(q)}{\partial q_i \partial q_j} = \frac{\partial^2 T(q)}{\partial q_j \partial q_i} \tag{15}$$

one has

$$\frac{\partial (T(q)\, M_j(q))}{\partial q_i} = \frac{\partial (T(q)\, M_i(q))}{\partial q_j}. \tag{16}$$

Expanding the derivatives on both sides we obtain

$$T(q) \left( M_i(q) M_j(q) + \frac{\partial M_j(q)}{\partial q_i} \right) = T(q) \left( M_j(q) M_i(q) + \frac{\partial M_i(q)}{\partial q_j} \right) \tag{17}$$

which, due to the fact that $T(q)$ is invertible (since $T(q_0) \in \mathrm{GL}_m(R)$), yields the condition (14).

*Sufficiency* The proof of this part of the demonstration can be easily derived from Theorem 2.3 as follows. Let $T_0 \in \mathrm{GL}_m(R)$ and denote by $(\Gamma_0^1, \ldots, \Gamma_0^n)$, $\Gamma_0^i$ the columns of matrix $T_0^{-1}$. Conditions (14) ensure the existence of a family of functions $\Gamma^k$ such that for all $k$ we have

$$\frac{\partial \Gamma^k}{\partial q_i} = -M_i \Gamma^k, \quad \Gamma^k(q_0) = \Gamma_0^k. \tag{18}$$

The matrix $\Gamma$ with columns $\Gamma^1, \ldots, \Gamma^n$ satisfies the equality:

$$\frac{\partial \Gamma}{\partial x_i} = -M_i \Gamma, \quad \Gamma(q_0) = T_0^{-1}. \tag{19}$$

Since $\Gamma(q_0) = T_0^{-1}$ which is non singular, we conclude that there exists a neighborhood $U$ of $q_0$ such that $\Gamma$ is non singular as a solution of (13), then for $T(q)$ we take the matrix $\Gamma^{-1}(q)$. $\square$

The above proof gives a condition of existence, but not a method allowing the construction of the solution; however, the control implementation needs the knowledge of a matrix $T(q)$.

In the sequel we will give another proof of the sufficient part of Lemma 4.4, based on a reasoning by induction which provides an explicit solution of (9). Moreover this solution is defined on the whole domain of definition of the matrices $M_i$ and not only locally.

*Alternative proof of the sufficient part of Lemma 4.4.* By induction on $n$ we show that if (14) holds, then we have the following property denoted by $\mathcal{P}(n)$.

For all $m \geq 1$, there exists an invertible matrix $T(q) \in \mathrm{GL}_m(R)$ such that equations (13) holds.

For $n = 1$: Equation (13) becomes

$$\frac{\partial T(q_1)}{\partial q_1} = T'(q_1) = T(q_1)M(q_1)$$

which admits solutions defined on the whole domain of definition of $M_1 \in \mathrm{M}_m(R)$ and so $\mathcal{P}(1)$ is true.

Assume that $\mathcal{P}(n)$ is true and let $M_1, \dots, M_{n+1} \in \mathrm{M}_m(R)$ be such that

$$M_j M_i - M_i M_j = \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j} \quad \text{for} \quad i, j = 1, \dots, n+1. \tag{21}$$

The induction hypothesis implies that there exists an invertible matrix $T_{q_{n+1}} = T_{q_{n+1}}(q_1, q_2, \dots, q_n)$ such that

$$\frac{\partial T_{q_{n+1}}}{\partial q_i} = T_{q_{n+1}} M_i, \quad i = 1, \dots, n.$$

We will show that there exists a solution of the form $T = \Psi_1(q_{n+1})T_{q_{n+1}}$. First, observe that

$$\frac{\partial T}{\partial q_i} = \Psi_1(q_{n+1})\frac{\partial T_{q_{n+1}}}{\partial q_i} = \Psi_1(q_{n+1})T_{q_{n+1}}M_i = TM_i,$$

for $i = 1, \dots, n$. Moreover $T$ satisfies the $(n+1)$-th equation if and only if

$$\frac{d\Psi_1}{dq_{n+1}}T_{q_{n+1}} + \Psi\frac{\partial T_{q_{n+1}}}{\partial q_{n+1}} = \Psi_1 T_{q_{n+1}} M_{n+1} \tag{22}$$

which is equivalent to

$$\frac{d\Psi_1}{dq_{n+1}} = \Psi_1(T_{q_{n+1}}M_{n+1} - \frac{\partial T_{q_{n+1}}}{\partial q_{n+1}})T_{q_{n+1}}^{-1}. \tag{23}$$

This equation with unknown function $\Psi_1$ depending only on $q_{n+1}$ admits a solution if and only if the term $\left(T_{q_{n+1}}M_{n+1} - \frac{\partial T_{q_{n+1}}}{\partial q_{n+1}}\right)T_{q_{n+1}}^{-1}$ does not depend on $q_1, \dots, q_n$. Now, taking into account that matrix $T_{q_{n+1}}$ satisfies equations (13), a straightforward calculation leads to the following expression for the derivative of this term in respect of $q_i$:

$$T_{q_{n+1}}\left(M_i\,M_{n+1}+\frac{\partial M_{n+1}}{\partial q_i}-\frac{\partial M_i}{\partial q_{n+1}}-M_{n+1}\,M_i\right)T^{-1}_{q_{n+1}}$$

which is zero because matrices $M_i$ satisfy equalities (21). □

**Proof of the main result.**
Since we have

$$\frac{d\,T(q)}{dt}=\sum_{i=1}^{n}\frac{\partial T(q)}{\partial q_i}\,v_i$$

and from the decomposition of the matrix $C(q,v)$ (see equality (5)), equation (9) is equivalent to the set of equations

$$\frac{\partial T(q)}{\partial q_i}=T(q)M_i(q)\quad i=1,\dots,n, \tag{24}$$

where $M_i(q)=M^{-1}(q)C_i(q)$. According to Lemma 4.4, we deduce that a solution of (9) exists if and only if

$$M_j(q)M_i(q)-M_i(q)M_j(q)=\frac{\partial M_j(q)}{\partial q_i}-\frac{\partial M_i(q)}{\partial q_j}.$$

Now,

$$\begin{aligned}\frac{\partial M_j}{\partial q_i}-\frac{\partial M_i}{\partial q_j}&=-M^{-1}(C_i+C_i^{\mathrm T})M^{-1}C_j+M^{-1}\frac{\partial C_j}{\partial q_i}-M^{-1}\frac{\partial C_i}{\partial q_j}+M^{-1}(C_j+C_j^{\mathrm T})M^{-1}C_i\\&=M_jM_i-M_iM_j+M^{-1}\left(\frac{\partial C_j}{\partial q_i}-\frac{\partial C_i}{\partial q_j}-C_i^{\mathrm T}M^{-1}C_j+C_j^{\mathrm T}M^{-1}C_i\right).\end{aligned}$$

It follows that a necessary and sufficient condition for the existence of a solution $T(q)$ of equation (9) is given by

$$\frac{\partial C_i}{\partial q_j}-\frac{\partial C_j}{\partial q_i}=C_j^{\mathrm T}M^{-1}C_i-C_i^{\mathrm T}M^{-1}C_j.$$

this concludes the proof. □

The preceding theorem gives an algebraic characterization of a family of Euler-Lagrange systems which can be transformed, with the help of a change of coordinates into some structure, affine in the unmeasured part of the state $v=\dot q$. The following one gives another characterization for the existence of a solution of equation (9).

**Theorem 4.2** *Consider an Euler-Lagrange system (1). The following conditions are equivalent.*

1. *There exists a matrix $T(q)$ such that (9) holds.*
2. *There exists a matrix $N(q)$ such that $M(q)=N^{\mathrm T}(q)N(q)$ and $N^{\mathrm T}(q)\dfrac{dN(q)}{dt}=C(q,v)$.*
3. *There exists a function $\Theta(q)\;R^n\to R^n$ and $N(q)$ nonsingular such that $M(q)=N^{\mathrm T}(q)N(q)$ and the Jacobian matrix of $\Theta$ is equal to $N(q)$.*

***Proof*** $1 \Rightarrow 2$

Suppose that (9) admits a solution; the computation of $\dfrac{d\overline{M}}{dt}$, where $\overline{M} = \left(T^{\mathrm{T}}\right)^{-1} M T^{-1}$, gives

$$\begin{aligned}\frac{d\overline{M}}{dt} &= -(T^{\mathrm{T}})^{-1}(C^{\mathrm{T}}M^{-1}T^{\mathrm{T}})(T^{\mathrm{T}})^{-1}M(T^{\mathrm{T}})^{-1} - (T^{\mathrm{T}})^{-1}MT^{-1}(TM^{-1}C)T^{-1} \\ &\quad + (T^{\mathrm{T}})^{-1}\frac{dM}{dt}T^{-1} = -(T^{\mathrm{T}})^{-1}(C^{\mathrm{T}} + C - \frac{dM}{dt})T^{-1} = 0\end{aligned}$$

because $C^{\mathrm{T}} + C = \dfrac{dM}{dt}$. So, $\overline{M} = \left(T^{\mathrm{T}}\right)^{-1} M T^{-1}$ is a constant symmetric positive definite matrix. Letting $N = \overline{M}^{\frac{1}{2}}T$, one can check easily that $M(q) = N^{\mathrm{T}}(q)N(q)$ and $N(q)^{\mathrm{T}}\dfrac{d}{dt}N(q) = C(q,v)$.

$2 \Rightarrow 1$ Suppose that conditions (2) are satisfied then $N(q)$ is nonsingular and an easy computation shows that this matrix is a solution of equation (9).

$2 \Rightarrow 3$ Let us denote the columns of matrix $N$ by $N^i$; $N(q)$ is the Jacobian matrix of a function $\Theta$ if and only if

$$\frac{\partial N^i}{\partial q_j} = \frac{\partial N^j}{\partial q_i}. \tag{25}$$

Now the equality $N^{\mathrm{T}}\dfrac{dN}{dt} = C$ is equivalent to

$$\frac{\partial N^i}{\partial q_j} = N^{\mathrm{T}}C^i_j, \quad i, j = 1, \ldots, n,$$

where $C^i_j$ denotes the $i$-th column of $C_j$. But from formula (3), we know that $C^j_i = C^i_j$; this proves formula (25).

$3 \Rightarrow 2$ Denoting by $N_{ij}$ the entries of matrix $N(q)$, conditions (3) imply that

$$\frac{\partial N_{ij}}{\partial q_k} = \frac{\partial N_{ik}}{\partial q_j}$$

for all triple $(i,j,k)$. From (4) and taking into account that $M(q) = N(q)^{\mathrm{T}}N(q)$, we have

$$\begin{aligned}2C_{ijk} &= \frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j} = \sum_{s=1}^{n}\left(\frac{\partial N_{sj}}{\partial q_i}N_{sk} + N_{sj}\frac{\partial N_{sk}}{\partial q_i}\right) \\ &\quad + \sum_{s=1}^{n}\left(\frac{\partial N_{sj}}{\partial q_k}N_{si} + N_{sj}\frac{\partial N_{si}}{\partial q_k}\right) - \sum_{s=1}^{n}\left(\frac{\partial N_{si}}{\partial q_j}N_{sk} + N_{si}\frac{\partial N_{sk}}{\partial q_j}\right) \\ &= \sum_{s=1}^{n}\left(N_{sj}\frac{\partial N_{sk}}{\partial q_i} + N_{sj}\frac{\partial N_{si}}{\partial q_k}\right) = 2\left(N^{\mathrm{T}}\frac{\partial N}{\partial q_i}\right)_{jk}\end{aligned}$$

so we have

$$C_i = N^{\mathrm{T}}\frac{\partial N}{\partial q_i}$$

which is equivalent to

$$C(q) = N^{\mathrm{T}}(q)\frac{d\,N(q)}{dt}.$$

□

### 4.3 The Riemmanian curvature

Now suppose that the conditions of theorem 4.5 are fulfilled, then there exists a function $\Theta\ R^n \to R^n$ such that, denoting by $N(q)$ the Jacobian matrix of $\Theta$,

$$M(q) = N^{\mathrm{T}}(q)N(q). \tag{26}$$

In terms of the new variables $Q = \Theta(q),\ V = N(q)v$ the Lagrangian dynamics equations (1) can be shown to reduce to

$$\dot{Q} = V, \tag{27}$$
$$\dot{V} = \dot{N}v + N\dot{v} = (N^{\mathrm{T}})^{-1}(\tau - V(q)). \tag{28}$$

Thus a double integrator model in terms of $Q$ is achieved by the much simpler inner loop feedback control law

$$\tau - V(q) = N^{\mathrm{T}}(q)\nu. \tag{29}$$

The point is that, in the new coordinates, the computation of the Coriolis and centrifugal terms in the inner loop is avoided. However, a necessary and sufficient condition for existence of the factorization (26) is that the Riemannian curvature of the metric defined by the robot inertia matrix be zero [5, 18]. More precisely we have the following theorem which summarizes our result and the result of papers [5, 18].

**Theorem 4.3** *Consider an Euler-Lagrange system (1). The following conditions are equivalent:*

1. *There exists a matrix $T(q)$ such that (9) holds.*
2. *The Riemmanian Curvature Tensor defined by*

$$\begin{aligned} R_{ijkl} = {} & \frac{\partial^2 M_{ik}(q)}{\partial q_l \partial q_j} + \frac{\partial^2 M_{jl}(q)}{\partial q_k \partial q_i} - \frac{\partial^2 M_{il}(q)}{\partial q_k \partial q_j} - \frac{\partial^2 M_{jk}(q)}{\partial q_l \partial q_i} \\ & + \frac{1}{2}\sum_{r,s=1}^{n} M_{r,s}^{-1}(q)[C_{rjl}C_{sik} - C_{ril}C_{sjk}] \end{aligned} \tag{30}$$

*are identically zero, where $M_{r,s}^{-1}(q)$ are the components of the inverse $M^{-1}(q)$ of the inertia matrix $M(q)$ and $C_{rjl}$ are the Christoffel symbols of the first kind defined by (3).*

### 4.4 Example: The cart pendulum system

As an example, we will consider the inverted pendulum. The Euler-Lagrange equations write:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = \tau_1, \\ ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mlg\sin\theta = 0, \end{aligned} \tag{31}$$

where $M$ and $x$ denote the mass and the position of the cart (which is moving horizontally), $m$, $l$ and $\theta$ denote the mass, the length and the angular derivation from the upward vertical position of the pendulum which is pivoting around a point fixed on the cart. We denote the state vector $(x, \theta, \dot{x}, \dot{\theta})^{\mathrm{T}}$ as $(q_1, q_2, v_1, v_2)^{\mathrm{T}}$. The output is $y = (q_1, q_2)^{\mathrm{T}}$.

The inertia matrix is

$$M(q) = \begin{pmatrix} a_1 & a_2 \cos q_2 \\ a_2 \cos q_2 & a_3 \end{pmatrix}$$

with $a_1 = M + m,\ a_2 = ml$ and $a_3 = ml^2$.

Using the Christoffel symbols, we obtain

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & -a_2 \sin(q_2) \\ 0 & 0 \end{pmatrix}.$$

One can check easily that condition (14) are verified so according to Theorem 4.3, equation (9) admits a solution that we will make explicit by using the method explained in the proof of Lemma 4.4.

First, if we denote by $T_{q_2}$ the 2-dimensional identity matrix, $T_{q_2}$ is obviously a solution of the differential equation

$$\frac{dT_{q_2}}{dq_1}(q_1) = T_{q_2} M^{-1} C1.$$

so we can find a solution of equations (9) under the form $\Psi(q_2)$, a 2-dimensional square matrix solution of the equation

$$\frac{d\Psi(q_2)}{dq_2} = \Psi(q_2) M^{-1} C_2. \tag{32}$$

An easy calculations shows that the solution of equation (32) with initial condition $\Psi(0) =$ the identity matrix is

$$\Psi(q_2) = \begin{pmatrix} 1 & \dfrac{a_2\beta(0)\cos(q_2) - a_2\beta(q_2)}{a_1\beta(0)} \\ 0 & \dfrac{\beta(q_2)}{\beta(0)} \end{pmatrix}, \tag{33}$$

where $\beta(q_2) = \sqrt{a_1 a_3 - a_2^2 \cos{(q_2)}^2}$.

Moreover the diffeomorphism $\Theta = (\Theta_1,\ \Theta_2)^{\mathrm{T}}$ defined by

$$\Theta_1 = q_1 + \int_0^{q_2} \frac{a_2\beta(0)\cos(s) - a_2\beta(s)}{a_1\beta(0)}\, ds,$$

$$\Theta_2 = \int_0^{q_2} \frac{\beta(s)}{\beta(0)}\, ds,$$

is such that Jacobian $(\Theta) = \Psi(q_2)$.

The following change of coordinates:

$$\Theta_1 = q_1 + \int_0^{q_2} \frac{a_2\beta(0)\cos(s) - a_2\beta(s)}{a_1\beta(0)}\, ds,$$
$$\Theta_2 = \int_0^{q_2} \frac{\beta(s)}{\beta(0)}\, ds,$$
$$z_1 = v_1 + \frac{a_2\beta(0)\cos(q_2) - a_2\beta(q_2)}{a_1\beta(0)} v_2,$$
$$z_2 = \frac{\beta(q_2)}{\beta(0)} v_2$$

transforms the dynamics of the Cart-Pendulum into a double integrator

$$\dot{\Theta} = p, \tag{34}$$
$$\dot{z} = T(q)M^{-1}(q)(\tau - V(q)) = u. \tag{35}$$

where $\tau = (\tau_1, 0)^{\mathrm{T}}$ and $V(q) = (0, -mlg\sin q_2)^{\mathrm{T}}$.

Clearly this system is linear in the unmeasured part of the state and an exponentially converging observer can be constructed.

## 5 Discussion about equation (10)

Let us consider the problem of finding $T$ such that (10) is satisfied.

Observe first that the matrix $M^{-1}C(q, v)v$ is quadratic in $v$ with coefficients depending only on $q$, i.e. there exists $R_i$ such that

$$M^{-1}(q)C(q, v)v = \sum_{i=1}^{n} v_i R_i v. \tag{36}$$

The matrices $R_i$ are not uniquely determined.

In the case of one degree of freedom, a solution of (10) always exists [2]. In the case of higher order system, equation (9) can admit no solution while equation (10) admits one: see the example of Remark 3.1. In this example, observe that there is no function $\Theta(q)$ such that the Jacobian matrix of $\Theta$ is equal to $T(q)$ and thus the Riemmanian curvature Tensor are not identically zero.

The following theorem gives a necessary and sufficient condition for equation (10) has a solution.

**Theorem 5.1** *Consider the Euler-Lagrange system (1), equation (10) admits a solution if and only if there exist matrices $R_i$ satisfying equality (36) and such that*

$$R_j R_i - R_i R_j = \frac{\partial R_j}{\partial q_i} - \frac{\partial R_i}{\partial q_j} \tag{37}$$

*for* $i, j = 1, \ldots, n$.

***Proof*** Let us suppose that there exist a family of matrices $R_k(q),\ k = 1\dots,n$, such that equality (36) holds. Condition (37) ensures the existence of an invertible matrix $T(q)$ such that

$$\frac{\partial T}{\partial q} = TR_k, \tag{38}$$

moreover a method of construction of solution is given by Lemma 4.4. So we get

$$\dot{T}(q)v = \sum_{i=1}^{n} v_i \frac{\partial T}{\partial q_k} v = \sum_{i=1}^{n} v_i T R_i v = T(q)M^{-1}(q)C(q,v)v.$$

For the necessary part assume that there exists a matrix $T(q)$ such that (10) and (36) hold. It follows that

$$\sum_{k=1}^{n} v_k T^{-1}(q)\frac{\partial T}{\partial q_k} v = M^{-1}C(q,v)v.$$

Let $R_k = T^{-1}(q)\dfrac{\partial T}{\partial q_k}$, it follows that

$$\begin{aligned}
\frac{\partial R_j}{\partial q_i} - \frac{\partial R_i}{\partial q_j} &= \frac{\partial}{\partial q_i}\left(T^{-1}\frac{\partial T}{\partial q_j}\right) - \frac{\partial}{\partial q_j}\left(T^{-1}\frac{\partial T}{\partial q_i}\right) \\
&= -T^{-1}\frac{\partial T}{\partial q_i}T^{-1}\frac{\partial T}{\partial q_j} + T^{-1}\frac{\partial^2 T}{\partial q_i \partial q_j} + T^{-1}\frac{\partial T}{\partial q_j}T^{-1}\frac{\partial T}{\partial q_i} - T^{-1}\frac{\partial^2 T}{\partial q_j \partial q_i} \\
&= R_j R_i - R_i R_j,
\end{aligned}$$

which proves the result. □

## 6 Triangular form for a particular family of Euler-Lagrange systems

It is now well-known that under certain conditions, we can carry out the transformation of a system, by a diffeomorphism into a state affine system in the velocity and carry out the synthesis of an observer. In the same way, we know that the necessary and sufficient conditions under which a system is transformable are very restrictive.

For that, we propose the triangular form in the unmeasured part of the state $\dot{q} = v$ for the analysis of observability. We will consider a particular family of Euler-lagrange systems, and we show that it can be transformed into some triangular structure for which an almost exponentially converging observer is given.

### 6.1 A family of Euler-Lagrange systems

In this section, we restrict ourselves to a particular family of Euler-Lagrange systems. We consider systems having two degrees of freedom and which satisfy the following properties.

**Property 6.1** The inertia matrix depend only on the variable $q_2$, this allows us to introduce the following notations:

$$M(q_2) = \begin{pmatrix} M_{11}(q_2) & M_{12}(q_2) \\ M_{12}(q_2) & M_{22}(q_2) \end{pmatrix}.$$

**Property 6.2** There exist three positive constants $m_1$, $m_2$ and $K$ such that for all $q$

$$m_1 I_2 \leq M(q) \leq m_2 I_2, \tag{39}$$
$$\|C(q,v)\| \leq K\|v\|, \tag{40}$$

where $I_2$ denotes the 2-dimensional identity matrix.

**Property 6.3** The function $\tau_1 - V_1$ is bounded in norm.

These properties are satisfied by many Euler-Lagrange systems with two degree of freedom: e.g. the cart-pole system [6, 19], the manipulator system [9]. In the problem under consideration, matrices $C_1$ and $C_2$ write

$$C_1(q_2) = \begin{pmatrix} 0 & \dfrac{1}{2}M'_{11}(q_2) \\ -\dfrac{1}{2}M'_{11}(q_2) & 0 \end{pmatrix}, \quad C_2(q_2) = \begin{pmatrix} \dfrac{1}{2}M'_{11}(q_2) & M'_{12}(q_2) \\ 0 & \dfrac{1}{2}M'_{22}(q_2) \end{pmatrix}$$

(the $'$ denotes the derivative). So, according to Theorem 4.3, equation (9) admits a solution iff

$$\frac{\partial C_1}{\partial q_2} - \frac{\partial C_2}{\partial q_1} = C_2^{\mathrm{T}} M^{-1} C_1 - C_1^{\mathrm{T}} M^{-1} C_2,$$

which is equivalent to

$$M''_{11}(q_2) = \frac{M'_{11}(q_2)\Delta'(q_2)}{2\Delta(q_2)}, \tag{41}$$

where $\Delta = M_{11}(q_2)M_{22}(q_2) - M_{12}(q_2)^2$, which is positive since matrix $M$ is positive definite.

An example of systems satisfying equation (41) is, for instance, the cart-pendulum system [6] and the tora system [20]. But, other systems such that the manipulator or the two links manipulator do not satisfy this conditions. In spite of this, we will show that, this class of systems can be turned with the help of an appropriate change of coordinates into some triangular form near to feedforward form.

More precisely, we have the following result.

**Proposition 6.1** *Under properties 2.1–6.3, the map*

$$\Phi\colon (q_1, v_1, q_2, v_2) \to (x_1, x_2, x_3, x_4)$$

*defined by*

$$\begin{aligned} x_1 &= q_1 + \int_0^{q_2} \frac{M_{12}(s)}{M_{11}(s)}\, ds, \\ x_2 &= M_{11}(q_2)v_1 + M_{12}(q_2)v_2, \\ x_3 &= q_2, \\ x_4 &= \alpha(q_2)v_2, \end{aligned}$$

*where*

$$\alpha(q_2) = \sqrt{\frac{\Delta(q_2)}{M_{11}(q_2)}}\,,$$

*defines a global change of coordinates which transforms system (1) into*

$$
\begin{aligned}
\dot{x}_1 &= \frac{x_2}{M_{11}(x_3)},\\
\dot{x}_2 &= u_1,\\
\dot{x}_3 &= \frac{x_4}{\alpha(x_3)},\\
\dot{x}_4 &= \frac{1}{\alpha(x_3)}\left(\frac{M_{11}'(x_3)}{2M_{11}^2(x_3)}x_2^2 + u_2\right),\\
Y &= (x_1, x_3)^{\mathrm{T}}.
\end{aligned}
\tag{42}
$$

*where* $u_1 = \tau_1 - V_1$ *and* $u_2 = \tau_2 - V_2 - \dfrac{M_{12}}{M_{11}}u_1$.

***Proof*** The proposed transformation is obviously one-to-one and onto, moreover its jacobian matrix is equal to

$$
\begin{pmatrix}
1 & 0 & \dfrac{M_{12}}{M_{11}} & 0\\
0 & M_{11} & M_{11}'v_1 + M_{12}'v_2 & M_{12}\\
0 & 0 & 1 & 0\\
0 & 0 & \alpha' v_2 & \alpha
\end{pmatrix}
$$

and we can see that this transformation is a global diffeomorphism.

On the other hand equations for $\dot{x}_1$ and $\dot{x}_3$ are obvious. One can determine the expression of $\dot{x}_2$ as follows; from

$$M(q_2)\dot{v} = -C(q_2, v)v + \tau - V$$

we have

$$M_{11}(q_2)\dot{v}_1 + M_{12}(q_2)\dot{v}_2 = -M_{11}'(q_2)\,v_1 v_2 - M_{12}'(q_2)\,v_2^2$$

and so

$$\dot{x}_2 = M_{11}(q_2)\dot{v}_1 + M_{12}(q_2)\dot{v}_2 + M_{11}'(q_2)\,v_1v_2 + M_{12}'(q_2)\,v_2^2 = \tau_1 - V_1 = u_1.$$

We will now compute the expression of $\dot{x_4}$. From

$$\dot{v} = -M(q_2)^{-1}(C(q_2, v) + \tau - V)$$

we have

$$
\begin{aligned}
\Delta(q_2)\dot{v}_2 &= \frac{1}{2}M_{11}M_{11}'\,v_1^2 + M_{12}M_{11}'\,v_1v_2 + \big(M_{12}M_{12}' - \frac{1}{2}M_{11}M_{22}'\big)v_2^2\\
&\quad - M_{12}(\tau_1 - V_1) + M_{11}(\tau_2 - V_2)\\
&= \frac{M_{11}'}{2M_{11}}\big(M_{11}^2\,v_1^2 + 2M_{1_2}M_{11}\,v_1v_2 + M_{12}^2\,v_2^2\big)\\
&\quad + \frac{2M_{11}M_{12}M_{12}' - M_{11}^2M_{12}' - M_{12}^2M_{11}'}{2M_{11}}\,v_2^2 - M_{12}(\tau_1 - V_1) + M_{11}(\tau_2 - V_2)\\
&= \frac{M_{11}'}{2M_{11}}\,x_2^2 + \frac{2M_{11}M_{12}M_{12}' - M_{11}^2M_{12}' - M_{12}^2M_{11}'}{2M_{11}\alpha^2}\,x_4^2 - M_{12}(\tau_1 - V_1)\\
&\quad + M_{11}(\tau_2 - V_2).
\end{aligned}
$$

Now

$$\dot{x}_4 = \alpha'(q_2)\, v_2^2 + \alpha(q_2)\, \dot{v}_2$$

and, taking into account that

$$\alpha' = \frac{1}{2\alpha} \frac{M_{11}^2 M_{22}' - 2M_{11}M_{12}M_{12}' + M_{12}^2 M_{11}'}{M_{11}^2},$$

we get the formula stated in the above proposition. □

### 6.2 Construction of observers

Let us point out some particular interests of the system exhibited in Proposition 6.5. System (42) is triangular with respect to the unmeasured part of the state. The difficulty in designing an observer for the above system lies in the presence of the nonlinearity $\dfrac{M_{11}'(x_3)}{2M_{11}^2(x_3)} x_2^2$, which depends on the unmeasured part of the state $x_2$. Moreover, due to the presence of term $x_2^2$ in the dynamics of $x_4$, hypothesis $[H2']$ of paper [3] is not satisfied. This fact precludes from applying the techniques of [3] to construct an observer.

However, from the $\dot{x}_1, \dot{x}_2$–equations in (42) we can see that the unmeasured state $x_4$ not appears in the derivative $\dot{x}_1, \dot{x}_2$.

Therefore, we can obtain the information about $x_2^2$ from the $x_1, x_2$–subsystem.

Consider $x_1, x_2$–subsystem constituted by the two first equations of system (42)

$$\begin{aligned} \dot{x}_1 &= \frac{x_2}{M_{11}(x_3)}, \\ \dot{x}_2 &= u_1, \\ Y_1 &= x_1. \end{aligned} \tag{43}$$

This subsystem does not depend on $x_4$. Moreover it is linear with respect to the unmeasured variable $x_2$. In fact it can be considered as a linear system with a time-varying coefficient $\dfrac{1}{M(x_3)}$. Consequently, one can easily determine a globally exponentially converging observer. More precisely we have,

**Proposition 6.2** *The auxiliary dynamical system*

$$\begin{aligned} \dot{\widehat{x}}_1 &= \frac{1}{M_{11}(x_3)} \left( \widehat{x}_2 + k_1(\widehat{x}_1 - x_1) \right), \\ \widehat{x}_2 &= \frac{1}{M_{11}(x_3)} k_2(\widehat{x}_1 - x_1) + u_1, \end{aligned} \tag{44}$$

*is a globally exponentially converging observer for system (43), provided that the parameters $k_1$ and $k_2$ are negative.*

***Proof*** Let $(\varepsilon_1,\ \varepsilon_2) = (\widehat{x}_1 - x_1,\ \widehat{x}_2 - x_2)$. The error equation is

$$\begin{aligned} \dot{\varepsilon}_1 &= \frac{1}{M_{11}(x_3)} (\varepsilon_2 - k_1\varepsilon_1), \\ \dot{\varepsilon}_2 &= \frac{1}{M_{11}(x_3)} k_2\varepsilon_1, \end{aligned} \tag{45}$$

or, in more compact form,

$$\begin{pmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\end{pmatrix} = \frac{1}{M_{11}(x_3)}\begin{pmatrix}k_1 & 1\\ k_2 & 0\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}.$$

The matrix in the system above is Hurwitz since $k_1$ and $k_2$ are negative. Moreover the inequality (39) in Assumption 6.3 implies the existence of two constants $\kappa_1$ and $\kappa_2$ such that

$$0 < \kappa_1 \le M_{11}(x_3) \le \kappa_2.$$

Consequently, we can find a positive definite quadratic Lyapunov function $V(\varepsilon_1, \varepsilon_2)$ whose derivative along the trajectories of system (45) satisfies

$$\dot{V} = -\frac{1}{M_{11}(x_3)}\, W(\varepsilon_1,\varepsilon_2) \le -\frac{1}{\kappa_2}\, W(\varepsilon_1,\varepsilon_2), \tag{46}$$

where $W(\varepsilon_1, \varepsilon_2)$ is a quadratic positive definite function. this implies that the system (44) is an exponential observer for the system (43). □

We are ready to give an observer for the system (42).

**Proposition 6.3** *Consider the following auxiliary dynamical system:*

$$\begin{aligned}
\dot{\widehat{x}}_3 &= \frac{\widehat{x}_4}{\alpha(x_4)} + k_3(\widehat{x}_1 - x_1) + \frac{k_4}{\alpha(x_3}(\widehat{x}_3 - x_3),\\
\dot{\widehat{x}}_4 &= \frac{1}{\alpha(x_3)}\left(\frac{M'_{11}(x_3)}{2M_{11}^2(x_3)}\widehat{x}_2^2 + u_2 + k_6(\widehat{x}_3 - x_3)\right) + k_5(\widehat{x}_1 - x_1),\\
\dot{\widehat{x}}_1 &= \frac{1}{M_{11}(x_3)}\left(\widehat{x}_2 + k_1(\widehat{x}_1 - x_1)\right),\\
\widehat{x}_2 &= \frac{1}{M_{11}(x_3)}k_2(\widehat{x}_1 - x_1) + u_1,
\end{aligned} \tag{47}$$

*where the parameters $k_1, k_2, k_4$ and $k_6$ are chosen negative. Under the Assumptions 6.2–6.4, system (47) is a globally converging observer for system (42).*

***Proof*** Let $\varepsilon_3 = \widehat{x}_3 - x_3$ and $\varepsilon_4 = \widehat{x}_4 - x_4$. The error equation writes:

$$\begin{aligned}
\dot{\varepsilon_3} &= \frac{\varepsilon_4}{\alpha(x_4)} + k_3\varepsilon_1 + \frac{k_4}{\alpha(x_3}\varepsilon_3,\\
\dot{x}_4 &= \frac{1}{\alpha(x_3)}\left(\frac{M'_{11}(x_3)}{2M_{11}^2(x_3)}(\widehat{x}_2^2 - x^2) + k_6\varepsilon_3\right) + k_6\varepsilon_1,\\
\dot{\varepsilon}_1 &= \frac{1}{M_{11}(x_3)}(\varepsilon_2 - k_1\varepsilon_1),\\
\dot{\varepsilon}_2 &= \frac{1}{M_{11}(x_3)}k_2\varepsilon_1,
\end{aligned} \tag{48}$$

From inequalities (39), (40) in property 6.3 and the positive definiteness of the inertia matrix, we can show easily that there exist $\alpha_1, \alpha_2$ and $c > 0$ such that

$$\alpha_1 \le \alpha(x_3) \le \alpha_2, \quad \left|\frac{m'_{11}(x_3)}{2k_2 m_{11}(x_3)^2}\right| \le c.$$

Moreover since $k_4$ and $k_6$ are negative, one can determine a positive definite quadratic function $Q(\varepsilon_3, \varepsilon_4)$ such that it's derivative along the trajectories of (48) satisfies

$$\begin{aligned} \dot{Q} &\le -\varepsilon_3^2 - \varepsilon_4^2 + c(|\varepsilon_3| + |\varepsilon_4|)(|\varepsilon_1| + \widehat{x}_2^2 - x_2^2) \\ &\le -\varepsilon_3^2 - \varepsilon_4^2 + c(|\varepsilon_3| + |\varepsilon_4|)(|\varepsilon_1| + |\varepsilon_2||\varepsilon_2 + 2x_2|) \\ &\le -\frac{1}{2}\varepsilon_3^2 - \frac{1}{2}\varepsilon_4^2 + 2c^2(|\varepsilon_1| + |\varepsilon_2||\varepsilon_2 + 2x_2|)^2. \end{aligned} \tag{49}$$

Now, property 6.4 ensures that $\dot{x}_2$ is bounded and (46) holds. It follows that there exist three constants $a, k, \beta$ such that for all $t \ge 0$,

$$\begin{aligned} |\varepsilon_1(t)| &\le k(|\varepsilon_1(0)| + |\varepsilon_2(0)|)e^{-\beta t}, \\ |\varepsilon_2(t)| &\le k(|\varepsilon_1(0)| + |\varepsilon_2(0)|)e^{-\beta t}, \\ |x_2(t)| &\le |x_2(0)| + at. \end{aligned} \tag{50}$$

It follows readily that there exists two constants $K_1, K_2$ which depends on $\varepsilon_1(0)$, $\varepsilon_2(0)$ and $x_2(0)$ such that

$$\dot{Q} \le -K_1 Q(\varepsilon_3, \varepsilon_4) + K_2 e^{-\frac{\beta}{2}t}, \tag{51}$$

which implies

$$Q(\varepsilon_3(t), \varepsilon_4(t)) \le -K_1 \int_0^t Q(\varepsilon_3(s), \varepsilon_4(s))ds + K_3 + Q(\varepsilon_3(0), \varepsilon_4(0)) \tag{52}$$

with $K_3 > 0$. It follows from Gronwall's Lemma that

$$Q(\varepsilon_3(t), \varepsilon_4(t)) \le (K_3 + Q(\varepsilon_3(0), \varepsilon_4(0)))\ e^{-K_1 t}. \tag{53}$$

This concludes the proof. □

### 6.3 Example

Consider the two-link manipulator studied in [4, 11]. The equations of motion are given by

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q,\ v)v + V(q) &= \tau, \end{aligned} \tag{54}$$

with $q = (q_1,\ q_2)^{\mathrm{T}}$, $\tau = (\tau_1,\ \tau_2)^{\mathrm{T}}$,

$$\begin{aligned} M(q) &= \begin{pmatrix} p_1 + 2p_3\cos q_2 & p_2 + p_3\cos q_2 \\ p_2 + p_3\cos q_2 & p_2 \end{pmatrix}, \\ C(q,\ v) &= \begin{pmatrix} -v_2 p_3 \sin q_2 & -(v_1 + v_2)p_3 \sin q_2 \\ v_1 p_3 \sin q_2 & 0 \end{pmatrix}, \end{aligned}$$

$V(q) = 0$ and $p_1 = 3.473$, $p_2 = 0.193$, $p_3 = 0.242$.

Easy calculations show that

$$M_{11}''(q_2) - \frac{M_{11}'(q_2)\Delta'(q_2)}{2\Delta(q_2)} = 2p_3\left(-1 - \frac{2p_3^2\sin^2 q_2}{-p_1p_2 + p_2^2 + p_3^2\cos^2 q_2}\right)\cos q_2$$

which is non zero. It yields that equation (9) does not admits any solution. But one can check readily that this fully-actuated system satisfies Assumptions 2.1–6.3. Thanks to Proposition 6.5 the change of coordinates

$$
\begin{aligned}
x_1 &= q_1 + \frac{p_2 + p_3\cos(s)}{p_1 + 2p_3\cos(s)}\,ds = q_1 + \frac{p_1 - 2p_2}{\sqrt{4p_3^2 - p_1^2}}\operatorname{arctanh}\left(\frac{2p_3 - p_1\tan(\frac{q_2}{2})}{\sqrt{4p_3^2 - p_1^2}}\right),\\
x_2 &= (p_1 + 2p_3\cos q_2)v_1 + (p_2 + p_3\cos q_2)\ v_2,\\
x_3 &= q_2,\\
x_4 &= \alpha(q_2)v_2
\end{aligned}
$$

with

$$
\alpha(q_2) = \sqrt{\frac{p_1p_2 - p_2^2 - p_3^2\cos^2(q_2)}{p_1 + 2p_3\cos(q_2)}}
$$

transforms (54) into

$$
\begin{aligned}
\dot{x}_1 &= \frac{x_2}{p_1 + 2p_3\cos q_2},\\
\dot{x}_2 &= u_1,\\
\dot{x}_3 &= \frac{x_4}{\alpha(x_3)},\\
\dot{x}_4 &= \frac{1}{\alpha(x_3)}\left(\frac{-p_3\sin x_3}{2(p_1 + 2p_3\cos x_3)^2}\,x_2^2 + u_2\right),\\
Y &= (x_1, x_3)^{\mathrm{T}},
\end{aligned}
\tag{55}
$$

where

$$
u_1 = (p_1 + 2p_3\cos q_2)\tau_1, \quad u_2 = \tau_2 - \frac{p_2 + p_3\cos q_2}{p_1 + 2p_3\cos q_2}\tau_1.
$$

According to Proposition 6.8, the following system

$$
\begin{aligned}
\dot{\widehat{x}}_1 &= \frac{\widehat{x}_2}{p_1 + 2p_3\cos q_2} + \frac{k_1}{p_1 + 2p_3\cos q_2}(\widehat{x}_1 - x_1),\\
\dot{\widehat{x}}_2 &= \tau_1 + \frac{k_2}{p_1 + 2p_3\cos q_2}(\widehat{x}_{-}x_1),\\
\dot{\widehat{x}}_3 &= \frac{\widehat{x}_4}{\alpha(x_3)} + k_3(\widehat{x}_1 - x_1) + k_4(\widehat{x}_3 - x_3),\\
\dot{\widehat{x}}_4 &= \frac{1}{\alpha(x_3)}\left(\frac{-p_3\sin x_3}{2(p_1 + 2p_3\cos x_3)^2}\widehat{x}_2^2 + u_2\right) + k_5(\widehat{x}_1 - x_1) + \frac{k_6}{\alpha(x_3)}(\widehat{x}_3 - x_3),
\end{aligned}
$$

is a global observer for (55) when the $k_i,\ i = 1, 2, 4, 6$, are negative.

## 7 Conclusion

A necessary and a sufficient condition for determining a state change of coordinate which transform an Euler-Lagrange system into an affine system in the unmeasured part of state was given. Obviously in the case of one degree of freedom, a solution always exists. A case of higher order system, is for instance, that of the cart-pendulum system [10], the tora system [20] and the overhead crane [7]. We conjecture the result several others problems in nonlinear control. Whereas, we know that these necessary and sufficient conditions so that a system is transformable are very restrictive. For that, we proposed the triangular forms in the unmeasured part of the state $\dot{q} = v$ for the analysis of observability. We have considered a particular family of Euler-lagrange systems, and we show that it can be transformed into some triangular structure for which an almost exponentially converging observer is given. Thanks to this triangular forms, a globally converging observer presented so called "two-link manipulator" system. Moreover the rate of convergence can be chosen arbitrary. Note also that our approach applies to the "cart-pendulum" system and an exponentially converging observers with an arbitrary rate of convergence can be constructed.

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# Stability Results for Large-Scale Difference Systems via Matrix-Valued Liapunov Functions

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**Abstract:** New results concerned with the Liapunov stability of composite or interconnected systems, described by linear difference equations are established. These results involve a matrix-valued Liapunov function. Furthemore, using a new approach for constructing Liapunov functions we obtain some results related to uniform asymptotic stability and compare our results with some know results which were obtained via vector Liapunov functions. The examples illustrating the efficiency of the proposed approach are given.

**Keywords:** *Large scale difference system; matrix-valued Liapunov function; uniform asymptotic stability.*

**Mathematics Subject Classification (2000):** 39A10, 93D05, 93D20, 93D30.

## 1 Introduction and Main Results

The aim of this paper is to study stability in the sense of Liapunov of a linear large-scale system of difference equations in the form

$$x_i(\tau+1) = A_{ii}x_i(\tau) + \sum_{j=1,\, j\neq i}^{m} A_{ij}(\tau)x_j(\tau), \quad i = 1, 2, \dots, m, \tag{1}$$

where $x = (x_1^{\mathrm{T}}, \dots, x_m^{\mathrm{T}})^{\mathrm{T}},\ \tau \in N_\tau^+ = \{\tau_0 + k,\ k = 0, 1, \dots\}\ \tau_0 > 0,\ x_i \in R^{n_i},\ x \in R^n,$ $n = \sum\limits_{i=1}^{m} n_i,\ A_{ii},\ i = 1, \dots, m,$ are constant matrices of appropriate dimensions, $A_{ij}(\tau),$ $i, j = 1, \dots, m,\ i \neq j,$ are determined on the set $N_\tau^+$.

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The transformation of initial systems to the form (1) is made by means of mathematical decomposition for the preassigned order of independent subsystems or in terms of some physical speculations formed in the description of real physical system by a system of difference equations.

For system (1) we construct the matrix-valued function $U(\tau, x)$ (for the details see [5]). The diagonal elements $v_{ii}(x_i)$ are taken as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1, 2, \dots, m, \tag{2}$$

where $P_{ii}$ are symmetric positive definite matrices. We assume that at least one of the matrices $A_{ij}$ or $A_{ji}$ is not equal to constant and takes the corresponding non-diagonal elements $v_{ij}(\tau, x_i, x_j)$ as the bilinear form

$$v_{ij}(\tau, x_i, x_j) = v_{ji}(\tau, x_i, x_j) = x_i^{\mathrm{T}} P_{ij}(\tau) x_j, \quad i, j = 1, 2, \dots, m, \quad i \neq j, \tag{3}$$

where the matrix $P_{ij}(\tau)$ satisfies difference equation

$$\begin{aligned} P_{ij}(\tau+1) - P_{ij}(\tau) + A_{ii}^{\mathrm{T}} P_{ij}(\tau+1) A_{jj} - P_{ij}(\tau+1) \\ = -\frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) - \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj}. \end{aligned} \tag{4}$$

Equation (4) can be solved in the explicit form. Consider two cases.

*Case 1.* Assume that the matrices $A_{ii}$ and $A_{jj}$ are such that

$$q = \max_{k,l} |\lambda_k(A_{ii}) \, \lambda_l(A_{jj})| < 1.$$

We consider the linear operators

$$F_{ij} \;\; R^{n_i \times n_j} \to R^{n_i \times n_j}, \quad F_{ij} X = A_{ii}^{\mathrm{T}} X A_{jj}.$$

and present equation (4) as

$$P_{ij}(\tau) = -F_{ij} P_{ij}(\tau+1) + \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj}. \tag{5}$$

Using the method of mathematical induction it is easy to show that

$$\begin{aligned} P_{ij}(\tau) = F_{ij}^{\nu} P_{ij}(\tau+\nu) + \sum_{k=0}^{\nu} F_{ij}^{k} \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau+k) \\ + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau+k) P_{jj} A_{jj}. \end{aligned} \tag{6}$$

for any positive integer $\nu$. It is shown (see [1]) that the eigenvalues of the operators $F_{ij}$ are $\lambda_k(A_{ii}) \, \lambda_l(A_{jj})$, therefore the norm of the operator $F_{ij}^{\nu}$ admits the estimate

$$\|F_{ij}^{\nu}\| = \left\| \frac{1}{2\pi i} \int\limits_{|z| = \frac{1+q}{2}} z^{\nu} R_z(F_{ij}) \, dz \right\| \leq \frac{c}{2\pi} \int\limits_{|z| = \frac{1+q}{2}} |z|^{\nu} \, dl = c\Big(\frac{1+q}{2}\Big)^{\nu},$$

where $c = \max\limits_{|z| = \frac{1+q}{2}} \|R_z(F_{ij})\|$, $R_z(F_{ij})$ is a resolvent of the operator $F_{ij}$. Taking into account $\frac{1+q}{2} < 1$, we get $\|F_{ij}^{\nu}\| \to 0$ as $\nu \to \infty$.

Further we are interested only in bounded solutions of equation (4). Passing to the limit in (6) as $\nu \to \infty$, we get

$$P_{ij}(\tau) = \sum_{k=\tau}^{\infty} F_{ij}^{-\tau+k} \left\{ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(k) P_{jj} A_{jj} \right\}. \tag{7}$$

Further it is assumed that the series in the right-side part of (7) converges.

*Case 2.* Assume that

$$q = \max_{k,l} |\lambda_k(A_{ii})\, \lambda_l(A_{jj})| \geq 1.$$

It is easy to notice that the operator $F_{ij}$ is non-degenerated. We present equation (4) as

$$P_{ij}(\tau+1) = F_{ij}^{-1} P_{ij}(\tau) - F_{ij}^{-1} \left\{ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau) P_{jj} A_{jj} \right\}. \tag{8}$$

Using the method of mathematical induction it is easy to show that

$$\begin{aligned} P_{ij}(\tau) &= F_{ij}^{-\tau+\tau_0} P_{ij}(\tau_0) \\ &\quad - \sum_{k=0}^{\tau-\tau_0-1} F_{ij}^{-\tau+\tau_0+k} \left[ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau_0+k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau_0+k) P_{jj} A_{jj} \right]. \end{aligned}$$

Setting $P_{ij}(\tau_0) = 0$ we find partial solution of equation (4) in the form

$$P_{ij}(\tau) = - \sum_{k=0}^{\tau-\tau_0-1} F_{ij}^{-\tau+\tau_0+k} \left[ \frac{\eta_i}{\eta_j} A_{ii} P_{ii} A_{ij}(\tau_0+k) + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}}(\tau_0+k) P_{jj} A_{jj} \right]. \tag{9}$$

Assuming that the matrices $P_{ij}(\tau)$ are bounded for all $\tau \geq \tau^*$ we introduce designations

$$\begin{aligned} \bar{c}_{ii} &= \lambda_M(P_{ii}), \quad \bar{c}_{ij} = \sup_{\tau \geq \tau^*} \|P_{ij}(\tau)\|, \\ \underline{c}_{ii} &= \lambda_m(P_{ii}), \quad \underline{c}_{ij} = -\sup_{\tau \geq \tau^*} \|P_{ij}(\tau)\|. \end{aligned}$$

In view of the results from [2, 4] the estimates for the elements matrix-valued function $U(\tau, x)$ are

$$\begin{gathered} \underline{c}_{ii} \|x_i\|^2 \leq v_{ii}(x_i) \leq \bar{c}_{ii} \|x_i\|^2, \quad i = 1, 2, \dots, m, \\ \underline{c}_{ij} \|x_i\|\, \|x_j\| \leq v_{ij}(\tau, x_i, x_j) \leq \bar{c}_{ij} \|x_i\|\, \|x_j\|, \quad i, j = 1, 2, \dots, m, \quad i \neq j. \end{gathered}$$

Therefore for scalar function $v(\tau, x, \eta) = \eta^{\mathrm{T}} U(\tau, x) \eta$, $\eta \in R_+^m$, $\eta > 0$, the bilateral inequality

$$w^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H w \leq v(\tau, x, \eta) \leq w^{\mathrm{T}} H^{\mathrm{T}} \overline{C} H w, \tag{10}$$

is satisfied, where

$$\begin{aligned} \overline{C} &= [\bar{c}_{ij}]_{i,j=1}^m, & \underline{C} &= [\underline{c}_{ij}]_{i,j=1}^m, \\ H &= \operatorname{diag}(\eta_1, \dots, \eta_m), & w &= (\|x_1\|, \dots, \|x_m\|)^{\mathrm{T}}. \end{aligned}$$

For the first difference of function $v(\tau,x,\eta)$ along solutions of system (1) in view of (4) one can get the estimate

$$\Delta v(\tau,x,\eta)\Big|_{(1)} \le w^{\mathrm T} S(\tau) w, \tag{11}$$

where $w=(\|x_1\|,\dots,\|x_m\|)^{\mathrm T}$, $S(\tau)=[\sigma_{ij}(\tau)]_{i,j=1}^m$. The elements of matrix $S(\tau)$ have the following structure

$$\begin{aligned}
\sigma_{ii}(\tau) = & -\lambda_m(G_{ii})\eta_i^2 + \sum_{j=1,\, j\ne i}^m A_{ji}\|^2\|P_{jj}\|\eta_j^2\\
& + \sum_{k,j=1,\, k\ne j}^m \lambda_M(A_{ki}^{\mathrm T}P_{kj}A_{ji} + A_{ji}^{\mathrm T}P_{jk}^{\mathrm T}A_{ki})\eta_k\eta_j,\\
\sigma_{ij}(\tau) = & \sum_{k=1,\, k\ne j,\, k\ne i}^m \eta_i^2\|A_{kj}\|\,\|P_{kk}\|\,\|A_{ki}\|\\
& + \sum_{k,l=1,\, k\ne i,\, k\ne j,\, l\ne j}^m \|A_{ki}\|\,\|P_{kl}\|\,\|A_{lj}\|\eta_j\eta_l, \quad i\ne j,
\end{aligned}$$

where $G_{ii}=-(A_{ii}^{\mathrm T}P_{ii}A_{ii}-P_{ii})$, $\|\cdot\|$ is a spectral norm of the corresponding matrix. Using the function $U(\tau,x)$, estimate (10) of the scalar function $v(\tau,x,\eta)$ and estimate (11) of the first difference of this function along solutions of system (1) we formulate sufficient conditions of stability and uniform asymptotic stability of the equilibrium state $x=0$ of system (1).

**Theorem 1.1** *Let system of equations (1) be such that*

(1) *matrices $\overline{C}$ and $\underline{C}$ in estimate (10) are positive definite;*

(2) *there exist negative semidefinite (negative definite) matrix $\overline{S}$ such that*

$$\frac12\big[S(\tau)+S^{\mathrm T}(\tau)\big] \le \overline{S} \quad \textit{for all} \quad \tau\ge\tau_0.$$

*Then the equilibrium state $x=0$ of system (1) is uniformly stable (uniformly asymptotically stable).*

***Proof*** Condition (2) of Theorem 1.1 ensures the existence of $\tau_1\in N_{\tau_0}^+$ such that for all $\tau\ge\tau_1$ for matrix $S(\tau)$ the generalized Silvester conditions are satisfied. So, for function $v(\tau,x,\eta)=\eta^{\mathrm T}U(\tau,x)\eta$ for all $\tau\ge\widetilde{\tau}=\max\{\tau_1,\tau^*\}$ all conditions of Theorem 16.3 from Hahn [3] are satisfied. Thus, the equilibrium state $x=0$ is stable (uniformly asymptotically stable) with respect to $N_{\widetilde{\tau}}^+$. Taking into account continuity of solutions $x(\tau,\tau_0,x_0)$ of system (1) in $x_0$ and discreteness of the set $N_{\tau_0}^+$ one can conclude on stability (uniform asymptotic stability) of the equilibrium state of system (1).

## 2 Examples

Consider the system

$$\begin{aligned}
x(\tau+1) &= \rho_1 x(\tau) + \alpha A(\omega,\tau)y(\tau),\\
y(\tau+1) &= \rho_2 y(\tau) + \beta A^{\mathrm T}(\omega,\tau)x(\tau),
\end{aligned} \tag{12}$$

where $x, y \in R^2$, and $\alpha, \beta, \rho_1, \rho_2 \in R$, $\omega \in [0, 2\pi)$,

$$A(\omega, \tau) = \begin{pmatrix} \cos\omega\tau & \sin\omega\tau \\ -\sin\omega\tau & \cos\omega\tau \end{pmatrix}, \qquad \tau \in N_0^+.$$

Moreover, we designate $q = \rho_1\rho_2$. Applying the approach proposed in Section 1 for system (12) we construct an auxiliary function

$$v(\tau, x, y) = x^{\mathrm{T}}x + y^{\mathrm{T}}y + 2x^{\mathrm{T}}P(\tau)y, \tag{13}$$

where

$$P(\tau) = \begin{cases} \dfrac{\alpha\rho_1 + \beta\rho_2}{1 - 2q\cos\omega + q^2} A(\omega, \tau - 1)(A(\omega, 1) - qI), & \text{if} \quad |q| \le 1; \\ -\dfrac{\alpha\rho_1 + \beta\rho_2}{1 - 2q\cos\omega + q^2}\Big[qA(\omega, \tau - 1) - A(\omega, \tau) - \\ \qquad q^{-\tau+1}A^{\mathrm{T}}(1) + q^{-\tau}I\Big], & \text{if} \quad |q| > 1, \end{cases}$$

and $I$ is an identify matrix of dimension 2. Theorem 1.1 allows us to establish sufficient stability conditions of system (12) in the form of a system of inequalities

$$\begin{gathered} |\alpha\rho_1 + \beta\rho_2| < \sqrt{1 - 2q\cos\omega + q^2}; \\ \sigma_{11} < 0, \quad \sigma_{11}\sigma_{22} - \sigma_{12}^2 > 0, \end{gathered} \tag{14}$$

where

$$\sigma_{11} = \rho_1^2 - 1 - \frac{2\rho_1\beta(\alpha\rho_1 + \beta\rho_2)(q - \cos\omega)}{1 - 2q\cos\omega + q^2} + \beta^2,$$
$$\sigma_{22} = \rho_2^2 - 1 - \frac{2\rho_2\alpha(\alpha\rho_1 + \beta\rho_2)(q - \cos\omega)}{1 - 2q\cos\omega + q^2} + \alpha^2,$$

and

$$\sigma_{21} = \sigma_{12} = |\alpha\beta|\frac{|\alpha\rho_1 + \beta\rho_2|}{\sqrt{1 - 2q\cos\omega + q^2}}.$$

It this case the equilibrium state $x = y = 0$ of system (12) is uniformly asymptotically stable, and the constructed function (13) is the Liapunov function.

In order to compare the obtained stability conditions with the conditions obtained in terms of vector Liapunov function we employ the results from [6]. Construct the vector function $V(x, y) = (v_1(x), v_2(y))^{\mathrm{T}}$ with the components $v_1(x) = x^{\mathrm{T}}x$ and $v_2(y) = y^{\mathrm{T}}y$. Applying Theorem 3.3.14 from [6] we present sufficient conditions of uniform asymptotic stability of system (12) in the form of the system of inequalities

$$\begin{gathered} \rho_1^2 + \beta^2 - 1 < 0, \\ (\rho_1^2 + \beta^2 - 1)(\rho_2^2 + \alpha^2 - 1) - 4|\alpha\beta||\rho_1\rho_2| > 0. \end{gathered} \tag{15}$$

To compare conditions (15) and (14) obtained in terms of Theorem 1.1 we consider a system of difference equations

$$\begin{aligned} x(\tau + 1) &= 0.95\, x + \alpha A\left(\frac{\pi}{3}, \tau\right) y, \\ y(\tau + 1) &= -0.95\, y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right) x \end{aligned} \tag{16}$$

and construct in the space of parameters $(\alpha, \beta)$ the domains of stability of the equilibrium space $x = y = 0$ of system (16). Figures 2.1 and 2.2 show that the domain constructed in terms of conditions (14) is wider than the domain constructed in terms of conditions (15).

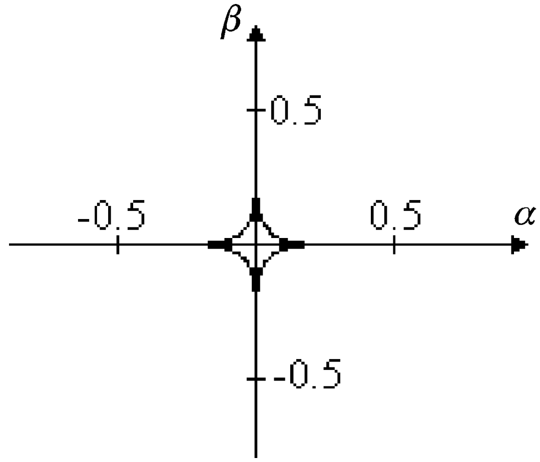


**Figure 2.1:** The domain of stability of (16) in the parameter space via Liapunov's vector function.

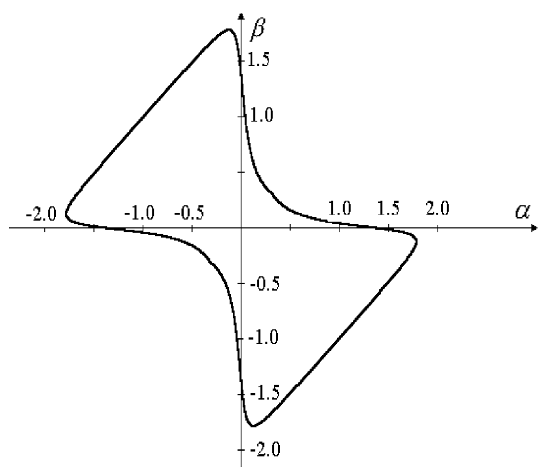


**Figure 2.2:** The domain of stability of (16) in the parameter space via Liapunov's matrix-valued function.

Note that for the system

$$\begin{aligned} x(\tau+1) &= 1.2\,x + \alpha A\left(\frac{\pi}{3}, \tau\right)y, \\ y(\tau+1) &= -0.8\,y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right)x \end{aligned} \tag{17}$$

it is impossible to apply the vector function, because subsystem $x(\tau+1) = 1.2\,x$ is not exponentially stable. Nevertheless conditions (14) allow us to construct for system (16) in the space $(\alpha, \beta)$ a domain of stability shown in Figure 2.3.

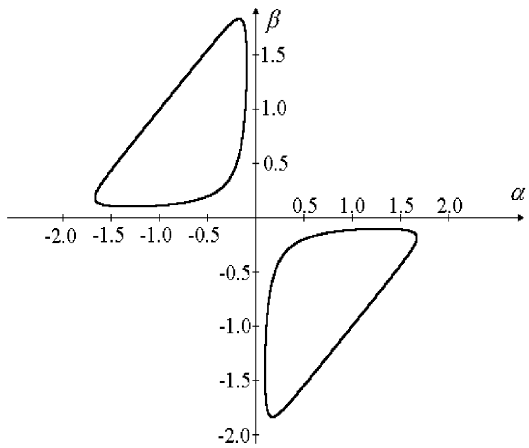


**Figure 2.3:** The domain of stability of (17) in the parameter space.

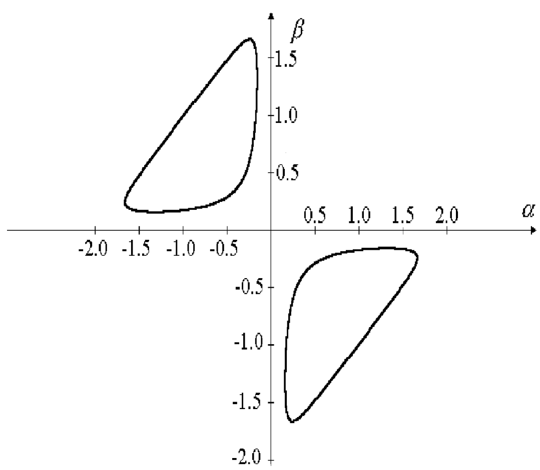


**Figure 2.4:** The domain of stability of (18) with exponentially unstable subsystem.

The system

$$
\begin{aligned}
x(\tau+1) &= 1.05\,x + \alpha A\left(\frac{\pi}{3}, \tau\right) y, \\
y(\tau+1) &= -1.05\,y + \beta A^{\mathrm{T}}\left(\frac{\pi}{3}, \tau\right) x
\end{aligned}
\tag{18}
$$

has exponentially unstable subsystems. However in this case as well conditions (14) allow us to construct for system (18) a domain of stability in the space of parameters shown in Figure 2.4.

## 3 Concluding Remarks

Generalized Liapunov function method for a class of large-scale difference systems (1) were developed. In particular, stability and uniform asymptotic stability theorems were presented. The efficiency of the proposed approach was demonstrated by two examples. An important aspect of the new results is that they account an estimation stability domain of parameters of the systems. In connection with the developed theory, there remain many open problems. Some of these include the following: to established guides for choosing "best" vector $\eta$ in the scalar function $v(\tau, x, \eta)$; to apply the developed theory to specific problems of uncertain systems. Because in general, one is not only interested in stability of systems (1), but also in trajectory bounds, it is desirable to investigate the behavior of systems (1) with respect to sub-sets of the state space.

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# Stability, Oscillations and Optimization of Systems

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If, for a process or a phenomenon, for example, atom oscillations or a supernova explosion, a mathematical model is constructed in the form of a system of differential equations, the investigation of the latter is possible either by a direct (numerical as a rule) integration of the equations or by its analysis by qualitative methods.

In XX century the fundamental works by Euler (1707 – 1783), Lagrange (1736 – 1813), Poincaré (1854 – 1912), Lyapunov (1857 – 1918) and others have been thoroughly developed and applied in stability and oscillations investigation of nature phenomena and solution of many problems of technical progress.

In particular, the problems of piloted space flights and those of astrodynamics were solved due to modern achievements of stability theory and motion control. The Poincaré and Lyapunov methods of qualitative investigation of solutions to nonlinear systems of differential equations in macro-world study have been refined to a great extend though not completed. On the other hand modeling and establishing stability conditions for micro-processes are still on the stage of accumulating ideas and facts and forming the principles. One of the examples is the fact that the stability problem of thermonuclear synthesis has not been solved yet.

Obviously, this is one of the areas for application of stability and control theory in XXI century. For the development of efficient methods and algorithms in this area the interaction and spreading of the ideas and results of various mathematical theories will be necessary as well as the co-operation of scientists specializing in different fields.

The mathematical theory of optimal control (of moving objects, water resources, global process in world economy, etc.) is being developed in terms of basic ideas and results obtained in 1956 – 1961 and formulated in the Pontryagin's principle of optimality and Bellman's principle of dynamical programming. Considering manufacturing and production engineering activities, due to the difficulties of description of discrete events and hybrid processes, various heuristic and soft computing approaches have been developed for solving optimization problems. The efforts of many scholars and engineers in the framework of these ideas resulted in the efficient methods of control for many concrete systems and technological processes.

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