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On the Non-Oscillation of Solutions of Some Nonlinear Differential Equations of Third Order

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Abstract: We present some new non-oscillation criteria for a class of non-linear differential equations of third order. Depending on these criteria, our results include and improve some well-known results in the literature.

Keywords: *Differential equation, third order, non-oscillation.*

Mathematics Subject Classification (2000): 34C11.

1 Introduction

We are concerned with non-oscillation of solutions of third-order nonlinear differential equations of the form

$$(r(t)y''(t))' + q(t)y'(t) + p(t)y^{\alpha}(g(t)) = f(t), \quad t \geq t_0 \tag{1}$$

and

$$(r(t)y''(t))' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t), \quad t \geq t_0, \tag{2}$$

where $t_0 \geq 0$ is a fixed real number, $f,\ p,\ q,\ r,\ g$ and $g_1 \in C\left([0,\infty), \Re\right)$ such that $r(t) > 0$ and $f(t) \geq 0$ for all $t \in [0, \infty)$. Throughout the paper, it is assumed, for all $g(t)$, $g_1(t)$, α and β appeared in (1) or (2), that $g(t) \leq t$ and $g_1(t) \leq t$ for all $t \geq t_0$; $\lim_{t\to\infty} g(t) = \infty$ and $\lim_{t\to\infty} g_1(t) = \infty$; both $\alpha > 0$ and $\beta > 0$ are quotients of odd integers.

In the relevant literature, till now, oscillation and non-oscillation behaviors of solutions of linear and non-linear second order, third order etc. differential equations have been the subject of intensive investigations for many authors. For instance, one can refer to [1–36] as some related papers or books on the subject. Now, to the best of our

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knowledge, some results obtained in the literature on the topic of this paper can be summarized, briefly, as follows: First, in 1974, Kusano and Onose [12] studied the oscillatory and asymptotic behavior of solutions of the differential equation

$$x^{(n)}(t) + p(t)f\left(x(g(t))\right) = g(t),$$

and they established two theorems on the topic. In the same year, Kartsatos and Manougian [10] provided some sets of criteria sufficient for oscillation of either all solutions of equation

$$x^{(n)}(t) + P(t)f\left(x(g(t))\right) = Q(t), \quad n \geq 2,$$

or all bounded solutions of the same equation. Later, in 1979, Singh [30] discussed the asymptotic oscillatory behavior of the solutions of the differential equations

$$\left(r(t)y'(t)\right)^{(n-1)} + F\left(h\left(y(g(t))\right), t\right) = 0, \quad n \geq 2,$$
$$\left(r(t)y'(t)\right)' + a(t)h\left(y(g(t))\right) = f(t)$$

and

$$\left(r(t)y'(t)\right)' + p(t)y(t) + a(t)h\left(y(g(t))\right) = f(t).$$

Afterward, in 1985, Grace and Lalli [7] established some oscillation and non-oscillation criteria for the n–order nonlinear differential equation

$$x^{(n)}(t) + f\left(t,\, x(t),\, x[g(t)]\right) = h(t).$$

In 1981, N. Parhi [16] and in 1983, 1985, 1986 and 1987, N. Parhi and S. Parhi [26–29] discussed the qualitative behavior, oscillation and non-oscillation of solutions of a third order differential equation of the form

$$\left(r(t)y''\right)' + q(t)\left(y'\right)^{\beta} + p(t)y^{\alpha} = f(t).$$

Similarly, in 1993, Parhi [18] established some sufficient conditions for oscillation of all solutions of the second order forced differential equation of the form

$$\left(r(t)y'(t)\right)' + p(t)y^{\alpha}(g(t)) = f(t)$$

and non-oscillation of all bounded solutions of the equations

$$\left(r(t)y'(t)\right)' + q(t)\left(y'(t)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t)$$

and

$$\left(r(t)y'(t)\right)' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t),$$

where the real-valued functions $f,\ p,\ q,\ r,\ g$ and g_1 are continuous on $[0, \infty)$ with $r(t) > 0$ and $f(t) \geq 0$; $g(t) \leq t$, $g_1(t) \leq t$ for $t \geq t_0$; $\lim_{t\to\infty} g(t) = \infty$, $\lim_{t\to\infty} g_1(t) = \infty$, and both $\alpha > 0$ and $\beta > 0$ are quotients of odd integers. In addition, in 1994, Parhi and Das [22] considered nonlinear third-order differential equations of the form

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)F\left(y(g(t))\right) = 0,$$

and they study the oscillatory and asymptotic behavior of solutions of this equation. In the same year, the same authors, Parhi and Das [22], also established some results for non-oscillation of solutions of equation

$$\left(r(t)y''(t)\right)' + q(t)y' + p(t)y = f(t)$$

and its associated homogeneous equation

$$(r(t)y''(t))' + q(t)y' + p(t)y = 0.$$

In 1996, Nayak and Choudhury [13] considered the differential equation

$$(r(t)y''(t))' - q(t)\left(y'(t)\right)^{\beta} - p(t)y^{\alpha}(g(t)) = f(t)$$

and they gave certain sufficient conditions on the functions involved for all bounded solutions of the above equation to be non-oscillatory.

Later, in 2001, Adamets and Lomtatidze [1] investigated oscillatory properties of solutions of the third order linear differential equation

$$u''' + p(t)u = 0,$$

where p is a locally integrable function on $[0, \infty)$ which is eventually of one sign.

In the same year, Parhi and Padhi [24] also gave sufficient conditions ensuring that all nontrivial solutions of the third-order linear differential equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0$$

are oscillatory.

After that, in 2002, the same authors, Parhi and Padhi [24] proved several theorems provided sufficient conditions for the above equation to have oscillatory solutions, and they also studied the nature of nonoscillatory solutions of the same equation. Namely, sufficient conditions were given for the set of nonoscillatory solutions of the above equation to form a one-dimensional subspace of the solution space.

In 2003, Candan and Dahiya [5] investigated oscillatory and asymptotic properties of solutions of the third order forced differential equation

$$\left(\left(b(t)\left(a(t)x'\right)^{\alpha}\right)'\right)' + q(t)f(x(g(t))) = r(t).$$

Finally, more recently, in 2005, Agarwal et al. [3] established some new criteria for the bounded oscillation of a fourth order functional differential equation. Besides, in the same year, Zhong et al. [35] considered a third order linear neutral delay difference equation with positive and negative coefficients. By using the Banach contraction principle the authors established some sufficient conditions which ensure that the equation considered has a nonoscillatory solution.

In this paper, we restrict our considerations to the real solutions of equations (1) and (2) which exist on the half-line $[T, \infty)$, where T (≥ 0) depends on the particular solution, are non-trivial in any neighborhood of infinity. It is well-known that a solution $y(t)$ of (1) or (2) is said to be non-oscillatory on $[T, \infty)$ if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; it is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; $y(t)$ is said to be a Z–type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

Now, it is reasonable to ask why the equations (1) and (2) have been investigated here. When one considers the papers and equations mentioned above, we think the importance of the investigation of behaviors of equations (1) and (2) may be acceptable.

2 Non-Oscillation Behaviors of Solutions of (1)

In this section, some sufficient conditions have been established for non-oscillation of all bounded solutions of (1). In order to reach our main results, first, we dispose of the following lemma.

Lemma 2.1 *Consider second order linear differential equation*

$$(r(t)z')' + q(t)z = 0, \tag{3}$$

where r and q are the same as in (1). If $z(t)$ is a non-oscillatory solution of equation (3) such that $z(t) > 0$ or $z(t) < 0$ for $t \in [a, \infty)$, $a > 0$, and if u is once continuously differentiable function on $[a, \infty)$, such that $u(b) = u(c) = 0$, $a < b < c$ and $u(t) \neq 0$ on $[b, c]$, then

$$\int_b^c \left[r(t)\ (u'(t))^2 - q(t)\,(u(t))^2 \right]\, dt > 0.$$

Proof See [28]. □

Next, in this section, we give the following four theorems.

Theorem 2.1 *Let us consider the equation (1), and let $f(t) - |p(t)| > 0$. If equation (3) is non-oscillatory, then all solution of equation (1), which are bounded above by 1, are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y > 0$, such that $|y(t)| \le 1$. Since $\lim_{t\to\infty} g(t) = \infty$, then there exists a $t_1 > t_0$ such that $g(t) \ge T_y$ for $t \ge t_1$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b ($T_y \le a < b$) such that $y(t) > 0$ for $t \in (a, b)$. So, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Multiplying equation (1) through by $y'(t)$, we obtain

$$(r(t)y'(t)y''(t))' = r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t). \tag{4}$$

Integrating (4) from a to c, we get

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 \right]\, dt + \int_a^c \left[f(t) - p(t)y^{\alpha}(g(t)) \right] y'(t)\, dt \\ &\ge \int_a^c \left[f(t) - |p(t)|\ |y^{\alpha}(g(t))| \right] y'(t)\, dt \ge \int_a^c \left[f(t) - |p(t)| \right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Next, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b ($T_y \le a < b$). Then, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Integrating (4) from c to b, we have

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 \right]\, dt + \int_c^b \left[f(t) - p(t)y^{\alpha}(g(t)) \right] y'(t)\, dt \\ &\ge \int_c^b \left[f(t) - |p(t)|\ |y^{\alpha}(g(t))| \right] y'(t)\, dt \ge \int_c^b \left[f(t) - |p(t)| \right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' ($T_y < a < b < a'$) such that $y'(a) \le 0$, $y'(b) \ge 0$, $y'(a') \le 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Therefore, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (3) from c to c' , we obtain

$$
\begin{aligned}
0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(t)\right)^2 \right] dt + \int_c^b \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\,dt \\
&\quad + \int_b^{c'} \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\,dt \\
&\ge \int_c^b \left[f(t) - |p(t)|\ |y^\alpha(g(t))|\right] y'(t)\,dt + \int_b^{c'} \left[f(t) - |p(t)|\ |y^\alpha(g(t))|\right] y'(t)\,dt \\
&\ge \int_c^b \left[f(t) - |p(t)|\right] y'(t)\,dt + \int_b^{c'} \left[f(t) - |p(t)|\right] y'(t)\,dt > 0,
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 2.1. □

Remark 2.1 It should be noted that there is no sign restriction on $p(t)$ and $q(t)$, which appear in equation (1), in Theorem 2.1. Our result, Theorem 2.1, improves the results established in N. Parhi [17; Theorem 2.4, Theorem 2.5] and N. Parhi and S. Parhi [28; Theorem 1.1].

Theorem 2.2 *If equation (3) admits a non-oscillatory solution and* $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, *then all bounded solution of (1) are non-oscillatory.*

Proof Because of the fact that $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, there exists a $t_2 \ge t_1$ such that $f(t) \ge M^\alpha |p(t)|$ for all $t \ge t_2$, where M is a positive constant and α is defined as in (1). The remaining of the proof of Theorem 2.2 follows a similar way as shown in proof of Theorem 2.1, except some minor modifications; hence we omit the detailed proof. □

Remark 2.2 It is interesting to note that there is no sign restriction on $p(t)$ and $q(t)$, which appear in equation (1), in Theorem 2.2. The author in [33], Tunç, proved a different result, when $g(t) = t$ in (1), under the conditions whenever equation (3) is non-oscillatory, $p(t) \le 0$ and $\lim\limits_{t\to\infty} \frac{f(t)}{-p(t)} = \infty$.

Theorem 2.3 *Consider the equation (1). If equation (3) admits a non-oscillatory solution and* $f(t) \ge K^\alpha |p(t)|$ *for large* t, *where* K *is a positive constant, then all* $y(t)$ *solutions of (1), which satisfy the inequality* $y(g(t)) \le K$ *in any interval where* $y(t) > 0$, *are non-oscillatory.*

Proof The proof of this theorem, Theorem 2.3, is similar to the proof of Theorem 2.1 and hence is omitted. □

Remark 2.3 The motivation for Theorem 2.3 has been inspired basically by N. Parhi and S. Parhi [26; Theorem 2.4], in which $g(t) = t$, $p(t) \ge 0$ and $q(t) \ge 0$. Next, there is no sign restriction on $p(t)$ and $q(t)$ in Theorem 2.3 proved here, and the inequality $f(t) \ge K^\alpha |p(t)|$ does not implies $\frac{f(t)}{|p(t)|} \to \infty$ as $t \to \infty$. Therefore, our conditions far less restrictive than those established in N. Parhi and S. Parhi [26; Theorem 2.4].

Theorem 2.4 *Consider the equation (1) with $\alpha \geq 1$. Suppose that $p(t) \geq 0$ and $q(t) \leq 0$, $q(t)$ once continuously differentiable such that $q'(t) \geq 0$. If $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$, then all bounded solutions of (1) are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of equation (1) on $[T_y, \infty)$, $T_y \geq 0$, such that $|y(t)| \leq M$ for all $t \geq T_y$, where M is a positive constant. Since $\lim\limits_{t\to\infty} g(t) = \infty$, then there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. In view of the assumption $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$, it follows that there exists a $t_2 \geq t_1$ such that $q'(t) \geq M^{\alpha-1}p(t)$ for $t \geq t_2$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(t_2 < a < b)$ such that $y(t) > 0$ for $t \in (a, b)$. Thus, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Clearly, $y''(a) \geq 0$ and $y''(c) \leq 0$. Thereby, integrating (1) from a to c, we obtain

$$
\begin{aligned}
0 &> -\int_a^c q(t)y'(t)\,dt - \int_a^c p(t)y^\alpha(g(t))\,dt \\
&\geq -q(c)y(c) + \int_a^c q'(t)y(t)\,dt - \int_a^c p(t)y^\alpha(g(t))\,dt \\
&\geq \int_a^c q'(t)y(t)\,dt - \int_a^c M^{\alpha-1}p(t)y(g(t))\,dt \\
&\geq \int_a^c \left[q'(t) - M^{\alpha-1}p(t)\right] y(g(t))\,dt > 0
\end{aligned}
$$

or

$$
\geq \int_a^c \left[q'(t) - M^{\alpha-1}p(t)\right] y(t)\,dt > 0
$$

a contradiction.

Similarly, it can be shown that $y(t)$ can not be of non-positive Z–type solution and oscillatory. Hence this completes the proof of the theorem. □

Remark 2.4 N. Parhi and S. Parhi [28; Theorem 6] proved a result, when $g(t) = t$ in (1), under the conditions $p(t) \geq 0$, $q(t) \leq 0$, $q'(t) \geq 0$ and $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$. Our conditions and equation (1) are different from the equation considered and the conditions established by N. Parhi and S. Parhi [28; Theorem 6].

3 Non-Oscillation Behaviors of Solutions of (2)

In this section, some results have been proved for non-oscillation of all bounded solutions of (2). The first one is the following.

Theorem 3.1 *Let $q(t) \leq 0$. If $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, the all bounded solutions of (2) are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of equation (2) on $[T_y, \infty)$, $T_y > 0$, such that $|y(t)| \leq M$. Because of $\lim\limits_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. Next, since $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, then it follows that there exists a $t_2 \geq t_1$ such that $f(t) > M^\alpha |p(t)|$ for $t \geq t_2$, where M is a positive constant and α is defined as the same in

(2). Now, if possible let $y(t)$ be of non-negative Z -type solution with consecutive double zeros at a and b $(t_2 < a < b)$ such that $y(t) > 0$ for $t \in (a,b)$. So, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Multiplying equation (2) through by $y'(t)$, we get

$$(r(t)y'(t)y''(t))' = r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t) - p(t)y^\alpha(g(t))y'(t) + f(t)y'(t). \quad (5)$$

Integrating (5) from a to c , we get

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)\beta y'(t)\right] dt + \int_a^c \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_a^c \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_a^c \left[f(t) - M^\alpha |p(t)|\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_2 < a < b)$. Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (5) from c to b, we have

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t)\right] dt + \int_c^b \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\,|y^\alpha(g(t))|\right] y'(t)\, dt \geq \int_c^b \left[f(t) - M^\alpha |p(t)|\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' $(t_2 < a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Therefore, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Now, integrating (5) from c to c', we obtain

$$\begin{aligned} 0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t)\right] dt + \int_c^{c'} \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt + \int_b^{c'} \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\,|y^\alpha(g(t))|\right] y'(t)\, dt + \int_b^{c'} \left[f(t) - |p(t)|\,|y^\alpha(g(t))|\right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - M^\alpha |p(t)|\right] y'(t)\, dt + \int_b^{c'} \left[f(t) - M^\alpha |p(t)|\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction. Hence $y(t)$ is non-oscillatory. □

Remark 3.1 For the special case $g(t) = g_1(t) = 0$ in (2), under the acceptations $p(t) \geq 0$, $q(t) \leq 0$ and $\lim_{t\to\infty} \frac{f(t)}{p(t)} = \infty$, Theorem 3.1 has been proved by N. Parhi and S. Parhi [26]. Our result improves the result established in N. Parhi and S. Parhi [26].

Theorem 3.2 *If* $\lim\limits_{t\to\infty} \dfrac{f(t)}{|p(t)|+|q(t)|} = \infty$, *the all solutions of equation (2), which are bounded together with their first derivatives, are non-oscillatory.*

Proof Let $y(t)$ be a solution of (2) on $[T_y, \infty)$, $T_y > 0$, such that $y(t)$ and $y'(t)$ are bounded. Hence, there exists positive constants M_1 and M_2 such that $|y(t)| \le M_1$ and $|y'(t)| \le M_2$ for all $t \ge T_y$. Further, in view of $\lim\limits_{t\to\infty} g(t) = \infty$ and $\lim\limits_{t\to\infty} g_1(t) = \infty$, it follows that there exists a $t_0 > 0$ such that $g(t) \ge T_y$ and $g_1(t) \ge T_y$ for $t \ge t_0$. Next, owing to the fact $\lim\limits_{t\to\infty} \dfrac{f(t)}{|p(t)|+|q(t)|} = \infty$, clearly, there exists a $t_1 > t_0$ such that $f(t) > L(|p(t)| + |q(t)|)$ for all $t \ge t_1$. Now, if possible let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$ such that $y(t) > 0$ for $t \in (a, b)$. Thus, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Integrating (5) from a to c, we obtain

$$\begin{aligned} 0 &= \int_a^c r(t)\,(y''(t))^2\, dt + \int_a^c \left[f(t) - \left\{q(t)\,(y'(g_1(t)))^\beta + p(t)y^\alpha(g(t))\right\}\right] y'(t)\, dt \\ &\ge \int_a^c \left[f(t) - \left\{M_2^\beta |q(t)| + M_1^\alpha |p(t)|\right\}\right] y'(t)\, dt \\ &\ge \int_a^c \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction, where $L = \max\left\{M_1^\alpha,\, M_2^\beta\right\}$.

Now, if possible, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$. Then, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$.

Integrating (5) from c to b, we have

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\,(y''(t))^2 - q(t)\,(y'(g_1(t)))^\beta\, y'(t) - p(t)y^\alpha(g(t))y'(t) + f(t)y'(t)\right] dt \\ &\ge \int_c^b \left[f(t) - \left\{M_2^\beta |q(t)| + M_1^\alpha |p(t)|\right\}\right] y'(t)\, dt \\ &\ge \int_a^c \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction, where $L = \max\left\{M_1^\alpha,\, M_2^\beta\right\}$.

Finally, if possible, let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' $(t_1 < a < b < a')$ such that $y'(a) \le 0$, $y'(b) \ge 0$, $y'(a') \le 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So, there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. Integrating (5) from c to c', we have

$$\begin{aligned} 0 &= \int_c^{c'} r(t)\,(y''(t))^2\, dt + \int_c^{c'} \left[f(t) - q(t)\,(y'(g_1(t)))^\beta - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\ge \int_c^{c'} \left[f(t) - \left\{M_2^\beta |q(t)| + M_1^\alpha |p(t)|\right\}\right] y'(t)\, dt \\ &\ge \int_c^{c'} \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction, where $L = \max\left\{M_1^{\alpha},\, M_2^{\beta}\right\}$. Therefore, we conclude that $y(t)$ is non-oscillatory. Thus the theorem is proved. □

Remark 3.2 For the special case $g(t) = g_1(t) = t$ and $\alpha = \beta$ in (2), subject to the assumptions $p(t) \geq 0$, $q(t) \geq 0$ and $\lim\limits_{t\to\infty} \dfrac{f(t)}{p(t)+q(t)} = \infty$, Theorem 3.2 has been proved by Tunç [34]. Our result improves the result established in Tunç [34].

Theorem 3.3 *Consider equation (2) with $\alpha = \beta$ and $g(t) = g_1(t)$. Let $p(t) \geq 0$. If $\lim\limits_{t\to\infty} \dfrac{f(t)}{|q(t)| + p(t)(g(t))^{\alpha}} = \infty$, then all bounded solutions of equation (2), for which their first derivatives, $y'(t)$, are also bounded for large t, are non-oscillatory.*

Proof Let $y(t)$ be a solution of equation (2) on $[T_y, \infty)$, $T_y > 0$, such that $y(t)$ and $y'(t)$ are bounded. So, there exists positive constants M_1 and M_2 such that $|y(t)| \leq M_1$ and $|y'(t)| \leq M_2$ for all $t \geq T_y$. Next, from the fact $\lim\limits_{t\to\infty} g(t) = \infty$, it follows that there exists a $t_0 > 0$ such that $g(t) \geq T_y$ for $t \geq t_0$. By virtue of $\lim\limits_{t\to\infty} \dfrac{f(t)}{|q(t)| + p(t)(g(t))^{\alpha}} = \infty$, evidently, there exists a $t_1 > t_0$ such that $f(t) > L[|q(t)| + p(t)(g(t))^{\alpha}]$ for all $t \geq t_1$, where $L = \max\{M_1^{\alpha},\, M_2^{\alpha}\}$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$ such that $y(t) > 0$ for $t \in (a, b)$. So, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Consequently, there exists $d \in (a, c)$ such that $y''(d) = 0$ and $y''(t) > 0$ for $t \in (a, d)$. Next, clearly, $y'(t) \geq \frac{y(t)}{t}$ for large t_1 and $t \in [a, d]$. Hence, $y'(g(t)) \geq \dfrac{y(g(t))}{g(t)}$ for large t_1 and $t \in [a, d]$. Now, integrating equality (5) from a to d, we obtain

$$
\begin{aligned}
0 &> \int_a^d r(t)\left(y''(t)\right)^2 dt + \int_a^d \left[f(t) - \left\{q(t)\left(y'(g_1(t))\right)^{\alpha} + p(t)y^{\alpha}(g(t))\right\}\right] y'(t)\, dt \\
&> \int_a^d \left[f(t) - \{|q(t)| + p(t)(g(t))^{\alpha}\}\left(y'(g(t))\right)^{\alpha}\right] y'(t)\, dt \\
&> \int_a^d \left[f(t) - M_2^{\alpha}\{|q(t)| + p(t)(g(t))^{\alpha}\}\right] y'(t)\, dt > 0,
\end{aligned}
$$

a contradiction.

If possible, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$. Then, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$.

Integrating (5) from c to b yields

$$
\begin{aligned}
0 &= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^{\alpha} y'(t) - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t)\right] dt \\
&\geq \int_c^b \left[f(t) - q(t)\left(y'(g_1(t))\right)^{\alpha}\right] y'(t)\, dt \geq \int_c^b \left[f(t) - |q(t)|\left(y'(g_1(t))\right)^{\alpha}\right] y'(t)\, dt \\
&\geq \int_c^b \left[f(t) - M_2^{\alpha}|q(t)|\right] y'(t)\, dt > 0,
\end{aligned}
$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' ($t_1 < a < b < a'$) such that $y'(a) \le 0$, $y'(b) \ge 0$, $y'(a') \le 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Thus, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (5) from c to b, we have

$$\begin{aligned}
0 &\ge r(b)y'(b)y''(b) \\
&= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^{\alpha} y'(t) - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t) \right] dt \\
&\ge \int_c^b \left[f(t) - q(t)\left(y'(g(t))\right)^{\alpha} \right] y'(t)\,dt \ge \int_c^b \left[f(t) - |q(t)|\left(y'(g(t))\right)^{\alpha} \right] y'(t)\,dt \\
&\ge \int_c^b \left[f(t) - M_2^{\alpha}|q(t)| \right] y'(t)\,dt > 0,
\end{aligned}$$

a contradiction. In conclusion, $y''(b) > 0$. Besides, since $y''(c') < 0$, there exists $d \in (b,c')$ such that $y''(d) = 0$ and $y''(t) > 0$ for $t \in [b,d)$. For that reason, $y'(t) \ge \frac{y(t)}{t}$ for $t \in [b,d]$ and for sufficiently large t_1. Again, integrating equality in (5) from b to d, we get

$$\begin{aligned}
0 &\ge -r(b)y'(b)y''(b) \\
&= \int_b^d r(t)\left(y''(t)\right)^2\,dt + \int_b^d \left[f(t) - q(t)\left(y'(g_1(t))\right)^{\alpha} - p(t)y^{\alpha}(g(t)) \right] y'(t)dt \\
&> \int_b^d \left[f(t) - \{\, |q(t)| + p(t)(g(t))^{\alpha} \}\left(y'(g(t))\right)^{\alpha} \right] y'(t)\,dt \\
&> \int_b^d \left[f(t) - M_2^{\alpha}\{\, |q(t)| + p(t)(g(t))^{\alpha} \} \right] y'(t)\,dt,
\end{aligned}$$

a contradiction. This completes the proof of the theorem. □

Remark 3.3 Theorem 3.3 includes, respectively, the results obtained by Tunç [33, Theorem 7] and improves the result established by of S. Parhi and N. Parhi [27, Theorem 2.5, Theorem 2.6].

Theorem 3.4 *Let $q(t) \le 0$. If $f(t) \ge K^{\alpha}|p(t)|$ for large t, where K is a positive constant and α is defined as the same in equation (2). Then all solutions $y(t)$ of (2), which satisfy the inequality $y(g(t)) \le K$ in any interval where $y(t) > 0$, are non-oscillatory.*

Proof The proof of the theorem is straightforward and hence is omitted. □

Remark 3.4 It should be noted that Theorem 3.4 is different than Theorem 2.3 just proved above because of $\beta \ne 1$ and $g_1(t) \ne t$.

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