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Comparison of Transfer Orbits in the Restricted Three and Four-Body Problems

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Abstract: The restricted three-body problem and the quasi-bicircular problem are the dynamical systems used as models in this paper. The first one describes the motion of one massless body in the potential field of the two massive bodies revolving in circular orbit around their center of mass. The quasi-bicircular problem (QBCP) is a variation of the restricted four-body problem, where the three massive primaries move in a quasi-circular motion around their center of mass. Here, we consider the Earth–Moon as primaries of the restricted three-body problem (RTBP) and the Earth–Moon-Sun as primaries of the QBCP. One of the spatial periodic solutions around the collinear point are known as halo orbits. Our objective is to determine, in both models, transfer orbits from a parking orbit around the Earth to a halo orbit. We apply the two-point boundary value problem, where the boundary points are on the parking and on the halo orbits. Since there is no Keplerian orbit involved in this transfer method, we have called it an adapted Lambert's problem. We compare the total velocity increment obtained with this method applied to both dynamical models. We find that there is a positive solar contribution decreasing the total impulse.

Keywords: *Restricted three-body problem; quasi-bicircular problem; halo orbits; impulsive transfer orbits; two point boundary value problem.*

Mathematics Subject Classification (2000): 70F07, 70F10, 0H12, 70F99, 30E25.

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1 Introduction

The restricted three-body problem is one of most studied problem in orbital dynamics. It has been investigated since Euler and Lagrange because of two important reasons: it is the simplest model of the N-body problem yielding a non-integrable system and it also fits, in first approximation, the motion of celestial bodies and artificial satellites. However, we have to consider other perturbations to describe accurately the real problem, such as the effect of other planets of the Solar System, the solar radiation pressure, the atmosphere drag (for objects close to the Earth) and the non circular orbit of the primary bodies. In this work we consider two cases: the planar circular restricted three-body problem and the four-body case when the circular orbits of the primaries are perturbed by the Sun, known as quasi-bicircular problem.

The five equilibrium points of the RTBP are the well known Lagrangian points. Three of them, L_1, L_2 and L_3, lie along the line joining both primaries. Usually L_1 denotes the solution located between the primaries while L_2 is behind the less massive primary, and L_3 is located in the opposite side of L_1, with respect to the center of mass. The other two points, L_4 and L_5, are on the plane of the motion and form an equilateral triangle with the primary bodies.

The collinear points are unstable for all mass ratios because the linear approximation has a pair of real eigenvalues. The other two are imaginary and span the linear center manifold. The full dynamics near the L_i's has two families of periodic orbits known as Lyapunov orbits, plane and vertical, which are continuation of the linear center manifold and tangent to it. When the horizontal and vertical frequencies attain a certain resonance, the plane Lyapunov family bifurcates into spatial orbits known, since Faquhar [1], as halo orbits. These orbits are such that an observer, placed on one of the primaries and looking towards the second primary, sees the massless body describing a halo around that body. All these periodic orbits can be calculated numerically or by perturbation methods, see for instance [2].

In the 1970s aerospatial engineers began the exploration of these orbits. They were proposed as good places to locate certain space observatories due to two main reasons. First, the point L_1 provides uninterrupted access to the solar visual field without occultation by the Earth; and second, in these places the solar wind is beyond the influence of the Earth's magnetosphere. The first satellite in a halo orbit was *Isee-3*, launched in 1978 by NASA. It was maintained in a halo orbit for nearly 4 years while observing the solar wind and cosmic rays, and then it undertook a complex trip to observe the tail of a comet in a heliocentric orbit. Since Isee-3 launch, five satellites were inserted into halo orbits of the Sun–Earth system. The second mission was the *Soho* telescope projected by ESA-NASA, launched at 1996 for solar observations; *Ace* satellite was launched at 1997 by NASA for solar wind observations. In 2001 two NASA satellites arrived at halo orbits: the *WMap* satellite, to observe cosmic microwave background radiation, and *Genesis*, another solar observatory whose re-entry occurred in 2004. Future missions are under development for launching in the next ten years.

For the Earth–Moon system the RTBP is just the first approximation since the presence of the Sun perturbs the Earth–Moon distance strongly. In 1998, Andreu [3] introduced a consistent model of the restricted four-body problem, named the Quasi-Bicircular Problem, where the motion of the three primary bodies is given by the solution of the non-restricted three body problem. In the QBCP the primaries are revolving around their center of mass in a quasi-circular motion and the massless body moves under the effect of their potential field. In [4] Andreu showed that the quasi-bicircular problem fits better the real case than the bicircular one.

Transference of space vehicles has been widely studied by many authors and this resulted in the development of a wide variety of methods. The most recent ones explore the unstable character of the libration point orbits to design low cost missions [5], using the stable/unstable manifold dynamics. However, the large time spent in these transfer orbits could not be appropriate for certain missions. On the other hand, transfers with the control of flight time, based on the optimization procedures, give paths with greater fuel consumption but with shorter transfer time. The choice of a specific method should be guided by the mission requirements.

In the case of the two body problem, the determination of a transfer orbit connecting the boundary conditions with a specified flight time, is the well-known Lambert's problem. This formulation has been applied by several authors who developed numerical tools for its resolution. The solution of the classical Lambert's problem, with a fixed flight time, has been undertaken by [6] and [7] who developed sophisticated algorithms and accurate methods, dealing with convergence techniques. A new version of Lambert's problem has been studied by [8], replacing the condition of a given transfer time by that of minimal fuel expenditure. We apply the latter conception to the restricted three and four-body problem and call it the adapted Lambert's problem.

In this paper we find transfer orbits from the Earth to a halo orbit in the vicinity of L_1 of the Earth–Moon system. We compare the total ΔV required by RTBP and QBCP models, showing that the presence of the Sun decreases the total impulse necessary to achieve the desired transfer.

2 Equations of Motion

2.1 The Restricted Three-Body Problem

In the restricted three-body problem, the mass of one of the bodies is supposed to be infinitely small when compared to the other two that move in circular motion around their center of mass. The reference frame is set according to the notation defined in [9], where the origin is on the center of mass, the positive x-direction is towards the biggest primary and rotates in the counterclockwise direction. The unit of length is the distance between the primaries and the unit of time is chosen so that the period of the primaries is 2π; consequently the gravitational constant is set to one. The potential function of the RTBP in this synodic coordinate system is given by

$$\Omega(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu), \tag{1}$$

where r_1 and r_2

$$r_1^2 = (x-\mu)^2 + y^2 + z^2,$$
$$r_2^2 = (x-(\mu-1))^2 + y^2 + z^2,$$

are the distances from the primary bodies (m_1, m_2) to the massless particle. The equations of motion are:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z. \end{aligned} \tag{2}$$

Defining the momenta as $p_x = \dot{x} - y$ and $p_y = \dot{y} + x$, the equations of motion can be written as an autonomous Hamiltonian system with three degrees of freedom derived from:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

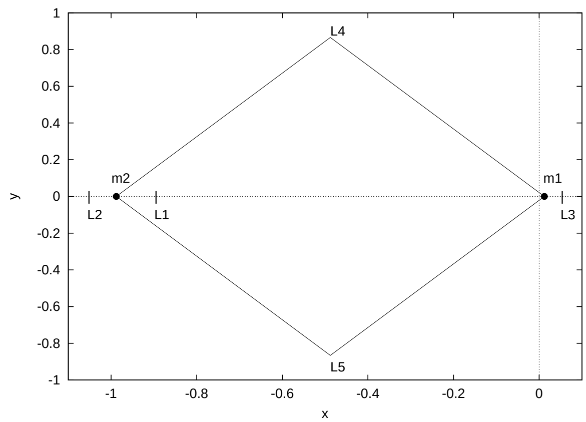

Figure 2.1: The restricted three-body problem configuration.

2.2 The Quasi-Bicircular Problem

The quasi-bicircular model is a restricted four-body problem where the three primaries are revolving in a quasi-bicircular motion. The fourth body, which has infinitely small mass, moves under the potential field generated by the primaries without disturbing their motion. In this case the equations of motion are time dependent with the frequency of the biggest primary. The Earth–Moon distance is no longer constant due to the Sun perturbation, therefore the equations of motion are written in a rotating pulsating reference frame to make the Earth–Moon distance constant. This coordinate system is centered on the barycenter of the Earth–Moon system and rotates with it.

In order to obtain a coherent formulation we first should find a solution for the three-body problem, where the three masses move in planar non-circular orbits around their common center of mass. This solution expressed as Fourier expansion is:

$$\alpha_k(t) = \alpha_{k_0} + \sum_{j\geq 1} \alpha_{kj} \cos(jnt) \quad \text{for} \quad k = 1, 3, 4, 6, 7,$$
$$\alpha_k(t) = \sum_{j\geq 1} \alpha_{kj} \sin(jnt) \quad \text{for} \quad k = 2, 5, 8,$$

where n is the mean relative angular velocity $n = 1 - n_s$ in inertial coordinates, n_s is the angular velocity of the Sun. We recall that the mean angular velocity of the Moon is unity in the inertial frame. The units of distance and time are the same as in the last section. Then, the Hamiltonian of the QBCP is:

$$H = \frac{1}{2}\alpha_1(p_x^2 + p_y^2 + p_z^2) + \alpha_2(p_x x + p_y y + p_z z) + \alpha_3(p_x y - p_y x) + \alpha_4 x + \alpha_5 y - \alpha_6\left(\frac{1-\mu}{q_{pe}} + \frac{\mu}{q_{pm}} + \frac{m_s}{p_{pe}}\right),$$

where q_{pe}, q_{pm} and q_{ps} are given by

$$\begin{aligned} q_{pe}^2 &= (x-\mu)^2 + y^2 + z^2, \\ q_{pm}^2 &= (x-\mu+1)^2 + y^2 + z^2, \\ q_{ps}^2 &= (x-\alpha_7)^2 + (y-\alpha_8)^2 + z^2, \end{aligned}$$

and are the distances of the particle to the Moon, Earth and Sun, respectively. We note that they are written in the synodical reference frame centered in the barycenter of the Earth–Moon system. For more details see [3]. The Hamiltonian equations of motion are:

$$\begin{aligned} \dot{x} &= \alpha_1 + \alpha_2 x + \alpha_3 y, \\ \dot{y} &= \alpha_1 p_y + \alpha_2 y - \alpha_3 x, \\ \dot{z} &= \alpha_1 p_z + \alpha_2 z, \\ \dot{p}_x &= -\alpha_2 p_x + \alpha_3 p_y - \alpha_4 - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}(x-\mu) + \frac{\mu}{q_{pm}^3}(x-\mu+1) + \frac{m_s}{q_{ps}^3}(x-\alpha_7)\right), \\ \dot{p}_y &= -\alpha_2 p_y + \alpha_3 p_x - \alpha_5 - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}y + \frac{\mu}{q_{pm}^3}y + \frac{m_s}{q_{ps}^3}(y-\alpha_8)\right), \\ \dot{p}_z &= -\alpha_2 p_z - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}z + \frac{\mu}{q_{pm}^3}z + \frac{m_s}{q_{ps}^3}z\right). \end{aligned} \tag{3}$$

3 Impulsive Transfer Orbit

When a system of differential equations is supposed to satisfy a set of initial and final conditions, it becomes a two point boundary value problem (TPBVP), where the time is a free variable. In this work, the boundary conditions are a point on the parking orbit (P_i) around the Earth and a point on the halo orbit (P_f). Without any time constraint, the problem of finding a trajectory that links the points P_i and P_f has infinite set of solutions with different flight times. However, if we add the flight time as a constraint, the set of solutions become finite. In this case, for each set of boundary conditions (P_i, P_f) with a fixed flight time (Δt), we have two solutions which are related via the Mirror Theorem [10].

As the restricted three and four-body models have no analytical solution, the boundary value problem has to be numerically solved. We use the following steps to find a solution of this time constrained TPBVP:

- guess an initial velocity $\vec{v}_i$. Together with the initial prescribed position $\vec{r}_i$ the complete initial state is known;
- guess a final time t_f and integrate the equations of motion from t_i to t_f;
- check the final position $\vec{r}_f$ obtained from the numerical integration with the prescribed final position and the final real time with the specified time of flight. If

there is an agreement (difference less than a specified error allowed) the solution is found and the process can stop here.

This is a simple shooting method described in reference [11], where an algorithm is also available.

As mentioned in the introduction, such a formulation: the two point boundary value problem plus a time constraint in the RTBP is a kind of Lambert's problem for the three-body problem. Thus the name adapted Lambert's problem, since there is no Keplerian orbit involved.

Our investigation begins by the search of transfer trajectories travelling between the points P_i and P_f with minimum velocity increment defined as follows. Let $\vec{V}_i$ and $\vec{V}_f$ be the velocity vectors on the parking orbit around the Earth and the halo orbit, respectively. The initial velocity increment (ΔV) is:

$$\Delta V_i = |\vec{V}_i - \vec{V}_T|, \tag{4}$$

where $\vec{V}_T$ is the transfer velocity given by the above numerical method, which satisfies the time and the boundary constraints. The second impulse, introduced to insert the space vehicle in the halo orbit, is given by:

$$\Delta V_f = |\vec{V}_T - \vec{V}_f|. \tag{5}$$

The total impulse is the sum of these impulses:

$$\Delta V = \Delta V_i + \Delta V_f. \tag{6}$$

4 Halo and Parking Orbits

As the QBCP is a three degrees of freedom time periodic Hamiltonian system, we can have an intuition of its periodic solutions considering it as a time periodic perturbation of the RTBP. So, the periodic solutions of QBCP are related to the period of the perturbation. Therefore, to compute halo orbits in the QBCP, one first looks for a halo perodic orbit in the RTBP which has the Solar period or a multiple of it. Then, a numerical continuation method can be used to find the corresponding QBCP halo orbit. As usual, we set the problem of continuation as follows:

$$H = H_{RTBP} + \epsilon(H_{QBCP} - H_{RTBP}),$$

where ϵ is a small parameter. When ϵ is equal to zero we have $H = H_{RTBP}$ and if ϵ is equal to unity, then $H = H_{QBCP}$. The halo initial conditions considered here are those labelled 01E and 1E in [3]. The period of the chosen halo in the QBCP is three times multiple of its equivalent orbit in the RTBP, which were determined in [12].

The parking orbits belong to the BD family of direct periodic orbits around the primary body (see [13]) and calculated them using a numerical continuation method described in [14]. The orbit Parking 1 is about 6.696 km from the Earth and the Parking 2 is 11.612 km. To make a clear identification of the selected points on the parking and the halo orbit, we choose angular coordinates θ_1 and θ_2 on the xy-projection of the former and on the yz-projection of the latter. The origin of these angles are the positive x and y axis, for θ_1 and θ_2, respectively, and the direction of rotation is taken to be counterclockwise (see Figure 4.1).

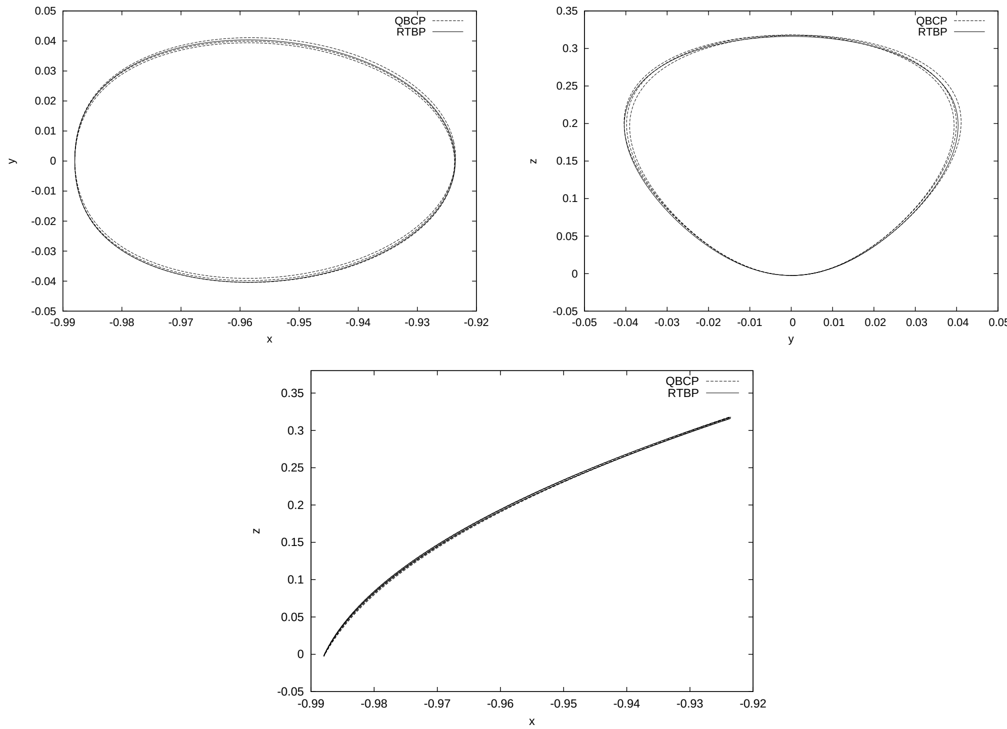

Figure 4.1: Projection of halo orbit in both dynamical systems.

5 Results

In this section we apply the adapted Lambert's method to the initial conditions shown in the table below. The first application considers only the RTBP while the second one considers both the RTBP and the QBCP.

Table 5.1: Initial Conditions

ORBIT	x_0	z_0	$\dot{y}_0$
Halo - RTBP	–9.87982557457513E-01	–2.48226957833054E-03	3.13139586195729E+00
Halo - QBCP	–9.879722904635280E-01	–2.462095060502300E-03	3.184909363998671E+00
Parking 1	–0.005270250000000E+0	0.0000E+0	7.547700390000000E+0
Parking 2	–0.018068060000000E+0	0.0000E+0	5.747748530000000E+0

5.1 Transfers in RTBP

To begin our simulations, we have chosen the parking orbit 0.0302 canonical units away from the center of the Earth (Parking 2 on Table 5.1) and the halo orbit. In these two orbits we take angular steps of approximately 6°, and the transfer method is applied considering the boundary conditions: the initial one on the parking orbit and the final one on the halo orbit. We stress that we explored all these possibilities of boundary conditions with a fixed time. This procedure was decisive to select which angular range on each orbit furnishes the lower total ΔV. Three time intervals, $t = 0.2$, 0.3 and 0.4, were used in this simulations. These preliminary study restricted the angular range to $[300°, 40°]$ for θ_1 and $[230°, 330°]$ for θ_2 (see Figure 5.1).

With this selected points we make simulations considering several time intervals, from 0.2 to 2.0 canonical units of time with step 0.05, as seen on Figure 5.2. As expected, the maximal contribution to the total ΔV comes from ΔV_i which is the departure impulse. The final impulse, which injects the vehicle into the halo orbit is, on average, one fourth of the initial impulse. The result of these simulations can be summarized as follows: the minimum ΔV is 4.7283 and occurs at $\theta_1 = 36°.3874$ and $\theta_2 = 261°.8722$ for $t = 0.85$. This can be seen in Figure 5.2. We also tested the corresponding retrograde parking orbits and the result is practically the same as for the direct ones.

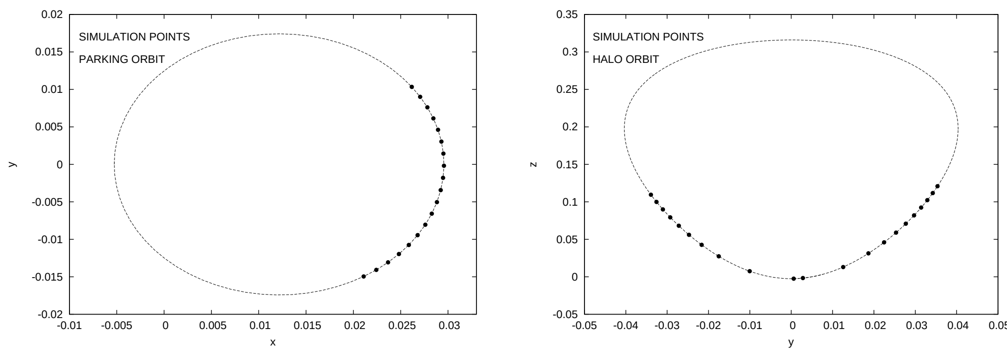

Figure 5.1: Boundary points on the parking and the halo orbit.

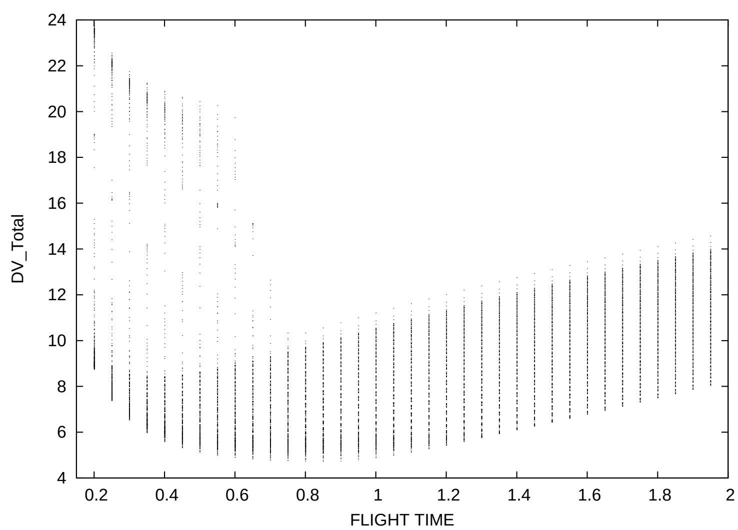

Figure 5.2: Total ΔV in the RTBP considering the boundary points on the Figure 5.1.

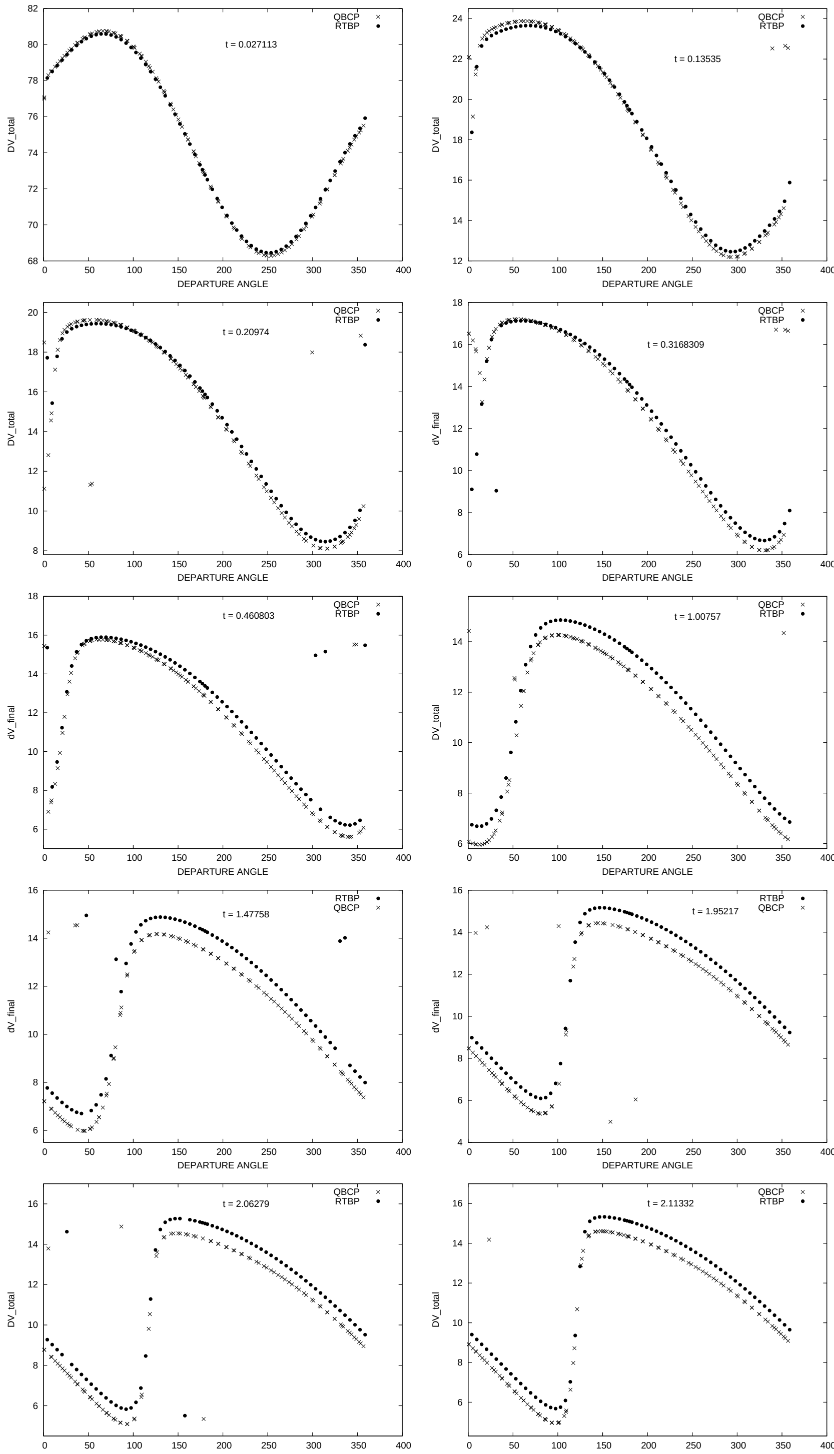

Figure 5.3: Comparison of total ΔV between the RTBP and QBCP considering many interval time, the departure angle is θ_1 whose origin is the positive x-axis on the parking orbit plane.

5.2 Comparison between the PRTC and QBCP

To obtain the transfer orbits in the QBCP, we apply a methodology similar to the one described above. Because the QBCP is a non-autonomous system, the set of initial conditions is time-dependent, implying that we have less freedom to vary the flight time as in the RTBP case. The transfer time must be the same in the state vector of the final boundary value if we begin the all the integrations at a common epoch.

Since our objective is to compare the total ΔV obtained in the RTBP and QBCP, we select the points on the halo orbit which are geometrically equivalent to the ones in the RTBP. The periodic parking orbits of the RTBP are, of course, no longer periodic in the QBCP. However, the satellite remains a short time on the parking orbit, so we take for it the same initial conditions as before. All simulations are done for the Parking 1 orbit.

The Figure 5.3 shows the total ΔV of both models for 10 different time intervals. In general, the simulations show that the Sun's presence improves the fuel consumption of approximately 9%.

6 Comments

It is difficult to show which parameter involved in the problem allows an optimal transfer, because the dynamical system is complex and very sensitive to initial conditions. If we change the flight time, it is possible that the economical boundary condition, selected for another flight time, will also change significantly. For this reason, we simulated many possibilities, varying the set of boundary conditions and flight time. However, it is possible that our results do not correspond to the exact global minimum, but just a discrete approximation. The graphics on the Figure 5.3 show that the presence of the Sun could contribute to decrease the total impulse (ΔV), specially for longer flight times.

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