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Elements of Stability Theory of A.M. Liapunov for Dynamic Equations on Time Scales

(Devoted to the 150th birthday of A.M. Liapunov)

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Abstract: Stability of dynamic equations on time scales is investigated in this paper. The main results are new conditions for stability, uniform stability, and uniform asymptotic stability of quasilinear and nonlinear systems.

Keywords: *Dynamic equation on time scales; stability; uniform stability; asymptotic stability; nonlinear integral inequality; Liapunov functions.*

Mathematics Subject Classification (2000): 34D20, 39A10.

1 Introduction

The sixth of June, 2007 is the 150th birthday anniversary of the outstanding Russian mathematician and mechanical scientist, Academician Liapunov. A brief outline of the life and activities of Alexander Mikhaylovich Liapunov is contained in [30], while a more detailed outline is given in [22]. The main directions of Liapunov's scientific activities are as follows:

– stability of equilibrium and motion of mechanical systems with a finite number of degrees of freedom;

– equilibrium figures of uniformly rotating liquids;

– stability of equilibrium figures of rotating liquids;

– equations of mathematical physics;

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– probability theory;

– lecture courses on theoretical mechanics.

For a detailed analysis of Liapunov's papers in the above mentioned directions see the survey [33].

Liapunov started publication of his works on the problems of motion stability of systems with a finite number of degrees of freedom in 1888. In 1892 he formulated a strict definition of stability which crowned his intensive work during 1889–1892. The notion of "Liapunov stability" adopted nowadays denotes stability of solutions with respect to perturbation of the initial data over infinite time intervals. The exact definition of stability was of principal importance for further determination of stability criteria of the equilibrium and/or motion of mechanical or other kinds of systems.

In 1892 the Kharkov Mathematical Society published Liapunov's paper "General Problem of Motion Stability" [11]. This work was defended by Liapunov as his doctoral thesis at Moscow University in 1892. In this paper Liapunov considered differential equations of perturbed motion in a quite general form and developed two general methods of analysis of their solutions. The first method is based on the integration of the above mentioned equations by special series. The second technique is based on the application of an auxiliary function whose properties together with properties of its total time derivative along solutions of the system under consideration allow the conclusion on dynamical behavior of solutions for the system.

Alongside these two methods of qualitative analysis of motion equations, Liapunov introduced the notion of a function's characteristic number and applied it to stability analysis of solutions for linear systems of differential equations with variable coefficients. Liapunov completely solved the problem of stability by the first approximation and studied stability of solutions to perturbed motion equations in some critical cases.

The list of references (see [9–23]) presents the papers by Liapunov published to date which deal with stability of systems with a finite number of degrees of freedom, general theory of ordinary differential equations, and classical mechanics. Note that many of Liapunov's papers still remain unpublished.

The aim of our paper is to present some results of stability analysis of solutions for a new class of perturbed motion equations referred to as *dynamic equations on time scales*. Equations on time scales provide a possibility for a simultaneous description of dynamics of continuous-time and discrete-time systems. Such two-mode systems occur in some problems on impulsive control in the description of some technological processes with discrete effects of a catalyst. Some necessary introduction to the mathematical analysis on time scales is presented here in accordance with [2, 3], with vast bibliography therein.

2 Elements of Calculus on Time Scales

2.1 Description of Time Scales

An arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ is referred to as a *time scale* and denoted by $\mathbb{T}$. Examples of time scales are the reals $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$. The most common time scales are $\mathbb{T} = \mathbb{R}$ for continuous calculus, $\mathbb{T} = \mathbb{Z}$ for discrete calculus, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : \ n \in \mathbb{N}_0\}$, where $q > 1$, for quantum calculus.

Definition 2.1

- The *forward* and *backward jump operators* σ and ρ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}:\ s > t\} \quad \text{for all} \quad t \in \mathbb{T}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T}:\ s < t\} \quad \text{for all} \quad t \in \mathbb{T},$$

respectively.

- By means of the operators $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$, the elements $t \in \mathbb{T}$ are classified as follows: If $\sigma(t) = t$, $\rho(t) = t$, $\sigma(t) > t$, and $\rho(t) < t$, then t is called *right-dense*, *left-dense*, *right-scattered*, and *left-scattered*, respectively. Here it is assumed that $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ contains the maximal element t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ contains the minimal element t).

- In addition to the set $\mathbb{T}$, the set $\mathbb{T}^\kappa$ is defined as follows. If $\mathbb{T}$ contains the left scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, and $\mathbb{T}^\kappa = \mathbb{T}$ in the other cases. Therefore,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

- The distance from an arbitrary element $t \in \mathbb{T}$ to its follower is called the *graininess* of the time scale $\mathbb{T}$ and is given by the formula

$$\mu(t) = \sigma(t) - t \quad \text{for all} \quad t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t = \rho(t)$ and $\mu(t) = 0$, and if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$.

In some cases for equations on a time scale $\mathbb{T}$, the principle of induction on time scales is applied. In the monograph [2], this principle is formulated as follows.

Theorem 2.1 *Let $t_0 \in \mathbb{T}$ and $\{S(t):\ t \in [t_0, \infty)\}$ be a set of assertions satisfying the conditions:*

1. *The statement $S(t)$ is true for $t = t_0$.*
2. *If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is true as well.*
3. *If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there exists a neighborhood W of t such that $S(s)$ is true for all $s \in W \cap (t, \infty)$.*
4. *If $t \in (t_0, \infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)$, then $S(t)$ is true.*

Then $S(t)$ is true for all $t \in [t_0, \infty)$.

2.2 Differentiation on Time Scales

Further we shall consider a function $f : \mathbb{T} \to \mathbb{R}$ and determine its Δ-derivative at a point $t \in \mathbb{T}^\kappa$.

Definition 2.2 • The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W-neighborhood of $t \in \mathbb{T}^\kappa$ satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| < \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in W.$$

In this case we shall write $f^\Delta(t) = \gamma$.

• If the function f is Δ-differentiable for any $t \in \mathbb{T}^\kappa$, then $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable on $\mathbb{T}^\kappa$.

Some useful properties of the derivative of a function f are found in the results below.

Theorem 2.2 *Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then the following assertions are true:*

(1) *if f is differentiable at t, then f is continuous at t;*

(2) *if f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

(3) *if t is right-dense, then f is differentiable at t iff there exists the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

as a finite number, and then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s};$$

(4) *if f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Note that, if $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, which is the Cauchy derivative of f, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, which is the forward difference of f.

Further we present the following result.

Theorem 2.3 *Assume that the functions $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then the following assertions are valid:*

(1) *the sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t and*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t);$$

(2) *for any constant α, the function $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t and*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t);$$

(3) *the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t));$$

(4) *if* $f(t)f(\sigma(t)) \neq 0$, *then the function* $1/f$ *is differentiable at* t *and*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))};$$

(5) *if* $g(t)g(\sigma(t)) \neq 0$, *then the function* f/g *is differentiable at* t *and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

2.3 Integration on Time Scales

Further we shall consider functions that are “integrable” on the time scale $\mathbb{T}$.

Definition 2.3 • A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

- A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.
- The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}} = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Theorem 2.4 *Assume that* $f : \mathbb{T} \to \mathbb{R}$. *Then the following assertions are true:*

(1) *If* f *is continuous on* $\mathbb{T}$, *then it is rd-continuous on* $\mathbb{T}$*;*

(2) *if* f *is rd-continuous on* $\mathbb{T}$, *then it is regulated on* $\mathbb{T}$*;*

(3) *the jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is rd-continuous;*

(4) *if* f *is regulated or rd-continuous on* $\mathbb{T}$, *then the function* $f \circ \sigma$ *possesses the same property;*

(5) *if* $f : \mathbb{T} \to \mathbb{R}$ *is continuous and* $g : \mathbb{T} \to \mathbb{R}$ *is regulated and rd-continuous, then the function* $f \circ g$ *possesses the same property.*

Definition 2.4 • A function $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\Delta} = f$ is called an *antiderivative* of the function f.

- If F is an antiderivative of f, then the integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

It is well known that any rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ possesses an antiderivative.

If $f^{\Delta}(t) \geq 0$ on $[a, b]$ and $s, t \in \mathbb{T}$ with $a \leq s \leq t \leq b$, then

$$f(t) = f(s) + \int_s^t f^{\Delta}(\tau)\Delta\tau \geq f(s),$$

i.e., the function f is increasing on $\mathbb{T}$.

Some properties of integration on $\mathbb{T}$ are presented next.

Theorem 2.5 *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in \mathrm{C_{rd}}(\mathbb{T})$. Then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;

(vi) $\int_a^a f(t)\Delta t = 0$;

(vii) $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$;

(viii) *if* $|f| \leq g$ *on* $[a, b)$, *then* $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t$;

(ix) *if* $f \geq 0$ *on* $[a, b)$, *then* $\int_a^b f(t)\Delta t \geq 0$.

Next we shall present some chain rules. We recall that if $f, g : \mathbb{R} \to \mathbb{R}$, then

$$(f \circ g)' = (f' \circ g)g'.$$

The following two chain rules hold.

Theorem 2.6 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{R} \to \mathbb{R}$ is continuous, and $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}$. Then there exists c in the real interval $[t, \sigma(t)]$ such that*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

Theorem 2.7 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}$. Then $(f \circ g) : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable, and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))\mathrm{d}h\right\} g^{\Delta}(t)$$

holds.

Definition 2.5 If $\sup \mathbb{T} = \infty$, then the *improper integral* is defined by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} F(t)\big|_a^b \quad \text{for} \quad a \in \mathbb{T}.$$

2.4 The Exponential Function on Time Scales

An rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}$$

(we write $f \in \mathcal{R}$) and *positively regressive* if

$$1 + \mu(t)f(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}$$

(we write $f \in \mathcal{R}^+$). For the operation $\oplus$ defined by

$$p \oplus q = p + q + \mu pq \quad \text{on} \quad \mathbb{T},$$

the couple $(\mathcal{R}, \oplus)$ is an Abelian group with inverse element

$$\ominus p = -\frac{p}{1 + \mu p} \quad \text{for} \quad p \in \mathcal{R}.$$

We also define $p \ominus q = p \oplus (\ominus q)$. We note that if $p, q \in \mathcal{R}$, then $\ominus p, \ominus q, p \oplus q, p \ominus q, q \ominus p \in \mathcal{R}$.

For the definition of the exponential function on a time scale $\mathbb{T}$, we follow [5] and shall consider for some $h > 0$ the strip

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : \ -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h} \right\}$$

and the set $\mathbb{C}_h$

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : \ z \neq -\frac{1}{h} \right\}.$$

For $h = 0$, we let $\mathbb{Z}_h = \mathbb{C} = \mathbb{C}_h$ be the set of complex numbers. Then for $h \geq 0$ we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by the formula

$$\xi_h = \begin{cases} \dfrac{1}{h}\mathrm{Log}(1 + zh) & \text{if} \quad h > 0, \\ z & \text{if} \quad h = 0. \end{cases}$$

where Log is the principal logarithm function. The inverse cylinder transformation $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ is given by

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h} = (\exp zh - 1)h^{-1}.$$

For a function $p \in \mathcal{R}$, the *exponential function* e_p is defined by the expression

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau \right) \quad \text{for all} \quad (t, s) \in \mathbb{T} \times \mathbb{T}. \tag{2.1}$$

The following properties of the exponential function (2.1) are known (see [2]).

Theorem 2.8 *Let $p, q \in \mathcal{R}$ and $t, r, s \in \mathbb{T}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(viii) *if* $\mathbb{T} = \mathbb{R}$, *then* $e_p(t,s) = e^{\int_s^t p(\tau)d\tau}$;

(ix) *if* $\mathbb{T} = \mathbb{R}$ *and* $p(t) \equiv \alpha$, *then* $e_p(t,s) = e^{\alpha(t-s)}$;

(x) *if* $\mathbb{T} = \mathbb{Z}$, *then* $e_p(t,s) = \prod_{\tau=s}^{t-1}(1+p(\tau))$;

(xi) *if* $\mathbb{T} = h\mathbb{Z}$ *with* $h > 0$ *and* $p(t) \equiv \alpha$, *then* $e_p(t,s) = (1+h\alpha)^{\frac{t-s}{h}}$.

2.5 Variation of Constants on Time Scales

In terms of the exponential function (2.1), there are two variation of constants formulas that read as follows.

Theorem 2.9 *Let* $f \in \mathrm{C_{rd}}$, $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, *and* $x_0 \in \mathbb{R}$. *Then the unique solution of the initial value problem*

$$x^\Delta(t) = -p(t)x(\sigma(t)) + f(t), \quad x(t_0) = x_0$$

is

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t,\tau)f(\tau)\Delta\tau,$$

and the unique solution of the initial value problem

$$x^\Delta(t) = p(t)x(t) + f(t), \quad x(t_0) = x_0$$

is

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

3 Method of Integral Inequalities on Time Scales

The method of integral inequalities for stability analysis of solutions of continuous systems is well developed and its main results are presented in a series of publications, of which we note [26,31]. The development of this method for stability analysis of solutions on a time scale $\mathbb{T}$ is associated with obtaining appropriate inequalities on time scales.

In this section we introduce the method of integral inequalities to study the behavior of solutions of the system of dynamic equations of the perturbed motion

$$x^\Delta = A(t)x + f(t,x), \quad f(t,0) = 0, \tag{3.1}$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ with $n \in \mathbb{N}$, $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, and $F(t) = f(t, x(t))$ satisfies $F \in \mathrm{C_{rd}}(\mathbb{T})$ whenever x is a differentiable function. These assumptions guarantee that the unique solution $x = x(\cdot; t_0, x_0)$ of (3.1) together with the initial condition $x(t_0) = x_0$, where $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$, may be written in the form (see Theorem 2.9 in Section 2.5)

$$x(t) = x(t; t_0, x_0) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau, x(\tau))\Delta\tau. \tag{3.2}$$

In this section, letting $m \in \mathbb{N}$ and subject to the two assumptions

$$\|f(t, x)\| \le a(t)\,\|x\|^m \quad \text{for} \quad t \ge t_0,\ x \in \mathbb{R}^n, \quad \text{where} \quad a \in \mathrm{C_{rd}}(\mathbb{T}) \tag{3.3}$$

and

$$\|e_A(t, s)\| \le \varphi(t)\psi(s) \quad \text{for} \quad t \ge s \ge t_0, \quad \text{where} \quad \varphi, \psi \in \mathrm{C_{rd}}(\mathbb{T}), \tag{3.4}$$

we derive sufficient criteria for stability, uniform stability, and asymptotical stability of the unperturbed motion of (3.1). In the next subsection below we consider the case $m = 1$ while we study the case $m > 1$ in the subsequent subsection. The case $m = 1$ uses the well-known Gronwall inequality on time scales while for the case $m > 1$, a dynamic version of Stachurska's inequality [34] is employed. This inequality is a new result for dynamic equations, so it will be proved in Section 3.2 below.

We will use the following standard definition of different types of stability.

Definition 3.1 The unperturbed motion of (3.1) is said to be

($\mathrm{S_1}$) *stable* if for each $\varepsilon > 0$ and $t_0 \in \mathbb{T}$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x_0\| < \delta \quad \text{implies} \quad \|x(t; t_0, x_0)\| < \varepsilon \quad \text{for all} \quad t \ge t_0;$$

($\mathrm{S_2}$) *uniformly stable* if the δ in ($\mathrm{S_1}$) is independent of t_0;

($\mathrm{S_3}$) *asymptotically stable* if it is stable and there exists δ_0 such that

$$\|x_0\| < \delta_0 \quad \text{implies} \quad \lim_{t\to\infty} x(t; t_0, x_0) = 0.$$

3.1 Stability via Gronwall's Inequality

We start by recalling Gronwall's inequality from [2, Theorem 6.4].

Theorem 3.1 (Gronwall's Inequality) *Let $y, f \in \mathrm{C_{rd}}$ and $p \ge 0$. Then*

$$y(t) \le f(t) + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau \quad \textit{for all} \quad t \ge t_0$$

implies

$$y(t) \le f(t) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau \quad \textit{for all} \quad t \ge t_0.$$

The version we will use is the following inequality from [2, Corollary 6.7].

Corollary 3.1 *Let* $y \in \mathrm{C}_{\mathrm{rd}}$, $p \geq 0$, *and* $\alpha \in \mathbb{R}$. *Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau \quad \text{for all} \quad t \geq t_0$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \quad \text{for all} \quad t \geq t_0.$$

The following main results in this subsection are given for $\mathbb{T} = \mathbb{R}$ in [31, Lemma 2 and Theorem 5].

Lemma 3.1 *Suppose that* (3.3) *for* $m = 1$ *and* (3.4) *hold. Then any solution of* (3.1) *satisfies the estimate*

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0)e_{\varphi\psi^\sigma a}(t, t_0) \|x_0\| \quad \text{for all} \quad t \geq t_0. \tag{3.5}$$

Proof First note that the assumptions of Corollary 3.1 are satisfied. Let x be a solution of (3.1) so that by (3.2) we have for all $t \geq t_0$ the estimate

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0) \|x_0\| + \int_{t_0}^{t} \varphi(t)\psi(\sigma(\tau))a(\tau) \|x(\tau; t_0, x_0)\| \Delta\tau.$$

Hence the function $y = \|x(\cdot; t_0, x_0)\| / \varphi$ satisfies

$$y(t) \leq \psi(t_0) \|x_0\| + \int_{t_0}^{t} \varphi(\tau)\psi(\sigma(\tau))a(\tau)y(\tau)\Delta\tau \quad \text{for all} \quad t \geq t_0.$$

By Corollary 3.1,

$$y(t) \leq \psi(t_0) \|x_0\| e_{\varphi\psi^\sigma a}(t, t_0) \quad \text{for all} \quad t \geq t_0.$$

Using the definition of y, the claim (3.5) follows. □

Theorem 3.2 *Suppose that* (3.3) *for* $m = 1$ *and* (3.4) *hold.*

(i) *If for all* $s \geq t_0$ *there exists* $K(s) > 0$ *such that*

$$\varphi(t)e_{\varphi\psi^\sigma a}(t, s) \leq K(s) \quad \text{for all} \quad t \geq s \geq t_0,$$

then the unperturbed motion of system (3.1) *is stable;*

(ii) *if there exists* $K > 0$ *such that*

$$\varphi(t)\psi(s)e_{\varphi\psi^\sigma a}(t, s) \leq K \quad \text{for all} \quad t \geq s \geq t_0,$$

then the unperturbed motion of system (3.1) *is uniformly stable;*

(iii) *if*

$$\lim_{t\to\infty} \{\varphi(t)e_{\varphi\psi^\sigma a}(t, s)\} = 0,$$

then the unperturbed motion of system (3.1) *is asymptotically stable.*

Proof First we prove (1). Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. Define

$$\delta(\varepsilon, t_0) = \varepsilon K^{-1}(t_0)\psi^{-1}(t_0)$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi^\sigma a}(t, t_0)\delta \le \psi(t_0)K(t_0)\delta = \varepsilon.$$

Now we prove (2). Let $\varepsilon > 0$. Define

$$\delta(\varepsilon) = \varepsilon K^{-1}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi^\sigma a}(t, t_0)\delta \le K\delta = \varepsilon.$$

Finally we prove (3). Since $\varphi e_{\varphi\psi^\sigma a}(\cdot, s)$ tends to zero, it is bounded. By (1), we have stability. Let $\delta_0 = 1$ and assume $\|x_0\| < \delta_0$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi^\sigma a}(t, t_0) \to 0$$

as $t \to \infty$. $\square$

3.2 Stability via Stachurska's Inequality

In preparation for Stachurska's inequality on time scales, we require the following two lemmas.

Lemma 3.2 *If $f \le g$ and $f, g \in \mathcal{R}^+$, then $\ominus f \ge \ominus g$.*

Proof Under the stated assumptions we calculate

$$(\ominus f) - (\ominus g) = -\frac{f}{1+\mu f} + \frac{g}{1+\mu g} = \frac{g-f}{(1+\mu f)(1+\mu g)} \ge 0. \ \square$$

Lemma 3.3 *If $f \ge 0$ and $g \in (0, 1]$, then $\ominus(f/g) \ge (\ominus f)/g$.*

Proof Under the stated assumptions we calculate

$$\left(\ominus \frac{f}{g}\right) - \frac{\ominus f}{g} = -\frac{f}{g+\mu f} + \frac{f}{g+\mu fg} = \frac{\mu f^2(1-g)}{(g+\mu f)(g+\mu fg)} \ge 0. \ \square$$

Theorem 3.3 (Stachurska's inequality) *Assume f, g, p are rd-continuous and nonnegative on $\mathbb{T}$. Let $m \in \mathbb{N} \setminus \{1\}$. If f/p is nondecreasing on $\mathbb{T}$, then each function x satisfying*

$$x(t) \le f(t) + p(t)\int_{t_0}^{t} q(s)x^m(s)\Delta s \quad \textit{for all} \quad t \ge t_0 \tag{3.6}$$

satisfies

$$x(t) \le \frac{f(t)}{\left\{1 + (m-1)\int_{t_0}^{t}(\ominus qpf^{m-1})(s)\Delta s\right\}^{1/(m-1)}} \tag{3.7}$$

on $[t_0, t_m)$, where t_m is the first point for which the denominator on the right-hand side of (3.7) is nonpositive.

Proof We prove the claim by induction. First we assume that (3.6) holds for $m = 2$. Define

$$v(t) := \int_{t_0}^{t} q(s)x^2(s)\Delta s + \frac{f(t)}{p(t)}.$$

Then $x \le pv$ and

$$v^\Delta = qx^2 + \left(\frac{f}{p}\right)^\Delta \le qp^2v^2 + \left(\frac{f}{p}\right)^\Delta$$

and therefore by [2, Theorem 6.1]

$$v(t) \le e_{qp^2v}(t,t_0)\left\{v(t_0) + \int_{t_0}^{t} e_{\ominus qp^2v}(\sigma(s),t_0)\left(\frac{f}{p}\right)^\Delta(s)\Delta s\right\} \le e_{qp^2v}(t,t_0)\frac{f(t)}{p(t)}$$

since $v(t_0) = f(t_0)/g(t_0)$, $(f/p)^\Delta \ge 0$, and $e_{\ominus qp^2v^2}(\sigma(s),t_0) \le 1$. Define now

$$V := e_{\ominus qp^2v}(\cdot,t_0)$$

so that $v \le f/(pV)$ and thus $qp^2v \le qpf/V$ and hence

$$\ominus qp^2v \ge \ominus\frac{qpf}{V} \ge \frac{\ominus qpf}{V},$$

where we used Lemmas 3.2 and 3.3. Hence

$$V^\Delta = (\ominus qp^2v)V \ge \frac{\ominus qpf}{V}V = \ominus qpf.$$

Thus

$$V(t) \ge V(t_0) + \int_{t_0}^{t}(\ominus qpf)(s)\Delta s = 1 + \int_{t_0}^{t}(\ominus qpf)(s)\Delta s$$

and therefore

$$v(t) \le \frac{f(t)}{p(t)V(t)} \le \frac{f(t)}{p(t)\left\{1 + \int_{t_0}^{t}(\ominus qpf)(s)\Delta s\right\}}.$$

Plugging this in the inequality $x \le pv$ yields (3.7) for $m = 2$.

Now we assume that the claim of the theorem holds for some $m \in \mathbb{N}\setminus\{1\}$. Suppose that (3.6) holds with m replaced by $m+1$. Then

$$x(t) \le f(t) + p(t)\int_{t_0}^{t} q(s)x(s)x^m(s)\Delta s$$

and using the induction hypothesis yields

$$x \le \frac{f}{\{1+(m-1)u\}^{1/(m-1)}}, \quad \text{where} \quad u(t) := \int_{t_0}^{t}(\ominus qxpf^{m-1})(s)\Delta s.$$

Now using again Lemmas 3.2 and 3.3, we find

$$u^\Delta = \ominus qxpf^{m-1} \ge \ominus\frac{qpf^m}{\{1+(m-1)u\}^{1/(m-1)}} \ge \frac{\ominus qpf^m}{\{1+(m-1)u\}^{1/(m-1)}}.$$

Thus

$$mu^{\Delta}\{1+(m-1)u\}^{1/(m-1)} \geq m(\ominus qpf^m).$$

Let $F(x)=(1+(m-1)x)^{m/(m-1)}$ for $x \geq 0$ so that $F'(x)=m(1+(m-1)x)^{1/(m-1)}$ is nondecreasing. By Keller's chain rule, Theorem 2.7, we have

$$\begin{aligned}\left\{(1+(m-1)u)^{m/(m-1)}\right\}^{\Delta} &= (F\circ u)^{\Delta} = u^{\Delta}\int_0^1 F'(u(1-h)+hu^{\sigma})\mathrm{d}h \\ &\geq u^{\Delta}\int_0^1 F'(u)\mathrm{d}h = u^{\Delta}F'(u) \geq m(\ominus qpf^m),\end{aligned}$$

where we used $u^{\Delta} \leq 0$ and its consequence $u^{\sigma} \leq u$. Integrating yields

$$\begin{aligned}\{1+(m-1)u\}^{m/(m-1)}(t) &= 1+\int_{t_0}^t \left\{(1+(m-1)u)^{m/(m-1)}\right\}^{\Delta}(s)\Delta s \\ &\geq 1+m\int_{t_0}^t (\ominus qpf^m)(s)\Delta s\end{aligned}$$

and therefore

$$\{1+(m-1)u(t)\}^{1/(m-1)} \geq \left\{1+m\int_{t_0}^t (\ominus qpf^m)(s)\Delta s\right\}^{1/m}.$$

Plugging this in $x \leq f/(1+(m-1)u)^{1/(m-1)}$ gives (3.7) with m replaced by $m+1$. □

The following main results in this subsection are given for $\mathbb{T}=\mathbb{R}$ in [31, Lemma 1 and Theorems 1–3].

Lemma 3.4 *Suppose that* (3.3) *for* $m>1$ *and* (3.4) *hold. Then any solution of* (3.1) *satisfies the estimate*

$$\|x(t;t_0,x_0)\| \leq \frac{\varphi(t)\psi(t_0)\|x_0\|}{\left\{1-(m-1)\|x_0\|^{m-1}\psi^{m-1}(t_0)D(t,t_0)\right\}^{1/(m-1)}} \tag{3.8}$$

for all $t \geq t_0$ *for which*

$$(m-1)\|x_0\|^{m-1}\psi^{m-1}(t_0)D(t,t_0) < 1,$$

where

$$D(t,t_0)=\int_{t_0}^t \phi^m(\tau)\psi(\sigma(\tau))a(\tau)\Delta\tau.$$

Proof First note that the assumptions of Theorem 3.3 are satisfied. Let x be a solution of (3.1) so that by (3.2) we have for all $t \geq t_0$ the estimate

$$\|x(t;t_0,x_0)\| \leq \varphi(t)\psi(t_0)\|x_0\| + \int_{t_0}^t \varphi(t)\psi(\sigma(\tau))a(\tau)\|x(\tau;t_0,x_0)\|^m\Delta\tau.$$

Hence the function $y=\|x(\cdot;t_0,x_0)\|/\varphi$ satisfies

$$y(t) \leq \psi(t_0)\|x_0\| + \int_{t_0}^t \varphi^m(\tau)\psi(\sigma(\tau))a(\tau)y^m(\tau)\Delta\tau \quad \text{for all} \quad t \geq t_0.$$

By Theorem 3.3, as long as the denominator remains positive,

$$y(t) \le \frac{\psi(t_0)\,\|x_0\|}{\left\{1+(m-1)\int_{t_0}^{t}(\ominus\phi^m\psi^\sigma a\psi^{m-1}(t_0)\,\|x_0\|^{m-1})(\tau)\Delta\tau\right\}^{1/(m-1)}}.$$

Since

$$\ominus g = -\frac{g}{1+\mu g} \ge -g \quad \text{for all} \quad g \ge 0,$$

and using the definition of y, the claim (3.8) follows. □

Theorem 3.4 *Suppose that* (3.3) *for* $m > 1$ *and* (3.4) *hold.*

(i) *If for all* $s \ge t_0$ *there exists* $K(s) > 0$ *such that*

$$\varphi(t) \le K(s) \quad \textit{for all} \quad t \ge s \ge t_0$$

and

$$D(t_0) := \lim_{t\to\infty} D(t,t_0) < \infty, \tag{3.9}$$

then the unperturbed motion of system (3.1) *is stable;*

(ii) *if there exist* $K_1, K_2 > 0$ *such that*

$$\varphi(t)\psi(s) \le K_1 \quad \textit{for all} \quad t \ge s \ge t_0$$

and

$$\psi^{m-1}(s)\left\{\lim_{t\to\infty} D(t,s)\right\} \le K_2 \quad \textit{for all} \quad s \ge t_0,$$

then the unperturbed motion of system (3.1) *is uniformly stable;*

(iii) *if* (3.9) *and*

$$\lim_{t\to\infty}\varphi(t) = 0$$

hold, then the unperturbed motion of system (3.1) *is asymptotically stable.*

Proof First we prove (1). Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. Define

$$\delta(\varepsilon,t_0) = \min\left\{[2(m-1)\psi^{m-1}(t_0)D(t_0)]^{-1/(m-1)}, \varepsilon\psi^{-1}(t_0)K^{-1}(t_0)2^{-1/(m-1)}\right\}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.4,

$$\begin{aligned}\|x(t;t_0,x_0)\| &< \frac{\varphi(t)\psi(t_0)\delta}{\{1-(m-1)\delta^{m-1}\psi^{m-1}(t_0)D(t,t_0)\}^{1/(m-1)}}\\ &\le \frac{\varphi(t)\psi(t_0)\varepsilon\psi^{-1}(t_0)K^{-1}(t_0)2^{-1/(m-1)}}{\{1-(m-1)2^{-1}(m-1)^{-1}\psi^{1-m}(t_0)D^{-1}(t_0)\psi^{m-1}(t_0)D(t,t_0)\}^{1/(m-1)}}\\ &\le \frac{\varepsilon 2^{-1/(m-1)}}{\{1-2^{-1}\}^{1/(m-1)}} = \varepsilon.\end{aligned}$$

Now we prove (2). Let $\varepsilon > 0$. Define

$$\delta(\varepsilon) = \min\left\{[2(m-1)K_2]^{-1/(m-1)}, \varepsilon K_1^{-1}2^{-1/(m-1)}\right\}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.4,

$$\begin{aligned}\|x(t;t_0,x_0)\| &< \frac{\varphi(t)\psi(t_0)\delta}{\{1-(m-1)\delta^{m-1}\psi^{m-1}(t_0)D(t,t_0)\}^{1/(m-1)}} \\ &\le \frac{\varphi(t)\psi(t_0)\varepsilon K_1^{-1}2^{-1/(m-1)}}{\left\{1-(m-1)2^{-1}(m-1)^{-1}K_2^{-1}\psi^{m-1}(t_0)D(t,t_0)\right\}^{1/(m-1)}} \\ &\le \frac{\varepsilon 2^{-1/(m-1)}}{\{1-2^{-1}\}^{1/(m-1)}} = \varepsilon.\end{aligned}$$

Finally we prove (3). Since φ tends to zero, it is bounded. By (1), we have stability. Let $\delta_0 > 0$ be such that the denominator in (3.8) is positive and assume $\|x_0\| < \delta_0$. Then by Lemma 3.4,

$$\|x(t;t_0,x_0)\| < \frac{\varphi(t)\psi(t_0)\delta_0}{\left\{1-(m-1)\delta_0^{m-1}\psi^{m-1}(t_0)D(t,t_0)\right\}^{1/(m-1)}} \to 0$$

as $t \to \infty$. □

4 Generalized Direct Liapunov Method on Time Scales

4.1 General Theorems

The direct method of investigation of motion stability of continuous systems with a finite number of degrees of freedom as developed by Liapunov is now extended for many classes of systems of equations. In this section we present the main theorems of the direct Liapunov method for dynamic equations on a time scale $\mathbb{T}$.

Corresponding to the time scale $\mathbb{T}$ we consider the following sets:

$$\begin{aligned}\mathcal{A} &= \{t \in \mathbb{T}:\ t \text{ left-dense and right-scattered}\},\\ \mathcal{B} &= \{t \in \mathbb{T}:\ t \text{ left-scattered and right-dense}\},\\ \mathcal{C} &= \{t \in \mathbb{T}:\ t \text{ left-scattered and right-scattered}\},\\ \mathcal{D} &= \{t \in \mathbb{T}:\ t \text{ left-dense and right-dense}\}.\end{aligned}$$

Assume that $\sup \mathbb{T} = a \in \mathcal{A} \cup \mathcal{D}$ and $\inf \mathbb{T} = b \in \mathcal{B} \cup \mathcal{D}$ and designate the Euler derivative of the state vector of system $x : \mathbb{T} \to \mathbb{R}^n$ in $t \in \mathbb{T}$ by $\dot{x}(t)$, should it exists.

We consider a system of perturbed motion equations

$$x^{\Delta}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{4.1}$$

where $x : \mathbb{T} \to \mathbb{R}^n$, $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, and

$$x^{\Delta}(t) = \begin{cases} \dfrac{x(\sigma(t)) - x(t)}{\mu(t)} & \text{if} \quad t \in \mathcal{A} \cup \mathcal{C}, \\ \dot{x}(t) & \text{in other points.} \end{cases}$$

Our assumptions on system (4.1) are as follows:

- H_1 The vector-valued function $F(t) = f(t, x(t))$ satisfies the condition $F \in C_{\mathrm{rd}}(\mathbb{T})$ whenever x is a differentiable function with its values in N, where $N \subset \mathbb{R}^n$ is an open connected neighborhood of the state $x = 0$.

H_2 The vector-valued function $f(t,x)$ is component-wise regressive, i.e.,

$$e^T + \mu(t)f(t,x) \neq 0 \text{ for all } t \in [t_0,\infty), \quad \text{where} \quad e^T = (1,\dots,1)^T \in \mathbb{R}^n.$$

H_3 $f(t,x) = 0$ for all $t \in [t_0,\infty)$ iff $x = 0$.

H_4 The graininess function μ satisfies $0 < \mu(t) \in M$ for all $t \in \mathbb{T}$, where M is a compact set.

For stability analysis of the state $x = 0$ of system (4.1), the matrix-valued function [24]

$$U(t,x) = [v_{ij}(t,x)], \quad i,j = 1,\dots,m \tag{4.2}$$

will be applied as an auxiliary function, where $v_{ii} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}_+$ for $i,j = 1,\dots,m$ and $v_{ij} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ for $i \neq j$, $i,j = 1,\dots,m$. The elements $v_{ij}(t,x)$ of the matrix-valued function (4.2) are assumed to satisfy the following conditions:

(1) $v_{ij}(t,x)$ are locally Lipschitzian in x for all $t \in \mathbb{T}$;

(2) $v_{ij}(t,x) = 0$ for all $t \in \mathbb{T}$ iff $x = 0$;

(3) $v_{ij}(t,x) = v_{ji}(t,x)$ for all $t \in \mathbb{T}$ and $i,j = 1,\dots,m$.

Along with the function (4.2) we shall use the scalar function

$$v(t,x,\theta) = \theta^T U(t,x)\theta, \quad \theta \in \mathbb{R}^m_+ \tag{4.3}$$

and comparison functions of class K. Recall that a real-valued function a belongs to the class K if it is definite continuous and strictly increasing on $[0,r_1]$ with $0 \leq r_1 < +\infty$ and $a(0) = 0$.

Definition 4.1 The matrix-valued function (4.2) is called

(1) *positive* (*negative*) *semidefinite* on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$, if $v(t,x,\theta) \geq 0$ ($v(t,x,\theta) \leq 0$) for all $(t,x,\theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$, respectively;

(2) *positive definite* on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$, if there exists a function $a \in K$ such that $v(t,x,\theta) \geq a(\|x\|)$ for all $(t,x,\theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$;

(3) *decrescent* on $\mathbb{T} \times N$ if there exists a function $b \in K$ such that $v(t,x,\theta) \leq b(\|x\|)$ for all $(t,x,\theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$;

(4) *radially unbounded* on $\mathbb{T} \times N$, if $v(t,x,\theta) \to +\infty$ for $\|x\| \to +\infty$, for $(t,x,\theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$.

Lemma 4.1 *The matrix-valued function* $U : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$ *is positive definite on* $\mathbb{T}$ *iff the function* (4.3) *can be represented as*

$$\theta^T U(t,x)\theta = \theta^T U_+(t,x)\theta + a(\|x\|), \quad t \in \mathbb{T},$$

where U_+ *is a positive semidefinite matrix-valued function and* $a \in K$.

Lemma 4.2 *The matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is decrescent on $\mathbb{T}$ iff the function* (4.3) *can be represented as*

$$\theta^T U(t,x)\theta = \theta^T U_-(t,x)\theta + b(\|x\|), \quad t \in \mathbb{T},$$

where U_- is a negative semidefinite matrix-valued function and $b \in K$.

Further we need the notion of the total Δ-derivative of the function (4.3) along solutions of system (4.1). It reads as

$$v_+^\Delta(t,x,\theta) = \theta^T U_+^\Delta(t,x)\theta, \quad \theta \in \mathbb{R}_+^m, \quad t \in \mathbb{T},$$

where $U_+^\Delta(t,x)$ is calculated element-wise by the formula

$$U_+^\Delta(t) = \begin{cases} \overline{\lim}\{[u_{ij}(t+h) - u_{ij}(t)]h^{-1} : \ h \to 0, \ h + t \in \mathbb{T}\} & \text{if} \quad t = \sigma(t), \\ [u_{ij}(\sigma(t)) - u_{ij}(t)]\mu^{-1}(t) & \text{if} \quad t < \sigma(t), \end{cases}$$

where $u_{ij}(t) = u_{ij}(t, x(t;t_0,x_0))$, $1 \le i,j \le m$.

We note that the calculation of the derivative is not easy in general. However, if the function (4.3) is independent of t, then it may be easy to calculate the Δ-derivative.

Example 4.1 Consider the function $v(t,x,\theta) = x^T x$, $x \in \mathbb{R}^n$. Then by Theorem 2.3 (3) we have

$$\begin{aligned} v_+^\Delta(t,x,\theta) &= (x^T x)^\Delta(t) = x^T(t)x^\Delta(t) + (x^T)^\Delta(t)x(\sigma(t)) \\ &= x^T(t)f(t,x(t)) + f^T(t,x(t))[x(t) + \mu(t)f(t,x(t))]. \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and

$$v_+^\Delta(t,x,\theta) = \frac{d}{dt}(x^T x) = x^T f(t,x) + f^T(t,x)x.$$

Example 4.2 Consider the function $U(t,x) = xx^T$, $x \in \mathbb{R}^n$. By Theorem 2.3 (3) we have

$$\begin{aligned} v_+^\Delta(t,x,\theta) &= \theta^T (xx^T)^\Delta(t)\theta \\ &= \theta^T\{x(t)f^T(t,x(t)) + f(t,x)x^T(t) + \mu(t)f(t,x(t))f^T(t,x(t))\}\theta. \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and

$$v_+^\Delta(t,x,\theta) = \theta^T \frac{d}{dt}(xx^T)\theta = \theta^T\{x(t)f^T(t,x) + f(t,x)x^T(t)\}\theta.$$

Next, we shall formulate a general Liapunov-type result on stability of the state $x = 0$ of system (4.1).

Theorem 4.1 *Assume that the vector-valued function $f(t,x)$ in system* (4.1) *satisfies assumptions H_1–H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Assume there exist*

(1) *a matrix-valued function $U : \mathbb{T} \times N \to \mathbb{R}^{m\times m}$ and a vector $\theta \in \mathbb{R}_+^m$ such that the function $v(t,x,\theta) = \theta^T U(t,x)\theta$ is locally Lipschitzian in x for all $t \in \mathbb{T}$;*

(2) *comparison functions* $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K$ *and symmetric* $m \times m$ *matrices* A_j, $j = 1, 2$, *such that for all* $(t, x) \in \mathbb{T} \times N$

(a) $\psi_1^T(\|x\|)A_1\psi_1(\|x\|) \le v(t, x, \theta)$;

(b) $v(t, x, \theta) \le \psi_2^T(\|x\|)A_2\psi_2(\|x\|)$;

(c) *there exists an* $m \times m$ *matrix* $A_3 = A_3(\mu(t))$ *such that*

$$v_+^\Delta(t, x, \theta) \le \psi_3^T(\|x\|)A_3\psi_3(\|x\|) \quad \textit{for all} \quad (t, x) \in \mathbb{T} \times N;$$

(d) *there exists* $\mu^* > 0$ *such that* $\mu^* \in M$ *and*

$$\frac{1}{2}\left[A_3^T(\mu(t)) + A_3(\mu(t))\right] \le A_3(\mu^*) \quad \textit{whenever} \quad 0 < \mu(t) < \mu^*.$$

Then, if the matrices A_1 *and* A_2 *are positive definite and the matrix* $A_3^* = A_3(\mu^*)$ *is negative semidefinite, then the state* $x = 0$ *of system* (4.1) *is stable under conditions 2(a), 2(b), 2(d) and uniformly stable under conditions 2(a)–2(d).*

Proof The fact that A_1 and A_2 are positive definite matrices implies that $\lambda_m(A_1) > 0$ and $\lambda_M(A_2) > 0$, where $\lambda_m(A_1)$ and $\lambda_M(A_2)$ are the minimal and maximal eigenvalues of the matrices A_1 and A_2, respectively. In view of this fact we present the estimates (a) and (b) from condition (2) as

$$\lambda_m(A_1)\bar{\psi}_1(\|x\|) \le v(t, x, \theta) \le \lambda_M(A_2)\bar{\psi}_2(\|x\|) \quad \text{for all} \quad (t, x) \in \mathbb{T} \times N,$$

where $\bar{\psi}_1, \bar{\psi}_2 \in K$ so that

$$\bar{\psi}_1(\|x\|) \le \psi_1^T(\|x\|)\psi_1(\|x\|),\ \bar{\psi}_2(\|x\|) \ge \psi_2^T(\|x\|)\psi_2(\|x\|) \quad \text{for all} \quad x \in N.$$

Let $\varepsilon > 0$. Let $S(t)$ be the following assertion:

There exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$.

Let

$$S^* = \{t \in [t_0, \infty) : \quad S(t) \text{ is false}\}.$$

Let us show that under our assumptions the set S^* is empty. Assume on the contrary $S^* \ne \varnothing$. The fact that S^* is closed and nonempty implies that $\inf S^* = t^* \in S^*$. First notice that $S(t_0)$ is true, since $\|x(t_0; t_0, x_0)\| < \varepsilon$ for $\|x_0\| < \varepsilon$ because $x(t_0; t_0, x_0) = x_0$. Therefore $t^* > t_0$. Then pick $\delta_1 = \delta_1(\varepsilon)$ such that

$$\lambda_M(A_2)\bar{\psi}_2(\delta_1) < \lambda_m(A_1)\bar{\psi}_1(\varepsilon).$$

Define $\delta = \min\{\varepsilon, \delta_1\}$ so that

$$\|x(t^*; t_0, x_0)\| = \varepsilon \quad \text{and} \quad \|x(t; t_0, x_0)\| < \varepsilon \text{ for } t \in [t_0, t^*) \text{ and } \|x_0\| < \delta.$$

By conditions 2(c) and 2(d) we have

$$v_+^\Delta(t, x, \theta) \le \lambda_M(A_3^*)\bar{\psi}_3(\|x\|) \le 0 \quad \text{for all} \quad (t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m.$$

Hence, for $t = t^*$,

$$\begin{aligned} \lambda_m(A_1)\bar{\psi}_1(\varepsilon) = \lambda_m(A_1)\bar{\psi}_1(\|x(t^*; t_0, x_0)\|) &\le v(t^*, x(t^*), \theta) \\ &\le v(t_0, x_0, \theta) < \lambda_M(A_2)\bar{\psi}_2(\delta) \end{aligned}$$

for $\|x_0\| < \delta$. This contradiction yields that $S(t^*)$ is true so that $t^* \notin S^*$. Hence $S^* = \varnothing$ and the proof is complete. □

Corollary 4.1 (cf. [7]) *Let the vector-valued function f in system* (4.1) *satisfy hypotheses H_1–H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose there exist at least one couple of indices $(p, q) \in [1, m]$ for which $(v_{pq}(t, x) \neq 0) \in U(t, x)$ and the function $v(t, x, \theta) = e^T U(t, x)e = v(t, x)$ for all $(t, x) \in \mathbb{T} \times N$ satisfies the conditions*

(a) $\psi_1(\|x\|) \le v(t, x)$;

(b) $v(t, x) \le \psi_2(\|x\|)$;

(c) $v^{\Delta}(t, x)\big|_{(4.1)} \le 0$ *for all* $0 < \mu(t) < \mu^* \in M$,

where ψ_1, ψ_2 are some functions of class K. Then the state $x = 0$ of system (4.1) *is stable under conditions (a) and (c) and uniformly stable under conditions (a)–(c).*

Theorem 4.2 *Assume that the vector-valued function $f(t, x)$ in system* (4.1) *satisfies hypotheses H_1–H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Assume there exist*

(1) *a matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$ and a vector $\theta \in \mathbb{R}^m_+$ such that the function $v(t, x, \theta) = \theta^T U(t, x)\theta$ is locally Lipschitzian in x for all $t \in \mathbb{T}$;*

(2) *comparison functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K$ and symmetric $m \times m$ matrices B_j, $j = 1, 2, 3$ such that*

  (a) $\psi_1^T(\|x\|)B_1\psi_1(\|x\|) \le v(t, x, \theta)$;

  (b) $v(t, x, \theta) \le \psi_2^T(\|x\|)B_2\psi_2(\|x\|)$ *for all* $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$;

  (c) *there exists an $m \times m$ matrix $B_3 = B_3(\mu(t))$ such that*

$$v_+^{\Delta}(t, x, \theta) \le \psi_3^T(\|x\|)B_3\psi_3(\|x\|) + w(t, \psi_3(\|x\|))$$

  for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}^m_+$, where $w(t, \cdot)$ satisfies the condition

$$\lim \frac{|w(t, \psi_3(\|x\|))|}{\|\psi_3\|} = 0 \quad \textit{as} \quad \|\psi_3\| \to 0$$

  uniformly with respect to $t \in \mathbb{T}$;

  (d) *there exists $\mu^* > 0$ such that $\mu^* \in M$ and*

$$\frac{1}{2}\left[B_3^T(\mu(t)) + B_3(\mu(t))\right] \le B_3(\mu^*) \quad \textit{for all} \quad 0 < \mu(t) < \mu^*.$$

Then, if the matrices B_1 and B_2 are positive definite and the matrix $B_3^* = B_3(\mu^*)$ is negative definite, then

(a) *under conditions 2(a) and 2(c) the state $x = 0$ of system* (4.1) *is asymptotically stable on $\mathbb{T}$;*

(b) *under conditions 2(a)–2(c) the state $x = 0$ of system* (4.1) *is uniformly asymptotically stable on $\mathbb{T}$.*

Proof Consider the assertion

$$\{S_1(t)\colon \quad S(t) \ \text{ for } \ t \in [t_0, \infty) \ \text{ and } \ \lim_{t\to\infty} \|x(t; t_0, x_0)\| = 0, \ \text{ if } \ \|x_0\| < \delta(t_0)\}.$$

Following considerations similar to those in the proof of Theorem 4.1, one can easily verify the assertions. □

Corollary 4.2 (cf. [7]) *Let the vector-function f in system (4.1) satisfy hypotheses H_1–H_4 on $\mathbb{T}\times N$, $N\subset\mathbb{R}^n$. Suppose there exist at least one couple of indices $(p,q)\in[1,m]$ for which $(v_{pq}(t,x)\neq 0)\in U(t,x)$ and the function $v(t,x,\theta)=e^TU(t,x)e=v(t,x)$ for all $(t,x)\in\mathbb{T}\times N$ satisfies the conditions*

(a) $\psi_1(\|x\|)\le v(t,x)$;

(b) $v(t,x)\le\psi_2(\|x\|)$;

(c) *for all* $0<\mu(t)<\mu^*\in M$

$$v^\Delta(t,x)|_{(4.1)}\le -\psi_3(\|x\|)+w(t,\psi_3(\|x\|))$$

and

$$\lim\frac{|w(t,\psi_3(\|x\|))|}{\psi_3(\|x\|)}\quad as\quad \psi_3\to 0$$

uniformly with respect to $t\in\mathbb{T}$, where ψ_1,ψ_2,ψ_3 are comparison functions of class K.

Then, under conditions (a) and (c) the state $x=0$ of system (4.1) is asymptotically stable and under conditions (a)–(c) the state $x=0$ of system (4.1) is uniformly asymptotically stable.

Theorem 4.3 *Assume that the vector-valued function $f(t,x)$ in system (4.1) satisfies hypotheses H_1–H_4 on $\mathbb{T}\times N$, $N\subset\mathbb{R}^n$. Suppose*

(1) *there exist a matrix-valued function $U:\mathbb{T}\times\mathbb{R}^n\to\mathbb{R}^{m\times m}$ and a vector $\theta\in\mathbb{R}^m_+$ such that the function $v(t,x,\theta)=\theta^TU(t,x)\theta$ is locally Lipschitzian in x for all $t\in\mathbb{T}$;*

(2) *there exist comparison functions $\psi_1,\psi_3\in K$ and a symmetric $m\times m$ matrix A_1 such that for $(t,x)\in\mathbb{T}\times N$*

   (a) $\psi_1^T(\|x\|)A_1\psi_1(\|x\|)\le v(t,x,\theta)$;

   (b) *there exists an $m\times m$ matrix $C_3=C_3(\mu(t))$ such that*

$$v_+^\Delta(t,x,\theta)\ge\psi_3^T(\|x\|)C_3\psi_3(\|x\|)\quad for\ all\quad (t,x,\theta)\in\mathbb{T}\times L\times\mathbb{R}^m_+,\ L\subset N;$$

   (c) *there exists an $m\times m$ matrix $C_3(\mu^*)\ge\frac{1}{2}[C_3^T(\mu(t))+C_3(\mu(t))]$ for some $\mu^*\in M$ at $t\in\mathbb{T}$;*

(3) *the point $x=0$ belongs to the boundary L;*

(4) $v(t,x,\theta)=0$ *on $\mathbb{T}\times(\partial L\cap B_\Delta)$, where $B_\Delta=\{x\in\mathbb{R}^n\colon\ \|x\|<\Delta\}$.*

Then, if the matrices A_1 and $C_3(\mu^*)$ are positive definite, then the state $x=0$ of system (4.1) is unstable.

Proof The proof is based on the assertion

$$\begin{aligned}\{S_2(t)\colon\ &\text{there exist } t_1\in[t_0,\infty)\ \text{ such that } \|x(t_1;t_0,x_0)\|>\varepsilon\\ &\text{for any }\ 0<\delta<\varepsilon,\ \text{for which }\ \|x_0\|<\delta\}\end{aligned}$$

and follows arguments similar to those of the proof of Theorem 4.1. □

Corollary 4.3 (cf. [7]) *Let the vector-function f in system* (4.1) *satisfy hypotheses H_1–H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose there exist at least one couple $(p, q) \in [1, m]$ such that for $(v_{pq}(t, x) \neq 0) \in U(t, x)$ and the function $v(t, x, e) = e^T U(t, x)e = v(t, x)$ for all $(t, x) \in \mathbb{T} \times N$ satisfies the conditions*

(a) $\psi_1(\|x\|) \leq v(t, x)$, $\psi_1 \in K$;

(b) *for all* $0 < \mu(t) < \mu^* < M$ *the inequality* $v_+^\Delta(t, x, \theta)\big|_{(4.1)} \geq \psi_3(\|x\|)$, $\psi_3 \in K$ *holds;*

(c) *the point* $(x = 0) \in \partial L$;

(d) $v(t, x) = 0$ *on* $\mathbb{T} \times (\partial\mathbb{T} \cap B_\Delta)$.

Then the state $x = 0$ of system (4.1) *is unstable.*

Example 4.3 Consider the perturbed motion equations on $\mathbb{T}$ with the graininess function $0 < \mu(t) < +\infty$

$$\begin{aligned} x^\Delta &= y(x + y), & x(t_0) &= x_0, \\ y^\Delta &= -x(x + y), & y(t_0) &= y_0. \end{aligned} \tag{4.4}$$

For the function $v(x, y) = x^2 + y^2$ we have

$$v_+^\Delta(x(t), y(t))\big|_{(4.4)} = \mu(t)(x + y)^2(x^2 + y^2) \tag{4.5}$$

which translates for the case $\mathbb{T} = \mathbb{R}$ to

$$\dot{v}(x(t), y(t)) = 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

Condition (4.5) implies that $x = y = 0$ of system (4.4) is stable when $\mathbb{T} = \mathbb{R}$, while $x = y = 0$ of system (4.4) is unstable whenever the graininess function satisfies $0 < \mu(t) < +\infty$.

Example 4.4 Let a system of dynamic equations

$$\begin{aligned} x^\Delta &= -x - y(x^2 + y^2), & x(t_0) &= x_0, \\ y^\Delta &= -y + x(x^2 + y^2), & y(t_0) &= y_0 \end{aligned} \tag{4.6}$$

be given. For the positive definite function $v(x, y) = x^2 + y^2$ we have

$$v_+^\Delta(x(t), y(t))\big|_{(4.6)} = -2(x^2 + y^2) + \mu(t)[x^2 + y^2 + (x^2 + y^2)^3] \tag{4.7}$$

which translates for the case $\mathbb{T} = \mathbb{R}$ to

$$\dot{v}(x(t), y(t)) = -2(x^2 + y^2) \quad \text{for all} \quad t \in \mathbb{R}.$$

The analysis of (4.7) shows that $x = y = 0$ of the system (4.6) is asymptotically stable when $\mathbb{T} = \mathbb{R}$. If the time scale $\mathbb{T}$ has the graininess $\mu(t) = 1$, i.e., $\mathbb{T} = \mathbb{Z}$, then for the initial values (x_0, y_0) from the domain $x_0^2 + y_0^2 < 1$, the zero solution of system (4.6) is asymptotically stable on $\mathbb{Z}$. If $\mu(t) = 2$, which corresponds to the time scale $\mathbb{T} = 2\mathbb{N}_0 = \{k_0, k_0 + 2, k_0 + 4, \dots\}$, then

$$v^\Delta(x(t), y(t))\big|_{(4.6)} = 2(x^2 + y^2)^3,$$

and the state $x = y = 0$ of system (4.6) is unstable.

4.2 Linear Systems

Consider a time scale $\mathbb{T}$ and a linear homogeneous dynamic system

$$x^\Delta(t) = A(t)x(t), \quad t \in \mathbb{T}, \tag{4.8}$$

where the matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is rd-continuous and regressive. Together with equation (4.8), we consider the initial value problem

$$x^\Delta(t) = A(t)x(t), \quad x(s) = x_0,$$

where $s \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

In some cases the behavior of the solution x of system (4.8) can be investigated by means of the function $v(x) = x^T x$ for which

$$v^\Delta(x(t))|_{(4.8)} = x^T(A^T \oplus A)(t)x,$$

where $(A^T \oplus A)(t) = A^T(t) + A(t) + \mu(t)A^T(t)A(t)$. We define the sets

$$\Lambda_s(\mathbb{T}) = \{A \in \mathcal{R}(\mathbb{T})\colon\ \exists\, c \in \mathcal{R}^+ \text{ for which } \ (A^T \oplus A)(t) \le 2cI < 0 \ \text{ for all } t \in \mathbb{T}\}$$

and

$$\Lambda_u(\mathbb{T}) = \{A \in \mathcal{R}(\mathbb{T})\colon\ \exists\, c > 0 \ \text{ for which } (A^T \oplus A)(t) \ge 2cI \ \text{ for all } t \in \mathbb{T}\},$$

where I is the $n \times n$ identity matrix and $\mathcal{R}^+$ is the set of positively regressive functions (see Section 2.4). Let the norm of the matrix M be defined by $\|M\| = \sup_{u\neq 0} \dfrac{|Mu|}{|u|}$.

The following results are known [1].

Theorem 4.4 *Consider system* (4.8). *If $A \in \Lambda_s(\mathbb{T})$, then*

(a) $\|e_A(t,s)\| \le e_c(t,s)$ *for all* $s \le t$*;*

(b) $\|e_A(t,s)\| \ge e_c(t,s)$ *for all* $s \ge t$*;*

(c) $\lim\limits_{t\to\infty} \|e_A(t,s)\| = 0$ *for every fixed* s *and* $\lim\limits_{s\to-\infty} \|e_A(t,s)\| = 0$ *for every fixed* t.

If $A \in \Lambda_u(\mathbb{T})$, then

(d) $\|e_A(t,s)\| \ge e_c(t,s)$ *for all* $t \le s$*;*

(e) $\|e_A(t,s)\| \le e_c(t,s)$ *for all* $t \ge s$*;*

(f) $\lim\limits_{t\to-\infty} \|e_A(t,s)\| = \infty$ *for every fixed* s *and* $\lim\limits_{s\to\infty} \|e_A(t,s)\| = 0$ *for every fixed* t.

The proof of these assertions is based on the analysis of the Δ-derivative of the function $v(x) = x^T x$:

$$v^\Delta(x(t))|_{(4.8)} = (2 \odot c)v(x(t)),$$

where $2 \odot c = c \oplus c = 2c + \mu(t)c^2$.

Now we apply Theorems 4.1, 4.2, 4.3 to system (4.8). Assume that in the matrix-valued function $U(t,x)$ the elements $v_{ij}(t,x)$, $i,j = 1,2,\ldots,n$ are such that $v_{ii}(t,x) = x_i^2$, $i = 1,2,\ldots,n$ and $v_{ij}(t,x) \equiv 0$ for $i \neq j$. In this case, the function (4.3) with $\theta = (1,1,\ldots,1)^T \in \mathbb{R}^n_+$ is of the form

$$v(t,x,\theta) = \theta^T U(t,x)\theta = x^T x. \tag{4.9}$$

Theorem 4.5 *Let the system* (4.1) *be of the form* (4.8) *and the function* (4.3) *be of the form* (4.9). *Then, if there exists* $\mu^* \in M$ *such that the matrix* $D_0(t,\mu(t))$ *in the expression*

$$v_+^\Delta(t,x(t)) = x^T(t)D_0(t,\mu(t))x(t), \quad \textit{where} \quad D_0(t,\mu(t)) = (A^T \oplus A)(t),$$

is negative semidefinite (negative definite) whenever $0 < \mu(t) \le \mu^*$, *then the equilibrium state* $x = 0$ *of system* (4.8) *is stable (asymptotically stable), respectively.*

Proof The statements of the theorem follow from Theorem 4.1. □

Next, we shall consider the case when

$$v(t,x,\theta) = \theta^T U(t,x)\theta = x^T H(t)x, \quad t \in \mathbb{T}^\kappa, \tag{4.10}$$

where $H \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}^\kappa, \mathbb{R}^{n\times n})$, and assume that the condition

$$\alpha\|x(t)\|^2 \le x^T H(t)x \le \beta\|x(t)\|^2 \quad \text{for all} \quad t \in \mathbb{T}^\kappa, \tag{4.11}$$

is satisfied, where $\alpha, \beta > 0$ are constants.

Theorem 4.6 (cf. [4]) *Let the system* (4.1) *be of the form* (4.8) *and suppose that the function* (4.10) *satisfies the estimate* (4.11). *Then, if there exists* $\mu^* \in M$ *such that the matrix* $D_1(t,\mu(t))$ *in the expression*

$$v_+^\Delta(t,x(t))\big|_{(4.8)} = x^T(t)D_1(t,\mu)x(t), \tag{4.12}$$

where

$$D_1(t,\mu) = (I+\mu A^T(t))H^\Delta(t)(I+\mu A(t)) + A^T(t)H(t) + H(t)A(t) + \mu A^T(t)H(t)A(t), \tag{4.13}$$

is negative semidefinite (negative definite) for all $0 < \mu(t) \le \mu^*$, *then the state* $x = 0$ *of system* (4.8) *is uniformly stable (uniformly asymptotically stable), respectively.*

Proof The statements of this theorem follow from Theorem 4.2. □

Remark 4.1 If in the expression (4.13) the Δ-derivative of the matrix $H(t)$ satisfies $H^\Delta(t) \equiv 0$ for all $t \in \mathbb{T}^\kappa$, then the analysis of $v_+^\Delta(t,x(t))\big|_{(4.8)}$ being of definite sign is simplified.

Now we assume that there exists a positive definite constant matrix Q, $Q = Q^T$, such that

$$A^T(t)H(t) + H(t)A(t) + \mu(t)A^T(t)H(t)A(t) = -Q. \tag{4.14}$$

Then the expression (4.12) becomes

$$v_+^\Delta(t,x(t))\big|_{(4.8)} = x^T(t)[(I+\mu(t)A^T(t))H^\Delta(t)(I+\mu(t)A(t)) - Q]x(t), \quad t \in \mathbb{T}^\kappa.$$

By the equation

$$(I+\mu(t)A^T(t))H^\Delta(t)(I+\mu(t)A(t)) - Q = 0$$

we define $\mu_{\max} = \max\{\mu(t) : \ t \in \mathbb{T}^\kappa\} \in M$.

Theorem 4.7 *Let system* (4.1) *be of form* (4.8) *and suppose that the function* (4.10) *satisfies condition* (4.11). *If for* $0 < \mu(t) < \mu_{\max}$,

$$A(t)H(t) + H(t)A(t) + \mu(t)A^T(t)H(t)A(t) \leq -Q,$$

then the state $x = 0$ *of system* (4.8) *is uniformly asymptotically stable.*

Proof All conditions of Theorem 4.2 from Section 4.1 are satisfied and thus the state $x = 0$ of system (4.8) is uniformly asymptotically stable. □

Remark 4.2 The matrix equation (4.14) is a generalization of the known matrix Liapunov equation [35]

$$A^T H + HA = -Q \tag{4.15}$$

for a stable linear autonomous system, whose solution is known in the form

$$H = \int_0^\infty \exp(A^T s) Q \exp(As) ds.$$

The matrix A in equation (4.15) is constant and stable.

In order to construct the solution H for equation (4.14) on $\mathbb{T}^\kappa$, we use the following result from [2].

Lemma 4.3 *Let be given* $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ *and* $C : \mathbb{T} \to \mathbb{R}^{n\times n}$. *If the matrix-valued function* C *is differentiable and is a solution of the dynamic matrix equation*

$$C^\Delta(\tau) = A(\tau)C(\tau) - C(\sigma(\tau))A(\tau),$$

then

$$C(\tau)e_A(\tau, s) = e_A(\tau, s)C(s).$$

Corollary 4.4 *Let* $A \in \mathcal{R}$. *If the constant matrix* C *commutes with* $A(t)$, *then* C *commutes with* $e_A(t)$. *In particular, if* A *is a constant matrix, then* A *commutes with* $e_A(t)$.

Using Lemma 4.3 and Corollary 4.4, the solution of equation (4.14) is obtained in [4] in the following form.

Theorem 4.8 *Assume that system* (4.8) *is such that all eigenvalues of the* $n \times n$ *matrix-valued function* A *are in the Hilger circle, i.e.,* $\{z \in \mathbb{C}\colon\ |z + \frac{1}{h}| = \frac{1}{h}\}$, $h > 0$ *for all* $t \geq t_0$. *Then for every* $t \in \mathbb{T}$ *there exists a time scale* $\mathbb{S}$ *such that the integration on* $\mathbb{T}_S = [0, \infty)$ *enables one to find the solution of equation* (4.14) *in the form*

$$H(t) = \int_{\mathbb{T}_S} e_{A^T}(s, 0) Q e_A(s, 0) \Delta s. \tag{4.16}$$

Besides, if the matrix Q *is positive definite, then the matrix* $H(t)$ *is also positive definite for all* $t \geq t_0$.

Proof This assertion is proved by direct substitution of expression (4.16) into the left-hand part of equation (4.14). Moreover, when $\mu(t) > 0$, then $\mathbb{S} = \mu(t)\mathbb{N}_0$, and when $\mu(t) = 0$, then $\mathbb{S} = \mathbb{R}$. □

Theorem 4.9 *Let system* (4.1) *be of form* (4.8) *and suppose the function* (4.10) *satisfies the estimate* $\alpha\|x(t)\|^2 \leq x^T H(t)x$*, where* $\alpha > 0$ *is constant, for all* $(t, x) \in \mathbb{T} \times \mathbb{R}^n$*,* $H \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$*. If there exists a value* $0 < \mu^* \in M$ *such that for at least one value of* $t^* \in \mathbb{T}$*, the matrix* $D_1(t^*, \mu(t^*))$ *in* (4.12) *is positive semidefinite (positive definite), then the state* $x = 0$ *of system* (4.8) *is unstable (strongly unstable).*

Strong instability is understood as exponential growth of solutions x on $\mathbb{T}$ of system (4.8).

In the end of this section we note that in [8] there is a result on the existence of a Liapunov function in the case of uniform exponential stability of the zero solution of system (4.8) in the form

$$v(t, x) = \sup_{\tau \in A_t} \|x(t + \tau;\, t, x)\| e^{c\tau}, \tag{4.17}$$

where $A_t = \{\tau \in [0, \infty) :\ t + \tau \in \mathbb{T}\}$. Conversion theorems with functions of type (4.17) for continuous systems are proved in [35, 36].

5 Concluding Remarks and Bibliography

The proofs of all assertions set out in Section 2 are found in [2, 3] (see also [5, 6]). The sufficient conditions of stability, uniform stability asymptotic stability and instability presented in the paper are obtained in terms of two general approaches set out in this paper. Namely, in Section 3, an approach is presented based on the application of integral inequalities on time scales. For stability analysis of the unperturbed motion of the quasilinear system (3.1), the known Gronwall inequality [2] and the nonlinear Stachurska inequality on time scales are applied, the latter being first established in this paper. This inequality is proved for the case of $m \in \mathbb{N} \setminus \{1\}$ in inequality (3.6).

In Section 4, stability analysis of system (4.1) is carried out in terms of the generalized direct Liapunov method. This generalization is associated with the application of a matrix-valued function for dynamic equations on time scales. Such investigations were undertaken in [29]. The application of matrix-valued functions for dynamic equations on time scales allows the construction of a heterogeneous Liapunov function [25], i.e., the functions consisting of continuous and discrete components, which is impossible to do in the framework of scalar Liapunov functions. Some concretization was made for the choice of Liapunov function in the investigation of linear dynamic equations on time scales.

In [1], the authors found new conditions on the coefficient matrix for certain perturbed linear dynamic equations (4.8) on time scales ensuring that there exists a bounded solution (which is explicitly given) to which all other solution converge, and similar conditions ensuring the existence of a bounded solution from which all other solutions diverge. In that paper, also periodic time scales and corresponding linear dynamic equations with periodic coefficients are considered and similar statements about periodic solutions to which all other solutions converge or from which all other solutions diverge are proved.

We note that in [8], the authors found conditions for the existence of a Liapunov function for the linear system (4.8) in the case of exponential stability of the state $x = 0$ on time scales. Thus, the versatility of the direct Liapunov method for dynamic equation on time scales was demonstrated.

We also remark that the construction of a general stability theory for dynamic equations on time scales is an *open problem* in the theory of this class of equations. The extension of the proposed approaches to the analysis of oscillatory systems [27, 28] as well as hybrid systems [32] containing continuous and discrete components is of undoubt interest for applications.

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Lyapunov Based on Cascaded Non-linear Control of Induction Machine

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Abstract: In this paper is developed Lyapunov based non-linear control to ensure the flux-speed tracking regime of voltage fed induction machine. The control law is determined in two steps, in the first the virtual control, based on Lyapunov function, is obtained in view to impose the flux-speed tracking. After this, is deduced the real control imposing the virtual control law. The simulation results of flux-speed tracking of induction machine show the validity of the proposed method in presence of strong parametric perturbations. Finally, an extension of the proposed method to most voltage alternating current (AC) machines is discussed. This allows to get a unified view for the control of electric AC machines.

Keywords: *Lyapunov method; virtual control; flux speed-tracking; induction machine; AC machines.*

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1 Introduction

One of the fundamental deficiencies of non linear control theory is the lack of a systematic design procedure for controllers synthesis. The earlier work of Lyapunov produced some of most powerful tools for control design that are still used up to date. In this work, the design problem is formulated in terms of finding a suitable state function (so called Lyapunov function) having some properties that guarantee boundedness of trajectories and convergence to an equilibrium point. Although this result is one of the most significant ones in control theory, there is no general theory for constructing such a Lyapunov

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function for a given non linear systems. Meanwhile, the backstepping methodology developed in [1] allows to construct recursively the non linear feedback law and its associated Lyapunov functions for a certain class of non linear systems. Beside this, a great effort was devoted to developing other methods for the control design of a certain class of non linear systems. Among these methods are found the following ones : a the diffeomorphic transformation, non linear feedback linearization, the sliding mode approach, and the dynamic linearization.

The analysis and control problem of cascaded non linear systems have been intensively studied during the last decades (see[2]-[7] and reference there in). In [2], based on the explicit construction of a Lyapunov function for a partially linear cascaded system, a stabilizing controller is designed for a special class of non linear cascaded systems. Sontag in [3] gave some sufficient conditions for asymptotic stabilization of two cascaded non linear systems. A passivity interpretation of this latter result is given in [4]. In [5], a wide class of time varying non linear systems is considered. The authors in [6] gave sufficient conditions under which an interconnected non linear system with parametric uncertainty is stabilizable. Singular H_∞ suboptimal control of a class of two blocks interconnected non linear systems is investigated [7].

Otherwise, the development of electrical machine drive growths more and more in order to follow the increasing need for various fields such as industry, electric cars, actuators, etc. By means of electrical machine drive, we can get high level of productivity in industry and product quality enhancement. Among, the most used electrical AC machine one can mention induction machine, permanent magnet synchronous machine, and synchronous machine. However, the induction machine is the machine of choice in many industrial applications due to its reliability, ruggedness and relatively low cost.

The control of electric machines has become an active domain of research over the last few years. Different control methods such as field oriented control, exact linearization, passivity approach and sliding mode control have been reported in literature. The field orientation control, which gives high dynamic response, ensures torque/flux decoupling of AC machines assuming exact knowledge of rotating field [8, 9, 10]. This assumption is difficult to realise in practice and the high performance of such strategy is often deteriorated due to significant plant uncertainties. These later include, in general, magnetic saturation or motor winding temperature change or motor internal parameters variance.

The control of AC machines can be decoupled and linear by means of non linear feedback linearization [11, 12]. However this method have some disadvantages:

i) the necessary and sufficient condition for linearization can't be held all the time,

ii) singular point exist,

iii) requires relatively complicated differential geometry to derive the control law.

Contrary, the passivity based control does not decoupling the system, but it has an outstanding advantage-simplicity, because it does not cancel all the nonlinearities. As the result it does not have any preliminary requirement or singular point. The passivity theory based on the control of AC machines is developed in [13] and experimental results for induction machines are given in [14, 15].

Due to its simplicity and attractive robustness properties, the sliding mode theory is widely applied in electrical drives. In [16], the fundamental principles of sliding mode control and its application to electrical machines are formulated. Example of real time sliding mode application involving induction motors is reported in [17, 18]. The cascaded structure is exploited in [19] to obtain a nonlinear predictive control of induction machine.

The authors in [20, 21], use the backstepping to derive the control for the torque and the field amplitude of induction motors in rotating (d, q) reference frame. As pointed out by the authors the proposed control law is not robust in face to parameter variations and necessitate the adaptive strategy for parameters involved in control law. Moreover, no information is given about the generalization of this method for other AC machines.

In this paper is developed Lyapunov based non-linear control to ensure the flux-speed tracking regime of voltage fed induction machine (see also [24]). The control law is determined in two steps, in the first the virtual control, based on Lyapunov function, is obtained in view to get the flux-speed tracking. After this, is deduced the real control imposing the virtual control law. From the fact that most voltage fed AC machines belongs to the same class of cascaded non linear systems, we show how to generalize the proposed method to these AC machines. This generalization allows to get a unified view for the control of most voltage fed AC machine.

This paper is organized as follows. The formulated problem is given in Section 2 where the induction motor model is seen in the cascaded system form and also for other electric machines. Section 3 is devoted to the development of the real control law in order to involve the flux-speed tracking objectives for the induction machines and some remarks are pointed out in the end of this section. The stability analysis of the induction motor under the proposed control law is discussed in Section 4. The application and simulation results appears in Section 5.

2 Formulation problem

In order to control induction machine, we give in first its model. In the stator reference frame, the state space model of voltage fed induction machine is obtained from Park's model. The state vector is composed of the stator current components (i_α, i_β), the rotor flux components $(\phi_\alpha, \phi_\beta)$ and the rotor rotating pulsation ω_r, whereas a vector control is composed of the stator voltage components (v_α, v_β) and the external disturbance is represented by the load torque Γ_r. By introducing our notation, the state vector and the control vector are respectively represented by :

$$\begin{pmatrix} \xi & \eta \end{pmatrix}^t = \begin{pmatrix} \xi_1 & \xi_2 & \eta_1 & \eta_2 & \eta_3 \end{pmatrix}^t = \begin{pmatrix} i_\alpha & i_\beta & \phi_\alpha & \phi_\beta & \omega_r \end{pmatrix}^t,$$
$$u^t = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^t = \begin{pmatrix} v_\alpha & v_\beta \end{pmatrix}^t.$$

Using these notations, the dynamic of voltage fed induction machine takes the form:

$$\begin{cases} \dot{\xi}_1 = f_1 + d_1 u_1, \ f_1 = -a_1\xi_1 + b_1\eta_1 + c_1\eta_2\eta_3, \\ \dot{\xi}_2 = f_2 + d_1 u_2, \ f_2 = -a_1\xi_2 + b_1\eta_2 - c_1\eta_1\eta_3, \\ \dot{\eta}_1 = F_1, \qquad F_1 = a_3\xi_1 - b_3\eta_1 - \eta_2\eta_3, \\ \dot{\eta}_2 = F_2, \qquad F_2 = a_3\xi_2 - b_3\eta_2 + \eta_1\eta_3, \\ \dot{\eta}_3 = F_3, \qquad F_3 = -a_5\eta_3 - c_5\Gamma_r + b_5(\eta_1\xi_2 - \eta_2\xi_1). \end{cases} \tag{1}$$

It is now well understood that flux reference can be used as an additional degree of freedom to improve motor efficiency (minimizing looses) or to maximize the delivered torque (minimum time). So, in this work, we are interested by the outputs represented by the rotor magnitude flux $\phi = \varphi_\alpha^2 + \varphi_\beta^2$ and the rotor rotating pulsation ω_r with $y = (y_1\ y_2)^t = (\phi\ \ \omega_r)^t$ it leads to :

$$\begin{cases} y_1 = h_1(\eta), \ h_1(\eta) = \eta_1^2 + \eta_2^2, \\ y_2 = h_2(\eta), \ h_2(\eta) = \eta_3, \end{cases} \tag{2}$$

where the positive coefficients $(a_1, ..., c_5)$ are given by

$a_1 = \frac{1}{\sigma T_s} + \frac{1-\sigma}{\sigma T_r}$, $b_1 = \frac{(1-\sigma)}{\sigma M T_r}$, $c_1 = \frac{(1-\sigma)}{\sigma M}$, $d_1 = \frac{1}{\sigma L_s}$, $a_3 = \frac{M}{T_r}$, $b_3 = \frac{1}{T_r}$,
$a_5 = \frac{k_f}{J}$, $b_5 = \frac{p^2 M}{J L_r}$, $c_5 = \frac{p}{J}$, $\sigma = 1 - \frac{M}{L_s L_r} < 1$

and related to the following machine parameters :

T_s, T_r :	the stator and rotor electric time constant;
σ:	the leakage coefficient;
L_s, L_r:	the cyclic stator inductance, the cyclic rotor inductance;
M :	the cyclic mutual inductance between stator and rotor;
k_f :	the friction coefficient and Γ_r is a load torque;
J :	the inertia and p is the pairs of poles.

The induction motor dynamic (1) with associated outputs (2) is square non linear system where the input u and the output y are that $u \in R^2$ and $y \in R^2$. The functions $f(.) = [f_1, \;\; f_2]^T$and $F(.) = [F_1 \; F_2 \; F_3]^T$ are continuous, moreover $h(.) = [h_1 \;\; h_2]^T$are continuous radially unbounded functions. Due to physical considerations, it is known that machine parameters are always positive and they may be constant or they change in continuous manner so, coefficients $(a_1...c_5)$ are positive bounded. The state vector ξ and the output $y_2 = \eta_3$ which represent respectively the stator current components and the rotor speed are in practice easily measured. On the other hand, the output y_1 which is the magnitude flux is derived from flux components (η_1,η_2). These later are generally observed and there exist enormous literature about this [9, 17].

We attach to the system (1), the outputs dynamic given by

$$\begin{cases} \dot{y}_1 = H_1(\xi,\eta) = \pi_1(\eta) + \psi_1(\xi,\eta), \\ \dot{y}_2 = H_2(\xi,\eta) = \pi_2(\eta) + \psi_2(\xi,\eta), \end{cases} \tag{3}$$

with

$$\begin{cases} \pi_1(\eta) = -2b_3(\eta_1^2 + \eta_2^2), \\ \pi_2(\eta) = -a_5\eta_3 - c_5\Gamma_r, \end{cases} \tag{4}$$

and

$$\begin{cases} \psi_1(\xi,\eta) = 2a_3(\eta_1\xi_1 + \eta_2\xi_2), \\ \psi_2(\xi,\eta) = b_5(\eta_1\xi_2 - \eta_2\xi_1). \end{cases} \tag{5}$$

Let us define the tracking errors e_1 and e_2 by

$$\begin{cases} e_1 = y_1 - y_{1d}, \\ e_2 = y_2 - y_{2d}, \end{cases} \tag{6}$$

their dynamics are:

$$\begin{cases} \dot{e}_1 = \pi_1(\eta) + \psi_1(\xi,\eta) - \dot{y}_{1d}, \\ \dot{e}_2 = \pi_2(\eta) + \psi_2(\xi,\eta) - \dot{y}_{2d}, \end{cases} \tag{7}$$

where y_{1d} and y_{2d} are desired trajectories.

The problem we are concerned with, consists of developing the control law u that allows the output $y_i (i = 1, 2)$ to track the desired trajectories $y_{id} (i = 1, 2)$. From the fact that the desired output trajectory may be defined by a signal external to the control system so that y_{id} and its time derivatives ($\ddot{y}_{id}$, $\dot{y}_{id}$, for $i = 1, 2$) may be measured or provided by a reference signal. Therefore, we assume that the reference signal y_{id} and its derivatives ($\ddot{y}_{id}$, $\dot{y}_{id}$, for $i = 1, 2$) are bounded and measurable. Our procedure to tackle the control problem is similar in spirit to the backsteeping methodology developed in [1]. In fact, the control problem is constructed in two steps :

i) Step1: For the tracking errors (e_1, e_2), we determine, based on the Lyapunov method, the desired values ψ_{1d} and ψ_{2d}, for respectively the functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$, which ensure the asymptotic convergence of the tracking errors e_1 and e_2 to zero. Therefore, functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ are seen as a virtual control signals for the output dynamic (7).

ii) Step 2: Based on the plant dynamic (1), we search for the real control signal u that constrain the functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$, to take respectively the desired values ψ_{1d} and ψ_{2d} and to ensure asymptotic converge of the tracking errors e_1 and e_2.

Remark 2.1 In stator reference frame, the dynamic of two phase symmetric induction machine and voltage fed is similarly modelled by system (1), when the outputs are chosen as rotor flux and rotor speed. For the case of permanent magnet synchronous machine, the developments are given in Appendix. In general, the dynamic models of most voltage fed machines can be put under a same general class of non linear system. This class is a cascade non linear of the underlying form :

$$\begin{cases} \dot{\xi} = f(\xi,\eta) + g(\xi,\eta).u, \\ \dot{\eta} = F(\xi,\eta), \\ y = h(\eta), \end{cases}$$

with the outputs dynamic given by:

$$\dot{y} = \pi(\eta) + \psi(\xi,\eta) = H(\xi,\eta).$$

The control input u and the measured output vector y are that $u \in R^m$ and $y \in R^m$. The state vectors $\xi \in R^p$, $\eta \in R^q$ with $p \geq 1$ and $q \geq 1$ are available by measure or by observation. Functions $f(.)$, $g(.)$, $F(.)$, $h(.)$ are known continuous and $h(.)$ is continuous radially unbounded functions.

3 Control Law Synthesis

Consider two continuous function $\Lambda(x)$ and $S(x)$ satisfying $\Lambda(x) > 0$, $\forall x \neq 0$ and $xS(x) > 0$, $\forall x \neq 0$, the following result can be established.

Proposition 3.1 *If the system (1) is in closed loop with the following real control law*

$$u = A^{-1}(\xi,\eta)\left(B(\xi,\eta) - e - k.S(z)\right) \quad with \quad k1, k2 > 0, \tag{8a}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A^{-1}(\eta,\xi)\left[\begin{pmatrix} B_1(\eta,\xi) \\ B_2(\eta,\xi) \end{pmatrix} - \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} - \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} S(z_1) \\ S(z_2) \end{pmatrix}\right], \tag{8b}$$

$$A(\eta,\xi) = \begin{pmatrix} 2a_3 d_1 \eta_1 & 2a_3 d_1 \eta_2 \\ -b_5 d_1 \eta_2 & b_5 d_1 \eta_1 \end{pmatrix}, \; z_i = \psi_i(\xi,\eta) - \psi_{id} \;\; with \;\; i = 1,2, \tag{8c}$$

$$\psi_{id} = -q_i e_i \Lambda(e_i) - \pi_i(\eta) + \dot{y}_{id} \;\; with \;\; i = 1,2, \tag{8d}$$

$$B_1(\eta,\xi) = -2a_3(\eta_1 f_1 + \eta_2 f_2 + \xi_1 F_1 + \xi_2 F_2) + \dot{\psi}_{1d}, \tag{8e}$$

$$B_2(\eta,\xi) = -b_5(\xi_2 F_1 + \eta_1 f_2 - \eta_2 f_1 - \xi_1 F_2) + \dot{\psi}_{2d}, \tag{8f}$$

then the outputs error (e_i, $i = 1,2$) *are bounded and converge at least asymptotically to the origin.*

Proof The proof is based on the two steps discussed in Section 2.

Step 1. Based on the dynamic (7), it is possible to search the desired values ψ_{1d} and ψ_{2d} that must take the functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ in order to force the asymptotic converge of the errors $(e_1,\, e_2)$. $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ constitute the virtual control laws for the tracking errors e_1 and e_2 so, they does not form the real control for the induction motor. To this end, let us consider the following Lyapunov function related to the system (7) :

$$V_i(e_i) = \frac{1}{2}(e_i)^2 \quad \text{for} \quad i = 1,2 \tag{9}$$

its time derivative is then

$$\dot{V}_i(e_i) = e_i \dot{e}_i \quad \text{for} \quad i = 1,2 \tag{10}$$

if the virtual control law $\psi_i(\xi,\eta)$ with $i = 1,2$ are equal to the desired value ψ_{id} the dynamic tracking error, given by expression (7), can be rewrite under the form :

$$\dot{e}_i = \pi_i(\eta) + \psi_{id} - \dot{y}_{1d} \quad \text{with} \quad i = 1,2. \tag{11}$$

By replacing ψ_{id} by its expression (8d), the tracking error dynamic (11) is reduced to:

$$\dot{e}_i = -q_i e_i \Lambda(e_i) \tag{12}$$

$$= \pi_i(\eta) + \psi_{id} - \dot{y}_{1d}. \tag{13}$$

With relation (12), the time derivative of Lyapunov function (10) becomes:

$$\dot{V}_i = -q_i(e_i)^2\Lambda(e_i) \quad with \quad i = 1,2. \tag{14}$$

To have $\dot{V}_i < 0\ \forall\ e_i \neq 0$ it is sufficient that $q_i > 0$ and $\Lambda(e_i) > 0,\ \forall e_i \neq 0$. Hence, e_i tend to zero at least asymptotically.

Step 2. Now, we must determine the real control input u, which, in same time, constrain the functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ to follow respectively the desired values ψ_{1d} and ψ_{2d} and the tracking errors (e_1, e_2) converge asymptotically to zero.

Indeed, adding and subtracting the desired values ψ_{1d} and ψ_{2d} in the equation (7), this latter becomes :

$$\begin{cases} \dot{e}_1 = \pi_1(\eta) + \psi_1(\xi,\eta) - \psi_{1d} + \psi_{1d} - \dot{y}_{1d}, \\ \dot{e}_2 = \pi_2(\eta) + \psi_2(\xi,\eta) - \psi_{2d} + \psi_{2d} - \dot{y}_{2d}, \end{cases} \tag{15}$$

and we define the error variables as:

$$\begin{cases} z_1 = \psi_1(\xi,\eta) - \psi_{1d}, \\ z_2 = \psi_2(\xi,\eta) - \psi_{2d}. \end{cases} \tag{16}$$

By introducing these two variables z_1 and z_2 in the system (15) it leads to :

$$\begin{cases} \dot{e}_1 = \pi_1(\eta) + \psi_{1d} - \dot{y}_{1d} + z_1, \\ \dot{e}_2 = \pi_2(\eta) + \psi_{2d} - \dot{y}_{2d} + z_2, \end{cases} \tag{17}$$

and replacing ψ_{1d} and ψ_{2d} by their expression (8d), the precedent relation becomes

$$\begin{cases} \dot{e}_1 = -q_1 e_1 \Lambda(e_1) + z_1, \\ \dot{e}_2 = -q_2 e_2 \Lambda(e_2) + z_2. \end{cases} \tag{18}$$

Besides, the time derivative of the variables z_1 and z_2 are obtained from relation (16) :

$$\begin{cases} \dot{z}_1 = 2a_3\left(\dot{\eta}_1\xi_1 + \eta_1\dot{\xi}_1 + \dot{\eta}_2\xi_2 + \eta_2\dot{\xi}_2\right) - \dot{\psi}_{1d}, \\ \dot{z}_1 = b_5\left(\dot{\eta}_1\xi_2 + \eta_1\dot{\xi}_2 - \dot{\eta}_2\xi_1 - \eta_2\dot{\xi}_1\right) - \dot{\psi}_{2d}. \end{cases} \tag{19}$$

By replacing the dynamics ($\dot{\eta}_1$, $\dot{\eta}_2$, $\dot{\xi}_1$, $\dot{\xi}_2$) by their respective expressions from (1), it leads to :

$$\begin{cases} \dot{z}_1 = 2a_3(F_1.\xi_1 + \eta_1.f_1 + F_2.\xi_2 + \eta_2 f_2) - \dot{\psi}_{1d} + 2a_3 d_1\eta_1 u_1 + \eta_2 d_1 u_2, \\ \dot{z}_2 = b_5(F_1\xi_2 + \eta_1 f_2 - F_2\xi_1 - \eta_2 f_1) - \dot{\psi}_{2d} - b_5 d_1\eta_2 u_1 + b_5 d_1\eta_1 u_2, \end{cases} \tag{20}$$

or in compact form :

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = -\begin{pmatrix} B_1(\xi,\eta) \\ B_2(\xi,\eta) \end{pmatrix} + A(\xi,\eta)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{21}$$

Let be the Lyapunov functions candidate V_{1a} and V_{2a} related to the systems (18) and (21) which are defined by :

$$\begin{cases} V_{1a} = \frac{1}{2}e_1^2 + \frac{1}{2}z_1^2, \\ V_{2a} = \frac{1}{2}e_2^2 + \frac{1}{2}z_2^2, \end{cases} \tag{22}$$

exploiting relation (18), the time derivative of expression (22) is then:

$$\begin{pmatrix} \dot{V}_{a,1}(e_1,z_1) \\ \dot{V}_{a,2}(e_2,z_2) \end{pmatrix} = -\begin{pmatrix} q_1 e_1^2\Lambda(e_1) \\ q_2 e_2^2\Lambda(e_2) \end{pmatrix} + \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\left[\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}\right] \tag{23}$$

introducing (21) in (23) induces

$$\begin{aligned} \begin{pmatrix} \dot{V}_{a,1}(e_1,z_1) \\ \dot{V}_{a,2}(e_2,z_2) \end{pmatrix} &= -\begin{pmatrix} q_1 e_1^2\Lambda(e_1) \\ q_2 e_2^2\Lambda(e_2) \end{pmatrix} \\ &\quad + \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\left[\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} + A(\xi,\eta)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]. \end{aligned} \tag{24}$$

By using the control law (8a) in (24) it leads to :

$$\begin{pmatrix} \dot{V}_{a,1}(e_1,z_1) \\ \dot{V}_{a,2}(e_2,z_2) \end{pmatrix} = -\begin{pmatrix} q_1 e_1^2\Lambda(e_1) \\ q_2 e_2^2\Lambda(e_2) \end{pmatrix} - \begin{pmatrix} k_1 z_1 S(z_1) \\ k_2 z_2 S(z_2) \end{pmatrix}. \tag{25}$$

The relation (25) allows to conclude that the variables (e_1, z_1, e_2, z_2) are bounded and they converge at least asymptotically to zero. So, the functions $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ follow respectively the desired value ψ_{1d} and ψ_{2d} and the outputs y_1 and y_2 track respectively their reference y_{1d} and y_{2d}. ■

Remark 3.1 $\Lambda(e_i)$ must be continuous satisfying $\Lambda_1(e_i) > 0$, $\forall e_i$ is realized by the function $\Lambda(e_i) = e_i^n$ for n even natural number or $\Lambda(e_i) = \cosh(e_i)$. The function $S(z_i)$ is continuous satisfying $z_i S(z_i) > 0$, $\forall z_i \neq 0$ can be implemented by any continuous function like a sign function by example smooth function defined by $S(z_i) = \frac{z_i}{|z_i| + \epsilon_i}$ with $i > 0$ and $i = i, 2$ or by $S(z_i) = \tanh(z_i)$.

Remark 3.2 The determination of the input vector u is possible only if the matrix $A(\eta,\xi)$ has an inverse. Its determinant given by $d_1 a_3 b_5(\eta_1^2 + \eta_2^2)$ is always positive if the rotor flux magnitude $\eta_1^2 + \eta_2^2$ is different from zero. This latter condition is verified since the machine is connected to the supply.

Remark 3.3 In the case where, the time derivative relating to the outputs y_1 and y_2 and the states η are used in place of their expressions from (1), for the determination of the control law, the vector $B(\eta, \xi)$ takes the form :

$$\begin{aligned} B_1(\eta,\xi) &= -2a_3(\eta_1 f_1 + \eta_2 f_2 + \xi_1\dot{\eta}_1 + \xi_2\dot{\eta}_2) + \dot{\psi}_{1d}, \\ B_2(\eta,\xi) &= -b_5(\xi_2\dot{\eta}_1 + \eta_1 f_2 - \eta_2 f_1 - \xi_1\dot{\eta}_2) + \dot{\psi}_{2d}, \end{aligned}$$

with

$$\dot{\psi}_{1d} = -q_1(\dot{y}_1 - \dot{y}_{1d}) + 2b_3\dot{y}_1 + \ddot{y}_{1d}, \qquad \dot{\psi}_{2d} = -q_2(\dot{y}_2 - \dot{y}_{1d}) + a_5\dot{y}_2 + c_5\dot{\Gamma}_r + \ddot{y}_{2d},$$

if function $\Lambda(x)$ is taken in its simplest form: $\Lambda(x) = 1$. Meanwhile in practice, the time derivative of signals included in control law are generally not used due to the unavoidable noise affecting the measured signals and may produce important spikes on the time derivative signals.

Remark 3.4 The global Lyapunov function and the global augmented Lyapunov function for the original system are :

$$V(e) = \sum_{i=1}^{2} V_i(e_i) \qquad \textit{and} \qquad V_a(e,z) = \sum_{i=1}^{2} V_{ai}(e_i, z_i) = V(e) + \frac{1}{2}\sum_{i=1}^{2} z_i^2.$$

Remark 3.5 In the general case of the cascade non linear system introduced in Remark 2.1, the control input can be derived in the following form :

$$u = A^{-1}(\eta,\xi).\left(B(\eta,\xi) - k.S(z)\right),$$

where

$$k.S(z) = \begin{pmatrix} k_1.S(z_1) \\ \vdots \\ k_m.S(z_m) \end{pmatrix},\ z_i = \psi_i(\xi,\eta) - \psi_{id} \ \text{ with } \quad i = 1,\dots,m,$$

$$B(\xi,\eta) = \begin{pmatrix} \dot{\psi}_{1d} \\ \vdots \\ \dot{\psi}_{md} \end{pmatrix} - \begin{pmatrix} \frac{\delta\psi_1}{\delta\eta} F(\xi,\eta,) \\ \vdots \\ \frac{\delta\psi_m}{\delta\eta} F(\xi,\eta) \end{pmatrix} - \begin{pmatrix} \frac{\delta\psi_1}{\delta\xi} f(\xi,\eta) \\ \vdots \\ \frac{\delta\psi_m}{\delta\xi} f(\xi,\eta) \end{pmatrix} - \begin{pmatrix} \frac{\partial V_1(e_1)}{\partial e_1} \\ \vdots \\ \frac{\partial V_m(e_m)}{\partial e_m} \end{pmatrix}.$$

4 Stability analysis

The convergence of e_i to zero does not implies that the state vector (ξ, η) remains bounded. As imposed by the control law the output $y_i(t)$ with $i = (1, 2)$ follows asymptotically its bounded reference $y_{id}(t)$ and from the fact that $h_i(\eta)$ is continuous function radially unbounded (see the expression form (2)) it induce to that the variable $\eta_i(t)$ takes a bounded values.

On the one hand, the functions ψ_{1d} and ψ_{2d} given by expression (8d) are bounded since the functions $\pi_1(\eta)$ and $\pi_2(\eta)$ are continuous radially unbounded, the desired trajectories $(y_{id},\ \dot{y}_{id},\ \ddot{y}_{id})$ and states are bounded. And in addition, the control input makes that the variables z_1 and z_2 bounded and they converge asymptotically to zero. So, according to the expression (16) it induces that the continuous functions $\psi_1(\eta,\xi)$ and $\psi_2(\eta,\xi)$ take bounded values.

From (5), the states ξ are forced to be the solution of the following system :

$$\begin{pmatrix} 2a_3\eta_1 & 2a_3\eta_2 \\ -b_5\eta_2 & b_5\eta_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{26}$$

Since ψ_1 and ψ_2 are bounded values and the determinant of (26), given by $2a_3b_5(\eta_{1d}^2+\eta_{2d}^2)$, is bounded positive scalar (see remark 3.2), then the states ξ are always bounded.

According to the control input expression (8a) and from the fact that the matrix $A(\eta,\xi)$ is non singular, the functions $f_i(\eta), F_i(\eta,\xi)$ and $\dot{\psi}_{id}$ are bounded moreover the desired trajectories (y_{id}, $\dot{y}_{id}$, $\ddot{y}_{id}$) and the state variable (η,ξ) are bounded it follows that the control input is bounded.

5 Application and simulations

For the application, we must in first choice the function $\Lambda(x)$ and its simplest form is:

$$\Lambda(x) = 1 \tag{27}$$

and applying relation (8d), therefore the desired values ψ_{1d} and ψ_{2d} are then given by:

$$\begin{cases} \psi_{1d} = -q_1\left(y_1 - y_{1d}\right) + 2b_3(\eta_1^2+\eta_2^2) + \dot{y}_{1d}, \\ \psi_{2d} = -q_2\left(y_2 - y_{2d}\right) + a_5\eta_3 + c_5\Gamma_r + \dot{y}_{2d}, \end{cases} \tag{28}$$

where y_{1d} and y_{2d} are respectively the desired flux and the desired speed.

Differentiating expression (28) gives:

$$\begin{cases} \dot{\psi}_{1d}(t) = -q_1\left(H_1 - \dot{y}_{1d}\right) + 2b_3H_1 + \ddot{y}_{1d}, \\ \dot{\psi}_{2d}(t) = -q_2\left(H_2 - \dot{y}_{2d}\right) + a_5F_3 + c_5\dot{\Gamma}_r + \ddot{y}_{2d}. \end{cases} \tag{29}$$

Therefore, the terms $B_1(\xi,\eta)$ and $B_2(\xi,\eta)$ given in (8e) and (8f) take the final expression :

$$B_1(\eta,\xi) = -2a_3(\eta_1f_1+\eta_2f_2+\xi_1F_1+\xi_2F_2) - q_1(H_1-\dot{y}_{1d}) + 2b_3H_1 + \ddot{y}_{1d}, \tag{30}$$

$$B_2(\eta,\xi) = -b_5(\xi_2F_1+\eta_1f_2-\eta_2f_1-\xi_1F_2) - q_2(H_2-\dot{y}_{2d}) + a_5F_3 + c_5\dot{\Gamma}_r + \ddot{y}_{2d}. \tag{31}$$

The simulations are performed for three phase induction machine characterised by :
$Pn = 3.7Kw$, 220/380, 8.54/14.8A,
$M = 0.048H$, $L_s = 0.17H$, $L_r = 0.015H$, $\sigma = 0.0964$,
$T_s = 0.151s$, $T_r = 0.136s$, $J = 0.135mN/rdS^{-2}$, $K_f = 0.0018mN/rdS^{-1}$.

The function $S(z_i)$ for $i = (1,2)$ is implemented by $S(z_i) = \frac{z_i}{|z_i|+\epsilon_i}$ where the threshold values ε_1 and ε_2 are fixed to unity. The desired flux and speed tracking are involved with the regulator coefficients tuned to :

$k_1 = 8000$; $k_2 = 2000$; $q_1 = 1000$; $q_2 = 2000$.

Figures 5.1 and 5.2 give the machine responses in tracking regime (for both $\omega_{ref} > 0$ and $\omega_{ref} < 0$). It appears clearly that the flux and speed track their references with a good accuracy. More over, the initial stator peak current are attenuated by reducing the control inputs only in the beginning of the transient stage (for time $t <= 0.175s$). This reduction affects the tracking during this interval of time. In order to maintain

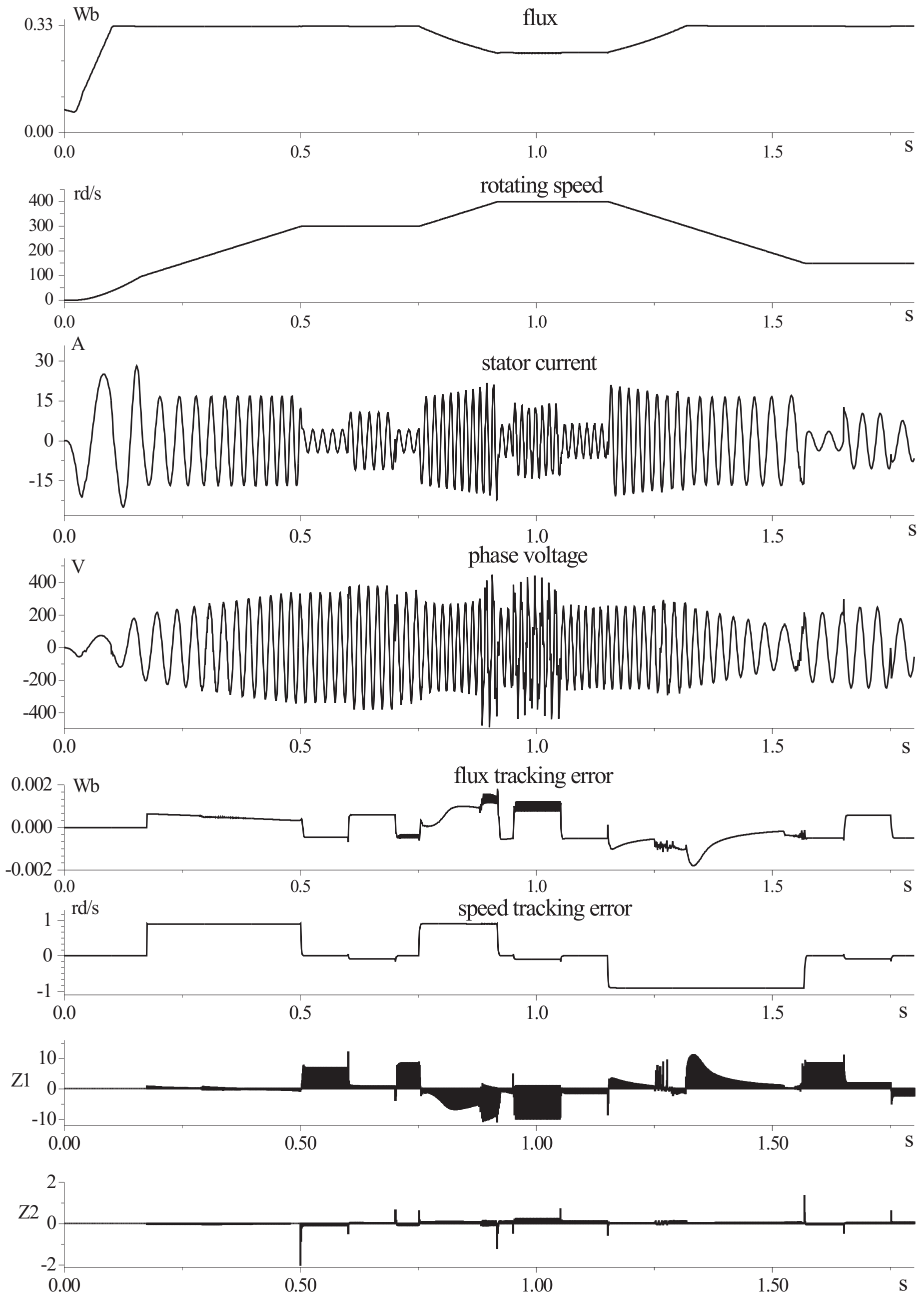

Figure 5.1: Induction machine responses in tracking regime for positive reference speed with the disturbances applied during only 0.1s respectively at time t=0.6, 0.95s and t=1.75s.

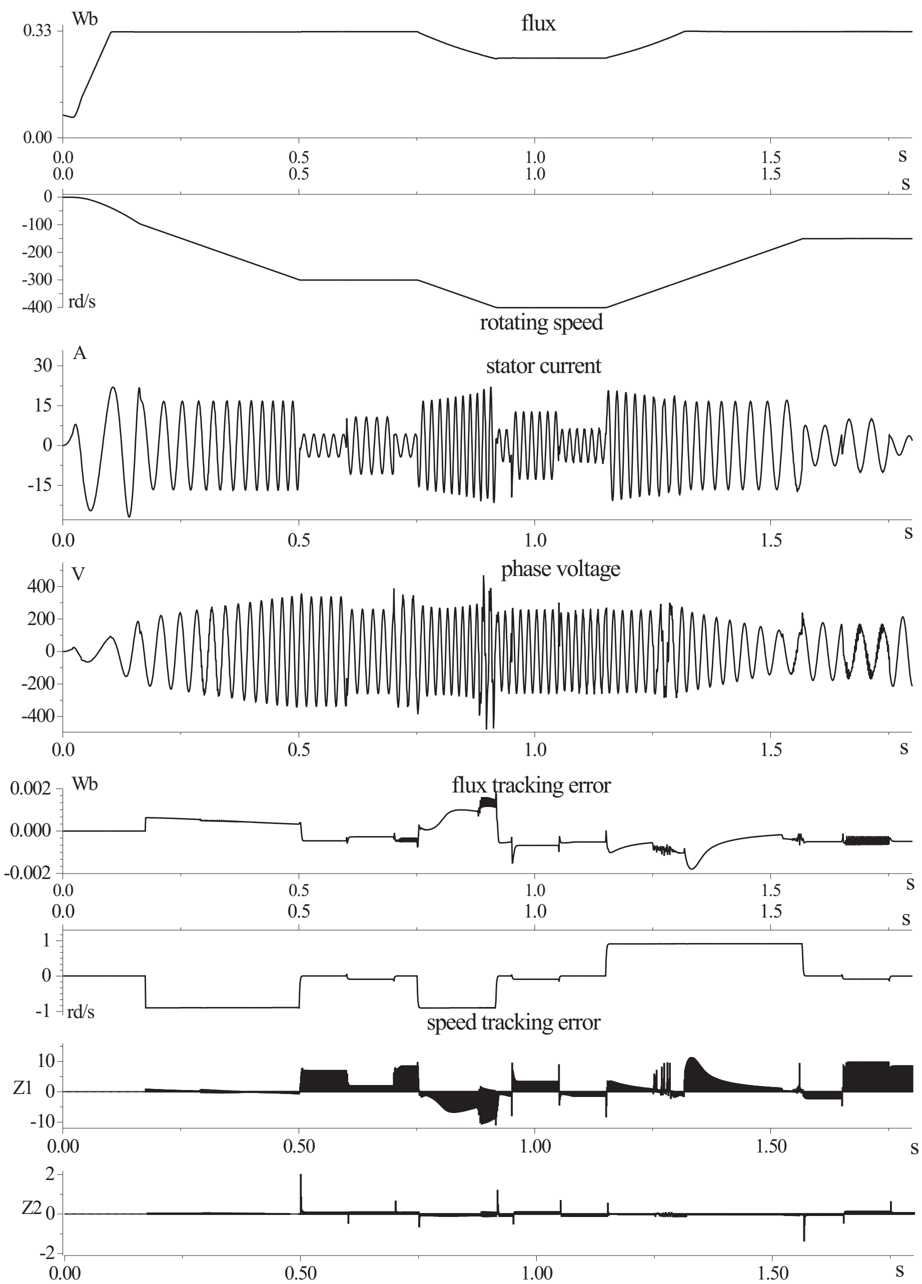

Figure 5.2: Induction machine responses in tracking regime for negative reference speed with the disturbances applied during only 0.1s at time t=0.6, 0.95s and t=1.75s.

the voltage in admissible range when the speed reference ω_{ref} grows up to nominal value $\omega_n = 300rd/s$, the reference flux ϕ_{ref} is reduced down to the nominal flux ϕ_n ($\phi_n = 0.33Wb$) as $\phi_{ref} = \phi_n \omega_n / \omega_{ref}$.

Further, it is noted that the speed and flux tracking reveal a good robustness against disturbances represented by parametric variations and nominal load torque occurring at the same time. These disturbances are applied during $0.1s$ respectively at the time $t = 0.6, 0.95s$ and $t = 1.75s$. The robustness tests are performed for the parameter variations around nominal values as that the stator and rotor resistors increase respectively by an amount of 50% and 100%, the stator and rotor inductors decrease respectively by an amount of 25% and 50%. Meanwhile, these variations affect only the machine model coefficients and who that appearing in the control (u_1, u_2), desired values (ψ_{1d}, ψ_{2d}) and variables (z_1, z_2), are maintained constant. The maximal absolute values of tracking errors (see Table 5.1) reveals that this control law is highly robust in face parameters variation when the state vector is completely known. Despite this highly disturbances, the stator voltage remains in admissible range.

Maximal Tracking error	$\lvert\phi_{ref} - \phi\rvert$	$\lvert\omega_{ref} - \omega\rvert$	$\lvert z_1\rvert$	$\lvert z_2\rvert$
Positive reference	$2.10^{-3}Wb$	$0.92rd/s$	12.1	2
Negative reference	$2.10^{-3}Wb$	$0.92rd/s$	11.3	2

Table 5.1: Maximal tracking error and maximal (z_1, z_2) values.

6 Conclusion

This paper develops a control design procedure for flux-speed tracking of voltage fed induction motor. This design procedure is based on the Lyapunov theory and is similar in spirit to the backsteeping methodology. So, in the first step, the virtual control law is derived as that flux and speed follow at last asymptotically their desired trajectory. Then, in second step, is deduced the real control, by imposing this virtual control law. Noticing that the proposed control law does not include the derivatives of states and outputs hence, it avoids the presence of spikes which often affect the derivative signals. The simulation results involving the flux-speed tracking are given a good results and highlight usability of the suggested approach. Moreover, the control law reveals a strong robustness in face to disturbances generated at the same moment by application of the nominal load torque and large parametric variations. The immediate interest of the proposed procedure comes from the fact that it can be easily extended to the most of voltage fed machines.

6.1 Appendix

In the field reference frame (i. e. the rotor), the state model of the permanent magnets synchronous machine (PMSM) and voltage fed is obtained from the Park equations [22, 23]. This model is derived using the state vector constituted by stator current components (i_{ds}, i_{qs}) and the rotor rotating pulsation ω_r, whereas a vector control is composed of the stator voltage components (v_{ds}, v_{qs}). It is known that the PMSM produce optimal electromagnetic torque when the stator current component i_{ds} takes a determined value i_{dref}. This latter must be zero ($i_{dref} = 0$) when the magnets are mounted on the rotor surface. So, the control objective is to constrain the component i_{ds} to take the value i_{dref} and to control the pulsation rotor rotation ω_r. So, the PMSM dynamic is separated into two interconnected systems : the first one concerns the control for the output ids and the second, which form cascaded non linear system, is related to the control output ω_r. Using the precedent notation for the state vector, the control input vector and the output vector :

$$\begin{pmatrix} \eta_1 & \eta_2 & \xi_1 \end{pmatrix} = \begin{pmatrix} i_{ds} & \omega_r & i_{qs} \end{pmatrix}, \quad u^T = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T = \begin{pmatrix} u_{ds} & u_{qs} \end{pmatrix}^T,$$

$$y^T = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T = (i_{ds}\, \omega_r)^T$$

and the PMSM dynamic takes the form :

$$\begin{cases} \dot{\eta}_1 = -a_1\eta_1 + b_1\eta_2\xi_1 + c_1u_1, \\ y_1 = \eta_1, \end{cases} \tag{32}$$

$$\begin{cases} \dot{\xi}_1 = -a_2\xi_1 - b_2\eta_1\eta_2 - c_2\eta_2 + d_2u_2, \\ \dot{\eta}_2 = -c_3\eta_2 - d_3\Gamma_r + (a_3\eta_1 + b_3)\xi_1, \\ y_2 = \eta_2. \end{cases} \tag{33}$$

It is obviously that the second subsystem has the same form as the studied one. Another way to control the PMSM is to regulate only the speed. In this case, the state vector, the input vector and the output are then respectively represented by :

$$\begin{pmatrix} \xi_1 & \xi_2 & \eta_1 \end{pmatrix} = \begin{pmatrix} i_{ds} & i_{qs} & w_r \end{pmatrix}, u^T = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T = \begin{pmatrix} v_{ds} & v_{qs} \end{pmatrix}^T, y_1 = \omega_r$$

and the PMSM state model takes the form :

$$\begin{cases} \dot{\xi}_1 = -a_1\xi_1 + b_1\xi_2\eta_1 + c_1u_1, \\ \dot{\xi}_2 = -a_2\xi_2 - b_2\xi_1\eta_1 - c_2\eta_1 + d_2u_2, \\ \dot{\eta}_1 = (a_3\xi_1 + b_3)\xi_2 - c_3\eta_1 - d_3\Gamma_r, \\ y_1 = \eta_1. \end{cases} \tag{34}$$

It appears that the precedent dynamic is the same class as indicated in 1. Meanwhile, the action on the speed is carried out by the two inputs (u_1 et u_2) so, this degree of freedom can be exploited in order to introduce another constraint.

The coefficients $(a_1, ..., c_4)$ are related to the machine parameters by :

$$a_1 = \frac{R_s}{L_d}\ , a_2 = \frac{L_q}{L_d}\ , a_3 = \frac{1}{L_d}\ , b_1 = \frac{R_s}{L_q}\ , b_2 = \frac{L_d}{L_q}\ , b_3 = \frac{\phi_f}{L_q}\ , b_4 = \frac{1}{L_q}\ ,$$
$$c_1 = \frac{3.(p)^2}{2.J}(L_d - L_q)\ , c_2 = \frac{3.(p)^2.\phi_f}{2.J}\ , c_3 = \frac{k_f}{J}\ , c_4 = \frac{p}{J}$$

and the physical parameters represent :

R_s : stator phase resistor,
L_d/L_q : cyclic stator/roto inductance related to (d, q) axe,
f : flux produced by rotor magnets,
J : inertia and p is the pairs of poles,
k_f : friction coefficient and Γ_r is a load torque.

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Comparison of Transfer Orbits in the Restricted Three and Four-Body Problems

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Abstract: The restricted three-body problem and the quasi-bicircular problem are the dynamical systems used as models in this paper. The first one describes the motion of one massless body in the potential field of the two massive bodies revolving in circular orbit around their center of mass. The quasi-bicircular problem (QBCP) is a variation of the restricted four-body problem, where the three massive primaries move in a quasi-circular motion around their center of mass. Here, we consider the Earth–Moon as primaries of the restricted three-body problem (RTBP) and the Earth–Moon-Sun as primaries of the QBCP. One of the spatial periodic solutions around the collinear point are known as halo orbits. Our objective is to determine, in both models, transfer orbits from a parking orbit around the Earth to a halo orbit. We apply the two-point boundary value problem, where the boundary points are on the parking and on the halo orbits. Since there is no Keplerian orbit involved in this transfer method, we have called it an adapted Lambert's problem. We compare the total velocity increment obtained with this method applied to both dynamical models. We find that there is a positive solar contribution decreasing the total impulse.

Keywords: *Restricted three-body problem; quasi-bicircular problem; halo orbits; impulsive transfer orbits; two point boundary value problem.*

Mathematics Subject Classification (2000): 70F07, 70F10, 0H12, 70F99, 30E25.

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1 Introduction

The restricted three-body problem is one of most studied problem in orbital dynamics. It has been investigated since Euler and Lagrange because of two important reasons: it is the simplest model of the N-body problem yielding a non-integrable system and it also fits, in first approximation, the motion of celestial bodies and artificial satellites. However, we have to consider other perturbations to describe accurately the real problem, such as the effect of other planets of the Solar System, the solar radiation pressure, the atmosphere drag (for objects close to the Earth) and the non circular orbit of the primary bodies. In this work we consider two cases: the planar circular restricted three-body problem and the four-body case when the circular orbits of the primaries are perturbed by the Sun, known as quasi-bicircular problem.

The five equilibrium points of the RTBP are the well known Lagrangian points. Three of them, L_1, L_2 and L_3, lie along the line joining both primaries. Usually L_1 denotes the solution located between the primaries while L_2 is behind the less massive primary, and L_3 is located in the opposite side of L_1, with respect to the center of mass. The other two points, L_4 and L_5, are on the plane of the motion and form an equilateral triangle with the primary bodies.

The collinear points are unstable for all mass ratios because the linear approximation has a pair of real eigenvalues. The other two are imaginary and span the linear center manifold. The full dynamics near the L_i's has two families of periodic orbits known as Lyapunov orbits, plane and vertical, which are continuation of the linear center manifold and tangent to it. When the horizontal and vertical frequencies attain a certain resonance, the plane Lyapunov family bifurcates into spatial orbits known, since Faquhar [1], as halo orbits. These orbits are such that an observer, placed on one of the primaries and looking towards the second primary, sees the massless body describing a halo around that body. All these periodic orbits can be calculated numerically or by perturbation methods, see for instance [2].

In the 1970s aerospatial engineers began the exploration of these orbits. They were proposed as good places to locate certain space observatories due to two main reasons. First, the point L_1 provides uninterrupted access to the solar visual field without occultation by the Earth; and second, in these places the solar wind is beyond the influence of the Earth's magnetosphere. The first satellite in a halo orbit was *Isee-3*, launched in 1978 by NASA. It was maintained in a halo orbit for nearly 4 years while observing the solar wind and cosmic rays, and then it undertook a complex trip to observe the tail of a comet in a heliocentric orbit. Since Isee-3 launch, five satellites were inserted into halo orbits of the Sun–Earth system. The second mission was the *Soho* telescope projected by ESA-NASA, launched at 1996 for solar observations; *Ace* satellite was launched at 1997 by NASA for solar wind observations. In 2001 two NASA satellites arrived at halo orbits: the *WMap* satellite, to observe cosmic microwave background radiation, and *Genesis*, another solar observatory whose re-entry occurred in 2004. Future missions are under development for launching in the next ten years.

For the Earth–Moon system the RTBP is just the first approximation since the presence of the Sun perturbs the Earth–Moon distance strongly. In 1998, Andreu [3] introduced a consistent model of the restricted four-body problem, named the Quasi-Bicircular Problem, where the motion of the three primary bodies is given by the solution of the non-restricted three body problem. In the QBCP the primaries are revolving around their center of mass in a quasi-circular motion and the massless body moves under the effect of their potential field. In [4] Andreu showed that the quasi-bicircular problem fits better the real case than the bicircular one.

Transference of space vehicles has been widely studied by many authors and this resulted in the development of a wide variety of methods. The most recent ones explore the unstable character of the libration point orbits to design low cost missions [5], using the stable/unstable manifold dynamics. However, the large time spent in these transfer orbits could not be appropriate for certain missions. On the other hand, transfers with the control of flight time, based on the optimization procedures, give paths with greater fuel consumption but with shorter transfer time. The choice of a specific method should be guided by the mission requirements.

In the case of the two body problem, the determination of a transfer orbit connecting the boundary conditions with a specified flight time, is the well-known Lambert's problem. This formulation has been applied by several authors who developed numerical tools for its resolution. The solution of the classical Lambert's problem, with a fixed flight time, has been undertaken by [6] and [7] who developed sophisticated algorithms and accurate methods, dealing with convergence techniques. A new version of Lambert's problem has been studied by [8], replacing the condition of a given transfer time by that of minimal fuel expenditure. We apply the latter conception to the restricted three and four-body problem and call it the adapted Lambert's problem.

In this paper we find transfer orbits from the Earth to a halo orbit in the vicinity of L_1 of the Earth–Moon system. We compare the total ΔV required by RTBP and QBCP models, showing that the presence of the Sun decreases the total impulse necessary to achieve the desired transfer.

2 Equations of Motion

2.1 The Restricted Three-Body Problem

In the restricted three-body problem, the mass of one of the bodies is supposed to be infinitely small when compared to the other two that move in circular motion around their center of mass. The reference frame is set according to the notation defined in [9], where the origin is on the center of mass, the positive x-direction is towards the biggest primary and rotates in the counterclockwise direction. The unit of length is the distance between the primaries and the unit of time is chosen so that the period of the primaries is 2π; consequently the gravitational constant is set to one. The potential function of the RTBP in this synodic coordinate system is given by

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu), \tag{1}$$

where r_1 and r_2

$$r_1^2 = (x-\mu)^2 + y^2 + z^2,$$
$$r_2^2 = (x-(\mu-1))^2 + y^2 + z^2,$$

are the distances from the primary bodies (m_1, m_2) to the massless particle. The equations of motion are:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z. \end{aligned} \tag{2}$$

Defining the momenta as $p_x = \dot{x} - y$ and $p_y = \dot{y} + x$, the equations of motion can be written as an autonomous Hamiltonian system with three degrees of freedom derived from:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

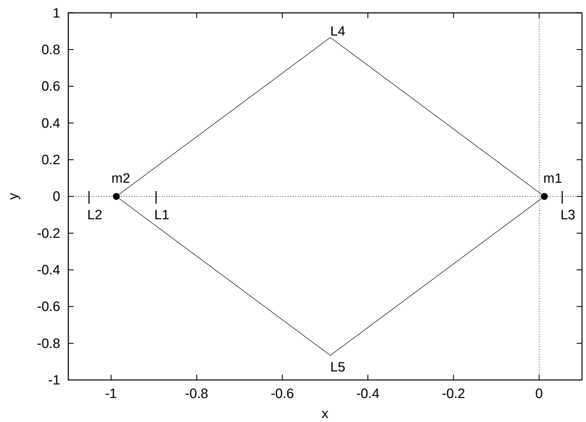

Figure 2.1: The restricted three-body problem configuration.

2.2 The Quasi-Bicircular Problem

The quasi-bicircular model is a restricted four-body problem where the three primaries are revolving in a quasi-bicircular motion. The fourth body, which has infinitely small mass, moves under the potential field generated by the primaries without disturbing their motion. In this case the equations of motion are time dependent with the frequency of the biggest primary. The Earth–Moon distance is no longer constant due to the Sun perturbation, therefore the equations of motion are written in a rotating pulsating reference frame to make the Earth–Moon distance constant. This coordinate system is centered on the barycenter of the Earth–Moon system and rotates with it.

In order to obtain a coherent formulation we first should find a solution for the three-body problem, where the three masses move in planar non-circular orbits around their common center of mass. This solution expressed as Fourier expansion is:

$$\alpha_k(t) = \alpha_{k_0} + \sum_{j\geq 1} \alpha_{kj} \cos(jnt) \quad \text{for} \quad k = 1, 3, 4, 6, 7,$$
$$\alpha_k(t) = \sum_{j\geq 1} \alpha_{kj} \sin(jnt) \quad \text{for} \quad k = 2, 5, 8,$$

where n is the mean relative angular velocity $n = 1 - n_s$ in inertial coordinates, n_s is the angular velocity of the Sun. We recall that the mean angular velocity of the Moon is unity in the inertial frame. The units of distance and time are the same as in the last section. Then, the Hamiltonian of the QBCP is:

$$H = \frac{1}{2}\alpha_1(p_x^2 + p_y^2 + p_z^2) + \alpha_2(p_x x + p_y y + p_z z) + \alpha_3(p_x y - p_y x) + \alpha_4 x + \alpha_5 y - \alpha_6\left(\frac{1-\mu}{q_{pe}} + \frac{\mu}{q_{pm}} + \frac{m_s}{p_{pe}}\right),$$

where q_{pe}, q_{pm} and q_{ps} are given by

$$q_{pe}^2 = (x-\mu)^2 + y^2 + z^2,$$
$$q_{pm}^2 = (x-\mu+1)^2 + y^2 + z^2,$$
$$q_{ps}^2 = (x-\alpha_7)^2 + (y-\alpha_8)^2 + z^2,$$

and are the distances of the particle to the Moon, Earth and Sun, respectively. We note that they are written in the synodical reference frame centered in the barycenter of the Earth–Moon system. For more details see [3]. The Hamiltonian equations of motion are:

$$\begin{aligned}
\dot{x} &= \alpha_1 + \alpha_2 x + \alpha_3 y, \\
\dot{y} &= \alpha_1 p_y + \alpha_2 y - \alpha_3 x, \\
\dot{z} &= \alpha_1 p_z + \alpha_2 z, \\
\dot{p}_x &= -\alpha_2 p_x + \alpha_3 p_y - \alpha_4 - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}(x-\mu) + \frac{\mu}{q_{pm}^3}(x-\mu+1) + \frac{m_s}{q_{ps}^3}(x-\alpha_7)\right), \\
\dot{p}_y &= -\alpha_2 p_y + \alpha_3 p_x - \alpha_5 - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}y + \frac{\mu}{q_{pm}^3}y + \frac{m_s}{q_{ps}^3}(y-\alpha_8)\right), \\
\dot{p}_z &= -\alpha_2 p_z - \alpha_6\left(\frac{1-\mu}{q_{pe}^3}z + \frac{\mu}{q_{pm}^3}z + \frac{m_s}{q_{ps}^3}z\right).
\end{aligned} \tag{3}$$

3 Impulsive Transfer Orbit

When a system of differential equations is supposed to satisfy a set of initial and final conditions, it becomes a two point boundary value problem (TPBVP), where the time is a free variable. In this work, the boundary conditions are a point on the parking orbit (P_i) around the Earth and a point on the halo orbit (P_f). Without any time constraint, the problem of finding a trajectory that links the points P_i and P_f has infinite set of solutions with different flight times. However, if we add the flight time as a constraint, the set of solutions become finite. In this case, for each set of boundary conditions (P_i, P_f) with a fixed flight time (Δt), we have two solutions which are related via the Mirror Theorem [10].

As the restricted three and four-body models have no analytical solution, the boundary value problem has to be numerically solved. We use the following steps to find a solution of this time constrained TPBVP:

- guess an initial velocity $\vec{v}_i$. Together with the initial prescribed position $\vec{r}_i$ the complete initial state is known;
- guess a final time t_f and integrate the equations of motion from t_i to t_f;
- check the final position $\vec{r}_f$ obtained from the numerical integration with the prescribed final position and the final real time with the specified time of flight. If

there is an agreement (difference less than a specified error allowed) the solution is found and the process can stop here.

This is a simple shooting method described in reference [11], where an algorithm is also available.

As mentioned in the introduction, such a formulation: the two point boundary value problem plus a time constraint in the RTBP is a kind of Lambert's problem for the three-body problem. Thus the name adapted Lambert's problem, since there is no Keplerian orbit involved.

Our investigation begins by the search of transfer trajectories travelling between the points P_i and P_f with minimum velocity increment defined as follows. Let $\vec{V}_i$ and $\vec{V}_f$ be the velocity vectors on the parking orbit around the Earth and the halo orbit, respectively. The initial velocity increment (ΔV) is:

$$\Delta V_i = |\vec{V}_i - \vec{V}_T|, \tag{4}$$

where $\vec{V}_T$ is the transfer velocity given by the above numerical method, which satisfies the time and the boundary constraints. The second impulse, introduced to insert the space vehicle in the halo orbit, is given by:

$$\Delta V_f = |\vec{V}_T - \vec{V}_f|. \tag{5}$$

The total impulse is the sum of these impulses:

$$\Delta V = \Delta V_i + \Delta V_f. \tag{6}$$

4 Halo and Parking Orbits

As the QBCP is a three degrees of freedom time periodic Hamiltonian system, we can have an intuition of its periodic solutions considering it as a time periodic perturbation of the RTBP. So, the periodic solutions of QBCP are related to the period of the perturbation. Therefore, to compute halo orbits in the QBCP, one first looks for a halo perodic orbit in the RTBP which has the Solar period or a multiple of it. Then, a numerical continuation method can be used to find the corresponding QBCP halo orbit. As usual, we set the problem of continuation as follows:

$$H = H_{RTBP} + \epsilon(H_{QBCP} - H_{RTBP}),$$

where ϵ is a small parameter. When ϵ is equal to zero we have $H = H_{RTBP}$ and if ϵ is equal to unity, then $H = H_{QBCP}$. The halo initial conditions considered here are those labelled 01E and 1E in [3]. The period of the chosen halo in the QBCP is three times multiple of its equivalent orbit in the RTBP, which were determined in [12].

The parking orbits belong to the BD family of direct periodic orbits around the primary body (see [13]) and calculated them using a numerical continuation method described in [14]. The orbit Parking 1 is about 6.696 km from the Earth and the Parking 2 is 11.612 km. To make a clear identification of the selected points on the parking and the halo orbit, we choose angular coordinates θ_1 and θ_2 on the xy-projection of the former and on the yz-projection of the latter. The origin of these angles are the positive x and y axis, for θ_1 and θ_2, respectively, and the direction of rotation is taken to be counterclockwise (see Figure 4.1).

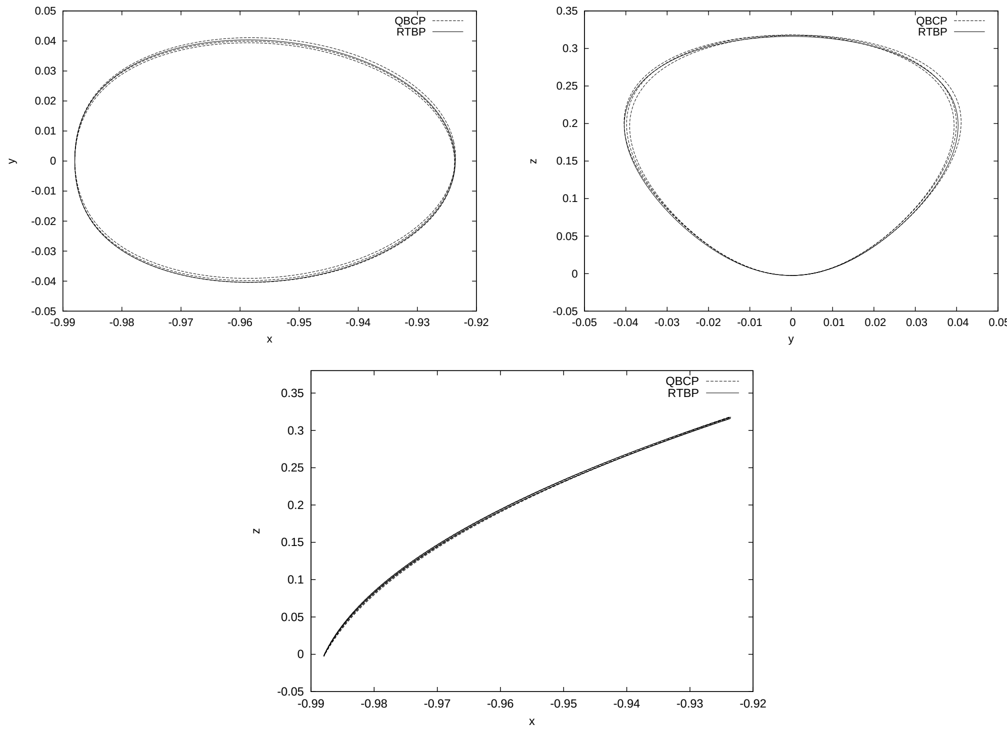

Figure 4.1: Projection of halo orbit in both dynamical systems.

5 Results

In this section we apply the adapted Lambert's method to the initial conditions shown in the table below. The first application considers only the RTBP while the second one considers both the RTBP and the QBCP.

Table 5.1: Initial Conditions

ORBIT	x_0	z_0	$\dot{y}_0$
Halo - RTBP	–9.87982557457513E-01	–2.48226957833054E-03	3.13139586195729E+00
Halo - QBCP	–9.879722904635280E-01	–2.462095060502300E-03	3.184909363998671E+00
Parking 1	–0.005270250000000E+0	0.0000E+0	7.547700390000000E+0
Parking 2	–0.018068060000000E+0	0.0000E+0	5.747748530000000E+0

5.1 Transfers in RTBP

To begin our simulations, we have chosen the parking orbit 0.0302 canonical units away from the center of the Earth (Parking 2 on Table 5.1) and the halo orbit. In these two orbits we take angular steps of approximately 6°, and the transfer method is applied considering the boundary conditions: the initial one on the parking orbit and the final one on the halo orbit. We stress that we explored all these possibilities of boundary conditions with a fixed time. This procedure was decisive to select which angular range on each orbit furnishes the lower total ΔV. Three time intervals, $t = 0.2$, 0.3 and 0.4, were used in this simulations. These preliminary study restricted the angular range to $[300°, 40°]$ for θ_1 and $[230°, 330°]$ for θ_2 (see Figure 5.1).

With this selected points we make simulations considering several time intervals, from 0.2 to 2.0 canonical units of time with step 0.05, as seen on Figure 5.2. As expected, the maximal contribution to the total ΔV comes from ΔV_i which is the departure impulse. The final impulse, which injects the vehicle into the halo orbit is, on average, one fourth of the initial impulse. The result of these simulations can be summarized as follows: the minimum ΔV is 4.7283 and occurs at $\theta_1 = 36°.3874$ and $\theta_2 = 261°.8722$ for $t = 0.85$. This can be seen in Figure 5.2. We also tested the corresponding retrograde parking orbits and the result is practically the same as for the direct ones.

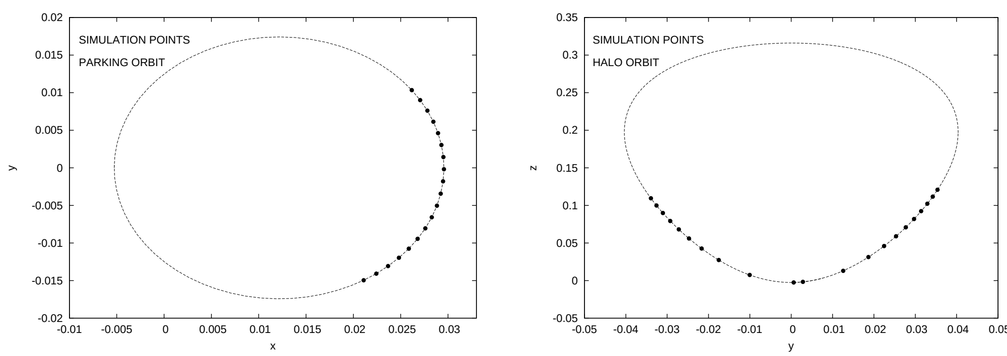

Figure 5.1: Boundary points on the parking and the halo orbit.

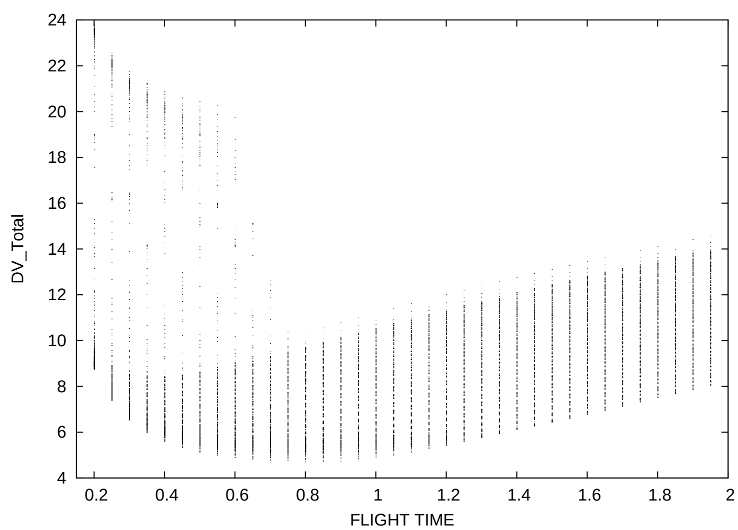

Figure 5.2: Total ΔV in the RTBP considering the boundary points on the Figure 5.1.

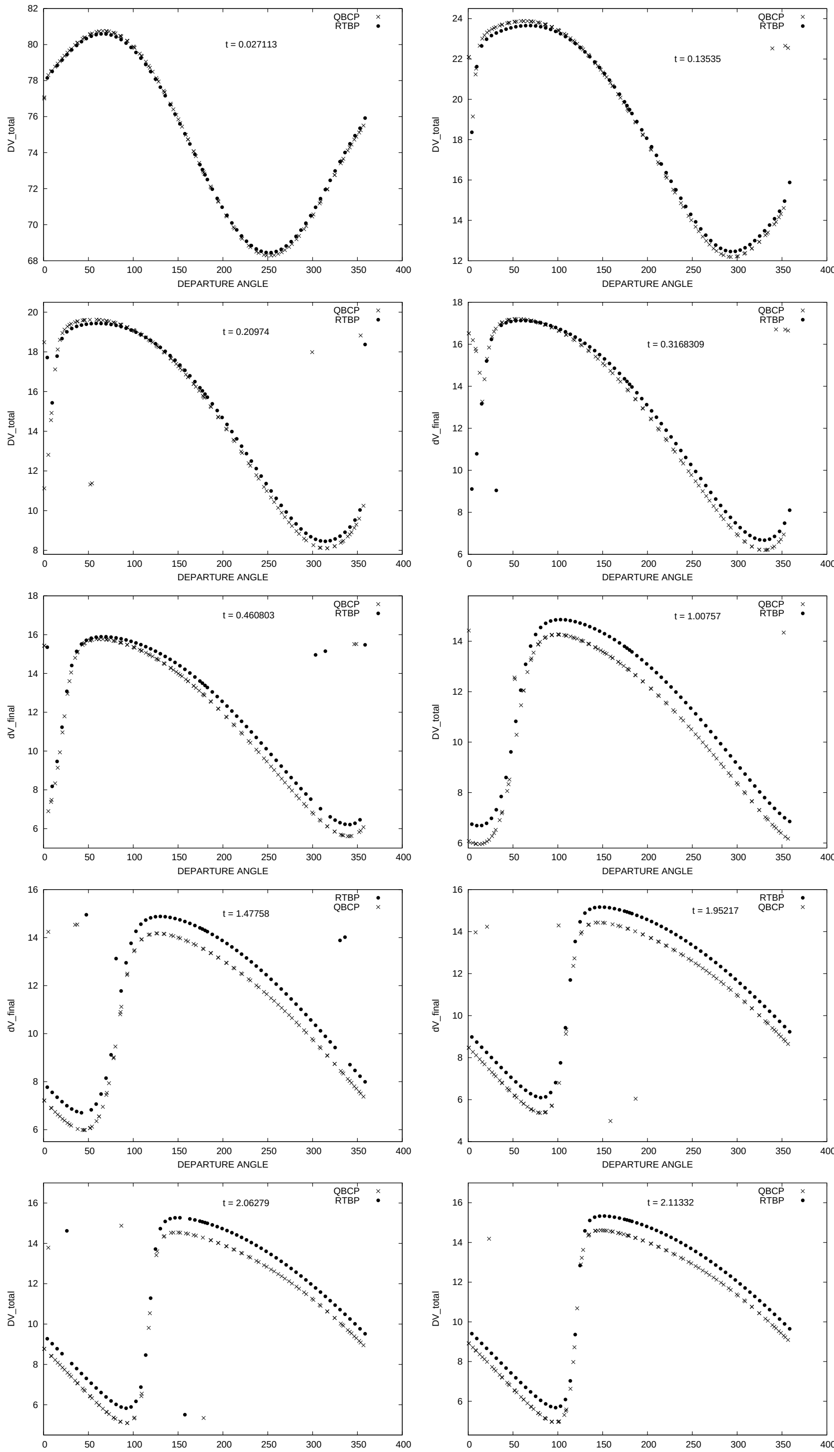

Figure 5.3: Comparison of total ΔV between the RTBP and QBCP considering many interval time, the departure angle is θ_1 whose origin is the positive x-axis on the parking orbit plane.

5.2 Comparison between the PRTC and QBCP

To obtain the transfer orbits in the QBCP, we apply a methodology similar to the one described above. Because the QBCP is a non-autonomous system, the set of initial conditions is time-dependent, implying that we have less freedom to vary the flight time as in the RTBP case. The transfer time must be the same in the state vector of the final boundary value if we begin the all the integrations at a common epoch.

Since our objective is to compare the total ΔV obtained in the RTBP and QBCP, we select the points on the halo orbit which are geometrically equivalent to the ones in the RTBP. The periodic parking orbits of the RTBP are, of course, no longer periodic in the QBCP. However, the satellite remains a short time on the parking orbit, so we take for it the same initial conditions as before. All simulations are done for the Parking 1 orbit.

The Figure 5.3 shows the total ΔV of both models for 10 different time intervals. In general, the simulations show that the Sun's presence improves the fuel consumption of approximately 9%.

6 Comments

It is difficult to show which parameter involved in the problem allows an optimal transfer, because the dynamical system is complex and very sensitive to initial conditions. If we change the flight time, it is possible that the economical boundary condition, selected for another flight time, will also change significantly. For this reason, we simulated many possibilities, varying the set of boundary conditions and flight time. However, it is possible that our results do not correspond to the exact global minimum, but just a discrete approximation. The graphics on the Figure 5.3 show that the presence of the Sun could contribute to decrease the total impulse (ΔV), specially for longer flight times.

Acknowledgements

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Rendezvous Maneuvers under Thrust Deviations and Mass Variation

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Abstract: The Rendezvous maneuvers are used in many important technological space missions. Today, the interception between space bodies (vehicles, stations, debris, etc.) is far from negligible, due the large number of such bodies in Earth orbit and the growth of the current rate space activities. The Rendezvous are realized during many satellites special formations, interception between space stations and satellites or spacecrafts, interception between this bodies and space debris, runaway maneuvers, Formation Flying, etc. In this paper, we study the Rendezvous maneuvers between one satellite and other space vehicle, considering the thrust direction deviations and the mass variation in the satellite, due to the non-ideal propulsion system. We found to the noncoplanar maneuvers, one nonlinear cause/effect relations between the position coordinates uncertainess of the vehicle-interceptor and the "pitch" and "yaw" deviations. Besides, this relation is weighed by time penalty functions, due the variation mass effect. This model is very close to the realistic case and can be implemented inside the technological missions range to the thrust deviations.

Keywords: *Rendezvous; thrust deviations; mass variation.*

Mathematics Subject Classification (2000): 70M20.

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1 Introduction

The Rendezvous is one completely constrained, with several applications, mainly for the space station and space debris. In this maneuver the orbit parameters and the distance between the two space objects can be divergent. The encounter between the two vehicles must occur without collisions between them, that is, the relative velocity must be null in same time. The Rendezvous coplanar solution, given the impulses and fixed time, was found by Clohessy-Wiltshire [1] in 1960. After him, many authors studied this maneuvers under many conditions and constraints. Stern [2] in 1984 approximated the Clohessy-Wiltshire equations to the small time transfer and obtained non-jouned rectilinear trajectories. Examples of applications of this result are the terminal and extra-vehicle Rendezvous and satellite operation service. The generalization to planar, minimum consumption Rendezvous case, in the general central force field was done by Humi [3]in 1993. In this year Abramovitz and Grunwald [4] developed an iterative graphical method to the optimal and planar Rendezvous inside many spacecraft of one space station environment, under several operational constraints saving more than 30 per cent fuel. Also in 1993, Lutze and Lawton [5] investigated optimal Rendezvous with free time, using regularized variables of the true anomaly, obtaining a simple form for the co-states equations during coast arcs. They established a new optimal necessary condition to the optimality problem. In 1994 Yuan and Hsu [6] proposed a new direction scheme to use the terminal Rendezvous phase. The solution related the fuel consumption to the new direction guidance law with propellant mass. They used the spacecraft variation mass and non-variation mass approach. Shaohua et all [7] also in 1994 applied a transverse propulsion to the Rendezvous trajectory, transforming it in an omni-direction and more fuel-economic trajectory with respect to the conventional cases and Jones and Bishop [8] developed one law target for the Rendezvous terminal phase, using a small Halo translunar orbit ratio with 3 bodies approach. They found 3D Rendezvous in terminal phase and a total minimum cost function for the transfer time, inclination angle and inital condition angle. Pardis and Carter [9] in 1995 considered the impulse saturation effects in optimal Rendezvous with limited power propulsion system and found that the saturation pointed to a degradation of the consumption performance index, which could be improved if the fly time was increased or if additional impulses were applied. In this year, Yu [10] showed that an stable equilibrium state can occur in the relative motion between two close spacecrafts to Rendezvous inside a local coordinate system. Prado [11] also in 1995, derived an algorithm to solve optimal Rendezvous maneuvers with two impulses for a mono-revolution transfer or a multi-revolutions transfer, coplanar or non-coplanar. He found fits of the fuel consumption as function of transfer time. In 2001 Prado and Felipe [12] used impulsive control to study the Rendezvous maneuvers ... . All this results were obtained to impulsive maneuvers and ideal propulsion system and the most with non-variable mass. Our approach is non-impulsive continuous Rendezvous maneuvers under thrust directions deviations and mass variations. We applied Rendezvous maneuvers between the control satellite and the interceptor satellite in one "Formation Flying".

2 Mathematical Model and Preliminaries

The mathematical model considers one control satellite in R ratio initial orbit with velocity $v = (GM_T/R)^{1/2}$ and one satellite interceptor in a transfer orbit with apogee close to R, conform Figure 2.1.

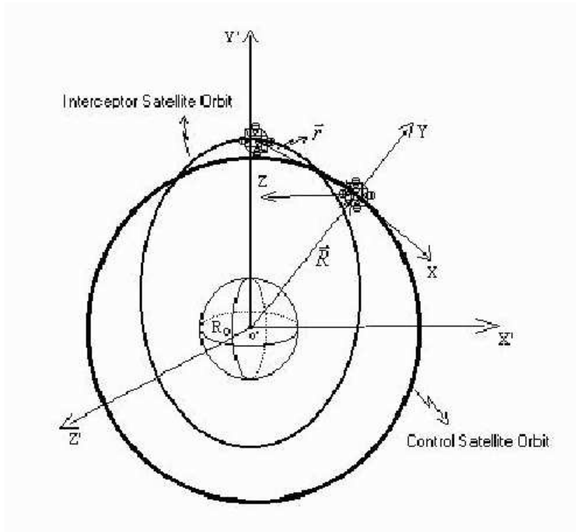

Figure 2.1: Space Satellites in Rendezvous.

In this Figure we show two reference systems: $\{X', Y', Z'\}$ - inertial, Earth-centered, ratio R_0 and $\{X, Y, Z\}$ - rotational, satellite control-centered. For our Rendezvous maneuver the condition that the distance r between the satellites compared with the distance R between the control satellite and the Earth is small must be satisfied. This condition can be satisfied, to technological purposes, with $\{(x(t)^2 + y(t)^2 + z(t)^2)\}^{1/2} \leq 200mi$. This condition allows us to neglect terms in higher order of the gravitational force expansion in serie. The movie equations for the satellite interceptor with respect the rotational system are

$$\ddot{x}(t) - 2W\dot{y}(t) = -v_{ex}\frac{d\{\ln[M(t)]\}}{dt}, \tag{1}$$

$$\ddot{y}(t) - 3W^2y(t) + 2W\dot{x}(t) = -v_{ey}\frac{d\{\ln[M(t)]\}}{dt}, \tag{2}$$

$$\ddot{z}(t) + W^2z(t) = -v_{ez}\frac{d\{\ln[M(t)]\}}{dt}. \tag{3}$$

These equations determine the Rendezvous dynamics between two satellites under thrusters and gravitational forces. In the right size of these equations are the propulsion force components, modeled as

$$\vec{f} = \left\{-\vec{v}_e\frac{dm}{dt}\right\}\frac{1}{M(t)}, \tag{4}$$

where $\vec{v}_e$ is the escape velocity vector of the fuel. The total satellite mass can be modeled as the sum of the satellite constant mass M, and the fuel variable mass m(t), that is,

$$M(t) = M + m(t). \tag{5}$$

Besides this, we consider that the satellite mass is proportional to the initial fuel mass So,

$$\chi \equiv \frac{M}{m(0)} = \frac{M}{m_0}. \tag{6}$$

The solution of the differential equations $(1), (2), (3)$ depend of the satellite time variation mass model.We consider in this paper the exponential model, that is,

$$M(t) = m_0(\chi + 1) + \dot{m}t. \tag{7}$$

In this equation $\dot{m}$ = constant < 0. If we suppose that $\chi \geq 1$ (technological approximation), we can expand the logarithms function. In this way, the solution of those equations are, after many algebraic manipulations,

$$x(t) = 2A\sin(Wt) - 2B\cos(Wt) + Et + \sum_{n=1}^{\infty} F_n e^{-n\gamma t} + G, \tag{8}$$

$$y(t) = A\cos(Wt) + B\sin(Wt) + \sum_{n=1}^{\infty} C_n e^{-n\gamma t} + D, \tag{9}$$

$$z(t) = H\cos(Wt) + I\sin(Wt) - \sum_{n=1}^{\infty} J_n e^{-n\gamma t}. \tag{10}$$

The constants A,B,D,E,G,H,I depend of the initial conditions and of the χ, γ, W. The constants C_n, F_n and J_n are sum in n.

For introduce the thrust direction "pitch", $\Delta\alpha(t)$, and "yaw", $\Delta\beta(t)$, deviations, we write the $\vec{v}_e$ components and the solutions $x(t)$, $y(t)$ and $z(t)$ with symbol $(*)$ and without it for these variables without deviations. So,

$$v_{ex}(t) = v\sin\alpha(t)\cos\beta(t), \tag{11}$$

$$v_{ey}(t) = v\cos\alpha(t)\cos\beta(t), \tag{12}$$

$$v_{ez}(t) = v\sin\beta(t). \tag{13}$$

And these variables with direction deviations,

$$v^*_{ex}(t) = v\sin[\alpha(t) + \Delta\alpha(t)]\cos[\beta(t) + \Delta\beta(t)], \tag{14}$$

$$v^*_{ey}(t) = v\cos[\alpha(t) + \Delta\alpha(t)]\cos[\beta(t) + \Delta\beta(t)], \tag{15}$$

$$v^*_{ez}(t) = v\sin[\beta(t) + \Delta\beta(t)]. \tag{16}$$

We define the difference between the both values, to coordinate y(t), for example,

$$y^*(t) - y(t) = \Delta y(t) = \frac{1}{W}\int_0^t [G^*(\tau) - G(\tau)]\sin[W(t-\tau)]d\tau, \tag{17}$$

where

$$G^*(\tau) = 2Wv_{ex}^* \ln[M(\tau)] - v_{ey}^* \frac{d[\ln M(\tau)]}{d\tau} - 2WC_1 \tag{18}$$

or, considering this result, we have

$$\Delta y(t) = \frac{1}{W}\int_0^t [2W(v_{ex}^* - v_{ex})\ln M(\tau) - (v_{ey}^* - v_{ey})\frac{d[\ln M(\tau)]}{d\tau}]\sin[W(t-\tau)]d\tau. \tag{19}$$

We adopted probabilistic approach, that is, we adopted the mean variables values, because we do not know about the final variables values. The thrust direction deviations were modeled through one uniform or gaussian probability distribution function. The expectation operator $\mathcal{E}$ is the mean in the assemble values. We consider that the stochastic processes are ergodic, so, the expectation operator commutes with the integral operator (in time). We consider too that the $\ln[M(\tau)]$ and $\sin[W(t-\tau)]$ functions are deterministic in time. So,

$$\mathcal{E}\{\Delta y(t)\} = \frac{1}{W}\int_0^t [2W\mathcal{E}\{(v_{ex}^* - v_{ex})\}\ln M(\tau) - \mathcal{E}\{(v_{ey}^* - v_{ey})\}\frac{d[\ln M(\tau)]}{d\tau}]\sin[W(t-\tau)]d\tau. \tag{20}$$

Equation (20) is general for any probability distribution deviations. We considered the uniform probability distribution.

3 Rendezvous under Direction "pitch" Deviations

To compute the means in Equation (20) in the fixed time and considering the random-bias deviations,that is, ($\Delta\alpha(t) = \Delta\alpha = constant$), we have

$$\mathcal{E}\{\Delta y(t)\} = K_1(t)\{\frac{\sin \Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\}, \tag{21}$$

where

$$K_1(t) = \{2v_{ex}(t_f)\int_0^t \ln M(\tau)\sin W(t-\tau)d\tau - \frac{v_{ef}(t_f)}{W}\int_0^t \frac{d\{\ln M(\tau)\}}{d\tau}\sin W(t-\tau)d\tau\}. \tag{22}$$

Equation (21) is the cause/effect relation very important between the thrust deviations through the "pitch" direction and the position satellite deviation y(t) coordinate. We observe too that in this relation there is one penalty time-function $K_1(t)$. This function depends of the mass parameters χ,γ and the control satellite angular velocity W. After the computation,

$$\mathcal{E}\{\Delta y(t)\} = \sum_{j=2}^{\infty} \frac{(-1)^{j+1}\{\Delta\alpha_{max}\}^{2(j-1)}}{(2j-1)!}[A'\cos(Wt) + B'\sin(Wt) + \sum_{n=1}^{\infty} C_n e^{-n\gamma t} + D'], \tag{23}$$

where

$$A' = \{-\frac{2v_{ex}(t_f)}{W}\ln(m_o\chi) - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\chi^n n}\{\frac{2v_{ex}(t_f)}{W} + \frac{n\gamma v_{ex}(t_f)}{W^2}\}\frac{1}{\{1+(\frac{n\gamma}{W})^2\}}, \tag{24}$$

and

$$B' = \{\frac{v_{ey}(t_f)}{W}\ln(\frac{\chi+1}{\chi}) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\chi^n n}\{-\frac{v_{ey}(t_f)}{W} + \frac{2n\gamma v_{ex}(t_f)}{W^2}\}\frac{1}{\{1+\{\frac{n\gamma}{W}\}^2\}}, \tag{25}$$

and

$$C_n = \frac{(-1)^{n+1}}{\chi^n n}\{\frac{2v_{ex}(t_f)}{W} + \frac{n\gamma v_{ey}(t_f)}{W^2}\}\frac{1}{\{1+\{\frac{n\gamma}{W}\}^2\}}, \tag{26}$$

and

$$D' = \{\frac{2v_{ex}(t_f)}{W}\ln(m_o\chi)\}. \tag{27}$$

The penalty function $K_1(t)$ weighted the cause/effect relation in time, besides the thrust deviations effects. Its effect is oscillate in the increasing time and the orbit will be damaged. But, the Rendezvous maneuvers under the realistic conditions are wanted realized in minimum time.

The similar mathematical proceedings to the x(t) coordinate, integrating the Equation (1), give

$$\dot{x}(t) = 2Wy(t) - v_{ex}\ln[M(t)] + C_1, \tag{28}$$

and with the thrust deviations

$$\dot{x}^*(t) = 2Wy^*(t) - v_{ex}^*\ln[M(t)] + C_1, \tag{29}$$

and

$$\Delta x(t) = 2W\int_0^t \Delta y(t')dt' - \int_0^t (v_{ex}^* - v_{ex})\ln[M(t')dt']. \tag{30}$$

Applying the expectation operator $\mathcal{E}$,

$$\mathcal{E}\{\Delta x(t)\} = 2W\int_0^t \mathcal{E}\{\Delta y(t')\}dt' - \int_0^t \mathcal{E}\{(v_{ex}^* - v_{ex})\}\ln[M(t')]dt'. \tag{31}$$

Taking deviations only in "pitch" direction,

$$\mathcal{E}\{\Delta x(t)\} = K_2(t)\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\}, \tag{32}$$

where

$$K_2(t) = 2W\int_0^t K_1(t')dt' - v_{ex}(t_f)\int_0^t \ln[M(t')]dt'. \tag{33}$$

Expanding in $\Delta\alpha_{max}$ power serie, and integrating,

$$\mathcal{E}\{\Delta x(t)\} = \sum_{j=2}^{\infty} \frac{(-1)^{j+1}\{\Delta\alpha_{max}\}^{2(j-1)}}{(2j-1)!}[2A'\sin(Wt) - 2B'\cos(Wt) - 2\sum_{n=1}^{\infty} C'_n e^{-n\gamma t} + D''t - L], \tag{34}$$

where

$$C'_n = \frac{(-1)^{n+1}}{\chi^n n^2\gamma}\{\{2v_{ex}(t_f) + \frac{n\gamma v_{ey}(t_f)}{W}\}\frac{1}{\{1+(\frac{n\gamma}{W})^2\}} - v_{ex}(t_f)\}, \tag{35}$$

and

$$D'' = \frac{3WD'}{2}, \tag{36}$$

and

$$L = v_{ex}(t_f)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\chi^n n^2\gamma}. \tag{37}$$

We observe, again, the nonlinear cause/effect relation in the "pitch" deviations and too the time penalty function K_2. This function presents a growing linear time term.

4 Rendezvous under Direction "yaw" Deviations

We consider the random-bias deviations in "yaw" direction, that is, ($\Delta\beta(t) = \Delta\beta =$ constant). With similar steps used previously, we obtain the results to the $\Delta y(t)$ and $\Delta x(t)$, that is,

$$\mathcal{E}\{\Delta y(t)\} = \sum_{j=2}^{\infty} \frac{(-1)^{j+1}\{\Delta\beta_{max}\}^{2(j-1)}}{(2j-1)!}[A'\cos(Wt) + B'\sin(Wt) + \sum_{n=1}^{\infty} C_n e^{-n\gamma t} + D'] \tag{38}$$

and

$$\mathcal{E}\{\Delta x(t)\} = \sum_{j=2}^{\infty} \frac{(-1)^{j+1}\{\Delta\beta_{max}\}^{2(j-1)}}{(2j-1)!}[2A'\sin(Wt) - 2B'\cos(Wt) - 2\sum_{n=1}^{\infty} C'_n e^{-n\gamma t} + D''t - L]. \tag{39}$$

But, in this case, we must consider the $z(t)$ coordinate, because the "yaw" deviation affects the movie in this direction. The velocity component in this direction depends only this angle. So, the solution for this coordinate with "yaw" deviation is

$$z^*(t) = C_1\cos(Wt) + C_2\sin(Wt) - \frac{1}{W}\int_0^t v^*_{ez}\frac{d\{\ln[M(\tau)]\}}{d\tau}\sin[W(t-\tau)]d\tau \tag{40}$$

and without this deviations is

$$z(t) = C_1 \cos(Wt) + C_2 \sin(Wt) - \frac{1}{W}\int_0^t v_{ez}\frac{d\{\ln[M(\tau)]\}}{d\tau}\sin[W(t-\tau)]d\tau. \tag{41}$$

Through the similar way, we can compute the difference between these function

$$\Delta z(t) = \frac{1}{W}\int_0^t [G^*(\tau) - G(\tau)]\sin[W(t-\tau)]d\tau, \tag{42}$$

where

$$G^*(\tau) = -v_{ez}^*\frac{d\{\ln[M(\tau)]\}}{d\tau}. \tag{43}$$

So,

$$\Delta z(t) = \frac{1}{W}\int_0^t \{(v_{ez} - v_{ez}^*)\frac{d\{\ln[M(\tau)]\}}{d\tau}\}\sin[W(t-\tau)]d\tau. \tag{44}$$

Applying the expectation operator $\mathcal{E}$,

$$\mathcal{E}\{\Delta z(t)\} = \frac{1}{W}\int_0^t \{\mathcal{E}\{(v_{ez} - v_{ez}^*)\}\frac{d\{\ln[M(\tau)]\}}{d\tau}\}\sin[W(t-\tau)]d\tau \tag{45}$$

and

$$\mathcal{E}\{\Delta z(t)\} = K_3(t)\{\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}} - 1\}, \tag{46}$$

where

$$\mathcal{E}\{\Delta z(t)\} = \sum_{j=2}^{\infty}\frac{(-1)^{j+1}\{\Delta\beta_{max}\}^{2(j-1)}}{(2j-1)!}[H'\cos(Wt) + I'\sin(Wt) - \sum_{n=1}^{\infty}J_n' e^{-n\gamma t}], \tag{47}$$

and

$$H' = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}v_{ez}(t_f)\gamma}{\chi^n W^2\{1 + (\frac{n\gamma}{W})^2\}}, \tag{48}$$

and

$$I' = -\frac{v_{ez}(t_f)}{W}\ln\{\frac{\chi+1}{\chi}\} + J_n', \tag{49}$$

and

$$J_n' = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}v_{ez}(t_f)}{n\chi^n W\{1 + (\frac{n\gamma}{W})^2\}}. \tag{50}$$

These results show that the the "yaw" deviations affect all the velocity components, that is, these deviations affects all the movie of the satellite. We observe too the presence of the time penalty function K_3 in this nonlinear relation.

5 Rendezvous under Superposed Direction "pitch" and "yaw" Deviations

Satellite trajectories under superposed direction "pitch" and "yaw" deviations is the general and realistic case, because, during the thrusters burns these deviations occur simultaneously. In this approach we consider deviations non-correlated, that is, they occur non affecting each in other. With the same approach and mathematical proceedings used previously,

$$\mathcal{E}\{\Delta y(t)\} = K_1(t)\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}}\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}} - 1\} \tag{51}$$

or

$$\mathcal{E}\{\Delta y(t)\} = \sum_{j=2}^{\infty}\sum_{s=2}^{\infty}\frac{(-1)^{j+s+2}\{\Delta\beta_{max}\}^{2(j-1)}\{\Delta\alpha_{max}\}^{2(s-1)}}{(2j-1)!(2s-1)!}[A'\cos(Wt) + B'\sin(Wt) + \sum_{n=1}^{\infty} C_n e^{-n\gamma t} + D'], \tag{52}$$

and

$$\mathcal{E}\{\Delta x(t)\} = K_2(t)\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}}\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}} - 1\} \tag{53}$$

or

$$\mathcal{E}\{\Delta x(t)\} = \sum_{j=2}^{\infty}\sum_{s=2}^{\infty}\frac{(-1)^{j+s+2}\{\Delta\beta_{max}\}^{2(j-1)}\{\Delta\alpha_{max}\}^{2(s-1)}}{(2j-1)!(2s-1)!}[2A'\sin(Wt) - 2B'\cos(Wt) - 2\sum_{n=1}^{\infty} C'_n e^{-n\gamma t} + D''t - L], \tag{54}$$

and

$$\mathcal{E}\{\Delta z(t)\} = K_3(t)\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}}\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}} - 1\} \tag{55}$$

or

$$\mathcal{E}\{\Delta z(t)\} = \sum_{j=2}^{\infty}\sum_{s=2}^{\infty}\frac{(-1)^{j+s+2}\{\Delta\beta_{max}\}^{2(j-1)}\{\Delta\alpha_{max}\}^{2(s-1)}}{(2j-1)!(2s-1)!}[H'\cos(Wt) + I'\sin(Wt) - \sum_{n=1}^{\infty} J'_n e^{-n\gamma t}]. \tag{56}$$

So, we obtain the nonlinear cause/effect relations for the superposed "pitch" and "yaw" direction deviations between these thrust deviations and the satellite position coordinates uncertainess. These relations are more realistic and are pondered under penalty time-functions $K_1(t)$, $K_2(t)$, $K_3(t)$ due the mass variation.These results are obtained when we considered the eject velocity components constants.

6 Conclusions

The results obtained in this study showed nonlinear cause/effect relations between the thrust direction deviations, "pitch" and "yaw", and the Rendezvous satellite position coordinates uncertainess. When the eject velocity components are constants, these relations are averaged by penalty time-functions, due the effects of the mass variation. This functions are like weight-functions over the nonlinear relations and the Rendezvous conditions are affected due two reasons: the thrust deviations and the mass variation. The time dependence in these functions shows that the wanted Rendezvous maneuvers are the minimum time maneuvers, because these dependence is linear. For the long time Rendezvous maneuver the penalty functions are time oscillate functions in the $y(t)$ and $z(t)$ coordinates and linear in the $x(t)$ coordinate. It means larger uncertainess in this coordinate in this time regime. Besides this, in general, the results showed that there is a satellite position probability region where it occurs the Rendezvous maneuvers. These uncertainess are due the deviations influence during the thrusters burns. If the eject velocity components were not constants in the time, the penalty functions would be quadratures.

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A Successive Approximation Algorithm to Optimal Feedback Control of Time-varying LPV State-delayed Systems

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Abstract: In this paper problem of finite-time optimal state feedback control for a class of time-varying linear parameter-varying (LPV) systems with a known delay in the state vector under quadratic cost functional is investigated via a successive approximation algorithm. The method of successive approximation algorithm results an iterative scheme, which successively improves any initial control law ultimately converging to the optimal state feedback control. On the other hand, by manipulating linear matrix inequalities imposed by Generalized-Hamiltonian-Jacobi-Bellman method and the polynomially parameter-dependent quadratic (PPDQ) functions, sufficient conditions with high precision are given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay.

Keywords: *Linear parameter-varying systems; time-delay; successive approximation algorithm; optimal control.*

Mathematics Subject Classification (2000): 34K50, 93B40, 93D15, 93D30.

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1 Introduction

The investigation of the optimal control is of importance in modern control theory. The theory and the application of optimal control for linear time invariant (LTI) systems have been developed perfectly. For the convenient implementation, many suboptimal control methods have risen which do not pursue the optimal control performance indexes. In the literature, some computational methods were stated to solve finite-time optimal control problem of the LTI systems, time varying systems, second order linear systems, singular perturbed systems, nonlinear systems with quadratic cost functions [15, 22, 28-30, 37].

Over the last three decades, considerable attention has been paid to robustness analysis and control of linear systems affected by structured real parameters. Linear parameter-varying (LPV) systems have gained a lot of interest as they provide a systematic means of computing gain-scheduled controllers, especially those related to vehicle and aerospace control [2-6, 9, 18, 21].

Generally speaking, a LPV system is a linear system in which the system matrices are fixed functions of a known parameter vector. A LPV system can be viewed as a non-linear system that is linearized along a trajectory determined by the parameter vector. Hence, the parameter vector of an LPV system corresponds to the operating point of the non-linear system. In the LPV framework, it is assumed that the parameter vector is measurable for control process. In many industrial applications, like flight control and process control, the operating point can indeed be determined from measurement, making the LPV approach viable, see for example [7, 32, 36, 39]. Concerning unknown parameter vector, an adaptive method has been presented for robust stabilization with performance of LPV systems in [27].

For LPV systems, establishing stability via the use of constant Lyapunov functions is conservative. To investigate the stability of LPV systems one needs to resort the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability, see [10, 12-14, 16, 17, 19, 33, 43]. Bliman proposed the problem of robust stability for LPV systems with scalar parameters in [13]. Also, he developed some conditions for robust stability in terms of solvability of some linear matrix inequalities (LMIs) without conservatism. Moreover, the existence of a polynomially parameter-dependent quadratic (PPDQ) Lyapunov function for systems, which are robustly stable, is investigated in [14]. However, as for LPV systems, synthesis problems that are solved by classic control theory lead to difficult computations. People have studied optimal control of LPV systems for decades.

On the other hand, time delays are often present in engineering systems, which have been generally regarded as a main source on instability and poor performance [11, 31]. Therefore, the stabilization of LPV state-delayed systems is a field of intense research [38-41, 44]. Generally, a way to ensure stability robustness with respect to the uncertainty in the delays is to employ stability criteria valid for any nonnegative value of the delays that is delay-independent results. This assumption that no information on the value of the delay is known is often coarse in practice. Recently, systematic ways of the use of PPDQ functions in the state feedback control and output feedback control for LPV systems with time-delay in the state vector were proposed in [23-26]. It was shown that the PPDQ Lyapunov-Krasovskii functions make some sufficient conditions to investigate robust stability analysis of LPV systems in LMIs.

In this paper, we provide a systematic way to finite-time state feedback control problem for time-varying LPV systems with a constant delay in the state vector under quadratic cost functional via a successive approximation algorithm (SAA). This paper is essentially an extension of the SAA of the linear and nonlinear systems presented in [8] to the optimal control problem of the time-varying LPV state-delayed systems. The method of SAA results an iterative scheme, which successively improves any initial control law ultimately converging to the optimal state feedback control. On the other hand, by manipulating LMIs imposed by Generalized-Hamiltonian-Jacobi-Bellman (GHJB) method and the PPDQ functions, sufficient conditions with high precision are given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay.

The notations used throughout the paper are fairly standard. The matrices I_n , 0_n and $0_{n\times p}$ are the identity matrix, the $n \times n$ and $n \times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes} = 1$, $M^{p\otimes} := M^{(p-1)\otimes} \otimes M$. Let $\hat{J}_k, \tilde{J}_k \in \Re^{k\times(k+1)}$, and $v^{[k]}$ be defined by $\hat{J}_k := [I_k, \quad 0_{k\times1}]$, $\tilde{J}_k := [0_{k\times1}, \quad I_k]$ and $v^{[k]} = [1, v, \ldots, v^{k-1}]^T$, respectively, which have essential roles for polynomial manipulations [11]. Finally given a signal $x(t)$,$\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; i.e., $\|x(t)\|_2^2 = \int_0^\infty x^T(t)x(t)\, dt$.

2 Problem Description

Consider in the following a class of time-varying LPV state-delayed system

$$\begin{cases} \dot{x}(t) = A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u(t), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1}$$

where the constant parameter h is time delay and $\phi(t)$ is the continuous vector valued initial function, also $x(t) \in \Re^n$ and $u(t) \in \Re^l$ are the state vector and the control input, respectively. Moreover, the parameter-dependent matrices $A(t;\rho)$, $A_d(t;\rho)$ and $B(t;\rho)$ are expressed as

$$[A(t;\rho)\ A_d(t;\rho)\ B(t;\rho)] = [A_0(t)\ A_{0d}(t)\ B_0(t)] + \sum_{i=1}^{m} \rho_i [A_i(t)\ A_{id}(t)\ B_i(t)],$$

where $A_0(t), \cdots, A_m(t), A_{0d}(t), \cdots, A_{md}(t)$ and $B_0(t), \cdots, B_m(t)$ are known constant matrices of appropriate dimensions. Furthermore, it is known that the vector $\rho = [\rho_1, \rho_2, \cdots, \rho_m] \in \Re^m$ is contained in a priori given set whereas the actual curve of the vector ρ is unknown but can be measured online for control process. In the sequel, we make the following definitions for the system (1).

Definition 2.1 A finite-time state feedback $u(t) = -K(t;\rho)x(t)$ for $t \in [0, T]$ with $K(t;\rho) \in \Re^{m\times n}$ is said to achieve global asymptotic stability of the system (1) if the closed-loop system

$$\dot{x}(t) = (A(t;\rho) - B(t;\rho)K(t;\rho))x(t) + A_d(t;\rho)x(t-h) \tag{2}$$

is globally asymptotic stable in the Lyapunov sense.

According to Definition 2.1, the main objective of the paper is to develop an iterative technique for finite-time optimal control problem of the time-varying LPV state-delayed system (1), which minimizes the following cost functional with respect to some $u^*(t;\rho)$:

$$J = \|x(T)\|_{Q_0}^2 + \int_0^T \|x(t)\|_Q^2 + \|u(t)\|_R^2 \ dt. \tag{3}$$

Definition 2.2 A polynomially parameter-dependent quadratic (PPDQ) function is said to any quadratic function $x^T(t)S(\rho)x(t)$ such $S(\rho)$ is defined as

$$S(\rho) := (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T S_k (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n) \tag{4}$$

for a certain $S_k \in \Re^{(k^m n)\times(k^m n)}$. The integer $k-1$ is called the degree of the PPDQ function of $S(\rho)$.

3 Finite-Time Optimal Control Problem

Before deriving the main results, a preliminary Lemma is reviewed.

Lemma 3.1 *(Schur Complement lemma) Given constant matrices Ψ_1, Ψ_2 and Ψ_3 where $\Psi_1 = \Psi_1^T$ and $\Psi_2 = \Psi_2^T > 0$, then $\Psi_1 + \Psi_3^T \Psi_2^{-1} \Psi_3 < 0$ if and only if*

$$\begin{bmatrix} \Psi_1 & \Psi_3^T \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0 \quad \textit{or equivalently,} \quad \begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^T & \Psi_1 \end{bmatrix} < 0.$$

In the literature, extensions of the Lyapunov method to the Lyapunov-Krasovskii method have been proposed for time-delayed systems [11, 31]. To investigate the delay-independent asymptotically stability analysis of the closed-loop system (2), we define in the following a class of PPDQ Lyapunov-Krasovskii functions of the degree $k-1$

$$V(x(t);\rho) = x^T(t)P_\rho(t)x(t) + \int_{t-h}^{t} x^T(\sigma)Q_\rho(\sigma)x(\sigma)\, d\sigma, \tag{5}$$

where the positive-definite matrices $P_\rho(t) := P(t;\rho) \in \Re^{n\times n}$ and $Q_\rho(t) := Q(t;\rho) \in \Re^{n\times n}$ with the following forms

$$P_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k(t) (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n), \tag{6}$$

$$Q_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Q_k(t) (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n), \tag{7}$$

where the positive-definite matrices $\{P_k(t), Q_k(t)\} \in \Re^{(k^m n)\times(k^m n)}$ are to be determined.

Definition 3.1 Given an admissible control $u(t;\rho)$, which ensures the asymptotic stability of the closed-loop system (2). The function $V(x(t);\rho)$ in (5) satisfies the Generalized-Hamiltonian-Jacobi-Bellman (GHJB) inequality, written $GHJB(V,u) < 0$, if

$$\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}(A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u(t)) + \|x(t)\|_Q^2 + \|u(t)\|_R^2 < 0, \tag{8}$$

where $V(T,x) = \|x(T)\|_{Q_0}^2$.

Remark 3.1 Generally, the Hamiltonian-Jacobi equation being nonlinear is very difficult to solve. Recently, a new approach for solving the Hamiltonian-Jacobi equation for a fairly large class of Hamiltonian systems has been studied in [1].

To improve the performance of an arbitrary control $u^{(0)}$ we minimize the following function

$$u^{(1)} = arg \min_{u\in A_J(D)} \{ \frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}(A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u^{(0)})$$

$$+\|x(t)\|_Q^2 + \left\|u^{(0)}\right\|_R^2\} = \frac{-1}{2}R^{-1}B^T(t;\rho)\frac{\partial V^{(0)}}{\partial x}$$
$$= -R^{-1}B^T(t;\rho)P_\rho^{(0)}(t)x(t), \qquad (9)$$

where $D := [0,T]\times\Omega$, and Ω is a compact set of $\Re^n$ containing a ball around the origin and $A_J(D)$ is the set of admissible controls. In the infinite-time case, the initial control law is required to be stabilizing for the SAA to converge. For the finite-time this is not the case; in particular we may chose $u^{(0)} = 0$. However, $u^{(0)}$ provides a degree of freedom which maybe judiciously chosen to speed the convergence of the algorithm.

The cost of $u^{(1)}$ is given by the solution of the equation $GHJB(V^{(1)}, u^{(1)}) < 0$. In [34], it has been shown that $V^{(1)}(t,x) \le V^{(0)}(t,x)$ for each $(t,x)\in D$ and the convergence does not get stuck in local minimum, i.e., if $V^{(i+1)}(t,x) = V^{(i)}(t,x)$ for a fixed i, then $V^i(t,x) = V^*(t,x)$. Based on this fact, we assume that a unique optimal control u^* exists and is an admissible control. Then the optimal cost is given by the solution to the GHJB inequality, i.e.,

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^{*T}}{\partial x}(A(t;\rho)x(t)+A_d(t;\rho)x(t-h)+B(t;\rho)u^*(t)) + \|x(t)\|_Q^2 + \|u^*(t)\|_R^2 < 0. \quad (10)$$

From the solution to the GHJB inequality (10) we obtain an optimal control law as

$$u^*(t) = \frac{-1}{2}R^{-1}B^T(t;\rho)\frac{\partial V^*}{\partial x}$$
$$:= -K(t;\rho)x(t), \quad t\in[0,T], \qquad (11)$$

where $K(t;\rho) = R^{-1}B^T(t;\rho)P_\rho^*(t)$ and the optimal cost is

$$J(x(0),u^*) = \phi^T(0)P_\rho^*(0)\phi(0).$$

Remark 3.2 For the finite-time version of the problem, there is generally a unique solution to GHJB (under appropriate conditions), which brings up the question of obtaining the solution relevant to the infinite-time problem as the limit of the unique solution of the finite-time one. This question is investigated in [42] for nonlinear systems affine in the control and the disturbance, and with a cost function quadratic in the control, where the control is not restricted to lie in a compact set. It establishes the existence of a well-defined limit, and also obtains a result on global asymptotic stability of closed-loop system under the H_∞ controller and the corresponding worst-case disturbance.

Noting to the expressions (5), (10) and (11), we find

$$x^T(t)(\dot P_\rho(t)+A^T(t;\rho)P_\rho(t)+P_\rho(t)A(t;\rho)-P_\rho(t)B(t;\rho)R^{-1}B^T(t;\rho)P_\rho(t)+Q_\rho(t)+Q)x(t)$$
$$-x^T(t-h)Q_\rho(t)x(t-h) + x^T(t)P_\rho(t)A_d(t;\rho)x(t-h) + (A_d(t;\rho)x(t-h))^TP_\rho(t)x(t) < 0. \qquad (12)$$

Then, the aforementioned inequality is rewritten as

$$X^T(t)M_\rho(t)X(t) < 0, \qquad (13)$$

where the new vector $X(t) = [x^T(t), x^T(t-h)]^T$ is an augmented state and the parameter-dependent matrix $M_\rho(t)$ is defined as

$$M_\rho(t) = \begin{bmatrix}\tilde\Sigma_{11} & P_\rho(t)A_{d\rho}(t)\\ * & -Q_\rho(t)\end{bmatrix}, \qquad (14)$$

where

$$\tilde{\Sigma}_{11} = \dot{P}_\rho(t) + A_\rho^T(t)P_\rho(t) + P_\rho(t)A_\rho(t) - P_\rho(t)B_\rho(t)R^{-1}B_\rho^T(t)P_\rho(t) + Q_\rho(t) + Q.$$

Remark 3.3 Stability of the time-varying LPV state-delayed system (1) can be provided by finding the positive-definite solutions $P_\rho(t)$ and $Q_\rho(t)$ to the associated parameter-dependent matrix inequality $M_\rho(t) < 0$.

In the iterative step, we assume a non-singular solution and that $V^{(i)}(x(t);\rho)$ has the form

$$V^{(i)}(x(t);\rho) = x^T(t)P_\rho^{(i)}(t)x(t) + \int_{t-h}^{t} x^T(\sigma)Q_\rho^{(i)}(\sigma)x(\sigma)\, d\sigma \tag{15}$$

and we let the new control be

$$u^{(i)}(t;\rho) := -K^{(i)}(t;\rho)x(t) = -R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t)x(t). \tag{16}$$

By substituting $V^{(i)}(x(t);\rho)$ and the control (16) into (10) and using Schur Complement Lemma, the following parameter-dependent LMI is easily obtained

$$\begin{bmatrix} \hat{\Sigma}_{11}^{(i)} & P_\rho^{(i-1)}(t)B_\rho(t) & P_\rho^{(i)}(t)A_{d\rho}(t) \\ * & -R & 0 \\ * & * & -Q_\rho^{(i)}(t) \end{bmatrix} < 0, \tag{17}$$

where

$$\hat{\Sigma}_{11}^{(i)} = \dot{P}_\rho^{(i)}(t) + P_\rho^{(i)}(t)(A_\rho(t) - B_\rho(t)R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t))$$
$$+(A_\rho(t) - B_\rho(t)R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t))^T P_\rho^{(i)}(t) + Q_\rho^{(i)}(t) + Q.$$

Remark 3.4 A general framework for relaxing parameter-dependent LMI problems into parameter-independent LMIs (conventional form) has been investigated in [5]. However, application of the PPDQ Lyapunov functions as a new tool for relaxing parameter dependency of the matrix inequalities will be stated in the next section.

4 Parameter-Dependent LMI Relaxations

In this section the PPDQ functions as the basis functions are used to relax parameter-dependent LMIs into conventional parameter-independent LMI problems by utilizing some positives-definite Lagrange multiplier matrices (see for instance [24,24]).

Lemma 4.1 *Let the degree of the PPDQ Lyapunov function $P_\rho^{(i)}(t)$ be $k-1$. The parameter-dependent matrix $P_\rho^{(i)}(t)B_\rho(t)$ can be represented as*

$$P_\rho^{(i)}(t)\, B_\rho(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T H_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_l), \tag{18}$$

where the matrix $H_k^{(i)}(t) \in \Re^{((k+1)^m n)\times((k+1)^m l)}$ which depends linearly on the matrix $P_k^{(i)}(t)$ is defined as

$$H_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes B_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t)). \tag{19}$$

Proof According to the structures of the parameter-dependent matrices $P_\rho^{(i)}(t)$ and $B_\rho(t)$, one has

$$P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k^{(i)}(t)(\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)(B_0(t) + \sum_{i=1}^{m} \rho_i B_i(t))$$

and using the property of $(\vartheta^{[k]} \otimes I_n)B_i(t) = (I_k \otimes B_i(t))(\vartheta^{[k]} \otimes I_l)$ one finds

$$P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]}\otimes\ldots\otimes\rho_1^{[k]}\otimes I_n)^T P_k^{(i)}(t)(I_{k^m}\otimes(B_0(t)+\sum_{i=1}^{m}\rho_i B_i(t)))(\rho_m^{[k]}\otimes\ldots\otimes\rho_1^{[k]}\otimes I_l)$$

or

$$\begin{aligned} P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k^{(i)}(t)((I_{k^m} \otimes B_0(t))(\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l) \\ +(I_{k^m} \otimes B_1(t))(\rho_m^{[k]} \otimes \ldots \otimes \rho_1\rho_1^{[k]} \otimes I_l) + \cdots \\ +(I_{k^m} \otimes B_m(t))(\rho_m\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l)), \end{aligned}$$

then by repeated use of the properties $\hat{J}_k\rho_i^{[k+1]} = \rho_i^{[k]}$ and $\tilde{J}_k\rho_i^{[k+1]} = \rho_i\rho_i^{[k]}$ the matrix $H_k^{(i)}(t)$ in (19) is obtained. □

According to Lemma 4.1 for the matrix $A_{d\rho}(t)$, we have:

$$P_\rho^{(i)}(t)\,A_{d\rho}(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n), \quad (20)$$

where the matrix $S_k^{(i)}(t)$ is expressed in the form

$$S_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes A_{0d}(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_{id}(t)). \quad (21)$$

Therefore, the PPDQ Lyapunov function of degree k for the positive-definite matrix $R_\rho^{(i)}(t) = A_\rho^T(t)P_\rho^{(i)}(t) + P_\rho^{(i)}(t)A_\rho(t)$ is written as

$$R_\rho^{(i)}(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T R_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n) \quad (22)$$

and from Lemma 4.1 the matrix $R_k^{(i)}(t)$ in (22), which depends linearly on the matrix $P_k^{(i)}(t)$ is obtained as follows:

$$\begin{aligned} R_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t)) \\ +(\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t))^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n). \end{aligned} \quad (23)$$

Similarly, the constant positive-definite matrices $R \in \Re^{l\times l}$, and $Q \in \Re^{n\times n}$ can be represented as

$$R = (\rho_m^{[k+1]}\otimes\ldots\otimes\rho_1^{[k+1]}\otimes I_l)^T(\hat{J}_k^{m\otimes}\otimes I_l)^T\bar{R}_k(\hat{J}_k^{m\otimes}\otimes I_l)(\rho_m^{[k+1]}\otimes\ldots\otimes\rho_1^{[k+1]}\otimes I_l), \quad (24)$$

and

$$Q = (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T (\hat{J}_k^{m\otimes} \otimes I_n)^T \bar{Q}_k (\hat{J}_k^{m\otimes} \otimes I_n)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n), \quad (25)$$

where the certain matrices $\bar{R}_k$ and $\bar{Q}_k$ are defined, respectively, as

$$\bar{R}_k = diag\,(R, \underbrace{0_l, \cdots, 0_l}_{(k^m-1)\ elements}),$$

and

$$\bar{Q}_k = diag\,(Q, \underbrace{0_n, \cdots, 0_n}_{(k^m-1)\ elements}).$$

We are now in the position to state our main result in the following Theorem.

Theorem 4.1 *For a given positive parameter k if there exist positive-definite matrices $P_k^{(i)}(t)$, $Q_k^{(i)}(t)$ and the set of positive definite Lagrange multipliers $\hat{Q}_{i,k}^{(1)}(t), \hat{Q}_{i,k}^{(2)}(t)$ and $\hat{Q}_{i,k}^{(3)}(t)$ for $i = 1, 2, \cdots, m$ to the following parameter-independent differential linear matrix inequality (DLMI),*

$$\begin{bmatrix} \Sigma_{11} & H_k^{(i-1)}(t) & S_k^{(i)}(t) \\ * & \Sigma_{22} & 0 \\ * & * & \Sigma_{33} \end{bmatrix} < 0, \quad (26)$$

where

$$\Sigma_{11} = (\hat{J}_k^{m\otimes} \otimes I_n)^T \dot{P}_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n) + \hat{R}_k^{(i)}(t)$$
$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T (Q_k^{(i)}(t) + \bar{Q}_k)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$

$$\Sigma_{22} = -(\hat{J}_k^{m\otimes} \otimes I_l)^T \bar{R}_k (\hat{J}_k^{m\otimes} \otimes I_l)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l}),$$

and

$$\Sigma_{33} = -(\hat{J}_k^{m\otimes} \otimes I_n)^T Q_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$

with

$$\hat{R}_k^{(i)}(t) = \{\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t) - (\hat{J}_k^{m\otimes} \otimes B_0(t)$$

$$+ \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t))R^{-1}H_k^{(i-1)^T}(t)\}^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n)$$

$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)\{\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t)$$

$$-(\hat{J}_k^{m\otimes} \otimes B_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t))R^{-1}H_k^{(i-1)^T}(t)\},$$

then the parameter-dependent finite-time state feedback control

$$u^{(i)}(t;\rho) = -R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t)x(t), \quad t \in [0,T] \tag{27}$$

achieves global asymptotic stability for the time-varying LPV state-delayed system (1) with the quadratic cost function (3).

Proof By substituting the relations (18)-(25) into the parameter-dependent LMI (17), one parameter-dependent matrix inequality is obtained which includes left- and right-multiplication of the (26) by

$$\begin{bmatrix} \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n & 0 & 0 \\ * & \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l & 0 \\ * & * & \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n \end{bmatrix},$$

and its transpose. Then, it can be concluded that the LMI (26), which included the positive-definite Lagrange multipliers $\hat{Q}_{1,k}^{(1)}, \cdots, \hat{Q}_{m,k}^{(1)}, \hat{Q}_{1,k}^{(2)}, \cdots, \hat{Q}_{m,k}^{(2)}$ and $\hat{Q}_{1,k}^{(3)}, \cdots, \hat{Q}_{m,k}^{(3)}$, is a sufficient condition to fulfil the parameter-dependent matrix inequality (17) for any vector ρ contained in a priori given set. □

It is essential in this result that the matrices $P_k^{(i)}(t)$ and $Q_k^{(i)}(t)$ are calculated independently from the parameter vector ρ and thereafter $P_\rho^{(i)}(t)$, $Q_\rho^{(i)}(t)$ and the control law are found analytically by (6), (7) and (27), respectively.

Remark 4.1 The solution to the DLMI in (26) can be obtained by discretizing the time interval $[0,T]$ into equally spaced time instances $\{t_j, j = 1, \cdots, N, t_N = T, t_0 = 0\}$ [35], where

$$t_j - t_{j-1} := \kappa = N^{-1}T, \quad j = 1, \cdots, N.$$

The discretized DLMI problem thus becomes one of finding, at each $\kappa \in [1,N]$, $P_k^{(i)_{j-1}}(:= P_k^{(i)}(t_{j-1}))$ that satisfies

$$\begin{bmatrix} \hat{\Sigma}_{11} & H_k^{(i-1)_j} & S_k^{(i)_j} \\ * & \hat{\Sigma}_{22} & 0 \\ * & * & \hat{\Sigma}_{33} \end{bmatrix} < 0 \tag{28}$$

with $P_k^{(i)_N} = Q_0$ and

$$\hat{\Sigma}_{11} = (\hat{J}_k^{m\otimes} \otimes I_n)^T(-P_k^{(i)_{j-1}} + P_k^{(i)_j})(\hat{J}_k^{m\otimes} \otimes I_n) + \kappa \hat{R}_k^{(i)_j}(t)$$
$$+\kappa(\hat{J}_k^{m\otimes} \otimes I_n)^T(Q_k^{(i)_j}(t) + \bar{Q}_k)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\kappa \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\kappa \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$
$$\hat{\Sigma}_{22} = -\kappa^{-1}(\hat{J}_k^{m\otimes} \otimes I_l)^T \bar{R}_k(\hat{J}_k^{m\otimes} \otimes I_l)$$
$$+\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})$$
$$-\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l}),$$
$$\hat{\Sigma}_{33} = -\kappa^{-1}(\hat{J}_k^{m\otimes} \otimes I_n)^T Q_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$
$$H_k^{(i-1)_j} := H_k^{(i-1)}(t_j)$$
$$= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i-1)_j}(\hat{J}_k^{m\otimes} \otimes B_0^j + \sum_{l=1}^{m} \hat{J}_k^{(m-l)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(l-1)\otimes} \otimes B_i^j),$$

and

$$S_k^{(i)_j} := S_k^{(i)}(t_j)$$
$$= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes A_{0d}^j + \sum_{l=1}^{m} \hat{J}_k^{(m-l)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(l-1)\otimes} \otimes A_{ld}^{(j)}),$$

with

$$\hat{R}_k^{(i)_j} := \hat{R}_k^{(i)}(t_j)$$
$$= \{\hat{J}_k^{m\otimes} \otimes A_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i^j - (\hat{J}_k^{m\otimes} \otimes B_0^j$$
$$+ \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i^j)R^{-1} H_k^{(i-1)_j^T}\}^T P_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)_j}\{\hat{J}_k^{m\otimes} \otimes A_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i^j$$
$$-(\hat{J}_k^{m\otimes} \otimes B_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i^j)R^{-1} H_k^{(i-1)_j^T}\},$$

where $A_i^j := A_i(t_j)$, $A_{ld}^j := A_{ld}(t_j)$ and $B_i^j := B_i(t_j)$.

Remark 4.2 From the DLMI (26) in Theorem 4.1, it can be concluded that a unique positive-definite solution to (8) exists, then $V^{(0)}(x(t);\rho) \geq V^{(1)}(x(t);\rho) \geq \cdots \geq V^{*}(x(t);\rho)$ with equality holding if and only if $V^{(i)}(x(t);\rho) \equiv V^{*}(x(t);\rho)$. Furthermore $V^{(i)}(x(t);\rho) \to V^{*}(x(t);\rho)$ and $u^{(i)}(x(t);\rho) \to u^{*}(x(t);\rho)$ pointwise for all $x(t), \rho$ and $t \in [0,T]$.

Remark 4.3 It is observed that the discretized DLMI (28) is linear in $P_k^{(i)_j}$, $Q_k^{(i)_j}, \hat{Q}_{1,k}^{(1)}, \cdots, \hat{Q}_{m,k}^{(1)}, \hat{Q}_{1,k}^{(2)}, \cdots, \hat{Q}_{m,k}^{(2)}$ and $\hat{Q}_{1,k}^{(3)}, \cdots, \hat{Q}_{m,k}^{(3)}$ thus the standard LMI techniques, [20], can be exploited to find the positive-definite solutions. It is also seen from the above results that the choice of appropriate parameter $k-1$ as the degree of the PPDQ Lyapunov functions of the matrix $P_k^{(i)}(t)$ and $Q_k^{(i)}(t)$ play the role of freedom of design in the control law.

5 Conclusion

A successive approximation algorithm was used to generate the finite-time optimal feedback gains for a class of time-varying LPV state-delayed systems under quadratic cost functional. The method of SAA was developed, which successively improves any initial control law ultimately converging to the optimal state feedback control. By manipulating LMIs imposed by Generalized-Hamiltonian-Jacobi-Bellman method and the PPDQ functions, sufficient conditions with high precision were given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay. In this paper, the results are presented on the delay-independent stability conditions case, and the extension of the results to delay-dependent stability conditions is a topic currently under study.

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The Sufficient Conditions of Local Controllability for Linear Systems with Random Parameters

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Abstract: This paper is concerned with the problem of local controllability for linear nonstationary systems with random parameters. In differ of well-known problem of controllability for the determinated systems, for systems with random parameters we must construct a non-predicting control when we use the information about system only before the current moment. We obtain the sufficient conditions of non-predicting controllability and estimation of the probability that the given system is a locally controllable on the fixed time segment. The algorithm of construction of the non-predicting control is developed.

Keywords: *Local controllability; non-predicting control; controllability set; stationary stochastic process.*

Mathematics Subject Classification (2000): 93B05, 93B52, 93E20, 37N35, 49J55.

1 Introduction

The problems of controllability, observability and stability of dynamical systems with random parameters was investigated in many works, for example, [1]-[7]. Notice, that for such type of systems we often have not the information about the systems behaviour in future, thats why is appeared a problem of existence of a non-predicting control. The term of the non-predicting control was introduced in Ekaterinburg school on the control theory (see [8, 9]), the problem of such control construction was investigated also in [10, 11]. The control $u(t,x)$ is called the *non-predicting* if for it construction in the moment $t = \tau$ we use the information about system only for $t \leqslant \tau$.

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In this paper we continue the investigation initiated in [12, 13], where we considered the linear systems with the stationary random parameters and obtained the sufficient conditions of existence of the non-predicting control for such systems. In [12, 13] we investigated the conditions of total controllability when we don't assumed any restrictions on the control $u \in \mathbb{R}^m$. Here we consider the system

$$\dot{x} = A(f^t\omega)x + B(f^t\omega)u, \quad (t, \omega, x, u) \in \mathbb{R} \times \Omega \times \mathbb{R}^n \times U, \tag{1}$$

where the function $t \to \xi(f^t\omega) \doteq \big(A(f^t\omega), B(f^t\omega)\big)$ of variable t is a piecewise constant for every $\omega \in \Omega$. We assume that $u \in U$, where U is a compact convex set in $\mathbb{R}^m$ and U contains the origin in their interior. The aim of this paper is to obtain the sufficient conditions of the non-predicting local controllability for system (1) on the segment $[0, T]$. We prove that in the case $u \in U$ for construction of the non-predicting control we must constantly hold the trajectories of the system (1) solutions in the neighbourhood of the origin, that lead to some additional conditions for the asymptotical behaviour of the system $\dot{x} = A(f^t\omega)x$ solutions.

2 The basic definitions and designations

Suppose $e_1 \doteq \mathrm{col}(1, 0, \dots, 0), \dots, e_n \doteq \mathrm{col}(0, \dots, 0, 1)$ is a standard basis in Euclidean space $\mathbb{R}^n$; $\|x\| = \sqrt{x^*x}$ is a norm in $\mathbb{R}^n$; $\mathrm{Lin}(q_1, \dots, q_r)$ is a linear hull of the vectors $q_1, \dots, q_r \in \mathbb{R}^n$; $O^n_\varepsilon(x_0)$ is an ε-neighbourhood of the point x_0 in $\mathbb{R}^n$, $O^n_\varepsilon \doteq O^n_\varepsilon(0)$; $\mathrm{int}\, U$ is an interior of the set U.

Let us consider the probability spaces $(\Omega_1, \mathfrak{F}_1, \mu_1)$ and $(\Omega_2, \mathfrak{F}_2, \mu_2)$, where Ω_1 is a space of number sequences $\theta = (\theta_1, \dots, \theta_k, \dots)$, $\theta_k \in (0, \infty)$, the space $\Omega_2 \doteq \{\varphi : \varphi = (\varphi_0, \varphi_1, \dots \varphi_k, \dots), \varphi_k \in \Psi\}$, $\Psi = \{\psi_j\}_{j=1}^s$ is a finite set of the matrix pairs $\psi_j \doteq (A_j, B_j)$, $\mathfrak{F}_i$ is a σ-algebra formed by the corresponding cylinder sets, μ_i is an extension of a measure $\widetilde{\mu}_i$ from the algebra of the cylinder sets to the σ-algebra $\mathfrak{F}_i$, $i = 1, 2$. We also consider the probability space $(\Omega, \mathfrak{F}, \mu)$, where $\Omega = \Omega_1 \times \Omega_2$. The construction of σ-algebra $\mathfrak{F}$ and the probability measure μ was described in [2].

On the space $(\Omega_2, \mathfrak{F}_2, \mu_2)$ for every $\theta \in \Omega_1$ we introduce the sequence of random variables $\zeta = (\zeta_0, \zeta_1, \dots)$ such that $\zeta_k(\omega) = \zeta_k(\varphi, \theta) = \varphi_k$, $\varphi_k \in \Psi$. We suppose that the sequence ζ forms the homogeneous Markov chain, which uniquely determines by the matrix of the transition probabilities $P = (p_{ij})_{i,j=1}^s$ and the initial distribution $\pi = (\pi_i)_{i=1}^s$ (see [14, p. 122]). We also suppose that the Markov chain ζ is a *stationary in the narrow sense* (see [14, p. 432]).

Let us introduce the sequence $\{\tau_k\}_{k=0}^\infty : \tau_0 = 0$, $\tau_k(\theta) = \sum_{i=1}^k \theta_i$, where $\theta \in \Omega_1$. We assume that $\theta_1, \theta_2, \dots$ are the independent positive random variables and $\theta_2, \theta_3, \dots$ have the equal distribution $F(t)$, $t \in (0, \infty)$ with the mathematical expectation m_θ. Denote by $\nu(t, \theta)$ a number of points of the sequence $\{\tau_k\}$, which lie left than t, that is

$$\nu(t, \theta) = \max\{k : \tau_k \leqslant t\}, \quad t \geqslant 0.$$

The variable $\nu(t)$ is called a recovery process. We assume that $\nu(t)$ is a stationary recovery process (that is this process have a stationary recovery speed), then the distribution of θ_1 satisfies the equality (see [15, p. 145–147])

$$F_1(t) = \frac{1}{m_\theta} \int_0^t (1 - F(x))dx, \quad t > 0. \tag{2}$$

Let us introduce the shift transformation $f_1^t\theta = (\tau_{\nu+1} - t, \theta_{\nu+2}, \theta_{\nu+3}, \dots)$, $t > 0$ on the probability space $(\Omega_1, \mathfrak{F}_1, \mu_1)$. The transformation f_1^t preserves the measure μ_1, because the sequence $\{\tau_k\}$ forms the stationary recovery process. We also introduce the shift transformation $f_2^t(\theta)\varphi = (\varphi_\nu, \varphi_{\nu+1}, \dots)$ on the space $(\Omega_2, \mathfrak{F}_2, \mu_2)$ for any $\theta \in \Omega_1$. From the stationarity of the Markov chain ζ it follows that the transformation f_2^t preserves the measure μ_2. In [16, p. 190] was proved that the shift transformation $f^t\omega = f^t(\theta, \varphi) = \big(f_1^t\theta, f_2^t(\theta)\varphi\big)$ on the space $(\Omega, \mathfrak{F}, \mu)$ preserves the measure μ.

Assume that $\xi(\omega) = \zeta_0(\omega)$ is a stochastic variable on the probability space $(\Omega, \mathfrak{F}, \mu)$. We introduce the random process $\xi(f^t\omega) = \big(A(f^t\omega), B(f^t\omega)\big)$ generated by the flow $f^t\omega$. Then $\xi(f^t\omega)$ receives the constant values φ_k for $t \in [\tau_k, \tau_{k+1})$. The function $\xi(f^t\omega)$ is a stationary in the narrow sense random process (see [14, p. 433], [16, p. 167], [17, p. 189]). We remind that the process $\xi(t, \omega)$ is called *stationary in the narrow sense* if the equality $\mu(f^tG) = \mu(G)$ satisfies for any cylinder set $G \in \mathfrak{F}$ (see [16, p. 174]).

We identify the system (1) with the function $\xi : \Omega \to \Psi$. For each fixed ω the function $\xi(f^t\omega)$ designates a linear determinate system. We say that *an admissible control* of the system ξ is any bounded and Lebesgue measurable function $u_\omega : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to U \in \mathbb{R}^m$. The control type $u_\omega(t, x_0)$ is said to be *program control* if it is not explicitly depends from x; the control type $u_\omega(t, x)$ is said to be *positional control*. The program control $u_\omega(t, x_0)$ is said to be *non-predicting* on the segment $[t_0, t_1]$ if for it construction in the moment $\tau \in [t_0, t_1]$ we use the information about matrices $A(f^t\omega)$ and $B(f^t\omega)$ only for $t \leqslant \tau$ (and not use the information for $t > \tau$).

Let us consider the intervals $[\tau_k, \tau_{k+1})$, where the function $\xi(f^t\omega)$ receives the constant values $\varphi_k \in \Psi$. On any interval $[\tau_k, \tau_{k+1})$ the system ξ coincides with one of the systems ξ_i, $i = 1, \dots, s$, where over ξ_i we denote the system

$$\dot{x} = A_i x + B_i u, \quad (x, u) \in \mathbb{R}^n \times U.$$

Here U is a compact convex set in $\mathbb{R}^m$ and U contains the origin in their interior. In this work we construct the non-predicting control in such form that on any interval $[\tau_k, \tau_{k+1})$, $k = 0, 1, \dots$ we apply either the positional control, or at first the program control for $t \in [\tau_k, \tau_k + \alpha)$, then the positional control for $t \in [\tau_k + \alpha, \tau_{k+1})$. Therefore let us improve in what sense we determine the solution of the system ξ under the fixed $\omega \in \Omega$. We introduce the sequence $\{\vartheta_k\}_{k=0}^{\infty}$, where $\vartheta_0 = 0$, $\vartheta_{k+1} > \vartheta_k$ such that on the intervals $[\vartheta_k, \vartheta_{k+1})$, $k = 1, \dots$ we apply either only the program control, or only the positional one in dependence from the number of system ξ_i that appeared in the corresponding time moment. If we construct the program control $u_\omega(t)$ on the interval $[\vartheta_k, \vartheta_{k+1})$, then the solution of the system ξ is an absolutely continuous function $x(t) = x(t, \vartheta_k, x_k, u_\omega)$, $x(\vartheta_k) = x_k$, which satisfies the corresponding system $\dot{x} = A_i x + B_i u_\omega(t)$ for almost all $t \in [\vartheta_k, \vartheta_{k+1})$. For the continuity of the solution we require that $x(\vartheta_k, \vartheta_{k-1}, x_{k-1}) = x_k$. Now we assume that on the interval $[\vartheta_k, \vartheta_{k+1})$ we must construct the positional control $u = u_\omega(t, x)$. Let us consider the system ξ_i closed by the control $u = u_\omega(t, x)$ and denote by $x(t) = x(t, \vartheta_k, x_k, u_\omega)$ the solution of this system. We require that $x(t)$ satisfies the conditions $x(\vartheta_k) = x_k$, $x(\vartheta_k, \vartheta_{k-1}, x_{k-1}) = x_k$. Let us denote $u_\omega(t) = u_\omega(t, x(t))$. Then for any initial point x_k the solution of the system $\dot{x} = A_i x + B_i u_\omega(t, x)$ we can also obtain as the solution of the control system ξ_i that corresponds the control $u_\omega(t)$, see [18, p. 431–433].

Definition 2.1 The state $x_0 \in \mathbb{R}^n$ of system $\xi(f^t\omega)$ is said to be *controllable* (*non-predicting controllable*) *on the segment* $[t_0, t_1]$ if there exists a control $u_\omega(t, x, x_0)$ (non-predicting control $u(f^t\omega, x, x_0))$, $t \in [t_0, t_1]$ such that the corresponding solution $x(t,\omega)$, $x(t_0,\omega)=x_0$ satisfies $x(t_1,\omega)=0$.

We denote by $D_{[t_0,t_1]}(\omega)$ the controllability set of the system $\xi(f^t\omega)$ on the segment $[t_0,t_1]$, that is the set of all points, which can be steered to zero on $[t_0,t_1]$ under the fixed $\omega \in \Omega$. We also denote by $\mathcal{D}_{[t_0,t_1]}(\omega)$ the set of all non-predicting controllable states of the system $\xi = \xi(f^t\omega)$ on the segment $[t_0,t_1]$.

Definition 2.2 The system ξ is said to be *locally controllable with the probability* μ_0 on the segment $[t_0,t_1]$ if $\mu\{\omega : 0 \in \operatorname{int} D_{[t_0,t_1]}(\omega)\} = \mu_0$ and *non-predicting locally controllable with the probability* μ_0 on the segment $[t_0,t_1]$ if the probability $\mu\{\omega : 0 \in \operatorname{int} \mathcal{D}_{[t_0,t_1]}(\omega)\} = \mu_0$.

3 The Construction of the Positional Control

Let us consider the system ξ_i and denote by K_i the matrix

$$K_i = (B_i, A_iB_i, \ldots, A_i^{n-1}B_i), \quad i = 1,\ldots,s,$$

by $D_{[t_0,t_1]}(\xi_i)$ the controllability set of the system ξ_i on the segment $[t_0,t_1]$, by $L(\xi_i) \doteq \operatorname{Lin} D_{[t_0,t_1]}(\xi_i)$ the controllability space of the system ξ_i, by $X_i(t,s) = X_i(t-s)$ the Cauchy matrix of this system. It is known that the controllability space $L(\xi_i)$ coincides with the subspace formed by the columns of the matrix K_i, that is $L(\xi_i) = \operatorname{Lin} K_i$. Therefore the condition $\operatorname{rank} K_i = n$ is the necessary and sufficient condition of the local controllability for system ξ_i (see [19, p. 140–145]).

Let us consider a determinate system ξ_0, which coincides with the system ξ_{i_ℓ} on any interval $[(\ell-1)\alpha, \ell\alpha)$, $\ell = 1,\ldots,k$, that is $\xi_0 = \psi_{i_\ell}$ for $t \in [(\ell-1)\alpha, \ell\alpha)$. We can consider the system ξ_0 as the system ξ under the fixed $\omega = (\theta,\varphi)$ with k first coordinates $\omega_\ell = (\alpha, \psi_{i_\ell})$.

Lemma 3.1 [20] *Assume that* $\xi_0 = \psi_{i_\ell}$ *for* $t \in [(\ell-1)\alpha, \ell\alpha)$, $\ell = 1,\ldots,k$. *Then the controllability space of system* ξ_0 *on the segment* $[(\ell-1)\alpha, k\alpha]$

$$L_{[(\ell-1)\alpha,k\alpha]}(\xi_0) = L(\xi_{i_\ell}) + X_{i_\ell}^{-1}(\alpha)L(\xi_{i_{\ell+1}}) + \ldots + X_{i_\ell}^{-1}(\alpha)\cdot\ldots\cdot X_{i_{k-1}}^{-1}(\alpha)L(\xi_{i_k}).$$

Suppose that for system ξ there exists $\omega \in \Omega$ such that the corresponding determinate system ξ_0 is a totally controllable on the segment $[0,k\alpha]$, that is the controllability space $L_{[0,k\alpha]}(\xi_0)$ coincides with $\mathbb{R}^n$. In the present work we investigate the next problem: is it possible to construct the non-predicting control for the system ξ and what is the probability that this system is the non-predicting controllable on the fixed time segment $[0,T]$ (in the process of construction of such control we assume that for the system ξ in the moment τ the moments of switching τ_k and the states of this system for $t > \tau$ are unknown). Further we propose the algorithm of construction of the non-predicting control when it is not sufficient the equality $L_{[0,k\alpha]}(\xi_0) = \mathbb{R}^n$. The subspaces $L_{[(\ell-1)\alpha,k\alpha]}(\xi_0)$, $\ell = 2,\ldots,k$ must satisfy some additional condition, that is the trajectory of the system under some control must retains in given subspace to the next moment of switching. In Lemma 3.2 we obtain the condition of such retaining when there are not any restrictions on the control.

Lemma 3.2 [12] *Let $\mathcal{M}$ be a subspace in $\mathbb{R}^n$ and M be a matrix formed from the vectors of basis $\mathcal{M}$. If for the system*

$$\dot{x} = Ax + Bu, \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{3}$$

we have $\operatorname{Lin} AM \subset \operatorname{Lin}(M, B)$, *then there exists a positional control $u(x)$ such that for any point $x_0 \in \mathcal{M}$ the trajectory of solution $x(t, t_0, x_0, u)$ contains in the subspace $\mathcal{M}$ for all $t \geqslant 0$.*

Further we consider the system $S : \dot{x} = Ax + Bu$, $(x, u) \in \mathbb{R}^n \times U$, where $U \subset \mathbb{R}^m$ is a compact convex set containing the origin in their interior. We denote by $L(S) \doteq \operatorname{Lin} D_{[t_0,t_1]}(S)$, then $L(S)$ is a controllability space for the system (3).

In the next statement we obtain the sufficient conditions of existence of the positional control $u(x) \in U$ for the system S. This control must retains the trajectory of solution $x(t, t_0, x_0, u)$ on the subspace $\mathcal{M}$ for $t \geqslant t_0$, if x_0 is a point located on this subspace in the moment t_0. Furthermore, for $u(x) \in U$ must exists $\varepsilon > 0$ such that from the inequality $\|x_0\| < \varepsilon$ follows that the solution $\|x(t, t_0, x_0, u)\| < \varepsilon$ for all $t \geqslant t_0$.

We denote by $\lambda_1, \dots, \lambda_p$ the eigenvalues of matrix A corresponding to the different Jordan cells (for this eigenvalues not required to be different), by m_k we denote the size of Jordan cell corresponding to the eigenvalue λ_k. We also denote by Λ the set of eigenvalues λ_k such that either $\mathrm{Re}\lambda_k > 0$ or $\mathrm{Re}\lambda_k = 0$ and the size of corresponding Jordan cell is more than one, that is

$$\Lambda \doteq \big\{\lambda_k : \lambda_k \in \{\mathrm{Re}\lambda_k > 0\} \cup \{\mathrm{Re}\lambda_k = 0, m_k > 1\}\big\}.$$

Lemma 3.3 *Let $\mathcal{M}$ be a subspace in $\mathbb{R}^n$ and M be a matrix from the vectors of basis $\mathcal{M}$. Suppose that the system S and the subspace $\mathcal{M}$ satisfy the conditions:*

(1) $\mathcal{M} \cap L(S) = \{0\}$;

(2) $\operatorname{Lin} AM \subset \operatorname{Lin}(M, B)$;

(3) *the controllability space $L(S)$ contains all rooted subspaces of matrix A, corresponding to the eigenvalues $\lambda_k \in \Lambda$.*

Then there exists the positional control $u(x) \in U$, for which we can find $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ such that for any point $x_0 \in \mathcal{M} \cap O_\delta$ the trajectory of solution $x(t, t_0, x_0, u)$ contains in $\mathcal{M} \cap O_\varepsilon$ for all $t \geqslant t_0$.

Proof Assume that for the system S a dimension of the controllability space $\dim L(S) = r$. Then there exists a linear transformation $x = Cy$ that reduce the system S to the system type $\widetilde{S} = (\widetilde{A}, \widetilde{B})$:

$$\dot{y}^1 = A_{11}y^1 + A_{12}y^2 + B_1\widetilde{u},$$

$$\dot{y}^2 = A_{22}y^2,$$

where $y^1 \in \mathbb{R}^r$, $y^2 \in \mathbb{R}^{n-r}$ and the controllability subspace $L(\widetilde{S})$ determines in $\mathbb{R}^n$ by the equation $y^2 = 0$ (see [18, p. 110]). From the equalities $\widetilde{A} = C^{-1}AC$ and $\widetilde{B} = C^{-1}B$ it is easy to verify that the controllability spaces of the systems S and $\widetilde{S}$ satisfy the condition

$$L(\widetilde{S}) = C^{-1}L(S). \tag{4}$$

Let us denote $\widetilde{\mathcal{M}} = C^{-1}\mathcal{M}$. Then from the conditions (1) and (4) follows that $\widetilde{\mathcal{M}} \cap L(\widetilde{S}) = \{0\}$. The conditions (2) and $\operatorname{Lin} C\widetilde{A}\widetilde{M} \subset \operatorname{Lin}(C\widetilde{M}, C\widetilde{B})$ are equivalent, therefore $\operatorname{Lin} \widetilde{A}\widetilde{M} \subset \operatorname{Lin}(\widetilde{M}, \widetilde{B})$.

It is known that the similar matrices A and $\widetilde{A}$ have the equal eigenvalues λ_k, $k = 1,\dots,p$. Let ℓ_i and $\widetilde{\ell}_i$ be the eigen and the adjoint vectors of matrices A and $\widetilde{A}$. If ℓ_i and $\widetilde{\ell}_i$ correspond to the equal λ_i, then we have $\ell_i = C\widetilde{\ell}_i$ (see [21, p. 31]). Therefore the condition (3) is equivalent the follow condition: the subspace $L(\widetilde{S})$ contains all rooted subspaces of matrix $\widetilde{A}$, corresponding to the $\lambda_k \in \Lambda$.

Note, that the vectors $\widetilde{\ell}_i$ type $\widetilde{\ell}_i = (\ell_i^1, 0)$, $\ell_i^1 \in \mathbb{R}^r$, $i = 1,\dots,r$, are contained in the controllability subspace $L(\widetilde{S}) = \mathrm{Lin}(e_1,\dots,e_r)$. The matrix $\widetilde{A}$ also have the rooted subspaces formed by the vectors $\widetilde{\ell}_i = (\ell_i^1, \ell_i^2)$, $\ell_i^2 \neq 0$, $i = r+1,\dots,n$ that dont lie in $L(\widetilde{S})$. Here $\ell_i^1 \in \mathbb{R}^r$, $\ell_i^2 \in \mathbb{R}^{n-r}$ and vectors ℓ_i^2 are the eigen or the adjoint ones of matrix A_{22} (vectors $\widetilde{\ell}_i$ and ℓ_i^2, $i = r+1,\dots,n$ correspond to the equal eigenvalues). Since (3), it follows that for these eigenvalues $\mathrm{Re}\lambda_k < 0$ or $\mathrm{Re}\lambda_k = 0$ and $m_k = 1$.

Using the conditions $L(\widetilde{S}) = \mathrm{Lin}(e_1,\dots,e_r)$ and $\widetilde{\mathcal{M}} \cap L(\widetilde{S}) = \{0\}$, we get that the subspace $\widetilde{\mathcal{M}}$ don't contains the unit vectors $e_1,\dots,e_r$. Therefore we can represent this subspace in the form

$$\widetilde{\mathcal{M}} = \mathrm{col}(\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2) = \mathrm{Lin}(h_1,\dots,h_j), \quad h_i = \mathrm{col}(h_i^1, h_i^2),$$

$$\widetilde{\mathcal{M}}_1 = \mathrm{Lin}(h_1^1,\dots,h_j^1), \quad \widetilde{\mathcal{M}}_2 = \mathrm{Lin}(h_1^2,\dots,h_j^2), \quad j \leqslant n-r,$$

where vectors $h_i^1 \in \mathbb{R}^r$, h_i^2 are the linear independent vectors in $\mathbb{R}^{n-r}$.

We denote by $y(t) = y(t,t_0,y_0,\widetilde{u}) = \mathrm{col}(y^1(t), y^2(t))$ the solution of the system $\widetilde{S}$ closed by the control $\widetilde{u}(y) \in U$. Here $y^1(t) = y^1(t,t_0,y_0^1,\widetilde{u})$ and $y^2(t) = y^2(t,t_0,y_0^2)$ is the solution of the system $\dot{y}^2 = A_{22}y^2$. Let us obtain the solution $y(t)$ such that its trajectory, going in the moment t_0 from the point $y_0 = (y_0^1, y_0^2) \in \widetilde{\mathcal{M}}$, remains in the subspace $\widetilde{\mathcal{M}}$ for all $t \geqslant t_0$. Note, that from the condition $\mathrm{Lin}\,\widetilde{A}\widetilde{M} \subset \mathrm{Lin}(\widetilde{M},\widetilde{B})$ follows the condition $\mathrm{Lin}\,A_{22}\widetilde{M}_2 \subset \mathrm{Lin}\,\widetilde{M}_2$, which means that for every point $y_0^2 \in \widetilde{\mathcal{M}}_2$ the trajectory of $y^2(t)$ contains in the subspace $\widetilde{\mathcal{M}}_2 = \mathrm{Lin}(h_1^2,\dots,h_j^2)$ for all $t \geqslant t_0$. Therefore we can represent the solution $y^2(t)$ in the form

$$y^2(t) = \alpha_1(t)h_1^2 + \dots + \alpha_j(t)h_j^2, \quad \alpha_i(t) = \sum_{l=1}^{q} e^{\lambda_l t} Q_{il}(t),$$

where the degree of polynomials $Q_{il}(t)$ not more than $m_l - 1$. The solution $y^2(t)$ is bounded for $t_0 \leqslant t < \infty$ because the eigenvalues λ_k of matrix A_{22} satisfy the condition $\mathrm{Re}\lambda_k < 0$ or $\mathrm{Re}\lambda_k = 0$ and $m_k = 1$.

Notice, that if $\mathrm{Lin}\,\widetilde{A}\widetilde{M} \subset \mathrm{Lin}(\widetilde{M},\widetilde{B})$, then for any basic vector $h_i \in \widetilde{\mathcal{M}}$, $i = 1,\dots,j$ there exists a vector $u_i \in \mathbb{R}^m$ such that $\widetilde{A}h_i + \widetilde{B}u_i \in \widetilde{\mathcal{M}}$. This means that there exists a vector $c_i = \mathrm{col}(c_{1i}\dots c_{ji})$ such that the system $\widetilde{M}c_i - \widetilde{B}u_i = \widetilde{A}h_i$ has the solution. Let us construct the positional control $\widetilde{u} = \alpha_1(t)u_1 + \ldots + \alpha_j(t)u_j$ and denote by $c = \alpha_1(t)c_1 + \ldots + \alpha_j(t)c_j$. Suppose that $y_0 \in \widetilde{\mathcal{M}}$ and $y(t) = y(t,t_0,y_0,\widetilde{u}) = \alpha_1(t)h_1 + \ldots + \alpha_j(t)h_j$ is the solution of the system $\widetilde{S}$ such that its trajectory lies in the subspace $\widetilde{\mathcal{M}}$. Then the vector $\mathrm{col}(c,\widetilde{u}) \in \mathbb{R}^{k+m}$ is the solution of the system

$$\widetilde{M}c - \widetilde{B}\widetilde{u} = \widetilde{A}y, \quad y \in \widetilde{\mathcal{M}}. \tag{5}$$

Combining $\widetilde{\mathcal{M}} = \mathrm{col}(\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2)$, $\widetilde{B} = \mathrm{col}(\widetilde{B}_1, 0)$, $\mathrm{rank}\,\widetilde{M}_2 = j$, $\mathrm{rank}\,\widetilde{B}_1 = m$ and condition (2), we obtain that $\mathrm{rank}(\widetilde{M},\widetilde{B}) = \mathrm{rank}(\widetilde{M},\widetilde{B},\widetilde{A}\widetilde{M}) = j + m$. This implies that the system (5) is compatible and has a unique solution $\widetilde{u}$, which we can represent in the form $\widetilde{u} = \widetilde{u}(y) = Dy$. Here D is some matrix sizes $m \times n$. Then there exists $\varepsilon > 0$ such that $\widetilde{u}(y) \in U$ for $\|y\| < \varepsilon$.

Since we can express the solution $y(t)$ over the same functions $\alpha_i(t)$, $i = 1, \dots, j$ that enters in $y^2(t)$, therefore this solution is also bounded for $t \geqslant t_0$. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y(t)\| < \varepsilon$ for all $t \geqslant t_0$ if $\|y_0\| < \delta$. Let us consider the phase trajectories of the system $\widetilde{S}$ closed by the control $\widetilde{u}(y) \in U$. Thus, we proved that if these trajectories go from the points $y_0 \in \widetilde{\mathcal{M}} \cap O_\delta$, then they lie in the set $\widetilde{\mathcal{M}} \cap O_\varepsilon$ for all $t \geqslant t_0$. Let us put $u(x) = \widetilde{u}(y)$, then the last condition is equivalent the next one: the phase trajectories of the system S, going from the points $x_0 \in \mathcal{M} \cap O_\delta$ under the control $u(x) \in U$, lie in the set $\mathcal{M} \cap O_\varepsilon$ for all $t \geqslant t_0$ (here ε and δ may be other).

In addition, note that the vector $\mathrm{col}(c, u)$ is the solution of the system

$$Mc - Bu = Ax, \quad x \in \mathcal{M}, \tag{6}$$

which is equivalent the system (5). Thus, the lemma is proved. □

Lemma 3.4 *Suppose $L(S)$ contains all rooted subspaces of matrix A, which correspond to the eigenvalues $\lambda_k \in \Lambda_k$. Then there exists the positional control $u(x) \in U$ type $u = Hx$, and for this control there exist $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ such that for any point $\|x_0\| < \delta$ the solution $\|x(t, t_0, x_0, u\| < \varepsilon$ for all $t \geqslant t_0$.*

Proof Assume that $\dim L(S) = r$. Let us reduce the system S to the system $\widetilde{S}$ by the linear transformation $x = Cy$. The matrix A_{22} of the system $\dot{y}^2 = A_{22}y^2$ have the eigenvalues λ_k such that $\mathrm{Re}\lambda_k < 0$ or $\mathrm{Re}\lambda_k = 0$ and $m_k = 1$. In [22, p. 30] was proved that there exists a control $u = Hx$ that gives to the matrix $A+BH$ of the closed system r predesigned eigenvalues and the rest eigenvalues of $A+BH$ coincide with the eigenvalues of matrix A_{22}. Therefore, we can choose the control $u = Hx$ such that all eigenvalues of matrix $A + BH$ satisfy the condition $\mathrm{Re}\lambda_k < 0$ or $\mathrm{Re}\lambda_k = 0$ and $m_k = 1$. Then there exists $\varepsilon > 0$ that $u(x) \in U$ for $\|x\| < \varepsilon$ and there exists $\delta = \delta(\varepsilon) > 0$ that for any point $\|x_0\| < \delta$ the solution $x(t) = x(t, t_0, x_0, u)$ satisfies the equality $\|x(t, x_0, u(\cdot))\| < \varepsilon$ for all $t \geqslant t_0$. The lemma is proved. □

4 The Conditions of the Non-predicting Local Controllability

We say that the finite sequence $V = (\psi_{i_1}, \dots, \psi_{i_k})$, where $\psi_{i_j} \in \Psi$ is called a word V. Let us put in correspondence to the word V the linear systems $\xi_{i_1}, \dots, \xi_{i_k}$, the controllability spaces of these systems $L(\xi_{i_1}), \dots, L(\xi_{i_k})$ and the controllability spaces $L_{[(\ell-1)\alpha, k\alpha]}(\xi_0)$, $\ell = 1, \dots, k$, constructed in Lemma 3.1.

Let us denote by $\mu(T)$ the probability of appearance the word V on the segment $[0, T]$.

Lemma 4.1 *Suppose that $0 < \alpha \leqslant \theta_k \leqslant \beta$ for all $k = 2, \dots,$ the set $\Psi = \{\psi_1, \psi_2\}$, the word $V = (\psi_{i_1}, \psi_{i_2})$. Then for $T \geqslant 2N\beta$, $N = 1, 2, \dots,$ the probability $\mu(T)$ satisfies the inequality*

$$\mu(T) \geqslant (1 - \pi_{i_2} p_{i_2 i_2}^{N-1})(1 - p_{i_1 i_1}^N). \tag{7}$$

Proof Here we consider the case, when the set Ψ contains two states, then the probability $\mu(T)$ equals to the probability of appearance the word $V = (\psi_{i_1}, \psi_{i_2})$ on the segment $[0,T]$. Notice that $\mu(T)$ not less than the probability of transition the system from any initial state to the state ψ_{i_1} over not more than N steps and then from ψ_{i_1} to ψ_{i_2} also not more than for N steps. It is clear that for such transition of the system on the segment $[0,T]$ must appeared not less than $2N$ jumps of the process, that always true for $T \geqslant 2N\beta$. Let us denote by $f_{i_1i_1}(N)$ the conditional probability of the first reaching of the system the state ψ_{i_1} from this own initial state not more than over N steps. The probability $f_{i_1i_1}(N)$ equals to the probability that the system either reaches the state ψ_{i_1} for one step or goes to ψ_{i_2}, then a few times goes again to ψ_{i_2} and then reaches the initial state ψ_{i_1}, hence

$$f_{i_1i_1}(N) = p_{i_1i_1} + p_{i_1i_2}p_{i_2i_1}(1 + p_{i_2i_2} + \ldots + p_{i_2i_2}^{N-2}) = 1 - p_{i_1i_2}p_{i_2i_2}^{N-1}.$$

Let $f_{i_2i_1}(N)$ be the conditional probability of the first reaching of the system the state ψ_{i_1} from the state ψ_{i_2} not more than over N steps. For this aim the system from the state ψ_{i_2} can reach the state ψ_{i_1} either over one step or at first it can go a few times to ψ_{i_2}, then it goes to ψ_{i_1}, therefore,

$$f_{i_2i_1}(N) = p_{i_2i_1}(1 + p_{i_2i_2} + \ldots + p_{i_2i_2}^{N-1}) = 1 - p_{i_2i_2}^{N}.$$

In the same way, we denote the probability $f_{i_1i_2}(N)$, then $f_{i_1i_2}(N) = 1 - p_{i_1i_1}^{N}$. Further note that the system can reach the state ψ_{i_1} either from ψ_{i_1} or from ψ_{i_2}, hence for $T \geqslant 2N\beta$ we have the inequality

$$\mu(T) \geqslant \Big(\pi_{i_1}f_{i_1i_1}(N) + \pi_{i_2}f_{i_2i_1}(N)\Big)f_{i_1i_2}(N) = \Big(1 - \pi_{i_1}p_{i_1i_2}p_{i_2i_2}^{N-1} - \pi_{i_2}p_{i_2i_2}^{N}\Big)(1 - p_{i_1i_1}^{N}).$$

It is well known that if the Markov chain is a stationary in the narrow sense, then the initial and transition probabilities satisfy the equations $\sum\limits_{j=1}^{s} \pi_j p_{jk} = \pi_k,\ k = 1,\ldots,s$. Hence in the case $s = 2$ we have $\pi_{i_1}p_{i_1i_2} + \pi_{i_2}p_{i_2i_2} = \pi_{i_2}$. Therefore $\mu(T) \geqslant (1 - \pi_{i_2}p_{i_2i_2}^{N-1})(1 - p_{i_1i_1}^{N})$. Thus, the lemma is proved. □

Let $p_{ij}^{(\ell)}$ be the probability of transition from the state ψ_i to the state ψ_j over ℓ steps. The state ψ_j is called an *attainable* from the state ψ_i if there exists $\ell \geqslant 0$ such that $p_{ij}^{(\ell)} > 0$. The states ψ_i and ψ_j are called the *connected* if the state ψ_j is attainable from the state ψ_i and the state ψ_i is attainable from ψ_j (see [14, p. 598]).

Theorem 4.1 *Suppose that for the system ξ the set $\Psi = \{\psi_1, \psi_2\}$, the states ψ_1, ψ_2 are connected and $0 < \alpha \leqslant \theta_k \leqslant \beta$ for all $k = 2, \ldots$ If there exist a word $V = (\psi_{i_1}, \psi_{i_2})$ and a subspace $\mathcal{M} \subset L(\xi_{i_2})$ such that:*

(1) $\mathcal{M} \cap L(\xi_{i_1}) = \{0\}$, $L(\xi_{i_1}) + \mathcal{M} = \mathbb{R}^n$;

(2) $\mathrm{Lin}A_{i_1}M \subset \mathrm{Lin}(M, B_{i_1})$;

(3) *the controllability space $L(\xi_{i_1})$ contains all rooted subspaces of matrix A_{i_1} and the controllability space $L(\xi_{i_2})$ contains all rooted subspaces of A_{i_2}, corresponding to the eigenvalues $\lambda_k \in \Lambda$,*

then the system ξ is non-predicting controlled on $[0,T]$ with probability $\mu(T)$ that satisfies (7) *for all $T \geqslant 2N\beta$, $N = 1, 2, \ldots$.*

The probability $\mu(T) \to 1$ as $T \to \infty$.

Proof Let us describe the construction of the non-predicting control for the system ξ that satisfies the conditions of the theorem.

1. First let us consider the case, when in the initial moment the system ξ is in the state ψ_{i_1}. The first task is to translate the points $x_0 \in O_\varepsilon$ to the set $\mathcal{M} \cap O_{\varepsilon_1}$ by the program control $u(t) \in U$ for time α. We denote by $D_{[t_0,t_1]}(S, M_0)$ the *controllability set of the system S to the set M_0* on the segment $[t_0, t_1]$. The point x_0 lies in the set $D_{[t_0,t_1]}(S, M_0)$ if and only if there exists an admissible control $u(t)$ such that the solution $x(t) = x(t, t_0, x_0, u)$ of the system S satisfies the condition $x(t_1) \in M_0$. It is known that the set $D_{[t_0,t_1]}(S, M_0)$ satisfies the equality

$$D_{[t_0,t_1]}(S, M_0) = D_{[t_0,t_1]}(S) + X^{-1}(t_1 - t_0)M_0.$$

Here under the algebraic sum of the sets A and B from $\mathbb{R}^n$ we intend the set $A + B = \{a + b : a \in A, b \in B\}$, by $X(t, s) = X(t - s)$ we denote the Cauchy matrix of the system $\dot{x} = Ax$. We obtain

$$\mathrm{Lin}D_{[0,\alpha]}(\xi_{i_1}, \mathcal{M} \cap O_{\varepsilon_1}) = \mathrm{Lin}\Big(D_{[0,\alpha]}(\xi_{i_1}) + X_{i_1}^{-1}(\alpha)(\mathcal{M} \cap O_{\varepsilon_1})\Big) =$$

$$= L(\xi_{i_1}) + X_{i_1}^{-1}(\alpha)\mathcal{M}.$$

In the work [20] was proved that the conditions $L(\xi_{i_1}) + X_{i_1}^{-1}(\alpha)\mathcal{M} = \mathbb{R}^n$ and $L(\xi_{i_1}) + \mathcal{M} = \mathbb{R}^n$ are equivalent, hence from the condition (1) it follows that $\mathrm{Lin}D_{[0,\alpha]}(\xi_{i_1}, \mathcal{M} \cap O_{\varepsilon_1}) = \mathbb{R}^n$. Since $\{0\} \in \mathrm{int}\mathcal{M}$ and $\{0\} \in \mathrm{int}D_{[0,\alpha]}(\xi_{i_1})$, then $\{0\} \in \mathrm{int}D_{[0,\alpha]}(\xi_{i_1}, \mathcal{M} \cap O_{\varepsilon_1})$. Therefore the set $D_{[0,\alpha]}(\xi_{i_1}, \mathcal{M} \cap O_{\varepsilon_1})$ contains some neighbourhood O_ε of the origin such that all points of O_ε reach the set $\mathcal{M} \cap O_{\varepsilon_1}$ by $u(t) \in U$ for time α.

Let us suppose that the system ξ have not the jumps for time $t = \alpha$, that is $\tau_1 \geqslant \alpha$. Since the system ξ_{i_1} and the subspace $\mathcal{M}$ satisfy the conditions of lemma 3.3, then there exists the positional control $u(x) \in U$, which retains the solution $x(t) = x(t, \alpha, x_\alpha, u)$, $x(\alpha) = x_\alpha$ on the subspace $\mathcal{M}$ for all $t \geqslant \alpha$. In this case for every $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that for all $\|x_0\| < \varepsilon_1$ the solution $\|x(t)\| < \varepsilon_2$ for all $t \geqslant \alpha$. Suppose that in the moment τ_1 the state ψ_{i_2} is appeared; then we can translate the points of $\mathcal{M} \cap O_{\varepsilon_2}$ to null for time α, because $\mathcal{M}$ contains in the controllability set $L(\xi_{i_2})$. In this case we choose ε_2 such that the program control $u(t) \in U$ for $t \in [\tau_1, \tau_1 + \alpha]$. The case $\tau_1 < \alpha$ considered further in item 3.

2. Suppose that in the initial moment the system ξ is in the state ψ_{i_2}. In this case we must wait for the moment of jump τ_1 during some unknown time and simultaneously choose the control $u(x) \in U$ that satisfy the follow condition: there exist $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ that all points from the neighbourhood O_δ contain in O_ε for any long time (to the moment τ_1). In Lemma 3.4 we prove the existence of such control $u(x) \in U$. If the state ψ_{i_2} appears again in the next moments of jumping $\tau_1, \ldots, \tau_k$, then we keep on restrain the trajectory of the system in the neighbourhood O_ε until the state ψ_{i_1} appeared in some moment τ_{k+1}. For $t \geqslant \tau_{k+1}$ we construct the control as in item 1.

3. Notice that in the initial moment we don't know about the time τ_1 of first jump of the process, thats why we cannot always reach the sets constructed above for this time. Therefore for $t < \tau_1$ we must construct the program control similarly as in the first or second item in dependence of the state of the system in the initial moment. Now suppose that the first jump of the process was in the moment $\tau_1 < \alpha$ and we don't reach the necessary sets for this time, then after the moment τ_1 we have a reserve time α without the next moment of jump τ_2 (because $\theta_k \in [\alpha, \beta]$). Thus, for $t \geqslant \tau_1$ we build the control as above in dependence from the number of state in the moment τ_1.

4. Finally let us prove that $\mu(T) \to 1$ as $T \to \infty$. The states ψ_1, ψ_2 are connected, hence $p_{11} \neq 1$, $p_{22} \neq 1$. Therefore from the inequality (7) we have that $\mu(T) \to 1$ as

$N \to \infty$. Notice, that in this case $T \to \infty$, because $T > \alpha(N-1)$, $\alpha > 0$. Thus, the theorem is proved. $\square$

5 Illustrative Example

Assume that the system ξ has two states $\psi_1 = (A_1, B_1)$, $\psi_2 = (A_2, B_2)$ with the next matrices:

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 1 & -2 \end{pmatrix}, B_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

It is also given the matrix of transition probabilities $P = \begin{pmatrix} 3/5 & 2/5 \\ 4/5 & 1/5 \end{pmatrix}$ and the initial distribution $\pi = (2/3, 1/3)$. Note, that the initial and transition probabilities satisfy the equations $\sum\limits_{j=1}^{2} \pi_j p_{ji} = \pi_i$, $i = 1, 2$. We also suppose that the length of intervals between the system jumps $\theta_k \in [0,5; 1]$, $k = 2, 3 \ldots$, then from (2) follows that $\theta_1 \in [0; 1]$.

It is easily shown that the controllability spaces of these systems $L(\xi_1) = \operatorname{Lin} B_1$, $L(\xi_2) = \operatorname{Lin} B_2$. We choose the word $V = (\psi_1, \psi_2)$ and the subspace $\mathcal{M} = L(\xi_2)$, then the subspaces $\mathcal{M}$ and $L(\xi_1)$ satisfy the equalities:

$$\mathcal{M} \cap L(\xi_1) = \{0\}, \quad L(\xi_1) + \mathcal{M} = \mathbb{R}^3, \quad \operatorname{Lin} A_1\mathcal{M} \subset \operatorname{Lin}(\mathcal{M}, B_1) = \mathbb{R}^3.$$

Further, the controllability space $L(\xi_1)$ contains the eigenvectors of matrix A_1, $v_1 = \operatorname{col}(0, 1, 0)$ and $v_2 = \operatorname{col}(1, 1, 0)$ that correspond to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$; the matrix A_1 also has the eigenvalue $\lambda_3 = -1$. The subspace $L(\xi_2)$ contains the eigenvector $v_1 = \operatorname{col}(1, 2, 1)$ of matrix A_2, corresponding $\lambda_1 = 1$, the other eigenvalues of A_2 are $\lambda_2 = -2$, $\lambda_3 = -3$. From the Theorem 4.1 it follows that the system ξ is the non-predicting controlled on the segment $[0, T]$ with the probability $\mu(T)$, which satisfies the next inequality for $T \geqslant 2N$:

$$\mu(T) \geqslant \Big(1 - \frac{1}{3} \cdot 0, 2^{N-1}\Big)(1 - 0, 6^N).$$

Let us describe the construction of the non-predicting control for this system and obtain the corresponding positional controls. Assume that in the initial moment the system ξ is in the state ψ_1. First we translate the points $x_0 \in O_\varepsilon$ to the set $\mathcal{M} \cap O_{\varepsilon_1}$ by the program control $u(t) \in U$ for the time $\alpha = 0, 5$. If the system has not the jumps during the time interval α, that is $\tau_1 \geqslant \alpha$, then we restrain the trajectories of the system ξ_1 in the set $\mathcal{M} \cap O_{\varepsilon_1}$ by the control $u(x)$ to the jump moment τ_k, when the system goes to the state ψ_2. For obtaining the control $u(x)$ we represent the vector $x \in \mathcal{M}$ in the form $x = \operatorname{col}(x_1, 2x_1, x_1)$, then from the system (6) we have $u(x) = \operatorname{col}(u_1, u_2) = \operatorname{col}(-3x_1, -7x_1)$. We obtain the solution $x(t, \alpha, x_0, u)$ of the system ξ_1, closed by the control $u(x)$, going from the point $x_0 = (x_0^1, 2x_0^1, x_0^1)$:

$$x(t, \alpha, x_0, u) = \operatorname{col}\Big(x_1^0 e^{-(t-\alpha)}, 2x_1^0 e^{-(t-\alpha)}, x_1^0 e^{-(t-\alpha)}\Big).$$

Note, that this solutions satisfies the inequality $\|x(t, \alpha, x_0, u)\| \leqslant \|x_0\| < \varepsilon_1$ and its trajectory contains in the subspace $\mathcal{M}$ for all $t \geqslant \alpha$. Further, when the state ξ_2 appears in the moment τ_k, we translated the points from $\mathcal{M} \cap O_{\varepsilon_1}$ to null by the corresponding program control.

Suppose that $\tau_1 < \alpha$ and in the moment τ_1 the state ψ_2 appears, then the trajectories of the system cannot always reach the set $\mathcal{M} \cap O_{\varepsilon_1}$ for the moment τ_1. In this case after τ_1 we must restrain the trajectories in some neighbourhood of the origin for the moment τ_q, when the system will be in the state ψ_1 again. For this aim we construct the positional control for the system $\xi_2 : u(x) = -x_1 - x_2$, such that all eigenvalues of the matrix of closed system are equal -2. Then there exist $\varepsilon > 0$ that $u(x) \in U$ for $\|x\| < \varepsilon$ and $\delta = \delta(\varepsilon) > 0$ that for any point $\|x_0\| < \delta$ the solution $\|x(t, \tau_1, x_0, u)\| < \varepsilon$ for all $t \geqslant \tau_1$. After appearing the state ψ_1 we deal as in the first case. In the same way, if in the moment $t = 0$ appears the state ψ_2, we must restrain the trajectories in some neighbourhood of the origin for the moment τ_q, when the system will be in the state ψ_1.

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Uniform Convergence to Global Attractors for Discrete Disperse Dynamical Systems

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Abstract: In this paper we study uniform convergence of trajectories of discrete disperse dynamical systems generated by set-valued mappings to their global attractors. In particular, we show that this convergence holds even in the presence of computational errors.

Keywords: *Compact metric space; set-valued mapping; trajectory; global attractor.*

Mathematics Subject Classification (2000): 37B99; 93C25.

1 Introduction

Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. A discrete-time dynamical system is described by a space of states and a transition operator which can be set-valued. Usually in the dynamical systems theory a transition operator is single-valued. In the present paper we study a class of dynamical systems introduced in [3] and studied in [4, 5] with a compact metric space of states and a set-valued transition operator. Such dynamical systems describe economical models [1, 2, 6].

Let (X, ρ) be a compact metric space and let $a\colon X \to 2^X \setminus \{\varnothing\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x, y) \in X \times X\colon\ y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x)\colon\ x \in E\} \quad \text{and} \quad a^0(E) = E.$$

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By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we study convergence of trajectories of the dynamical system generated by the set-valued mapping a. Following [3, 4] this system is called a discrete disperse dynamical system.

First we define a trajectory of this system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Put

$$\begin{aligned} \Omega(a) = \{z \in X\colon\ & \text{for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \\ & \text{such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \epsilon\}. \end{aligned} \tag{1.1}$$

Clearly, $\Omega(a)$ is closed subset of (X, ρ). In the present paper the set $\Omega(a)$ will be called a global attractor of a. Note that in [3–5] $\Omega(a)$ was called a turnpike set of a. This terminology was motivated by mathematical economics [1, 2, 6].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y)\colon\ y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have $\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0$.

It is not difficult to see that if for a nonempty closed set $B \subset X$

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

for each trajectory $\{x_t\}_{t=0}^{\infty}$, then $\Omega(a) \subset B$.

In the present paper we study uniform convergence of trajectories to the global attractor $\Omega(a)$.

The following useful result will be proved in Section 2.

Proposition 1.1 *Let $\epsilon > 0$. Then there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$*

$$\min\{\rho(x_t, \Omega(a)):\ t = 0, \ldots, T(\epsilon)\} \leq \epsilon.$$

The following theorem provides necessary and sufficient conditions for uniform convergence of trajectories to the global attractor.

Theorem 1.1 *The following properties are equivalent:*

(1) *For each $\epsilon > 0$ there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ and each integer $t \geq T(\epsilon)$ we have $\rho(x_t, \Omega(a)) \leq \epsilon$.*

(2) *If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ for all integers t, then $\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$.*

(3) *For each $\epsilon > 0$ there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \leq \delta$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq 0$.*

Theorem 1.1 will be proved in Section 3.

The following two theorems show that convergence of trajectories to the global attractor holds even in the presence of computational errors. These theorems will be proved in Section 5.

Theorem 1.2 *Let $\epsilon > 0$. Then there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \le \delta$ for each integer $t \ge 0$ the following inequality holds:*

$$\min\{\rho(x_t, \Omega(a)) : t = 0, \dots, T(\epsilon)\} \le \epsilon.$$

Theorem 1.3 *Assume that property (2) from Theorem 1.1 holds. Then for each $\epsilon > 0$ there exist $\delta > 0$ and a natural number $T(\epsilon)$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying*

$$\rho(x_{t+1}, a(x_t)) \le \delta \text{ for all integers } t \ge 0$$

the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ holds for each integer $t \ge T(\epsilon)$.

Some examples of set-valued mappings are considered in Section 6. In Section 7 we obtain generic convergence results for certain classes of set-valued mappings.

2 Proof of Proposition 1.1

Let us assume the converse. Then for each natural number n there exists a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\min\{\rho(x_t^{(n)}, \Omega(a)) : t = 0, \dots, n\} \ge \epsilon. \tag{2.1}$$

It is easy to see that there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer $t \ge 0$ there exists

$$x_t = \lim_{k \to \infty} x_t^{(n_k)}. \tag{2.2}$$

Since graph(a) is a closed subset of $X \times X$ equality (2.2) implies that $\{x_t\}_{t=0}^{\infty}$ is a trajectory. It follows from (2.1) and (2.2) that for each integer $t \ge 0$ the inequality $\rho(x_t, \Omega(a)) \ge \epsilon$ holds. This contradicts the definition of $\Omega(a)$. The contradiction we have reached proves Proposition 1.1.

3 Proof of Theorem 1.1

We will show that property (1) implies property (2). Assume that property (1) holds. Let a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfy $x_{t+1} \in a(x_t)$ for all integers t. Let τ be an integer, ϵ be a positive number and let a natural number $T(\epsilon)$ be as guaranteed by property (1). Define

$$y_t = x_{t+\tau-T(\epsilon)} \quad \text{for each integer} \quad t \ge 0. \tag{3.1}$$

It is clear that $\{y_t\}_{t=0}^{\infty}$ is a trajectory. By property (1), the choice of $T(\epsilon)$ and (3.1)

$$\rho(x_\tau, \Omega(a)) = \rho(y_{T(\epsilon)}, \Omega(a)) \le \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that $x_\tau \in \Omega(a)$ for each integer τ. Thus property (1) implies property (2).

Let us show that property (2) implies property (3). Assume that property (2) holds. Let $\epsilon \in (0, 1)$. We show that there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \le \delta$ the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ holds for all integers $t \ge 0$.

Let us assume the converse. Then for each integer $n \geq 1$ there exists a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\rho(x_0^{(n)}, \Omega(a)) \leq (2n)^{-1}\epsilon \;\text{ and }\; \sup\{\rho(x_t^{(n)}, \Omega(a)) \colon\; t \geq 0 \;\text{ is an integer}\} > \epsilon. \tag{3.2}$$

In view of (3.2) for each natural number n there exists a natural number T_n such that

$$\rho(x_{T_n}^{(n)}, \Omega(a)) > \epsilon. \tag{3.3}$$

Assume that the sequence $\{T_n\}_{n=1}^{\infty}$ is not bounded. Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that $T_n \to \infty$ as $n \to \infty$. For each integer $n \geq 1$ set

$$y_t^{(n)} = x_{t+T_n}^{(n)} \;\text{ for all integers }\; t \geq -T_n. \tag{3.4}$$

Evidently there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer t there exists

$$y_t = \lim_{k\to\infty} y_t^{(n_k)}. \tag{3.5}$$

Since the graph of a is closed it follows from (3.4) and (3.5) that $y_{t+1} \in a(y_t)$ for each integer t. By property (2), $\{y_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$. On the other hand by (3.3)-(3.5)

$$\rho(y_0, \Omega(a)) = \lim_{k\to\infty} \rho(y_0^{(n_k)},\ \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_{n_k}}^{(n_k)},\ \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves that our assumption is wrong and the sequence $\{T_n\}_{n=1}^{\infty}$ is bounded. Extracting a subsequence and re-indexing we may assume without loss of generality that

$$T_n = T_1 \;\text{ for all integers }\; n \geq 1. \tag{3.6}$$

Let n be a natural number. It follows from (3.2) that there is $z_n \in \Omega(a)$ such that

$$\rho(x_0^{(n)}, z_n) \leq (2n)^{-1}\epsilon. \tag{3.7}$$

By the definition of $\Omega(a)$ there exists a trajectory $\{y_t^{(n)}\}_{t=0}^{\infty}$ such that

$$\liminf_{t\to\infty} \rho(y_t^{(n)}, z_n) \leq (8n)^{-1}\epsilon. \tag{3.8}$$

In view of (3.8) there exists a natural number $S_n > n$ such that

$$\rho(y_{S_n}^{(n)}, z_n) < (4n)^{-1}\epsilon. \tag{3.9}$$

Relations (3.7) and (3.9) imply that

$$\rho(y_{S_n}^{(n)}, x_0^{(n)}) \leq \rho(y_{S_n}^{(n)}, z_n) + \rho(z_n, x_0^{(n)}) < \frac{\epsilon}{n}. \tag{3.10}$$

Set

$$\xi_t^{(n)} = y_{t+S_n}^{(n)}, \quad t = -S_n, \dots, -1, 0, \quad \xi_t^{(n)} = x_t^{(n)}, \quad t = 1, 2, \dots\,. \tag{3.11}$$

Clearly, there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer t there exists

$$\xi_t = \lim_{k\to\infty} \xi_t^{(n_k)} \tag{3.12}$$

and also there exists

$$x_0 = \lim_{k\to\infty} x_0^{(n_k)}. \tag{3.13}$$

Since the graph of a is closed it follows from (3.11) and (3.12) that

$$\xi_{t+1} \in a(\xi_t)$$

for each integer $t \geq 1$ and for each integer $t \leq -1$.

We will show that $\xi_1 \in a(\xi_0)$. Since the graph of a is closed it follows from (3.11)–(3.13) that $\xi_1 \in a(x_0)$. By (3.10)–(3.13) and the inclusion above

$$\rho(x_0, \xi_0) = \lim_{k\to\infty} \rho(x_0^{(n_k)}, \xi_0^{(n_k)}) = \lim_{k\to\infty} \rho(x_0^{(n_k)}, y_{S_{n_k}}^{(n_k)}) = 0,$$
$$x_0 = \xi_0 \ \text{ and } \ \xi_1 \in a(\xi_0).$$

Thus we have shown that

$$\xi_{t+1} \in a(\xi_t) \ \text{ for all integers } \ t. \tag{3.14}$$

In view of property (2) $\xi_t \in \Omega(a)$ for all integers t. On the other hand it follows from (3.12), (3.11), (3.6) and (3.3) that

$$\rho(\xi_{T_1}, \Omega(a)) = \lim_{k\to\infty} \rho(\xi_{T_1}^{(n_k)}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_1}^{(n_k)}, \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves that there exists $\delta > 0$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \Omega(a)) \leq \delta$ the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all integers $t \geq 0$. Thus property (2) implies property (3).

Let us show that property (3) implies property (1). Assume that property (3) holds. Let $\epsilon > 0$ and let $\delta > 0$ be as guaranteed by property (3). By Proposition 1.1 there exists a natural number T_0 such that for each trajectory $\{x_t\}_{t=0}^{\infty}$

$$\min\{\rho(x_t, \Omega(a)) \colon \ t = 0, \dots, T_0\} \leq \delta. \tag{3.15}$$

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory. By the choice of T_0 there is an integer $j \in [0, T_0]$ such that

$$\rho(x_j, \Omega(a)) \leq \delta.$$

In view of this inequality and the choice of δ $\rho(x_t, \Omega(a)) \leq \epsilon$ for all integers $t \geq j$ and property (1) holds. Thus property (3) implies property (1). Theorem 1.1 is proved.

4 An auxiliary result

Lemma 4.1 *Let T be a natural number and let $\epsilon > 0$. Then there exists a number $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{T} \subset X$ satisfying*

$$\rho(x_{t+1}, a(x_t)) \leq \delta, \quad t = 0, \dots, T-1$$

there is a sequence $\{y_t\}_{t=0}^{T} \subset X$ such that

$$y_{t+1} \in a(y_t), \quad t = 0, \dots, T-1, \tag{4.1}$$
$$\rho(y_t, x_t) \leq \epsilon, \quad t = 0, \dots, T. \tag{4.2}$$

Proof Let us assume the converse. Then for each natural number n there exists a sequence $\{x_t^{(n)}\}_{t=0}^T \subset X$ such that

$$\rho(x_{t+1}^{(n)}, a(x_t^{(n)})) \leq 1/n, \quad t = 0, \dots, T-1 \tag{4.3}$$

and that for each sequence $\{y_t\}_{t=0}^T \subset X$ satisfying (4.1)

$$\sup\{\rho(y_t, x_t^{(n)}) \colon \ t = 0, \dots, T\} > \epsilon. \tag{4.4}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that for $t = 0, \dots, T$ there exists

$$x_t = \lim_{n \to \infty} x_t^{(n)}. \tag{4.5}$$

By (4.3) for $t = 0, \dots, T-1$ and each integer $n \geq 1$ there is

$$z_{t+1}^{(n)} \in a(x_t^{(n)}) \tag{4.6}$$

such that

$$\rho(x_{t+1}^{(n)}, z_{t+1}^{(n)}) \leq 1/n. \tag{4.7}$$

Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that for $t = 0 \dots, T-1$ there is

$$z_{t+1} = \lim_{n \to \infty} z_{t+1}^{(n)}. \tag{4.8}$$

Since the graph of a is closed it follows from (4.5) and (4.8) that for each $t = 0, \dots, T-1$

$$z_{t+1} \in a(x_t). \tag{4.9}$$

By (4.5), (4.7) and (4.8) for each $t = 0, \dots, T-1$ we have $x_{t+1} = z_{t+1}$. Together with (4.9) this equality implies that $x_{t+1} \in a(x_t)$ for $t = 0, \dots, T-1$. In view of (4.5) there is a natural number n_0 such that

$$\rho(x_t, x_t^{(n_0)}) \leq \epsilon/4, \quad t = 0, \dots, T.$$

This contradicts the choice of $\{x_t^{(n_0)}\}_{t=0}^T$ (see (4.4)). The contradiction we have reached proves Lemma 4.1. □

5 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2 By Proposition 1.1 there exists a natural number $T(\epsilon)$ such that for each trajectory $\{x_t\}_{t=0}^\infty$ of a

$$\min\{\rho(x_t, \Omega(a)) \colon \ t = 0, \dots, T(\epsilon)\} \leq \epsilon/4. \tag{5.1}$$

By Lemma 4.1 there exists a number $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{T(\epsilon)} \subset X$ satisfying

$$\rho(x_{t+1}, a(x_t)) \leq \delta, \quad t = 0, \dots, T(\epsilon) - 1, \tag{5.2}$$

there exists a sequence $\{y_t\}_{t=0}^{T(\epsilon)} \subset X$ such that

$$y_{t+1} \in a(y_t), \quad t = 0, \dots, T(\epsilon) - 1, \tag{5.3}$$

$$\rho(y_t, x_t) \leq \epsilon/4, \quad t = 0, \dots, T(\epsilon). \tag{5.4}$$

Assume that a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfies

$$\rho(x_{t+1}, a(x_t)) \leq \delta \ \text{ for all integers } \ t \geq 0. \tag{5.5}$$

It follows from (5.5) and the choice of δ that there exists a sequence $\{y_t\}_{t=0}^{T(\epsilon)} \subset X$ such that (5.3) and (5.4) hold. By (5.3) and the choice of $T(\epsilon)$ (see (5.1)) there is $j \in \{0, \dots, T(\epsilon)\}$ such that $\rho(y_j, \Omega(a)) \leq \epsilon/4$. Combined with (5.4) this inequality implies that

$$\rho(x_j, \Omega(a)) \leq \rho(x_j, y_j) + \rho(y_j, \Omega(a)) \leq \epsilon/2.$$

Theorem 1.2 is proved. □

Proof of Theorem 1.3 Let $\epsilon > 0$. By Theorem 1.1, property (1) holds and there exists a natural number $T(\epsilon) \geq 4$ such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ of a and each integer $t \geq T(\epsilon)$

$$\rho(x_t, \Omega(a)) \leq \epsilon/8. \tag{5.6}$$

By Lemma 4.1 there exists a number $\delta > 0$ such that for each sequence $\{y_t\}_{t=0}^{4T(\epsilon)} \subset X$ satisfying

$$\rho(y_{t+1}, a(y_t)) \leq \delta, \quad t = 0, \dots, 4T(\epsilon) - 1 \tag{5.7}$$

there is a sequence $\{z_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that

$$z_{t+1} \in a(z_t), \quad t = 0, \dots, 4T(\epsilon) - 1, \tag{5.8}$$

$$\rho(y_t, z_t) \leq \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.9}$$

Assume that a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfies

$$\rho(x_{t+1}, a(x_t)) \leq \delta \ \text{ for each integer } \ t \geq 0. \tag{5.10}$$

In view of (5.10) and the choice of δ (see (5.7)-(5.9)) there is a sequence $\{z_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that (5.8) is true and

$$\rho(x_t, z_t) \leq \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.11}$$

By (5.8) and the choice of $T(\epsilon)$ (see (5.6))

$$\rho(z_t, \Omega(a)) \leq \epsilon/8, \quad t = T(\epsilon), \ \dots, 4T(\epsilon). \tag{5.12}$$

Relations (5.11) and (5.12) imply that for $t = T(\epsilon), \dots, 4T(\epsilon)$

$$\rho(x_t, \Omega(a)) \leq \rho(x_t, z_t) + \rho(z_t, \Omega(a)) \leq \epsilon/4. \tag{5.13}$$

We show that $\rho(x_t, \Omega(a)) \leq \epsilon$ for all integers $t \geq T(\epsilon)$.

Let us assume the converse. Then there is an integer $j \geq T(\epsilon)$ such that

$$\rho(x_j, \Omega(a)) > \epsilon, \tag{5.14}$$

$$\text{if an integer } \ t \ \text{ satisfies } \ T(\epsilon) \leq t < j, \ \text{ then } \ \rho(x_t, \Omega(a)) \leq \epsilon. \tag{5.15}$$

In view of (5.13)

$$j > 4T(\epsilon). \tag{5.16}$$

For $t = 0, \dots, 4T(\epsilon)$ set

$$y_t = x_{t+j-2T(\epsilon)}. \tag{5.17}$$

By (5.10) and (5.17) for $t = 0, \dots, 4T(\epsilon) - 1$

$$\rho(y_{t+1}, a(y_t)) = \rho(x_{t+j-2T(\epsilon)+1}, a(x_{t+j-2T(\epsilon)})) \le \delta.$$

In view of this relation and the choice of δ (see (5.7)–(5.9)) there is a sequence $\{\xi_t\}_{t=0}^{4T(\epsilon)} \subset X$ such that

$$\xi_{t+1} \in a(\xi_t), \quad t = 0, \dots, 4T(\epsilon) - 1, \tag{5.18}$$

$$\rho(\xi_t, y_t) \le \epsilon/8, \quad t = 0, \dots, 4T(\epsilon). \tag{5.19}$$

It follows from (5.18) and the choice of $T(\epsilon)$ (see (5.6)) that

$$\rho(\xi_t, \Omega(a)) \le \epsilon/8, \quad t = T(\epsilon), \ \dots, 4T(\epsilon).$$

Together with (5.19) this inequality implies that for $t = T(\epsilon), \dots, 4T(\epsilon)$

$$\rho(y_t, \Omega(a)) \le \rho(y_t, \xi_t) + \rho(\xi_t, \Omega(a)) \le \epsilon/4.$$

Together with (5.17) this inequality implies that

$$\rho(x_j, \Omega(a)) = \rho(y_{2T(\epsilon)}, \Omega(a)) \le \epsilon/4.$$

This relation contradicts (5.14). The contradiction we have reached proves that

$$\rho(x_t, \Omega(a)) \le \epsilon \ \text{ for all integers } \ t \ge T(\epsilon).$$

Theorem 1.3 is proved. □

6 Examples

Denote by $\Pi(X)$ the set of all nonempty closed subsets of (X, ρ). For each $A, B \in \Pi(X)$ set

$$H(A, B) = \max\{\sup_{x \in A} \rho(x, B), \ \sup_{y \in B} \rho(y, A)\}.$$

Clearly the space $(\Pi(X), H)$ is a complete metric space.

Example 6.1 Let $a\colon X \to X$ satisfy $\rho(a(x), a(y)) \le \rho(x, y)$ for each $x, y \in X$. Since the mapping a is single-valued it is not difficult to see that $a(\Omega(a)) \subset \Omega(a)$ and property (3) from Theorem 1.1 holds.

Example 6.2 Let $a\colon X \to X$ satisfy the following condition:

(C1) for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that for each $x, y \in X$ satisfying $\rho(x, y) \le \delta$ we have $\rho(a^n x, a^n y) \le \epsilon$ for all natural numbers n.

Define

$$\rho_1(x, y) = \sup\{\rho(a^n x, a^n y)\colon \ n = 0, 1, \dots\}, \quad x, y \in X.$$

Clearly, (X, ρ_1) is a complete metric space and for each $x, y \in X$ we have $\rho(x, y) \leq \rho_1(x, y)$. Let $\epsilon > 0$ and let $\delta \in (0, \epsilon)$ be as guaranteed by (C1). It is clear that $\rho_1(x, y) \leq \epsilon$ for each $x, y \in X$ satisfying $\rho(x, y) \leq \delta$.

Thus the metrics ρ and ρ_1 induce in X the same topology. It is clear that $\rho_1(a(x), a(y)) \leq \rho_1(x, y)$ for each $x, y \in X$. Thus in view of Example 1, property (3) from Theorem 1.1 holds.

Example 6.3 Let $a\colon X \to 2^X \setminus \varnothing$ have a closed graph. Assume that

$$H(a(x), a(y)) \leq c\rho(x, y) \quad \text{for all} \quad x, y \in X$$

with a constant $c \in (0, 1)$. We will show that property (3) from Theorem 1.1 holds. Clearly, it is sufficient to show that

$$a(\Omega(a)) = \cup\{a(z) : \ z \in \Omega(a)\} \subset \Omega(a).$$

Let $z \in a(E_1)$ and let ϵ be a positive number. There exist $x \in E_1$ such that $z \in a(x)$ and $y \in E_2$ such that $\rho(x, y) \leq \rho(x, E_2) + \epsilon$. It is not difficult to see that

$$\rho(z, a(E_2)) \leq \rho(z, a(y)) \leq H(a(x), a(y)) \leq c\rho(x, y) \leq c\rho(x, E_2) + c\epsilon \leq cH(E_1, E_2) + c\epsilon.$$

Since ϵ is an arbitrary positive number we conclude that

$$\rho(z, a(E_2)) \leq cH(E_1, E_2) \quad \text{for all} \quad z \in a(E_1).$$

Analogously we can show that

$$\rho(y, a(E_1)) \leq cH(E_1, E_2) \quad \text{for all} \quad y \in a(E_2).$$

Hence

$$H(a(E_1), a(E_2)) \leq cH(E_1, E_2) \quad \text{for all} \quad E_1, E_2 \in \Pi(X).$$

By this inequality and Banach fixed point theorem there is a unique $\Omega_* \in \Pi(X)$ such that $a(\Omega_*) = \Omega_*$ and that for each $E \in \Pi(X)$

$$a^n(E) \to \Omega_* \quad \text{as} \quad n \to \infty \quad \text{in} \quad (\Pi(X), H). \tag{6.1}$$

Clearly, $\Omega(a) \subset \Omega_*$. It is sufficient to show that $\Omega(a) = \Omega_*$.

Denote by S_a the set of all continuous functions $s : X \to R^1$ such that $\sup_{y \in a(x)} s(y) \leq s(x)$ for all $x \in X$. For each $s \in S_a$ put

$$W_s = \{x \in X\colon \sup_{y \in a(x)} s(y) = s(x)\}.$$

Set

$$W_a = \cap_{s \in S_a} W_a.$$

By Theorem 1 of [5]

$$W_a = \Omega(a).$$

It is sufficient to show that $\Omega_* \subset W_a$.

Let $s \in S_a$. There $x_* \in X$ such that $s(x_*) \leq s(x)$ for all $x \in X$. It is clear that $s(y) = s(x_*)$ for each $y \in \cup_{n=1}^{\infty}\{a^n(x) : \ n = 1, 2, \dots\}$. Together with (6.1) this implies that $s(y) = s(x_*)$ for each $y \in \Omega_*$ and that $\Omega_* \subset W_s$. Since this inclusion holds for any $s \in S_a$ we obtain that $\Omega_* \subset W_a$.

Example 6.4 Let $X = [0, 1]$, $a(x) = x^2$, $x \in [0, 1]$. It is clear that $\Omega(a) = \{0, 1\}$ and that $a(\Omega(a)) = \Omega(a)$. It is not difficult to see that for any $z \in (0, 1)$ there exists a sequence $\{x_i\}_{i=-\infty}^{\infty} \subset (0, 1)$ such that $x_0 = z$ and $x_{i+1} = a(x_i)$ for all integers i. Therefore property (2) of Theorem 1.1 does not hold.

7 Spaces of set-valued mappings

In this section we consider classes of discrete disperse dynamical systems whose global attractors are a singleton.

Denote by $\mathcal{A}$ the set of all mappings $a\colon X \to \Pi(X)$ with closed graphs. For each $a_1, a_2 \in \mathcal{A}$ set

$$d_{\mathcal{A}}(a_1, a_2) = \sup\{H(a_1(x), a_2(x))\colon x \in X\}. \tag{7.1}$$

It is clear that the metric space $(\mathcal{A}, d_{\mathcal{A}})$ is complete.

Denote by $\mathcal{A}_c$ the set of all continuous mappings $a : X \to \Pi(X)$ which belong to $\mathcal{A}$, by $\mathcal{A}_f$ the set of all $a \in \mathcal{A}$ such that $a(x)$ is a singleton for each $x \in X$ and set $\mathcal{A}_{fc} = \mathcal{A}_f \cap \mathcal{A}_c$. Clearly $\mathcal{A}_f$, $\mathcal{A}_c$ and $\mathcal{A}_{fc}$ are closed subsets of $(\mathcal{A}, d_{\mathcal{A}})$.

Let $\mathcal{M}$ be one of the following spaces: $\mathcal{A}$; $\mathcal{A}_c$; $\mathcal{A}_f$; $\mathcal{A}_{fc}$. The space $\mathcal{M}$ is equipped with the metric $d_{\mathcal{A}}$.

Denote by $\mathcal{M}_{reg}$ the set of all $a \in \mathcal{M}$ such that $\Omega(a)$ is a singleton and that properties (1–3) from Theorem 1.1 hold.

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $(\mathcal{M}, d_{\mathcal{A}})$. In this section we will establish the following result which shows that most elements of $\bar{\mathcal{M}}_{reg}$ (in the sense of Baire category) belong to $\mathcal{M}_{reg}$.

Theorem 7.1 *The set $\mathcal{M}_{reg}$ contains a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$.*

Proof For each $a \in \mathcal{M}_{reg}$ there is $x_a \in X$ such that

$$\Omega(a) = \{x_a\}. \tag{7.2}$$

Let $a \in \mathcal{M}_{reg}$ and let n be a natural number. Since the mapping a has property (2) from Theorem 1.1 it follows from Theorem 1.3 that there exist a natural number $T(a, n)$ and $\delta(a, n) > 0$ such that the following property holds:

(P1) for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\rho(x_{t+1}, a(x_t)) \le \delta(a, n)$, $t = 0, 1, \ldots$ and each integer $t \ge T(a, n)$ we have

$$\rho(x_t, x_a) \le 1/n.$$

Let $\mathcal{U}(a, n)$ be an open neighborhood of a in $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$ such that

$$H(a(x), b(x)) \le \delta(a, n)/2 \ \text{ for each } \ x \in X \ \text{ and each } \ b \in \mathcal{U}(a, n). \tag{7.3}$$

It follows from property (P1) and (7.3) that the following property holds:

(P2) for each $b \in \mathcal{U}(a, n)$ and each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $x_{t+1} \in b(x_t)$, $t = 0, 1, \ldots$

$$\rho(x_t, x_a) \le 1/n \ \text{ for all integers } \ t \ge T(a, n).$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(a, n)\colon a \in \mathcal{M}_{reg}\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{M}}_{reg}, d_{\mathcal{A}})$. In order to complete the proof it is sufficient to show that $\mathcal{F} \subset \mathcal{M}_{reg}$.

Let $b \in \mathcal{F}$ and $\epsilon > 0$. Choose a natural number n such that

$$n > 8(\min\{1, \epsilon\})^{-1}. \tag{7.4}$$

By the definition of $\mathcal{F}$ there exists $a \in \mathcal{M}_{reg}$ such that

$$b \in \mathcal{U}(a, n). \tag{7.5}$$

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory of b. By (7.5) and property (P2)

$$\rho(x_t, x_a) \leq 1/n < \epsilon/8 \;\text{ for all integers }\; t \geq T(a, n). \tag{7.6}$$

Since ϵ is an arbitrary positive number we conclude that $\{x_t\}_{t=0}^{\infty}$ is a Cauchy sequence. Therefore there exists $\lim_{t\to\infty} x_t \in X$. By (7.6),

$$\rho(\lim_{t\to\infty} x_t, x_a) \leq \epsilon/8. \tag{7.7}$$

Since ϵ is an arbitrary positive number and $\{x_t\}_{t=1}^{\infty}$ is an arbitrary trajectory of b we conclude that there exists $x_b \in X$ such that $\lim\limits_{t\to\infty} x_t = x_b$ for each trajectory $\{x_t\}_{t=0}^{\infty}$ of b. By (7.7)

$$\rho(x_a, x_b) \leq \epsilon/8. \tag{7.8}$$

By (7.6) and (7.8) for each trajectory $\{x_t\}_{t=0}^{\infty}$ of b and all integers $t \geq T(a, n)$

$$\rho(x_t, x_b) \leq \rho(x_t, x_a) + \rho(x_a, x_b) \leq \epsilon/4.$$

Theorem 7.1 is proved. □

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A Study on Stabilization of Nonholonomic Systems Via a Hybrid Control Method

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Abstract: In this paper, we consider a hybrid control strategy for stabilization of nonholonomic systems. In particular, we deal with a typical nonholonomic system, namely a two-wheeled vehicle. We first rewrite the system in a chained form, and then transform it into a nonholonomic integrator (NHI) system. Finally, we apply and modify the hybrid control method for the NHI system, so that the entire system is exponentially stable. We provide a simulation example to demonstrate the effectiveness of the transformation and the control, and give some analysis together with an example for the case where there are constraints on control inputs. We also extend the discussion to the case of four-wheeled vehicles.

Keywords: *Nonholonomic system; two(four)-wheeled vehicle; hybrid control; chained form; nonholonomic integrator; exponential stability/stabilization; switching strategy.*

Mathematics Subject Classification (2000): 93B50, 93B51, 93C10, 93C95, 93D15, 93D20.

1 Introduction

It is known that many mechanical systems are subject to nonholonomic velocity constraints (for example, wheeled mobile robots [2], tractor-trailer (or car-trailor) systems [3], free-floating space [4], etc.), and these constraints can be modelled as symmetrically affine systems [5,6]. Since such nonholonomic systems do not satisfy the so-called Brockett's stabilizability condition [7], they can not be asymptotically stabilized to their equilibrium points by any continuously differentiable, time invariant, state feedback control laws [7,8]. For this reason, there have been a large quantity of works on the stabilization

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problem of nonholonomic systems in the last two decades, including the efforts of finding continuous, time varying control laws [9,10], discontinuous ones [8,11,12] and middle strategies (discontinuous and time varying) [13,14].

In this paper, we consider a hybrid control strategy for this challenging problem. In particular, we deal with a typical nonholonomic system, namely a two-wheeled vehicle. The significant difference from the abovementioned existing work is that we seek the possibility of achieving exponential stabilization for such a nonholonomic system. For this purpose, we first rewrite the model of the two-wheeled vehicle in a chained form, and then transform it into a nonholonomic integrator (NHI) system [7]. Finally, we apply and modify the hybrid control method, which was originally proposed in [1], for the NHI system. We demonstrate by a simulation example the effectiveness of the transformation and the control. After that, we discuss the case where there are constraints on the control inputs and propose using the idea of bounded functions in the hybrid control method. We also extend our discussion to the case of four-wheeled vehicles. By choosing an alternative control input, we can reduce the stabilization of four-wheeled vehicles to the same control problem as for two-wheeled vehicles, and thus can apply the same approach.

The remainder of this paper is organized as follows. In Section 2, we describe the system of a two-wheeled vehicle and then transform it into an NHI system. In Section 3, we present the hybrid control strategy and the simulation result, and give two important remarks, which concern the switching time interval and the method of dealing with singularities. Section 4 considers the case where there exist constraints on the control inputs, and Section 5 discusses the extension to the case of four-wheeled vehicles. Finally, we give some concluding remarks in Section 6.

2 System Description and Transformation

We deal with a two-wheeled vehicle as depicted in Figure 2.1, which is known as a typical nonholonomic system. Let (x, y) denote the position of the vehicle, let θ be the angle with respect to the x-axis and let $\bar{v}_1$ be the velocity of the vehicle in its body direction. If we view $\bar{v}_1 = u_1$, $\dot{\theta} = u_2$ as control inputs, we obtain the vehicle's system described by

$$\left\{ \begin{array}{rcl} \dot{x} &=& u_1 \cos\theta, \\ \dot{y} &=& u_1 \sin\theta, \\ \dot{\theta} &=& u_2. \end{array} \right. \tag{1}$$

Note that this is a three-dimensional symmetrically affine system with two control inputs.

In this paper, we propose transforming the system (1) into a chained form, and then transforming the chained form into an NHI system. More precisely, we first let $\mu_1 = u_1 \cos\theta$, $\mu_2 = u_2$ to rewrite (1) as

$$\left\{ \begin{array}{rcl} \dot{x} &=& \mu_1, \\ \dot{y} &=& \mu_1 \tan\theta, \\ \dot{\theta} &=& \mu_2. \end{array} \right. \tag{2}$$

In (2), we let $z_1 = x$, $z_2 = \tan\theta$, $z_3 = y$, $v_1 = \mu_1$, $v_2 = (\sec^2\theta)\mu_2$ to obtain

$$\left\{ \begin{array}{rcl} \dot{z}_1 &=& v_1, \\ \dot{z}_2 &=& v_2, \\ \dot{z}_3 &=& z_2 v_1, \end{array} \right. \tag{3}$$

which is a chained form.

Next, we apply the idea in [6] to transform the chained form (3) into an NHI system. More precisely, if we define the new variables

$$x_1 = z_1\,, \quad x_2 = z_2\,, \quad x_3 = -2z_3 + z_1 z_2\,, \tag{4}$$

then the NHI system

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{x}_2 = v_2, \\ \dot{x}_3 = x_1 v_2 - x_2 v_1, \end{cases} \tag{5}$$

is obtained from (3) easily.

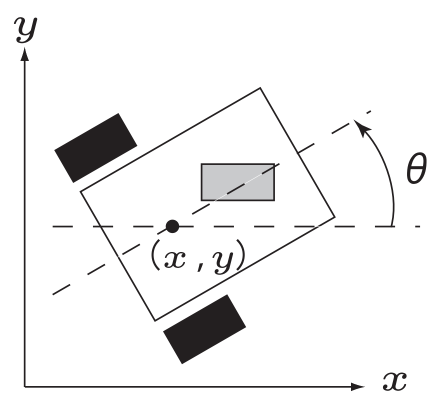

Figure 2.1: A two-wheeled vehicle.

It is not difficult to obtain the relation between (x, y, θ) in (1) and (x_1, x_2, x_3) in (5) as

$$x = x_1\,, \quad y = \frac{-x_3 + x_1 x_2}{2}\,, \quad \theta = \tan^{-1} x_2 \tag{6}$$

and the relation between (u_1, u_2) in (1) and (v_1, v_2) in (5) as

$$u_1 = \frac{v_1}{\cos\theta}\,, \qquad u_2 = v_2 \cos^2\theta\,. \tag{7}$$

These relations imply that if we can design a controller $v = [v_1 \;\; v_2]^T$ to make the NHI system (5) asymptotically/exponentially stable, then the controller u computed by (7) stabilizes the original nonholonomic system (1) asymptotically/exponentially.

3 Hybrid Control and Simulation

Since the control problem has been reduced to stabilizing the NHI system (5), we propose applying the hybrid control method in [1]. Define the functions

$$\pi_1(w) = 0.5(1 - e^{-\sqrt{w}}), \quad \pi_2(w) = 1.7\pi_1(w), \quad \pi_3(w) = 2.5\pi_1(w), \quad \pi_4(w) = 4\pi_1(w), \tag{8}$$

and the overlapping regions

$$\begin{aligned} R_1 &= \left\{x \in R^3 : 0 \leq x_1^2 + x_2^2 \leq \pi_2(x_3^2)\right\}, \\ R_2 &= \left\{x \in R^3 : \pi_1(x_3^2) \leq x_1^2 + x_2^2 \leq \pi_4(x_3^2)\right\}, \\ R_3 &= \left\{x \in R^3 : \pi_3(x_3^2) \leq x_1^2 + x_2^2\right\}, \\ R_4 &= \{0\}. \end{aligned} \tag{9}$$

Then, we define the control strategy

$$v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T = g_\sigma(x), \tag{10}$$

where σ is a piecewise constant switching signal, which is continuous from the right at every point and takes value on a finite set $J = \{1, 2, 3, 4\}$, and

$$\begin{aligned} g_1(x) &\triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_2(x) \triangleq \begin{bmatrix} x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \\ g_3(x) &\triangleq \begin{bmatrix} -x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \quad g_4(x) \triangleq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

The switching signal σ is defined recursively by

$$\begin{aligned} \sigma &= \phi(x, \sigma^-), \\ \phi(x, j) &= \begin{cases} j, & \text{if } x \in R_j \\ \max\{i \in J : x \in R_i\}, & \text{if } x \notin R_j . \end{cases} \end{aligned} \tag{12}$$

Therefore, the above control strategy is a hybrid control which is composed of four continuous-time controllers and a state-dependent switching law. It has been shown in [1] that exponential stability is obtained for the NHI system (5) by using the above hybrid control method. Therefore, as explained before, the original nonholonomic system (1) is also exponentially stabilized by the hybrid controller defined by (7) and (10)-(12).

The simulation results are described in Figures 3.1 – 3.5, where the initial state is $x_1 = 0.25$, $x_2 = 0.15$, $x_3 = 1.0$. Figure 3.1 describes how the switching signal changes with $w_1 = x_3^2$, $w_2 = x_1^2 + x_2^2$. Figure 3.2 and Figure 3.3 respectively show that both the NHI system (5) and the original system (1) are exponentially stable. Figure 3.4 and Figure 3.5 depict the switchings in control inputs.

In the end of this section, we give two important remarks concerning the discussion in this section.

First, as also pointed out in [1], the time interval between consecutive switchings in the switching law is bounded away from zero, not only on any finite time interval but also as time goes to infinity. Therefore, chattering phenomena will not happen. Here, we give more precise description, though similar to that appeared in [1], so that the readers can follow the design precept.

Let $\bar{t}$ denote any time instant at which σ switches from Mode 2 (Controller 2) to Mode 3 (Controller 3). Then one must have

$$w_2(\bar{t}) = \pi_4(w_1(\bar{t})). \tag{13}$$

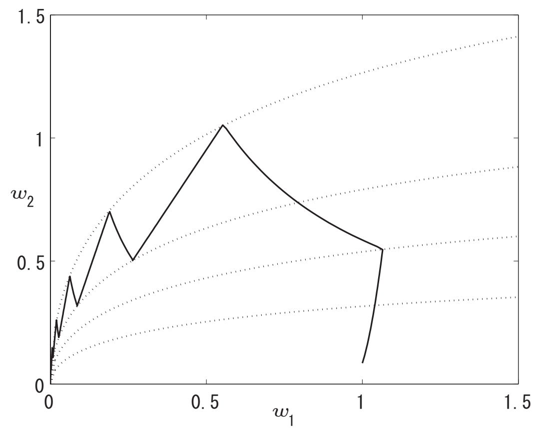

Figure 3.1: Switchings.

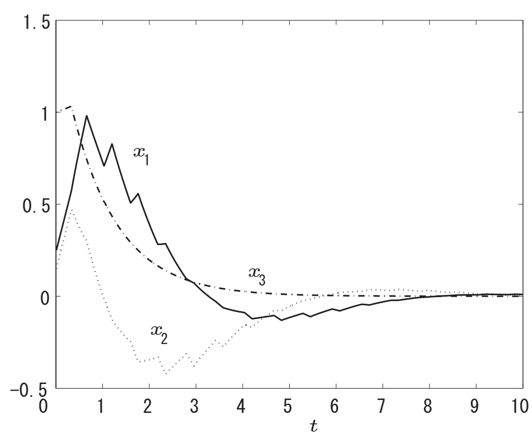

Figure 3.2: The states of the NHI (5).

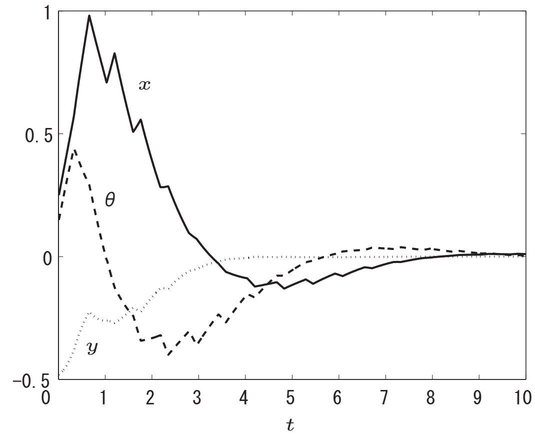

Figure 3.3: The states of the the vehicle (1).

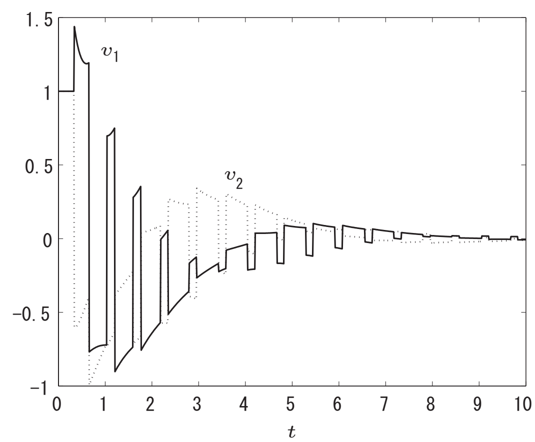

Figure 3.4: The control inputs of the NHI (5).

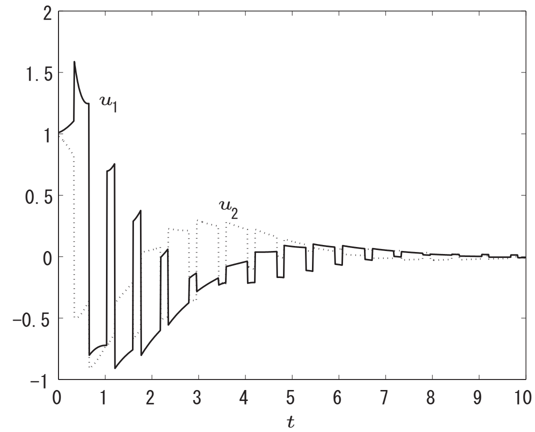

Figure 3.5: The control inputs of the vehicle (1).

Suppose that σ switches back to Mode 2 after some time interval Δt, which implies that

$$w_2(\bar{t} + \Delta t) = \pi_3(w_1(\bar{t} + \Delta t))\,. \tag{14}$$

Since $\dot{w_2} = -2w_2$ on the interval $[\bar{t}, \bar{t} + \Delta t)$, we obtain

$$w_2(\bar{t} + \Delta t) = w_2(\bar{t})e^{-2\Delta t}\,, \tag{15}$$

and thus

$$\Delta t = \frac{1}{2} \log \frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t} + \Delta t))}\,. \tag{16}$$

Noting the fact that w_1 is decreasing for all $t \geq \bar{t}$ and π_3 is monotone nondecreasing, which leads to $\pi_3(w_1(\bar{t} + \Delta t)) \leq \pi_3(w_1(\bar{t}))$, we conclude that

$$\Delta t \geq \frac{1}{2} \log \frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t}))}\,. \tag{17}$$

Therefore, we can adjust the switching time interval by choosing the ratio between π_4 and π_3 (we used the ratio $\frac{4}{2.5} = 1.6$ in (8)). For example, if we desire $\Delta t \geq 2$, then we choose $\pi_4(w_1) = \exp(4)\pi_3(w_1)$. In this case, the switchings are done as described in Figure 3.6.

Next, we give a remark on the system transformation from the original system (1) to the NHI system (5). In (2) or (7), it is easy to understand that we can't obtain the original control input u in singular points such as $\theta = \pm\frac{\pi}{2}$ (which means that the vehicle is located towards the vertical direction). To say it in other words, the consideration for the NHI system (5) does not cover the case of $\theta = \pm\frac{\pi}{2}$. To overcome this difficulty, we can use an alternative transformation which doesn't result in singular points. That is, we let $\mu_1 = u_2$, $\mu_2 = u_1$ to rewrite (1) as

$$\dot{x} = \mu_2 \cos\theta, \qquad \dot{y} = \mu_2 \sin\theta, \qquad \dot{\theta} = \mu_1. \tag{18}$$

Then, we let $z_1 = \theta$, $z_2 = -x\cos\theta - y\sin\theta$, $z_3 = -x\sin\theta + y\cos\theta$, $v_1 = \mu_1$, $v_2 = (x\sin\theta - y\cos\theta)\mu_1 - \mu_2$ in (18) to obtain

$$\dot{z}_1 = v_1, \qquad \dot{z}_2 = v_2, \qquad \dot{z}_3 = z_2 v_1, \tag{19}$$

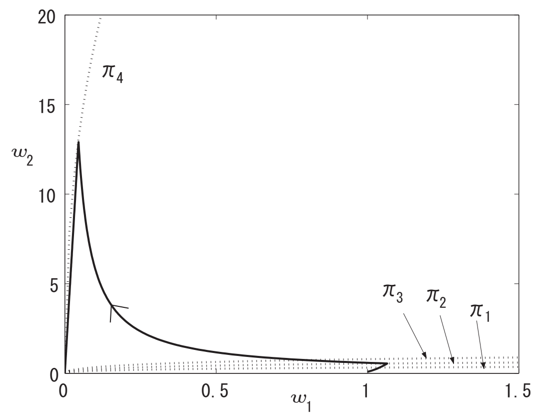

Figure 3.6: Switchings when $\pi_4(w_1) = \exp(4)\pi_3(w_1)$.

which is also a chained form. The remaining discussion is the same as before.

4 Constrained Control Input

In this section, we give some analysis and simulation in the case where there exist constraints on the control inputs. Ref. [15] considered the asymptotic stabilization problem for nonholonomic mobile robots under constraints on control inputs, but it is found that the convergence rate is very slow there (only asymptotic stability is guaranteed there). Here, we suggest using the bounded function proposed in [15] for the hybrid controller (11).

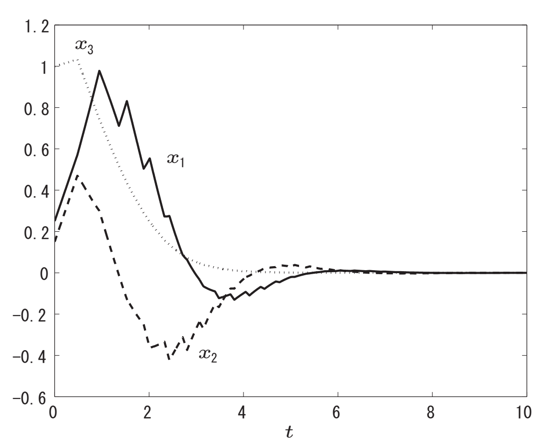

Figure 4.1: The states in the saturated case ($x_1(0) = 0.25$, $x_2(0) = 0.15$, $x_3(0) = 1.0$).

Suppose that due to physical environment and/or actuator capability limitation, we need imposing certain constraints on the control inputs. For simplicity, we consider here

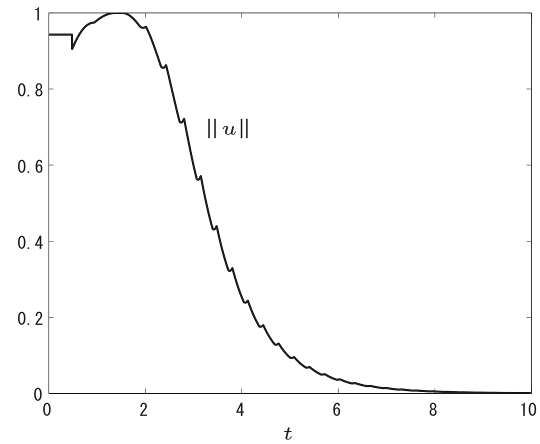

Figure 4.2: The norm of the control input in the saturated case ($x_1(0) = 0.25$, $x_2(0) = 0.15$, $x_3(0) = 1.0$).

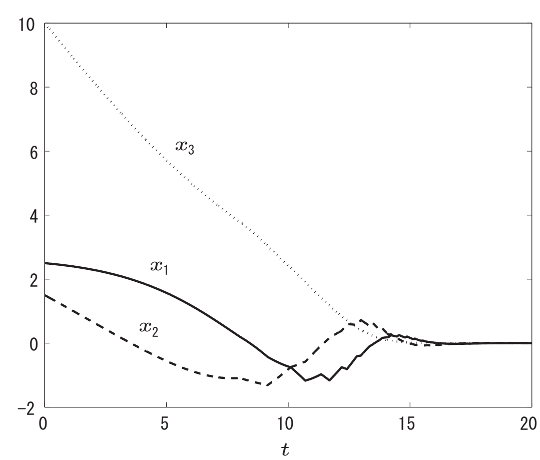

Figure 4.3: The states in the saturated case ($x_1(0) = 2.5$, $x_2(0) = 1.5$, $x_3(0) = 10$).

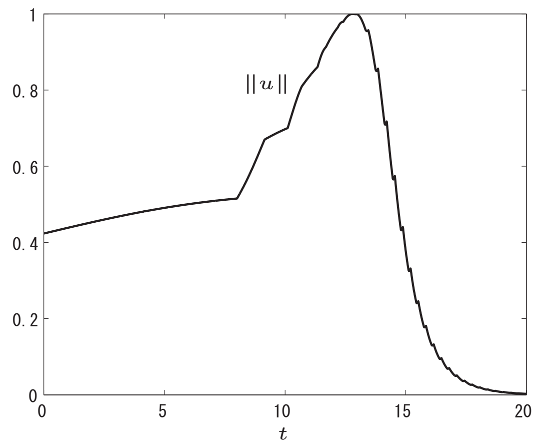

Figure 4.4: The norm of the control input in the saturated case ($x_1(0) = 2.5$, $x_2(0) = 1.5$, $x_3(0) = 10$).

the case where the constraints can be imposed directly on the NHI system (5) as

$$\left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\| \le r\,, \tag{20}$$

where r is a positive scalar indicating the constraint bound. Then, utilizing the idea of bounded function in the control input vector (11), we propose the new controller candidates

$$\bar{g}_i = \frac{2r}{1 + g_i^T g_i} g_i\,, \quad i = 1, 2, 3, 4 \tag{21}$$

instead of (11). Note that the constraints are satisfied with the above controllers since

$$\bar{g}_i^T \bar{g}_i = \frac{4r^2}{(1 + g_i^T g_i)^2} g_i^T g_i \le r^2\,. \tag{22}$$

Now, we consider the same system with constrained controller under $r = 1$. The simulation result with the same initial state ($x_1 = 0.25$, $x_2 = 0.15$, $x_3 = 1.0$) is shown in Figure 4.1 and Figure 4.2. Figure 4.1 tells that the system is also exponentially stabilized, and Figure 4.2 tells that the constraint on the control inputs is satisfied.

Since the controller switchings depend on the initial state significantly, we increase the initial state to $x_1 = 2.5$, $x_2 = 1.5$, $x_3 = 10$. Then, the simulation result is shown in Figure 4.3 and Figure 4.4. We see that we have also obtained desired exponential stability under the constrained control inputs.

5 Extension to Four-Wheeled Vehicles

In this section, we extend our consideration to the case of four-wheeled vehicles, which are depicted in Figure 5.1.

We let (x, y) and (x_f, y_f) be the coordinates of the middle point of the rear tire axle and that of the front tire, respectively, and let L be the length from (x, y) to (x_f, y_f).

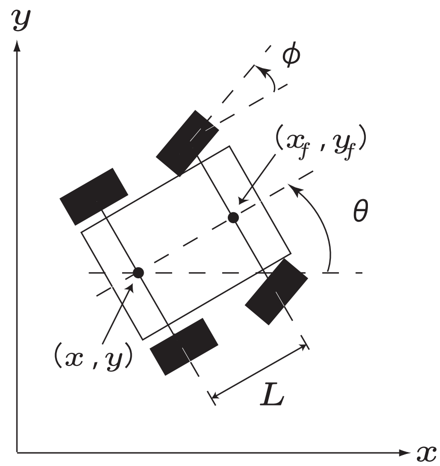

Figure 5.1: A four-wheeled vehicle.

Define θ and $\bar{v}_1$ as in the two-wheeled vehicle, and let ϕ be the angle with respect to its body direction. If we view $\bar{v}_1 = u_1$, $\dot{\phi} = u_2$ as control inputs, we obtain the vehicle's system described by

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_1 \dfrac{\tan\phi}{L}, \\ \dot{\phi} = u_2. \end{cases} \tag{23}$$

Note that this is a four-dimensional symmetrically affine system with two inputs. We can transform this system into four-dimensional chained form. However, we find that it is hard to apply the aforementioned control method since the obtained chained form can not be transformed further into the NHI system. For this reason, we propose choosing the control inputs as $\bar{v}_1 = u_1$, $\bar{v}_1 \tan\phi = u_2$, and rewrite the vehicle's system as

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = \dfrac{u_2}{L}. \end{cases} \tag{24}$$

Since (24) is a three-dimensional symmetrically affine system with two inputs, we can use the same approach as in Section II to transform this system into an NHI system, and then apply the aforementioned hybrid control strategy for the system. Note that the relation between (x, y, θ) in (24) and (x_1, x_2, x_3) in (5) is the same as in the case of two-wheeled vehicles, and the relation between (u_1, u_2) in (24) and (v_1, v_2) in (5) is

$$u_1 = \frac{v_1}{\cos\theta}\,, \qquad u_2 = v_2 L \cos^2\theta\,. \tag{25}$$

6 Concluding Remarks

We have considered a hybrid control strategy for stabilization of a class of nonholonomic systems, namely two(four)-wheeled vehicle systems. We first rewrite the system in a chained form, and then transform it into a nonholonomic integrator (NHI) system. Finally, we have applied the hybrid control method proposed in [1] for the obtained NHI system. The key point is that the transformations are returnable and the switching time interval can be adjusted easily. We have shown that it is possible to extend the results to the case involving constrained control inputs.

Future research includes the hybrid control for extended NHI forms (for example, those with even dimension) and for robust performance of nonholonomic systems.

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