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Uniform Convergence to Global Attractors for Discrete Disperse Dynamical Systems

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Abstract: In this paper we study uniform convergence of trajectories of discrete disperse dynamical systems generated by set-valued mappings to their global attractors. In particular, we show that this convergence holds even in the presence of computational errors.

Keywords: *Compact metric space; set-valued mapping; trajectory; global attractor.*

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1 Introduction

Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. A discrete-time dynamical system is described by a space of states and a transition operator which can be set-valued. Usually in the dynamical systems theory a transition operator is single-valued. In the present paper we study a class of dynamical systems introduced in [3] and studied in [4, 5] with a compact metric space of states and a set-valued transition operator. Such dynamical systems describe economical models [1, 2, 6].

Let (X, ρ) be a compact metric space and let $a\colon X \to 2^X \setminus \{\varnothing\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in X \times X\colon\ y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x)\colon\ x \in E\} \ \text{ and } \ a^0(E) = E.$$

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By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we study convergence of trajectories of the dynamical system generated by the set-valued mapping a. Following [3, 4] this system is called a discrete disperse dynamical system.

First we define a trajectory of this system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Put

$$\begin{aligned} \Omega(a) = \{z \in X\colon\ & \text{for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \\ & \text{such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \epsilon\}. \end{aligned} \tag{1.1}$$

Clearly, $\Omega(a)$ is closed subset of (X, ρ). In the present paper the set $\Omega(a)$ will be called a global attractor of a. Note that in [3–5] $\Omega(a)$ was called a turnpike set of a. This terminology was motivated by mathematical economics [1, 2, 6].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y)\colon\ y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have $\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0$.

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

In the present paper we study uniform convergence of trajectories to the global attractor $\Omega(a)$.

The following useful result will be proved in Section 2.

Proposition 1.1 *Let $\epsilon > 0$. Then there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$*

$$\min\{\rho(x_t, \Omega(a)) :\ t = 0, \dots, T(\epsilon)\} \leq \epsilon.$$

The following theorem provides necessary and sufficient conditions for uniform convergence of trajectories to the global attractor.

Theorem 1.1 *The following properties are equivalent:*

(1) *For each $\epsilon > 0$ there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ and each integer $t \geq T(\epsilon)$ we have $\rho(x_t, \Omega(a)) \leq \epsilon$.*

(2) *If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ for all integers t, then $\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$.*

(3) *For each $\epsilon > 0$ there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \leq \delta$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq 0$.*

Theorem 1.1 will be proved in Section 3.

The following two theorems show that convergence of trajectories to the global attractor holds even in the presence of computational errors. These theorems will be proved in Section 5.

Theorem 1.2 *Let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \leq \delta$ for each integer $t \geq 0$ the following inequality holds:*

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T(\epsilon)\} \leq \epsilon.$$

Theorem 1.3 *Assume that property (2) from Theorem 1.1 holds. Then for each $\epsilon > 0$ there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying*

$$\rho(x_{t+1}, a(x_t)) \leq \delta \textit{ for all integers } t \geq 0$$

the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for each integer $t \geq T(\epsilon)$.

Some examples of set-valued mappings are considered in Section 6. In Section 7 we obtain generic convergence results for certain classes of set-valued mappings.

2 Proof of Proposition 1.1

Let us assume the converse. Then for each natural number n there exists a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\min\{\rho(x_t^{(n)}, \Omega(a)) : \ t = 0, \dots, n\} \geq \epsilon. \tag{2.1}$$

It is easy to see that there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer $t \geq 0$ there exists

$$x_t = \lim_{k \to \infty} x_t^{(n_k)}. \tag{2.2}$$

Since graph(a) is a closed subset of $X \times X$ equality (2.2) implies that $\{x_t\}_{t=0}^{\infty}$ is a trajectory. It follows from (2.1) and (2.2) that for each integer $t \geq 0$ the inequality $\rho(x_t, \Omega(a)) \geq \epsilon$ holds. This contradicts the definition of $\Omega(a)$. The contradiction we have reached proves Proposition 1.1.

3 Proof of Theorem 1.1

We will show that property (1) implies property (2). Assume that property (1) holds. Let a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfy $x_{t+1} \in a(x_t)$ for all integers t. Let τ be an integer, ϵ be a positive number and let a natural number $T(\epsilon)$ be as guaranteed by property (1). Define

$$y_t = x_{t+\tau-T(\epsilon)} \quad \text{for each integer} \quad t \geq 0. \tag{3.1}$$

It is clear that $\{y_t\}_{t=0}^{\infty}$ is a trajectory. By property (1), the choice of $T(\epsilon)$ and (3.1)

$$\rho(x_\tau, \Omega(a)) = \rho(y_{T(\epsilon)}, \Omega(a)) \leq \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that $x_\tau \in \Omega(a)$ for each integer τ. Thus property (1) implies property (2).

Let us show that property (2) implies property (3). Assume that property (2) holds. Let $\epsilon \in (0,1)$. We show that there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \leq \delta$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq 0$.

Let us assume the converse. Then for each integer $n \geq 1$ there exists a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\rho(x_0^{(n)}, \Omega(a)) \leq (2n)^{-1}\epsilon \ \text{ and } \ \sup\{\rho(x_t^{(n)}, \Omega(a)) \colon \ t \geq 0 \ \text{ is an integer}\} > \epsilon. \tag{3.2}$$

In view of (3.2) for each natural number n there exists a natural number T_n such that

$$\rho(x_{T_n}^{(n)}, \Omega(a)) > \epsilon. \tag{3.3}$$

Assume that the sequence $\{T_n\}_{n=1}^{\infty}$ is not bounded. Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that $T_n \to \infty$ as $n \to \infty$. For each integer $n \geq 1$ set

$$y_t^{(n)} = x_{t+T_n}^{(n)} \ \text{ for all integers } \ t \geq -T_n. \tag{3.4}$$

Evidently there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer t there exists

$$y_t = \lim_{k\to\infty} y_t^{(n_k)}. \tag{3.5}$$

Since the graph of a is closed it follows from (3.4) and (3.5) that $y_{t+1} \in a(y_t)$ for each integer t. By property (2), $\{y_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$. On the other hand by (3.3)-(3.5)

$$\rho(y_0, \Omega(a)) = \lim_{k\to\infty} \rho(y_0^{(n_k)}, \ \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_{n_k}}^{(n_k)}, \ \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves that our assumption is wrong and the sequence $\{T_n\}_{n=1}^{\infty}$ is bounded. Extracting a subsequence and re-indexing we may assume without loss of generality that

$$T_n = T_1 \ \text{ for all integers } \ n \geq 1. \tag{3.6}$$

Let n be a natural number. It follows from (3.2) that there is $z_n \in \Omega(a)$ such that

$$\rho(x_0^{(n)}, z_n) \leq (2n)^{-1}\epsilon. \tag{3.7}$$

By the definition of $\Omega(a)$ there exists a trajectory $\{y_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\liminf_{t\to\infty} \rho(y_t^{(n)}, z_n) \leq (8n)^{-1}\epsilon. \tag{3.8}$$

In view of (3.8) there exists a natural number $S_n > n$ such that

$$\rho(y_{S_n}^{(n)}, z_n) < (4n)^{-1}\epsilon. \tag{3.9}$$

Relations (3.7) and (3.9) imply that

$$\rho(y_{S_n}^{(n)}, x_0^{(n)}) \leq \rho(y_{S_n}^{(n)}, z_n) + \rho(z_n, x_0^{(n)}) < \frac{\epsilon}{n}. \tag{3.10}$$

Set

$$\xi_t^{(n)} = y_{t+S_n}^{(n)}, \quad t = -S_n, \dots, -1, 0, \quad \xi_t^{(n)} = x_t^{(n)}, \quad t = 1, 2, \dots\ . \tag{3.11}$$

Clearly, there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer t there exists

$$\xi_t = \lim_{k\to\infty} \xi_t^{(n_k)} \tag{3.12}$$

and also there exists

$$x_0 = \lim_{k\to\infty} x_0^{(n_k)}. \tag{3.13}$$

Since the graph of a is closed it follows from (3.11) and (3.12) that

$$\xi_{t+1} \in a(\xi_t)$$

for each integer $t \geq 1$ and for each integer $t \leq -1$.

We will show that $\xi_1 \in a(\xi_0)$. Since the graph of a is closed it follows from (3.11)–(3.13) that $\xi_1 \in a(x_0)$. By (3.10)–(3.13) and the inclusion above

$$\rho(x_0, \xi_0) = \lim_{k\to\infty} \rho(x_0^{(n_k)}, \xi_0^{(n_k)}) = \lim_{k\to\infty} \rho(x_0^{(n_k)}, y_{S_{n_k}}^{(n_k)}) = 0,$$
$$x_0 = \xi_0 \quad \text{and} \quad \xi_1 \in a(\xi_0).$$

Thus we have shown that

$$\xi_{t+1} \in a(\xi_t) \quad \text{for all integers} \quad t. \tag{3.14}$$

In view of property (2) $\xi_t \in \Omega(a)$ for all integers t. On the other hand it follows from (3.12), (3.11), (3.6) and (3.3) that

$$\rho(\xi_{T_1}, \Omega(a)) = \lim_{k\to\infty} \rho(\xi_{T_1}^{(n_k)}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_1}^{(n_k)}, \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves that there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \leq \delta$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq 0$. Thus property (2) implies property (3).

Let us show that property (3) implies property (1). Assume that property (3) holds. Let $\epsilon > 0$ and let $\delta > 0$ be as guaranteed by property (3). By Proposition 1.1 there exists a natural number T_0 such that for each trajectory $\{x_t\}_{t=0}^{\infty}$

$$\min\{\rho(x_t, \Omega(a)) \colon\ t = 0, \dots, T_0\} \leq \delta. \tag{3.15}$$

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory. By the choice of T_0 there is an integer $j \in [0, T_0]$ such that

$$\rho(x_j, \Omega(a)) \leq \delta.$$

In view of this inequality and the choice of δ $\rho(x_t, \Omega(a)) \leq \epsilon$ for all integers $t \geq j$ and property (1) holds. Thus property (3) implies property (1). Theorem 1.1 is proved.

4 An auxiliary result

Lemma 4.1 *Let T be a natural number and let $\epsilon > 0$. Then there exists a number $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{T} \subset X$ satisfying*

$$\rho(x_{t+1}, a(x_t)) \leq \delta, \quad t = 0, \dots, T-1$$

there is a sequence $\{y_t\}_{t=0}^{T} \subset X$ such that

$$y_{t+1} \in a(y_t), \quad t = 0, \dots, T-1, \tag{4.1}$$
$$\rho(y_t, x_t) \leq \epsilon, \quad t = 0, \dots, T. \tag{4.2}$$

Proof Let us assume the converse. Then for each natural number n there exists a sequence $\{x_t^{(n)}\}_{t=0}^T \subset X$ such that

$$\rho(x_{t+1}^{(n)}, a(x_t^{(n)})) \leq 1/n, \quad t = 0, \dots, T-1 \tag{4.3}$$

and that for each sequence $\{y_t\}_{t=0}^T \subset X$ satisfying (4.1)

$$\sup\{\rho(y_t, x_t^{(n)}) \colon \ t = 0, \dots, T\} > \epsilon. \tag{4.4}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that for $t = 0, \dots, T$ there exists

$$x_t = \lim_{n \to \infty} x_t^{(n)}. \tag{4.5}$$

By (4.3) for $t = 0, \dots, T-1$ and each integer $n \geq 1$ there is

$$z_{t+1}^{(n)} \in a(x_t^{(n)}) \tag{4.6}$$

such that

$$\rho(x_{t+1}^{(n)}, z_{t+1}^{(n)}) \leq 1/n. \tag{4.7}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that for $t = 0 \dots, T-1$ there is

$$z_{t+1} = \lim_{n \to \infty} z_{t+1}^{(n)}. \tag{4.8}$$

Since the graph of a is closed it follows from (4.5) and (4.8) that for each $t = 0, \dots, T-1$

$$z_{t+1} \in a(x_t). \tag{4.9}$$

By (4.5), (4.7) and (4.8) for each $t = 0, \dots, T-1$ we have $x_{t+1} = z_{t+1}$. Together with (4.9) this equality implies that $x_{t+1} \in a(x_t)$ for $t = 0, \dots, T-1$. In view of (4.5) there is a natural number n_0 such that

$$\rho(x_t, x_t^{(n_0)}) \leq \epsilon/4, \quad t = 0, \dots, T.$$

This contradicts the choice of $\{x_t^{(n_0)}\}_{t=0}^T$ (see (4.4)). The contradiction we have reached proves Lemma 4.1. □

5 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2 By Proposition 1.1 there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^\infty$ of a

$$\min\{\rho(x_t, \Omega(a)) \colon \ t = 0, \dots, T(\epsilon)\} \leq \epsilon/4. \tag{5.1}$$

By Lemma 4.1 there exists a number $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{T(\epsilon)} \subset X$ satisfying

$$\rho(x_{t+1}, a(x_t)) \leq \delta, \quad t = 0, \dots, T(\epsilon) - 1, \tag{5.2}$$

there exists a sequence $\{y_t\}_{t=0}^{T(\epsilon)} \subset X$ such that

$$y_{t+1} \in a(y_t), \quad t = 0, \dots, T(\epsilon) - 1, \tag{5.3}$$

$$\rho(y_t, x_t) \le \epsilon/4, \quad t = 0, \dots, T(\epsilon). \tag{5.4}$$

Assume that a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfies

$$\rho(x_{t+1}, a(x_t)) \le \delta \ \text{ for all integers } \ t \ge 0. \tag{5.5}$$

It follows from (5.5) and the choice of δ that there exists a sequence $\{y_t\}_{t=0}^{T(\epsilon)} \subset X$ such that (5.3) and (5.4) hold. By (5.3) and the choice of $T(\epsilon)$ (see (5.1)) there is $j \in \{0, \dots, T(\epsilon)\}$ such that $\rho(y_j, \Omega(a)) \le \epsilon/4$. Combined with (5.4) this inequality implies that

$$\rho(x_j, \Omega(a)) \le \rho(x_j, y_j) + \rho(y_j, \Omega(a)) \le \epsilon/2.$$

Theorem 1.2 is proved. □

Proof of Theorem 1.3 Let $\epsilon > 0$. By Theorem 1.1, property (1) holds and there exists a natural number $T(\epsilon) \ge 4$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ of a and each integer $t \ge T(\epsilon)$

$$\rho(x_t, \Omega(a)) \le \epsilon/8. \tag{5.6}$$

By Lemma 4.1 there exists a number $\delta > 0$ such that for each sequence $\{y_t\}_{t=0}^{4T(\epsilon)} \subset X$ satisfying

$$\rho(y_{t+1}, a(y_t)) \le \delta, \quad t = 0, \dots, 4T(\epsilon) - 1 \tag{5.7}$$

there is a sequence $\{z_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that

$$z_{t+1} \in a(z_t), \quad t = 0, \dots, 4T(\epsilon) - 1, \tag{5.8}$$

$$\rho(y_t, z_t) \le \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.9}$$

Assume that a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfies

$$\rho(x_{t+1}, a(x_t)) \le \delta \ \text{ for each integer } \ t \ge 0. \tag{5.10}$$

In view of (5.10) and the choice of δ (see (5.7)-(5.9)) there is a sequence $\{z_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that (5.8) is true and

$$\rho(x_t, z_t) \le \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.11}$$

By (5.8) and the choice of $T(\epsilon)$ (see (5.6))

$$\rho(z_t, \Omega(a)) \le \epsilon/8, \quad t = T(\epsilon), \ \dots, 4T(\epsilon). \tag{5.12}$$

Relations (5.11) and (5.12) imply that for $t = T(\epsilon), \dots, 4T(\epsilon)$

$$\rho(x_t, \Omega(a)) \le \rho(x_t, z_t) + \rho(z_t, \Omega(a)) \le \epsilon/4. \tag{5.13}$$

We show that $\rho(x_t, \Omega(a)) \le \epsilon$ for all integers $t \ge T(\epsilon)$.

Let us assume the converse. Then there is an integer $j \ge T(\epsilon)$ such that

$$\rho(x_j, \Omega(a)) > \epsilon, \tag{5.14}$$

$$\text{if an integer } \ t \ \text{ satisfies } \ T(\epsilon) \le t < j, \ \text{ then } \ \rho(x_t, \Omega(a)) \le \epsilon. \tag{5.15}$$

In view of (5.13)

$$j > 4T(\epsilon). \tag{5.16}$$

For $t = 0, \dots, 4T(\epsilon)$ set

$$y_t = x_{t+j-2T(\epsilon)}. \tag{5.17}$$

By (5.10) and (5.17) for $t = 0, \dots, 4T(\epsilon) - 1$

$$\rho(y_{t+1}, a(y_t)) = \rho(x_{t+j-2T(\epsilon)+1},\, a(x_{t+j-2T(\epsilon)})) \leq \delta.$$

In view of this relation and the choice of δ (see (5.7)–(5.9)) there is a sequence $\{\xi_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that

$$\xi_{t+1} \in a(\xi_t), \quad t = 0, \dots, 4T(\epsilon) - 1, \tag{5.18}$$

$$\rho(\xi_t, y_t) \leq \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.19}$$

It follows from (5.18) and the choice of $T(\epsilon)$ (see (5.6)) that

$$\rho(\xi_t, \Omega(a)) \leq \epsilon/8, \quad t = T(\epsilon),\ \dots, 4T(\epsilon).$$

Together with (5.19) this inequality implies that for $t = T(\epsilon), \dots, 4T(\epsilon)$

$$\rho(y_t, \Omega(a)) \leq \rho(y_t, \xi_t) + \rho(\xi_t, \Omega(a)) \leq \epsilon/4.$$

Together with (5.17) this inequality implies that

$$\rho(x_j, \Omega(a)) = \rho(y_{2T(\epsilon)}, \Omega(a)) \leq \epsilon/4.$$

This relation contradicts (5.14). The contradiction we have reached proves that

$$\rho(x_t, \Omega(a)) \leq \epsilon \text{ for all integers } t \geq T(\epsilon).$$

Theorem 1.3 is proved. □

6 Examples

Denote by $\Pi(X)$ the set of all nonempty closed subsets of (X, ρ). For each $A, B \in \Pi(X)$ set

$$H(A, B) = \max\{\sup_{x \in A} \rho(x, B),\ \sup_{y \in B} \rho(y, A)\}.$$

Clearly the space $(\Pi(X), H)$ is a complete metric space.

Example 6.1 Let $a\colon X \to X$ satisfy $\rho(a(x), a(y)) \leq \rho(x, y)$ for each $x, y \in X$. Since the mapping a is single-valued it is not difficult to see that $a(\Omega(a)) \subset \Omega(a)$ and property (3) from Theorem 1.1 holds.

Example 6.2 Let $a\colon X \to X$ satisfy the following condition:

(C1) for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that for each $x, y \in X$ satisfying $\rho(x, y) \leq \delta$ we have $\rho(a^n x, a^n y) \leq \epsilon$ for all natural numbers n.

Define

$$\rho_1(x, y) = \sup\{\rho(a^n x, a^n y)\colon\ n = 0, 1, \dots\}, \quad x, y \in X.$$

Clearly, (X, ρ_1) is a complete metric space and for each $x, y \in X$ we have $\rho(x,y) \le \rho_1(x,y)$. Let $\epsilon > 0$ and let $\delta \in (0, \epsilon)$ be as guaranteed by (C1). It is clear that $\rho_1(x,y) \le \epsilon$ for each $x, y \in X$ satisfying $\rho(x,y) \le \delta$.

Thus the metrics ρ and ρ_1 induce in X the same topology. It is clear that $\rho_1(a(x), a(y)) \le \rho_1(x,y)$ for each $x, y \in X$. Thus in view of Example 1, property (3) from Theorem 1.1 holds.

Example 6.3 Let $a\colon X \to 2^X \setminus \varnothing$ have a closed graph. Assume that

$$H(a(x), a(y)) \le c\rho(x,y) \ \text{ for all } \ x, y \in X$$

with a constant $c \in (0,1)$. We will show that property (3) from Theorem 1.1 holds. Clearly, it is sufficient to show that

$$a(\Omega(a)) = \cup\{a(z) : \ z \in \Omega(a)\} \subset \Omega(a).$$

Let $z \in a(E_1)$ and let ϵ be a positive number. There exist $x \in E_1$ such that $z \in a(x)$ and $y \in E_2$ such that $\rho(x,y) \le \rho(x, E_2) + \epsilon$. It is not difficult to see that

$$\rho(z, a(E_2)) \le \rho(z, a(y)) \le H(a(x), a(y)) \le c\rho(x,y) \le c\rho(x, E_2) + c\epsilon \le cH(E_1, E_2) + c\epsilon.$$

Since ϵ is an arbitrary positive number we conclude that

$$\rho(z, a(E_2)) \le cH(E_1, E_2) \ \text{ for all } \ z \in a(E_1).$$

Analogously we can show that

$$\rho(y, a(E_1)) \le cH(E_1, E_2) \ \text{ for all } \ y \in a(E_2).$$

Hence

$$H(a(E_1), a(E_2)) \le cH(E_1, E_2) \ \text{ for all } \ E_1, E_2 \in \Pi(X).$$

By this inequality and Banach fixed point theorem there is a unique $\Omega_* \in \Pi(X)$ such that $a(\Omega_*) = \Omega_*$ and that for each $E \in \Pi(X)$

$$a^n(E) \to \Omega_* \ \text{ as } \ n \to \infty \ \text{ in } \ (\Pi(X), H). \tag{6.1}$$

Clearly, $\Omega(a) \subset \Omega_*$. It is sufficient to show that $\Omega(a) = \Omega_*$.

Denote by S_a the set of all continuous functions $s : X \to R^1$ such that $\sup_{y \in a(x)} s(y) \le s(x)$ for all $x \in X$. For each $s \in S_a$ put

$$W_s = \{x \in X\colon \sup_{y \in a(x)} s(y) = s(x)\}.$$

Set

$$W_a = \cap_{s \in S_a} W_a.$$

By Theorem 1 of [5]

$$W_a = \Omega(a).$$

It is sufficient to show that $\Omega_* \subset W_a$.

Let $s \in S_a$. There $x_* \in X$ such that $s(x_*) \le s(x)$ for all $x \in X$. It is clear that $s(y) = s(x_*)$ for each $y \in \cup_{n=1}^{\infty}\{a^n(x) : \ n = 1, 2, \dots\}$. Together with (6.1) this implies that $s(y) = s(x_*)$ for each $y \in \Omega_*$ and that $\Omega_* \subset W_s$. Since this inclusion holds for any $s \in S_a$ we obtain that $\Omega_* \subset W_a$.

Example 6.4 Let $X = [0,1]$, $a(x) = x^2$, $x \in [0,1]$. It is clear that $\Omega(a) = \{0,1\}$ and that $a(\Omega(a)) = \Omega(a)$. It is not difficult to see that for any $z \in (0,1)$ there exists a sequence $\{x_i\}_{i=-\infty}^{\infty} \subset (0,1)$ such that $x_0 = z$ and $x_{i+1} = a(x_i)$ for all integers i. Therefore property (2) of Theorem 1.1 does not hold.

7 Spaces of set-valued mappings

In this section we consider classes of discrete disperse dynamical systems whose global attractors are a singleton.

Denote by $\mathcal{A}$ the set of all mappings $a\colon X \to \Pi(X)$ with closed graphs. For each $a_1, a_2 \in \mathcal{A}$ set

$$d_{\mathcal{A}}(a_1, a_2) = \sup\{H(a_1(x), a_2(x))\colon\ x \in X\}. \tag{7.1}$$

It is clear that the metric space $(\mathcal{A}, d_{\mathcal{A}})$ is complete.

Denote by $\mathcal{A}_c$ the set of all continuous mappings $a : X \to \Pi(X)$ which belong to $\mathcal{A}$, by $\mathcal{A}_f$ the set of all $a \in \mathcal{A}$ such that $a(x)$ is a singleton for each $x \in X$ and set $\mathcal{A}_{fc} = \mathcal{A}_f \cap \mathcal{A}_c$. Clearly $\mathcal{A}_f$, $\mathcal{A}_c$ and $\mathcal{A}_{fc}$ are closed subsets of $(\mathcal{A}, d_{\mathcal{A}})$.

Let $\mathcal{M}$ be one of the following spaces: $\mathcal{A}$; $\mathcal{A}_c$; $\mathcal{A}_f$; $\mathcal{A}_{fc}$. The space $\mathcal{M}$ is equipped with the metric $d_{\mathcal{A}}$.

Denote by $\mathcal{M}_{reg}$ the set of all $a \in \mathcal{M}$ such that $\Omega(a)$ is a singleton and that properties (1–3) from Theorem 1.1 hold.

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $(\mathcal{M}, d_{\mathcal{A}})$. In this section we will establish the following result which shows that most elements of $\bar{\mathcal{M}}_{reg}$ (in the sense of Baire category) belong to $\mathcal{M}_{reg}$.

Theorem 7.1 *The set $\mathcal{M}_{reg}$ contains a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$.*

Proof For each $a \in \mathcal{M}_{reg}$ there is $x_a \in X$ such that

$$\Omega(a) = \{x_a\}. \tag{7.2}$$

Let $a \in \mathcal{M}_{reg}$ and let n be a natural number. Since the mapping a has property (2) from Theorem 1.1 it follows from Theorem 1.3 that there exist a natural number $T(a, n)$ and $\delta(a, n) > 0$ such that the following property holds:

(P1) for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \le \delta(a, n)$, $\quad t = 0, 1, \ldots$ and each integer $t \ge T(a, n)$ we have

$$\rho(x_t, x_a) \le 1/n.$$

Let $\mathcal{U}(a, n)$ be an open neighborhood of a in $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$ such that

$$H(a(x), b(x)) \le \delta(a, n)/2 \ \text{ for each } \ x \in X \ \text{ and each } \ b \in \mathcal{U}(a, n). \tag{7.3}$$

It follows from property (P1) and (7.3) that the following property holds:

(P2) for each $b \in \mathcal{U}(a, n)$ and each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $x_{t+1} \in b(x_t)$, $t = 0, 1, \ldots$

$$\rho(x_t, x_a) \le 1/n \ \text{ for all integers } \ t \ge T(a, n).$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(a, n)\colon\ a \in \mathcal{M}_{reg}\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$. In order to complete the proof it is sufficient to show that $\mathcal{F} \subset \mathcal{M}_{reg}$.

Let $b \in \mathcal{F}$ and $\epsilon > 0$. Choose a natural number n such that

$$n > 8(\min\{1, \epsilon\})^{-1}. \tag{7.4}$$

By the definition of $\mathcal{F}$ there exists $a \in \mathcal{M}_{reg}$ such that

$$b \in \mathcal{U}(a, n). \tag{7.5}$$

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory of b. By (7.5) and property (P2)

$$\rho(x_t, x_a) \le 1/n < \epsilon/8 \ \text{ for all integers } \ t \ge T(a, n). \tag{7.6}$$

Since ϵ is an arbitrary positive number we conclude that $\{x_t\}_{t=0}^{\infty}$ is a Cauchy sequence. Therefore there exists $\lim_{t\to\infty} x_t \in X$. By (7.6),

$$\rho(\lim_{t\to\infty} x_t, x_a) \le \epsilon/8. \tag{7.7}$$

Since ϵ is an arbitrary positive number and $\{x_t\}_{t=1}^{\infty}$ is an arbitrary trajectory of b we conclude that there exists $x_b \in X$ such that $\lim\limits_{t\to\infty} x_t = x_b$ for each trajectory $\{x_t\}_{t=0}^{\infty}$ of b. By (7.7)

$$\rho(x_a, x_b) \le \epsilon/8. \tag{7.8}$$

By (7.6) and (7.8) for each trajectory $\{x_t\}_{t=0}^{\infty}$ of b and all integers $t \ge T(a, n)$

$$\rho(x_t, x_b) \le \rho(x_t, x_a) + \rho(x_a, x_b) \le \epsilon/4.$$

Theorem 7.1 is proved. □

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