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Strange Attractors and Classical Stability Theory

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Abstract: Definitions of global attractor, B-attractor and weak attractor are introduced. Relationships between Lyapunov stability, Poincare stability and Zhukovsky stability are considered. Perron effects are discussed. Stability and instability criteria by the first approximation are obtained. Lyapunov direct method in dimension theory is introduced. For the Lorenz system necessary and sufficient conditions of existence of homoclinic trajectories are obtained.

Keywords: *Attractor, instability, Lyapunov exponent, stability, Poincaré section, Hausdorff dimension, Lorenz system, homoclinic bifurcation.*

Mathematics Subject Classification (2000): 34C28, 34D45, 34D20.

1 Introduction

In almost any solid survey or book on chaotic dynamics, one encounters notions from classical stability theory such as Lyapunov exponent and characteristic exponent. But the effect of sign inversion in the characteristic exponent during linearization is seldom mentioned. This effect was discovered by Oscar Perron [1], an outstanding German mathematician. The present survey sets forth Perron's results and their further development, see [2]–[4]. It is shown that Perron effects may occur on the boundaries of a flow of solutions that is stable by the first approximation. Inside the flow, stability is completely determined by the negativeness of the characteristic exponents of linearized systems.

It is often said that the defining property of strange attractors is the sensitivity of their trajectories with respect to the initial data. But how is this property connected with the classical notions of instability? For continuous systems, it was necessary to remember the almost forgotten notion of Zhukovsky instability. Nikolai Egorovich Zhukovsky, one of the founders of modern aerodynamics and a prominent Russian scientist, introduced

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his notion of stability of motion in 1882, see [5, 6] — ten years before the publication of Lyapunov's investigations [7]. The notion of Zhukovsky instability is adequate to the sensitivity of trajectories with respect to the initial data for continuous dynamical systems. In this survey we consider the notions of instability according to Zhukovsky, Poincaré, and Lyapunov, along with their adequacy to the sensitivity of trajectories on strange attractors with respect to the initial data.

In order to investigate Zhukovsky stability, a new research tool — a moving Poincaré section — is introduced. With the help of this tool, extensions of the widely-known theorems of Andronov–Witt and Demidovic are carried out.

At the present time, the problem of justifying nonstationary linearizations for complicated, nonperiodic motions on strange attractors bears a striking resemblance to the situation that occurred 120 years ago.

J.C. Maxwell [8] and I.A. Vyshnegradskii [9], the founders of automatic control theory, courageously used linearization in a neighborhood of stationary motions, leaving the justification of such linearization to H. Poincaré [10] and A.M. Lyapunov [7]. Now many specialists in chaotic dynamics believe that the positiveness of the largest characteristic exponent of a linear system of the first approximation implies the instability of solutions of the original system. Moreover, there is a great number of computer experiments in which various numerical methods for calculating characteristic exponents and Lyapunov exponents of linear systems of the first approximation are used. As a rule, authors largely ignore the justification of the linearization procedure and use the numerical values of exponents thus obtained to construct various numerical characteristics of attractors of the original nonlinear systems (Lyapunov dimensions, metric entropies, and so on). Sometimes computer experiments serve as arguments for partial justification of the linearization procedure. For example, computer experiments in [11, 12] show the coincidence of the Lyapunov and Hausdorff dimensions of the attractors of Henon, Kaplan–Yorke, and Zaslavskii. But for B-attractors of Henon and Lorenz, such coincidence does not hold, see [13, 14]).

So linearizations along trajectories on strange attractors require justification. This problem gives great impetus to the development of the nonstationary theory of instability by the first approximation. The present survey describes the contemporary state of the art of the problem of justifying nonstationary linearizations.

The method of Lyapunov functions — Lyapunov's so-called direct method — is an efficient research device in classical stability theory. It turns out that even in the dimension theory of strange attractors one can progress by developing analogs of this method. This interesting line of investigation is also discussed in the present survey.

When the parameters of a dynamical system are varied, the structure of its minimal global attractor can change as well. Such changes are the subject of bifurcation theory. Here we describe one of these, namely the homoclinic bifurcation.

The first important results concerning homoclinic bifurcation in dissipative systems were obtained in 1933 by the outstanding Italian mathematician Francesco Tricomi [15]. Here we give Tricomi's results along with similar theorems for the Lorenz system.

For the Lorenz system, necessary and sufficient conditions for the existence of homoclinic trajectories are obtained.

2 Definitions of Attractors

The attractor of a dynamical system is an attractive closed invariant set in its phase space.

Consider the dynamical systems generated by the differential equations

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1, \; x \in \mathbb{R}^n, \tag{2.1}$$

and by the difference equations

$$x(t+1) = f\left(x(t)\right), \quad t \in \mathbb{Z}, \; x \in \mathbb{R}^n. \tag{2.2}$$

Here $\mathbb{R}^n$ is a Euclidean space, $\mathbb{Z}$ is the set of integers, and $f(x)$ is a vector-function: $\mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.1. We say that (2.1) or (2.2) *generates a dynamical system* if for any initial data $x_0 \in \mathbb{R}^n$ the trajectory $x(t, x_0)$ is uniquely determined for $t \in [0, +\infty)$. Here $x(0, x_0) = x_0$.

It is well known that the solutions of dynamical system (2.1) satisfy the semigroup property

$$x(t+s, x_0) = x\left(t, x(s, x_0)\right) \tag{2.3}$$

for all $t \geq 0$, $s \geq 0$.

For equation (2.1) on $[0, +\infty)$ there are many existence and uniqueness theorems [16]–[19] that can be used for determining the corresponding dynamical system with the phase space $\mathbb{R}^n$. The partial differential equations, generating dynamical systems with different infinite-dimensional phase spaces, can be found in [20]–[24]. The classical results of the theory of dynamical systems with a metric phase space are given in [25].

For (2.2) it is readily shown that in all cases the trajectory, defined for all $t = 0, 1, 2, \ldots$, satisfying (2.3), and having initial condition x_0, is unique. Thus (2.2) always generates a dynamical system with phase space $\mathbb{R}^n$.

A dynamical system generated by (2.1) is called *continuous*. Equation (2.2) generates a *discrete dynamical system.*

The definitions of attractors are, as a rule, due to [14, 23, 24, 26].

Definition 2.2. We say that K *is invariant* if $x(t, K) = K$, $\forall t \geq 0$. Here

$$x(t, K) = \{x(t, x_0) \mid x_0 \in K\}.$$

Definition 2.3. We say that the invariant set K is *locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of K the relation

$$\lim_{t \to +\infty} \rho\left(K, x(t, x_0)\right) = 0, \quad \forall\, x_0 \in K(\varepsilon)$$

is satisfied. Here $\rho(K, x)$ is the distance from the point x to the set K, defined by

$$\rho(K, x) = \inf_{z \in K} |z - x|.$$

Recall that $|\cdot|$ is a Euclidean norm in $\mathbb{R}^n$, and $K(\varepsilon)$ is the set of points x such that $\rho(K, x) < \varepsilon$.

Definition 2.4. We say that the invariant set K is *globally attractive* if

$$\lim_{t \to +\infty} \rho\big(K, x(t, x_0)\big) = 0, \quad \forall\, x_0 \in \mathbb{R}^n.$$

Definition 2.5. We say that the invariant set K is *uniformly locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of it and for any $\delta > 0$ and bounded set B there exists $t(\delta, B) > 0$ such that

$$x\big(t, B \cap K(\varepsilon)\big) \subset K(\delta), \quad \forall t \geq t(\delta, B).$$

Here

$$x\big(t, B \cap K(\varepsilon)\big) = \{x(t, x_0) \mid x_0 \in B \cap K(\varepsilon)\}.$$

Definition 2.6. We say that the invariant set K is *uniformly globally attractive* if for any $\delta > 0$ and bounded set $B \subset \mathbb{R}^n$ there exists $t(\delta, B) > 0$ such that

$$x(t, B) \subset K(\delta), \quad \forall t \geq t(\delta, B).$$

Definition 2.7. We say that the invariant set K is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x(t, K(\delta)) \subset K(\varepsilon), \quad \forall t \geq 0.$$

Note that if K consists of one trajectory, then the last definition coincides with the classical definitions of the Lyapunov stability of solution. If such K is locally attractive, then we have asymptotic stability in the sense of Lyapunov.

Definition 2.8. We say that K is

1) an *attractor* if it is an invariant closed and locally attractive set;

2) a *global attractor* if it is an invariant closed and globally attractive set;

3) a *B-attractor* if it is an invariant, closed, and uniformly locally attractive set;

4) a *global B-attractor* if it is an invariant, closed, and uniformly globally attractive set.

A trivial example of an attractor is the whole phase set $\mathbb{R}^n$ if the trajectories are defined for all $t \geq 0$. This shows that it is sensible to introduce the notion of a *minimal attractor*, namely the minimal invariant set possessing the attractive property.

We give the simplest examples of attractors.

Example 2.1. Consider the equations of pendulum motion:

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \beta \sin\theta, \end{aligned} \tag{2.4}$$

where α and β are positive. The trajectories have a well-known asymptotic behavior (Figure 2.1).

Any solution of (2.4) tends to a certain equilibrium as $t \to +\infty$. Therefore the minimal global attractor of (2.4) is a stationary set.

Consider now a ball B of small radius centered on the separatrix of the saddle. As $t \to +\infty$ the image $x(t, B)$ of this small ball tends to the set consisting of a saddle equilibrium and of two separatrices, leaving this point and tending to an asymptotically stable equilibrium (Figure 2.2) as $t \to +\infty$.

Thus, a global minimal B-attractor is a union of a stationary set and the separatrices, leaving the saddle points (unstable manifolds of the saddle points) (Figure 2.3).

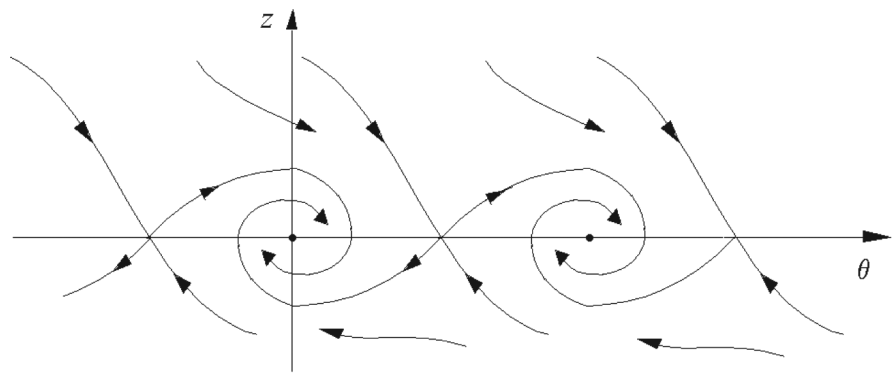

Figure 2.1:

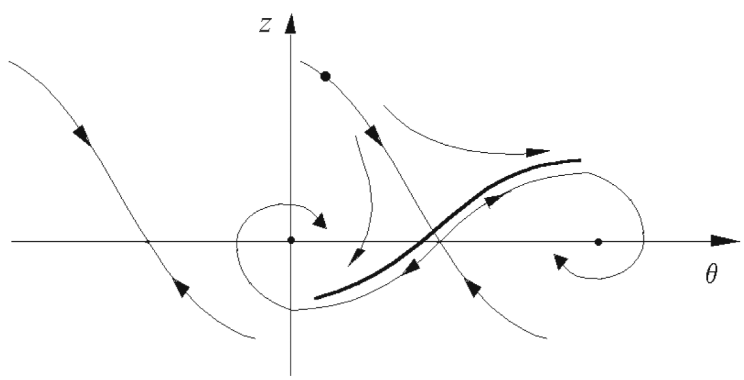

Figure 2.2:

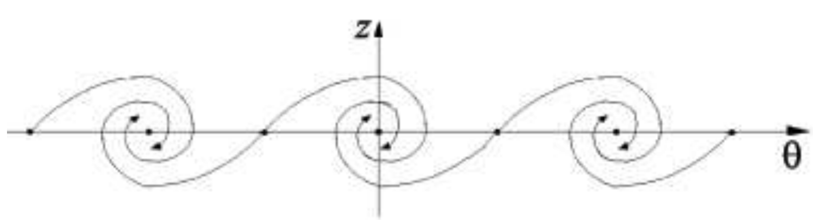

Figure 2.3:

In more general situations the B-attractor involves the unstable manifolds of saddle (hyperbolic) points. This fact is often used for estimating the topological dimension of attractors from below [20]. □

We remark that the natural generalization of the notion of attractor is to weaker requirements of attraction: on sets of positive Lebesgue measure, almost everywhere, and so on. As an illustration of such an approach we give a definition of weak attractor [26].

Definition 2.9. We say that K is a *weak attractor* if K is an invariant closed set for which there exists a set of positive Lebesgue measure $U \subset \mathbb{R}^n$ satisfying the following relation:

$$\lim_{t\to+\infty} \rho(K, x(t, x_0)) = 0, \quad \forall x_0 \in U.$$

Note that for each concrete system it is necessary to detail the set U.

3 Strange Attractors and the Classical Definitions of Instability

One of the basic characteristics of a strange attractor is the sensitivity of its trajectories to the initial data.

We consider the correlation of such "sensitivity" with a classical notion of instability. We recall first the basic definitions of stability.

Consider the system

$$\frac{dx}{dt} = F(x, t), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where $F(x, t)$ is a continuous vector-function, and

$$x(t + 1) = F(x(t), t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Denote by $x(t, t_0, x_0)$ the solution of (3.1) or (3.2) with initial data t_0, x_0:

$$x(t_0, t_0, x_0) = x_0.$$

Definition 3.1. The solution $x(t, t_0, x_0)$ is said to be *Lyapunov stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0)$ such that

1. all the solutions $x(t, t_0, y_0)$, satisfying the condition
$$|x_0 - y_0| \leq \delta,$$
are defined for $t \geq t_0$,
2. for these solutions the inequality
$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \leq \varepsilon, \quad \forall t \geq t_0$$
is valid.

If $\delta(\varepsilon, t_0)$ is independent of t_0, the Lyapunov stability is called *uniform.*

Definition 3.2. The solution $x(t, t_0, x_0)$ is said to be *asymptotically Lyapunov stable* if it is Lyapunov stable and for any $t_0 \geq 0$ there exists $\Delta(t_0) > 0$ such that the solution $x(t, t_0, y_0)$, satisfying the condition $|x_0 - y_0| \leq \Delta$, has the following property:

$$\lim_{t\to+\infty} |x(t, t_0, x_0) - x(t, t_0, y_0)| = 0.$$

Definition 3.3. The solution $x(t, t_0, x_0)$ is said to be *Krasovsky stable* if there exist positive numbers $\delta(t_0)$ and $R(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t, t_0, y_0)$ is defined for $t \ge t_0$ and satisfies

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le R(t_0)|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ and R are independent of t_0, then Krasovsky stability is called *uniform*.

Definition 3.4. The solution $x(t, t_0, x_0)$ is said to be *exponentially stable* if there exist the positive numbers $\delta(t_0)$, $R(t_0)$, and $\alpha(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t, t_0, y_0)$ is defined for all $t \ge t_0$ and satisfies

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le R(t_0) \exp(-\alpha(t_0)(t- t_0))|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ, R, and α are independent of t_0, then exponential stability is called *uniform*.

Consider now dynamical systems (3.1) and (3.2). We introduce the following notation:

$$L^+(x_0) = \big\{x(t, x_0) \mid \; t \in [0, +\infty)\big\}.$$

Definition 3.5. The trajectory $x(t, x_0)$ of a dynamical system is said to be *Poincaré stable* (or *orbitally stable*) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the relation

$$\rho\big(L^+(x_0),\, x(t, y_0)\big) \le \varepsilon, \quad \forall\, t \ge 0$$

is satisfied. If, in addition, for a certain number δ_0 and for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta_0$, the relation

$$\lim_{t\to+\infty} \rho\big(L^+(x_0),\, x(t, y_0)\big) = 0$$

holds, then the trajectory $x(t, x_0)$ is said to be *asymptotically Poincaré stable* (or *asymptotically orbitally stable*).

Note that for continuous dynamical systems we have $t \in \mathbb{R}^1$, and for discrete dynamical systems $t \in \mathbb{Z}$.

We now introduce the definition of Zhukovsky stability for continuous dynamical systems. For this purpose we must consider the following set of homeomorphisms:

$$\mathrm{Hom} = \{\tau(\cdot) \mid \tau : [0, +\infty) \to [0, +\infty), \tau(0) = 0\}.$$

The functions $\tau(t)$ from the set Hom play the role of the reparametrization of time for the trajectories of system (3.1).

Definition 3.6 [5, 6, 27, 28]. The trajectory $x(t, x_0)$ of system (3.1) is said to be *Zhukovsky stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the function $\tau(\cdot) \in \mathrm{Hom}$ can be found such that the inequality

$$|x(t, x_0) - x(\tau(t), y_0)| \le \varepsilon, \quad \forall t \ge 0$$

is valid. If, in addition, for a certain number $\delta_0 > 0$ and any y_0 from the ball $\{y|\,|x_0 - y| \le \delta_0\}$ the function $\tau(\cdot) \in \mathrm{Hom}$ can be found such that the relation

$$\lim_{t\to+\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

holds, then the trajectory $x(t, x_0)$ is *asymptotically stable in the sense of Zhukovsky.*

This means that Zhukovsky stability is Lyapunov stability for the suitable reparametrization of each of the perturbed trajectories.

Recall that, by definition, Lyapunov instability is the negation of Lyapunov stability. Analogous statements hold for Krasovsky, Poincaré, and Zhukovsky instability.

The following obvious assertions can be formulated.

Proposition 3.1. *For continuous dynamical systems, Lyapunov stability implies Zhukovsky stability, and Zhukovsky stability implies Poincaré stability.*

Proposition 3.2. *For discrete dynamical systems, Lyapunov stability implies Poincaré stability.*

Proposition 3.3. *For equilibria, all the above definitions due to Lyapunov, Zhukovsky, and Poincaré are equivalent.*

Proposition 3.4. *For periodic trajectories of discrete dynamical systems with continuous $f(x)$, the definitions of Lyapunov and Poincaré stability are equivalent.*

Proposition 3.5. *For the periodic trajectories of continuous dynamical systems with differentiable $f(x)$, the definitions of Poincaré and Zhukovsky stability are equivalent.*

Also well known are examples of periodic trajectories of continuous systems that happen to be Lyapunov unstable but Poincaré stable.

Now we proceed to compare the definitions given above with the effect of trajectory sensitivity to the initial data for strange attractors.

Lyapunov instability cannot characterize the “mutual repulsion” of continuous trajectories due to small variations in initial data. Neither can Poincaré instability characterize this repulsion. In this case, the perturbed solution can leave the ε-neighborhood of a certain segment of the unperturbed trajectory (the effect of repulsion) while simultaneously entering the ε-neighborhood of another segment (the property of Poincaré stability). Thus, mutually repulsive trajectories can be Poincaré stable. Let us consider these effects in more detail.

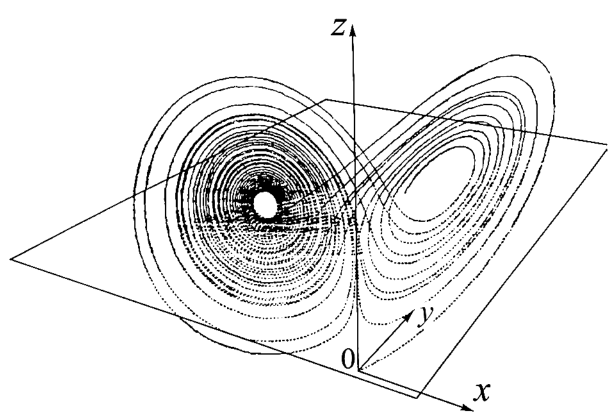

Figure 3.1: Unstable manifold of the saddle of the Lorenz system. The first fifty turns.

In computer experiments it often happens that the trajectories, situated on the unstable manifold of a saddle singular point, everywhere densely fill the B-attractor (or that portion of it consisting of the bounded trajectories). This can be observed on the B-attractor of the Lorenz system [29]

$$\begin{aligned}\dot{x} &= -\sigma(x-y),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned}$$

where $\sigma = 10$, $r = 28$, and $b = 8/3$ (Figure 2.1).

Example 3.1. Consider the linearized equations of two decoupled pendula:

$$\begin{aligned}\dot{x}_1 &= y_1, \quad \dot{y}_1 = -\omega_1^2 x_1,\\ \dot{x}_2 &= y_2, \quad \dot{y}_2 = -\omega_2^2 x_3.\end{aligned} \tag{3.3}$$

The solutions are

$$\begin{aligned}x_1(t) &= A\sin(\omega_1 t + \varphi_1(0)),\\ y_1(t) &= A\omega_1\cos(\omega_1 t + \varphi_1(0)),\\ x_2(t) &= B\sin(\omega_2 t + \varphi_2(0)),\\ y_2(t) &= B\omega_2\cos(\omega_2 t + \varphi_2(0)).\end{aligned}$$

For fixed A and B, the trajectories of system are situated on two-dimensional tori

$$\omega_1^2 x_1^2 + y_1^2 = A^2, \qquad \omega_2^2 x_2^2 + y_2^2 = B^3.$$

When ω_1/ω_2 is irrational, the trajectories are everywhere densely situated on the tori for any initial data $\varphi_1(0)$ and $\varphi_2(0)$.

This implies asymptotic Poincaré stability of the trajectories of the dynamical system on tori. However, the motion of the points $x(t,x_0)$ and $x(t,y_0)$ along the trajectories occurs in such a way that they do not tend toward each other as $t \to +\infty$. Neither are the trajectories "pressed" toward each other. Hence the intuitive conception of asymptotic stability as a convergence of objects toward each other is in contrast to the formal definition of Poincaré.

It is clear that a similar effect is lacking for the notion of Zhukovsky stability: in the case under consideration, asymptotic Zhukovsky stability does not occur. □

Example 3.2. We reconsider the dynamical system (3.3) with ω_1/ω_2 irrational. Change the flow of trajectories on the tori as follows. Cut the toroidal surface along a certain segment of the fixed trajectory from the point z_1 to the point z_2. Then the surface is stretched diffeomorphically along the torus so that a cut is mapped into the circle with the fixed points z_1 and z_2 (Fig. 3.2). Denote by H the interior of the circle.

Change the dynamical system so that z_1 and z_2 are saddle stationary points and the semicircles connecting z_1 and z_2 are heteroclinic trajectories, tending as $t \to +\infty$ and $t \to -\infty$ to z_2 and z_1, respectively (Figure 3.2).

Outside the "hole" H, after the diffeomorphic stretching, the disposition of trajectories on the torus is the same.

Consider the behavior of the system trajectories from the Poincaré and Zhukovsky points of view.

Outside the hole H, the trajectories are everywhere dense on torus. They are therefore, as before, asymptotically Poincaré stable.

Now we consider a certain δ-neighborhood of the point z_0, situated on the torus and outside the set H. The trajectory leaving z_0 is either everywhere dense or coincides with

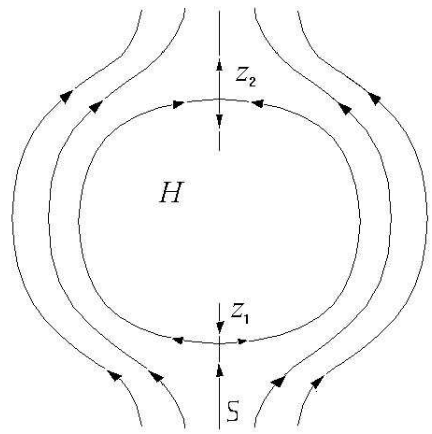

Figure 3.2:

the separatrix S of the saddle z_1, tending to z_1 as $t \to +\infty$ (Figure 3.2). Then there exists a time t such that some trajectories, leaving the δ-neighborhood of z_0, are situated in a small neighborhood of z_1 to the right of the separatrix S. At time t the remaining trajectories, leaving this neighborhood of z_0, are situated to the left of S. It is clear that in this case the trajectories, situated to the right and to the left of S, envelope the hole H on the right and left, respectively. It is also clear that these trajectories are repelled from each other; hence, the trajectory leaving z_0 is Zhukovsky unstable.

Thus, a trajectory can be asymptotically Poincaré stable and Zhukovsky unstable.

This example shows that the trajectories are sensitive to the initial data and can diverge considerably after some time. The notion of Zhukovsky instability is adequate to such a sensitivity.

Note that the set of such sensitive trajectories is situated on the smooth manifold, named "a torus minus the hole H". Thus, the bounded invariant set of trajectories, which are sensitive to the initial data, do not always have a noninteger Hausdorff dimension or the structure of the Cantor set.

Hence, from among the classical notions of instability for studying strange attractors, the most adequate ones are Zhukovsky instability (in the continuous case) and Lyapunov instability (in the discrete case).

4 Characteristic Exponents and Lyapunov Exponents

Definition 4.1. The number (or the symbol $+\infty, -\infty$), defined by the formula

$$\lambda = \limsup_{t \to +\infty} \frac{1}{t} \ln |f(t)|,$$

is called a *characteristic exponent* of the vector-function $f(t)$.

Definition 4.2. The characteristic exponent λ of the vector-function $f(t)$ is said to be *sharp* if there exists the following finite limit:

$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \ln |f(t)|.$$

The value

$$\lambda = \liminf_{t\to+\infty} \frac{1}{t} \ln |f(t)|$$

is often called a *lower characteristic exponent* of $f(t)$.

Consider the linear system

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n, \tag{4.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0, +\infty)$. Let $X(t) = (x_1(t), \dots, x_n(t))$ be a fundamental matrix of (4.1) (i.e. $\det X(0) \neq 0$). It is well known that under the above conditions the characteristic exponents λ_j of the solutions $x_j(t)$ are numbers.

Definition 4.3. Fundamental matrix $X(t)$ is said to be *normal* if the sum $\sum\limits_{j=1}^{n} \lambda_j$ of the characteristic exponents of the vector-functions $x_j(t)$ is minimal in comparison to other fundamental matrices.

The following substantial and almost obvious results are well-known.

Theorem 4.1. *For all normal fundamental matrices $(x_1(t), \dots, x_n(t))$ the number of solutions $x_j(t)$ having the same characteristic exponent is the same.*

We can now introduce the following definitions.

Definition 4.4. The set of characteristic exponents $\lambda_1, \dots, \lambda_n$ of the solutions $x_1(t), \dots, x_n(t)$ of certain normal fundamental matrices $X(t)$ is called the *complete spectrum* of linear system (4.1), and the numbers λ_j are called the *characteristic exponents* of (4.1).

Thus, any normal fundamental matrix realizes the complete spectrum of the system (4.1).

In the sequel, by $\Sigma = \sum\limits_{j=1}^{n} \lambda_j$ is denoted the sum of characteristic exponents of system (4.1).

The Lyapunov inequality

$$\Sigma \geq \limsup_{t\to+\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau \tag{4.3}$$

is well known. Here Tr is a spur of the matrix A.

Definition 4.5. If the relation

$$\Sigma = \limsup_{t\to+\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau$$

is satisfied, then system (4.1) is called *regular*.

It is well-known that each characteristic exponent of a regular system is sharp.

Definition 4.6. The number

$$\Gamma = \Sigma - \liminf_{t\to+\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau$$

is called the *coefficient of irregularity* for (4.1).

We assume further that $\lambda_1 \geq \cdots \geq \lambda_n$. The number λ_1 is called a *largest characteristic exponent*.

Let $X(t)$ be a fundamental matrix of system (4.1). We introduce the singular values $\alpha_1(X(t)) \geq \cdots \geq \alpha_n(X(t)) \geq 0$ of $X(t)$. Recall that the singular values $\alpha_j(X(t))$ of a matrix $X(t)$ are square roots of eigenvalues of the matrix $X(t)^* X(t)$. Geometrically, the $\alpha_j(X(t))$ coincide with the principal axes of the ellipsoid $X(t)B$, where B is the unit ball.

Definition 4.7 [22]. The *Lyapunov exponent* μ_j is the number

$$\mu_j = \limsup_{t \to +\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

We say that μ_j is *sharp* if there exists the finite limit

$$\lim_{t \to +\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

Proposition 4.1. *The largest characteristic exponent λ_1 and the Lyapunov exponent μ_1 coincide.*

5 Perron Effects

In 1930, O. Perron [1] showed that the negativity of the largest characteristic exponent of the system of the first approximation does not always imply the stability of the zero solution of the original system. In addition, in an arbitrary small neighborhood of zero the solutions of the original system with positive characteristic exponent can exist. Perron's results impressed the specialists in the theory of motion stability.

The effect of sign reversal for the characteristic exponent of solutions of the system of the first approximation, and of the original system under the same initial data, we shall call the *Perron effect*.

We cite the outstanding result of Perron. Consider a system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1, \\ \frac{dx_2}{dt} &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_2 + x_1^2, \end{aligned} \tag{5.1}$$

where a satisfies

$$1 < 2a < 1 + \frac{1}{2}\exp(-\pi). \tag{5.2}$$

The solution of the equation of the first approximation takes the form

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_2(0). \end{aligned}$$

It is obvious that for the system of the first approximation under condition (5.2) we have $\lambda_1 < 0$.

Now we write the solution of (5.1):

$$
\begin{aligned}
&x_1(t) = \exp[-at]x_1(0),\\
&x_2(t) = \exp[(t+1)\sin(\ln(t+1)) - 2at]\bigg(x_2(0)+\\
&+x_1(0)^2\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau\bigg).
\end{aligned}
\tag{5.3}
$$

Letting $t = \exp[(2k+\frac{1}{2})\pi] - 1$, where k is an integer, we obtain

$$
\begin{aligned}
&\exp[(t+1)\sin(\ln(t+1)) - 2at] = e(\exp[(1-\\
&-2a)t]),\ \ (1+t)e^{-\pi} - 1 > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau >\\
&\quad > \int_{f(k)}^{g(k)} \exp[-(\tau+1)\sin(\ln(\tau+1))]\,d\tau\\
&\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\right]\,d\tau\\
&\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\exp(-\pi)\right]\,d\tau\\
&\quad = \exp\left[\frac{1}{2}(t+1)\exp(-\pi)\right](t+\\
&\quad +1)\left(\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&f(k) = (1+t)\exp[-\pi] - 1,\\
&g(k) = (1+t)\exp\left[-\frac{2\pi}{3}\right] - 3.
\end{aligned}
$$

This implies the estimate

$$
\begin{aligned}
&\exp[(t+1)\sin(\ln(t+1)) - 2at]\int_0^t \exp[-(\tau+\\
&+1)\sin(\ln(\tau+1))]d\tau\\
&> \exp\left[\frac{1}{2}(2+\exp(-\pi))\right]\left(\exp\left(-\frac{2\pi}{3}\right) -\right.\\
&\left.-\exp(-\pi)\right)\cdot\exp\left[\left(1-2a+\frac{1}{2}\exp(-\pi)\right)t\right].
\end{aligned}
\tag{5.4}
$$

This and condition (5.2) imply that the characteristic exponent λ of the solutions of system (5.1) for $x_1(0) \neq 0$ is positive.

Thus, all characteristic exponents of the system of the first approximation are negative, and almost all solutions of the original system (5.1) tend exponentially to infinity as $k \to +\infty$. □

We consider the similar effect of the sign reversal of characteristic exponents but "on the contrary", namely the solution of the system of the first approximation has a positive characteristic exponent while the solution of the original system with the same initial data has a negative exponent [2, 3, 4]. Consider a system

$$
\begin{aligned}
&\dot{x}_1 = -ax_1,\\
&\dot{x}_2 = -2ax_2,\\
&\dot{x}_3 = [\sin(\ln(t+1)) + \cos(\ln(t+1)) -\\
&-2a]x_3 + x_2 - x_1^2,
\end{aligned}
\tag{5.5}
$$

on the invariant manifold

$$M = \{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2\}.$$

Here a satisfies (5.2). The solutions of (5.5) on the set M are

$$
\begin{aligned}
x_1(t) &= \exp[-at]x_1(0),\\
x_2(t) &= \exp[-2at]x_2(0),\\
x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_3(0),\\
x_1(0)^2 &= x_2(0).
\end{aligned}
$$

Obviously, these have negative characteristic exponents.

Consider now the system of the first approximation in the neighborhood of the zero solution of system (5.5):

$$
\begin{aligned}
\dot{x}_1 &= -ax_1,\\
\dot{x}_2 &= -2ax_2,\\
\dot{x}_3 &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_3 + x_2.
\end{aligned}
\tag{5.6}
$$

Its solutions have the form

$$
\begin{aligned}
x_1(t) &= \exp[-at]x_1(0),\\
x_2(t) &= \exp[-2at]x_2(0),\\
x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]\Bigg(x_3(0) +\\
&\quad + x_2(0)\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]\,d\tau\Bigg).
\end{aligned}
\tag{5.7}
$$

Comparing (5.7) with (5.3) and applying (5.4), we find that for $x_2(0) \neq 0$ the relation

$$\limsup_{t\to+\infty} \frac{1}{t}\ln|x_3(t)| > 0$$

holds. It is easily shown that for the solutions of systems (5.5) and (5.6) we have

$$(x_1(t)^2 - x_2(t))^\bullet = -2a(x_1(t)^2 - x_2(t)).$$

Then

$$x_1(t)^2 - x_2(t) = \exp[-2at](x_1(0)^2 - x_2(0)).$$

It follows that M is a global attractor for the solutions of (5.5) and (5.6). This means that the relation $x_1(0)^2 = x_2(0)$ yields $x_1(t)^2 = x_2(t)$ for all $t \in \mathbb{R}^1$ and that for any initial data we have

$$|x_1(t)^2 - x_2(t)| \le \exp[-2at]|x_1(0)^2 - x_2(0)|.$$

Thus, systems (5.5) and (5.6) have the same global attractor M on which almost all the solutions of the system of the first approximation (5.6) have a positive characteristic exponent and all the solutions of original system (5.5) have negative characteristic exponents.

Here the Perron effect occurs on the two-dimensional manifold, namely

$$\{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2 \neq 0\}.$$

To construct the exponentially stable system for which the first approximation has a positive characteristic exponent, we change (5.5) to

$$\begin{aligned} &\dot{x}_1 = F(x_1, x_2), \\ &\dot{x}_2 = G(x_1, x_2), \\ &\dot{x}_3 = [\sin\ln(t+1) + \cos\ln(t+1)- \\ &-2a]x_3 + x_2 - x_1^3. \end{aligned} \tag{5.8}$$

Here the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have the form

$$\begin{aligned} F(x_1, x_2) &= \pm 2x_2 - ax_1, \\ G(x_1, x_2) &= \mp x_1 - \varphi(x_1, x_2), \end{aligned}$$

in which case the upper sign is taken for $x_1 > 0$, $x_2 > x_1^2$ and for $x_1 < 0$, $x_2 < x_1^2$, the lower one for $x_1 > 0$, $x_2 < x_1^2$ and for $x_1 < 0$, $x_2 > x_1^2$. The function $\varphi(x_1, x_2)$ is defined as

$$\varphi(x_1, x_2) = \begin{cases} 4ax_2, & |x_2| > 2x_1^2, \\ 2ax_2, & |x_2| < 2x_1^3. \end{cases}$$

The solutions of system (5.8) are credited to A.F. Filippov [19]. Then for the given functions F and G, on the lines of discontinuity $\{x_1 = 0\}$ and $\{x_2 = x_1^2\}$ the system

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2), \\ \dot{x}_2 &= G(x_1, x_2), \end{aligned} \tag{5.9}$$

has the sliding solutions, which are defined as

$$x_1(t) \equiv 0, \quad \dot{x}_2(t) = -4ax_2(t),$$

and

$$\begin{aligned} &\dot{x}_1(t) = -ax_1(t), \quad \dot{x}_2(t) = -2ax_2(t), \\ &x_2(t) \equiv x_1(t)^2. \end{aligned}$$

In this case the solutions of system (5.9) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$ attain the curve $\{x_2 = x_1^2\}$ in a finite time, which does not exceed 2π. The phase picture of such a system is shown in Figure 5.1.

From the above it follows that for the solutions of system (5.8) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$, $x_3(0) \in \mathbb{R}^1$ for $t \ge 2\pi$ we have the relations $F(x_1(t), x_2(t)) =$

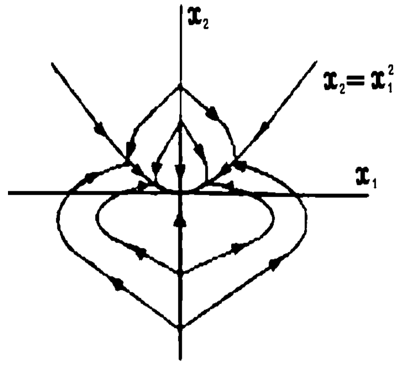

Figure 5.1:

$-ax_1(t)$, $G(x_1(t), x_2(t)) = -2ax_2(t)$. Therefore on these solutions, for $t \geq 2\pi$ system (5.6) is the system of the first approximation.

This system, as we have shown earlier, has a positive characteristic exponent. At the same time all the solutions of system (5.8) tend exponentially to zero. $\square$

The technique considered here permits us to construct the different classes of nonlinear systems for which Perron effects occur.

6 Stability Criteria by the First Approximation

We now describe the most famous stability criteria by the first approximation for the system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{6.1}$$

Here $A(t)$ is a continuous $n \times n$ matrix bounded for $t \geq 0$, and $f(t,x)$ is a continuous vector-function, satisfying in some neighborhood $\Omega(0)$ of the point $x = 0$ the condition

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0). \tag{6.2}$$

Here κ and ν are certain positive numbers, $\nu \geq 1$.

We refer to

$$\frac{dx}{dt} = A(t)x \tag{6.3}$$

as the *system of the first approximation*. Suppose that there exist $C > 0$ and a piecewise continuous function $p(t)$ such that Cauchy matrix $X(t)X(\tau)^{-1}$ of (6.3) satisfies

$$|X(t)X(\tau)^{-1}| \leq C \exp \int_\tau^t p(s)\, ds, \quad \forall t \geq \tau \geq 0.$$

Theorem 6.1. *If condition (6.2) with $\nu = 1$ and the inequality*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t p(s)\, ds + C\kappa < 0$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Theorem 6.1 shows that for the equation of the first order the negativity of the characteristic exponent of the system of the first approximation implies the asymptotic Lyapunov stability of the zero solution. (Here $\nu > 1$ or $\nu = 1$ and κ is sufficiently small.)

Let us now assume that $X(t)X(\tau)^{-1}$ satisfies

$$\begin{aligned} &|X(t)X(\tau)^{-1}| \le C\exp[-\alpha(t-\tau)+\gamma\tau], \\ &\forall t \ge \tau \ge 0, \end{aligned} \tag{6.4}$$

where $\alpha > 0$ and $\gamma \ge 0$.

Theorem 6.2 [30]. *If conditions (6.4) with $\gamma = 0$ and (6.2) with $\nu = 1$ and sufficiently small κ are valid, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.*

Theorem 6.2 results from Theorem 6.1 for $p(t) \equiv -\alpha$.

Theorem 6.3 [31, 32, 33]. *If conditions (6.4), (6.2), and the inequality*

$$(\nu - 1)\alpha - \gamma > 0 \tag{6.5}$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Consider a system

$$\frac{dx}{dt} = F(x,t), \quad t \ge 0, \quad x \in \mathbb{R}^n, \tag{6.6}$$

where $F(x,t)$ is a twice continuously differentiable vector-function. Suppose that for the solutions of system (6.6) with the initial data $y = x(0,y)$ from a certain domain Ω, the following condition is satisfied. The maximal singular value $\alpha_1(t,y)$ of the fundamental matrix $X(t,y)$ of the linear system

$$\frac{dz}{dt} = A(t)z \tag{6.7}$$

satisfies the inequality

$$\alpha_1(t,y) \le \alpha(t), \quad \forall t \ge 0, \quad \forall y \in \Omega.$$

Here

$$A(t) = \left.\frac{\partial F(x,t)}{\partial x}\right|_{x=x(t,y)}$$

is the Jacobian matrix of the vector-function $F(x,t)$ on the solution $x(t,y)$, $X(0,y) = I$.

Theorem 6.4 [34]. *Let the function $\alpha(t)$ be bounded on the interval $(0, +\infty)$. Then the solution $x(t,y)$, $y \in \Omega$, is Lyapunov stable. If, in addition, we have*

$$\lim_{t\to+\infty} \alpha(t) = 0,$$

then the solution $x(t,y)$, $y \in \Omega$, is asymptotically Lyapunov stable.

Consider now the hypotheses of Theorem 6.4. The theorem establishes the asymptotic Lyapunov stability of solutions with the initial data from Ω if the corresponding equations (6.7) have negative Lyapunov exponents (or negative characteristic exponents). In this case the requirement that the negativity of Lyapunov exponents is uniform by Ω replaces the requirement in Theorem 6.3 that the coefficient of irregularity is small.

Thus, the Perron effects, considered in Section 5, are possible on the boundaries of the flow stable by the first approximation only.

7 Instability Criteria

Consider a system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0,\infty)$. We assume that the vector-function $f(t,x)$ is continuous and in some neighborhood $\Omega(0)$ of the point $x = 0$ the inequality

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0) \tag{7.2}$$

holds. Here $\kappa > 0$ and $\nu > 1$.

Consider the normal fundamental matrix

$$Z(t) = (z_1(t), \ldots, z_n(t)), \tag{7.3}$$

consisting of the linearly independent solutions $z_j(t)$ of the following linear system of the first approximation:

$$\frac{dz}{dt} = A(t)z. \tag{7.4}$$

Theorem 7.1 [35]. *If*

$$\begin{aligned} &\sup_k \liminf_{t\to+\infty} \left[\frac{1}{t}\left(\int_0^t TrA(s)\ ds - \right.\right. \\ &\left.\left. - \sum_{j\neq k} \ln|z_j(t)|\ \right)\right] > 0, \end{aligned} \tag{7.5}$$

then the solution $x(t) \equiv 0$ of system (7.1) is Krasovsky unstable.

The condition for Krasovsky instability (7.5) was obtained by [31] under the additional requirement of the analyticity of $f(t,x)$.

Theorem 7.2 [4]. *Assume that for some numbers $C > 0$, $\beta > 0$, and $\alpha_j < \beta$ $(j = 1, \ldots, n-1)$ the following conditions are valid:*

1. *for $n > 2$*

$$\prod_{j=1}^n |z_j(t)| \leq C \exp \int_0^t TrA(s)\, ds, \ \ \forall t \geq 0.$$

2.

$$|z_j(t)| \leq C \exp(\alpha_j(t-\tau))|z_j(\tau)|,$$

$$\forall\, t \geq \tau \geq 0, \quad \forall j = 1, \ldots, n-1.$$

3.

$$\frac{1}{(t-\tau)} \int_\tau^t TrA(s)\, ds > \beta + \sum_{j=1}^{n-1} \alpha_j,$$

$$\forall\, t \geq \tau \geq 0.$$

Then the zero solution of system (7.1) is Lyapunov unstable.

Let us reconsider the ensemble of solutions $x(t, t_0, x_0)$ of the system

$$\frac{dx}{dt} = F(x,t), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.6}$$

where $F(x,t)$ is a continuously differentiable function. Here $x_0 \in \Omega$, where Ω is a certain bounded open set in $\mathbb{R}^n$, and t_0 is a certain fixed nonnegative number.

Assume that for the fundamental matrix $X(t,t_0,x_0)$ of the system

$$\frac{dz}{dt} = \left(\frac{\partial F(x,t)}{\partial x} \bigg|_{x=x(t,t_0,x_0)} \right) z$$

with the initial data $X(t_0,t_0,x_0) = I$ and a certain vector-function $\xi(t)$ the relations

$$|\xi(t)| = 1, \quad \inf_{\Omega} |X(t,t_0,x_0)\xi(t)| \geq \alpha(t), \quad \forall t \geq t_0$$

are valid.

Theorem 7.3 [4]. *Suppose that the function $\alpha(t)$ satisfies*

$$\limsup_{t \to +\infty} \alpha(t) = +\infty.$$

Then any solution $x(t,t_0,x_0)$ with the initial data $x_0 \in \Omega$ is Lyapunov unstable.

Conclusion

Let us summarize the investigations of stability by the first approximation, considered in Sections 5–7.

Theorems 6.4 and 7.3 give a complete solution to the problem for the flows of solutions in the noncritical case when for small variations of the initial data of the original system, the system of the first approximation preserves its stability (or the instability in the certain "direction" $\xi(t)$).

Thus, here the classical problem on the stability by the first approximation of time-varying motions is completely proved in the generic case [32].

The Perron effects, described in Section 5, are possible on the boundaries of flows that are either stable or unstable by the first approximation only. From this point of view here we have a nongeneric case.

Progress in the generic case became possible since the theorem on finite increments permits us to reduce the estimate of the difference between perturbed and unperturbed solutions to the analysis of the system of the first approximation, linearized along a certain "third" solution of the original system. Such an approach renders the proof of the theorem "almost obvious".

8 Zhukovsky Stability

Zhukovsky stability is simply the Lyapunov stability of reparametrized trajectories. To study it, we may apply the arsenal of methods and devices that were developed for the study of Lyapunov stability.

The reparametrization of trajectories permits us to introduce another tool for investigation, the *moving Poincaré section*. The classical Poincaré section is the transversal $(n-1)$-dimensional surface S in the phase space $\mathbb{R}^n$, which possesses a recurring property. The latter means that for the trajectory of a dynamical system $x(t,x_0)$ with the initial data $x_0 \in S$, there exists a time instant $t = T > 0$ such that $x(T,x_0) \in S$. The transversal property means that

$$n(x)^* f(x) \neq 0, \quad \forall x \in S.$$

Here $n(x)$ is a normal vector of the surface S at the point x, and $f(x)$ is the right-hand side of the differential equation

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{8.1}$$

generating a dynamical system.

We now "force" the Poincaré section to move along the trajectory $x(t, x_0)$. We assume further that the vector-function $f(x)$ is twice continuously differentiable and that the trajectory $x(t, x_0)$, whose the Zhukovsky stability (or instability) will be considered, is wholly situated in a certain bounded domain $\Omega \subset \mathbb{R}^n$ for $t \geq 0$. Suppose also that $f(x) \neq 0$, $\forall x \in \overline{\Omega}$. Here $\overline{\Omega}$ is a closure of the domain Ω. Under these assumptions there exist positive numbers δ and ε such that

$$f(y)^* f(x) \geq \delta, \quad \forall y \in S(x, \varepsilon), \quad \forall x \in \overline{\Omega}.$$

Here

$$S(x, \varepsilon) = \{y \mid (y - x)^* f(x) = 0, \quad |x - y| < \varepsilon\}.$$

Definition 8.1. The set $S(x(t, x_0), \varepsilon)$ is called a *moving Poincaré section.*

Note that for small ε it is natural to restrict oneself to the family of segments of the surfaces $S(x(t, x_0), \varepsilon)$ rather than arbitrary surfaces. From this point of view a more general consideration does not give new results. It is possible to consider the moving Poincaré section more generally by introducing the set

$$S(x, q(x), \varepsilon) = \{y \mid (y - x)^* q(x) = 0, \quad |x - y| < \varepsilon\},$$

where the vector-function $q(x)$ satisfies the condition $q(x)^* f(x) \neq 0$. Such a consideration can be found in [27]. We treat the most interesting and descriptive case $q(x) \equiv f(x)$.

The classical Poincaré section allows us to clarify the behavior of trajectories using the information at their discrete times of crossing the section. Reparametrization makes it possible to organize the motion of trajectories so that at time t all trajectories are situated on the same moving Poincaré section $S(x(t, x_0), \varepsilon)$:

$$x(\varphi(t), y_0) \in S(x(t, x_0), \varepsilon). \tag{8.2}$$

Here $\varphi(t)$ is a reparametrization of the trajectory $x(t, y_0)$, $y_0 \in S(x_0, \varepsilon)$. This consideration has, of course, a local property and is only possible for t satisfying

$$|x(\varphi(t), y_0) - x(t, x_0)| < \varepsilon. \tag{8.3}$$

Let as consider system of the first approximation

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))w \tag{8.4}$$

System (8.4) has the one null characteristic exponent λ_1. Denote by $\lambda_2 \geq \cdots \geq \lambda_n$ the remaining characteristic exponents, γ is coefficient of irregularity.

Theorem 8.1.[36] *If for system (8.4) the inequality*

$$\lambda_2 + \gamma < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable.

This result generalizes the well-known Andronov–Witt theorem.

Theorem 8.2 (Andronov, Witt). *If the trajectory $x(t, x_0)$ is periodic, differs from equilibria, and for system (8.4) the inequality*

$$\lambda_2 < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable (asymptotically Poincaré stable).

Theorem 8.2 is a corollary of Theorem 8.1 since system (8.4) with the periodic matrix

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

is regular.

Recall that for periodic trajectories, asymptotic stability in the senses of Zhukovsky and Poincaré are equivalent.

The theorem of Demidovich is also a corollary of Theorem 8.3.

Theorem 8.3 [37]. *If system (8.4) is regular (i.e. $\gamma = 0$) and $\lambda_2 < 0$, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable.*

9 Lyapunov Functions in the Estimates of Attractor Dimension

Harmonic oscillations are characterized by an amplitude, period, and frequency, and periodic oscillations by a period. Numerous investigations have shown that more complex oscillations have also numerical characteristics. These are the dimensions of attractors, corresponding to ensembles of such oscillations.

The theory of topological dimension [38, 39], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor.

For investigating attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value, and on such customary objects in Euclidean space as a smooth curve, a surface, or a countable set of points, it coincides with the topological dimension. Let us proceed to the definition of Hausdorff dimension.

Consider a compact metric set X with metric ρ, a subset $E \subset X$, and numbers $d \geq 0$, $\varepsilon > 0$. We cover E by balls of radius $r_j < \varepsilon$ and denote

$$\mu_H(E, d, \varepsilon) = \inf \sum_j r_j^d,$$

where the infimum is taken over all such ε-coverings E. It is obvious that $\mu_H(E, d, \varepsilon)$ does not decrease with decreasing ε. Therefore there exists the limit (perhaps infinite), namely

$$\mu_H(E, d) = \lim_{\varepsilon \to 0} \mu_H(E, d, \varepsilon).$$

Definition 9.1. The function $\mu_H(\cdot, d)$ is called the *Hausdorff d-measure.*

For fixed d, the function $\mu_H(E, d)$ possesses all properties of outer measure on X. For a fixed set E, the function $\mu_H(E, \cdot)$ has the following property. It is possible to find $d_{kp} \in [0, \infty]$ such that

$$\mu_H(E, d) = \infty, \quad \forall\, d < d_{kp},$$
$$\mu_H(E, d) = 0, \quad \forall\, d > d_{kp}.$$

If $X \subset \mathbb{R}^n$, then $d_{kp} \leq n$. Here $\mathbb{R}^n$ is an Euclidean n-dimensional space.

We put

$$\dim_H E = d_{kp} = \inf\{d \mid \mu_H(E, d) = 0\}.$$

Definition 9.2. We call $\dim_H E$ the *Hausdorff dimension* of the set E.

Example 9.1. Consider the Cantor set

$$E = \bigcap_{j=0}^{\infty} E_j,$$

where $E_0 = [0, 1]$ and E_j consists of 2^j segments of length 3^{-j}, obtained from the segments belonging to E_{j-1} by eliminating from them the open middle segments of length 3^{-j}. In the classical theory of topological dimension it is well known that $\dim_T E = 0$. From the definitions of Hausdorff dimension we deduce easily that $\mu_H(E, d) = 1$ for $d = \log 2/\log 3 = 0.63010\ldots$ and, therefore,

$$\dim_H E = \frac{\log 2}{\log 3}. \qquad \square$$

Topological dimension is invariant with respect to homeomorphisms. Hausdorff dimension is invariant with respect to diffeomorphisms, and noninteger Hausdorff dimension is not invariant with respect to homeomorphisms [38].

In studying the attractors of dynamical systems in phase space, the smooth change of coordinates is often used. Therefore, in such considerations it is sufficient to assume invariance with respect to diffeomorphisms.

It is well known that $\dim_T E \leq \dim_H E$. The Cantor set E shows that this inequality can be strict.

We give now two equivalent definitions of fractal dimension. Denote by $\mathcal{N}_\varepsilon(E)$ the minimal number of balls of radius ε needed to cover the set $E \subset X$. Consider the numbers $d \geq 0$, $\varepsilon > 0$ and put

$$\mu_F(E, d, \varepsilon) = \mathcal{N}_\varepsilon(E)\varepsilon^d,$$
$$\mu_F(E, d) = \limsup_{\varepsilon \to 0} \mu_F(E, d, \varepsilon).$$

Definition 9.3. The *fractal dimension* of the set E is the value

$$\dim_F E = \inf\{d \mid \mu_F(E, d) = 0\}.$$

Note that this definition is patterned after that for Hausdorff dimension. However in this case the covering is by the balls of the same radius ε only.

Definition 9.4. The fractal dimension of E is the value

$$\dim_F E = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}_\varepsilon(E)}{\log(1/\varepsilon)}.$$

It is easy to see that

$$\dim_H E \leq \dim_F E.$$

Example 9.2. For $X = [0, 1]$ and $E = \{0, 1, 2^{-1}, 3^{-1}, \ldots\}$ we have

$$\dim_H E = 0, \quad \dim_F E = \frac{1}{2}. \qquad \square$$

The extension of the scheme for introducing the Hausdorff and fractal measures and dimensions and the definitions of different metric dimensional characteristics can be found in [40]. It turns out [41]–[44] that the upper estimate of the Hausdorff and fractal dimension of invariant sets is the Lyapunov dimension, which will be defined below.

Consider the continuously differentiable map F of the open set $U \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Denote by T_xF the Jacobian matrix of the map F at the point x. The continuous differentiability of F gives

$$F(x+h) - F(x) = (T_xF)h + o(h).$$

We shall assume further that the set $K \subset U$ is invariant with respect to the transformation F: $F(K) = K$.

Consider the singular values of the $n \times n$ matrix A

$$\alpha_1(A) \geq \cdots \geq \alpha_n(A).$$

Recall that a singular value of A is a square root of an eigenvalue of the matrix A^*A. Here the asterisk denotes either transposition (in the real case) or Hermitian conjugation. Further we shall often write

$$\omega_d(A) = \alpha_1(A) \cdots \alpha_j(A)\alpha_{j+1}(A)^s,$$

where $d = j + s$, $s \in [0, 1]$, j is an integer from the interval $[1, n]$.

Definition 9.5. The *local Lyapunov dimension* of the map F at the point $x \in K$ is the number

$$\dim_L(F, x) = j + s,$$

where j is the largest integer from the interval $[1, n]$ such that

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF) \geq 1$$

and s is such that $s \in [0, 1]$ and

$$\alpha_1(T_xF) \cdots \alpha_j(T_xF)\alpha_{j+1}(T_xF)^s = 1.$$

By definition in the case $\alpha_1(T_xF) < 1$ we have $\dim_L(F, x) = 0$ and in the case

$$\alpha_1(T_xF) \cdots \alpha_n(T_xF) \geq 1$$

we have $\dim_L(F, x) = n$.

Definition 9.6. The *Lyapunov dimension* of the map F of the set K is the number

$$\dim_L(F, K) = \sup_K \dim_L(F, x).$$

Definition 9.7. The local Lyapunov dimension of the sequence of the maps F^i at the point $x \in K$ is the number

$$\dim_L x = \limsup_{i \to +\infty} \dim_L(F^i, x).$$

Definition 9.8. The Lyapunov dimension of the sequence of the maps F^i of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

For the maps F_t, depending on the parameter $t \in \mathbb{R}^1$, we can introduce the following analog of Definitions 9.7 and 9.8.

Definition 9.9. The local Lyapunov dimension of the map F_t at the point $x \in K$ is the number

$$\dim_L x = \limsup_{t \to +\infty} \dim_L(F_t, x).$$

Definition 9.10. The Lyapunov dimension of the map F_t of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

Again, the inequality[41]–[44] $\dim_F K \leq \dim_L K$ is an important property of Lyapunov dimension. Its proof can be found in [43, 44].

Thus, we have

$$\dim_T K \leq \dim_H K \leq \dim_F K \leq \dim_L K.$$

Note that the Lyapunov dimension can be used as the characteristic of the inner instability of the dynamical system, defined on the invariant set K and generated by the family of the maps F^i or F_t.

The Lyapunov dimension is not a dimensional characteristic in the classical sense. However, it does permit us to estimate from above a topological, Hausdorff, or fractal dimension. It is also the characteristic of instability of dynamical systems. Finally, it is well "adapted" for investigations by the methods of classical stability theory. We shall demonstrate this, introducing the Lyapunov functions in the estimate of Lyapunov dimension. The idea of introducing Lyapunov functions in the estimate of dimensional characteristics first appeared in [45], and was subsequently developed in [46]–[60]. Here we follow, in the main, these ideas.

Consider the $n \times n$ matrices $Q(x)$, depending on $x \in \mathbb{R}^n$. We assume that

$$\det Q(x) \neq 0, \quad \forall\, x \in U,$$

and that there exist c_1 and c_2 such that

$$\sup_K \omega_d(Q(x)) \leq c_1, \qquad \sup_K \omega_d(Q^{-1}(x)) \leq c_2.$$

Theorem 9.1. *Let $F(K) = K$ and suppose that for some matrix $Q(x)$*

$$\sup_K \omega_d\big(Q(F(x)) T_x F Q^{-1}(x)\big) < 1. \tag{9.1}$$

Then

$$\dim_L(F^i, K) \leq d \tag{9.2}$$

for sufficiently large natural numbers i.

Proof For the matrix $T_x F^i$ we have

$$T_x F^i = (T_{F^{i-1}(x)} F)(T_{F^{i-2}(x)} F) \cdots (T_x F).$$

This relation can be represented as

$$\begin{aligned}
&T_x F^i = Q(F^i(x))^{-1} \big(Q(F^i(x)) T_{F^{i-1}(x)} F Q(F^{i-1}(x))^{-1}\big) \cdot \\
&\cdot (Q(F^{i-1}(x)) T_{F^{i-2}(x)} F Q(F^{i-2}(x))^{-1}) \cdot \\
&\cdots (Q(F(x)) T_x F Q(x)^{-1}) Q(x).
\end{aligned}$$

From this and the well-known property [60]

$$\omega_d(AB) \le \omega_d(A)\omega_d(B)$$

we obtain

$$\omega_d(T_x F^i) \le c_1 c_2 \left[\sup_K \omega_d(Q(F(x))T_x F Q(x)^{-1} \right]^i .$$

This estimate, the condition (10.1) of the theorem, and the definitions of Lyapunov dimension imply the estimate (9.2).

Condition (9.1) is easily seen to be invariant with respect to the linear nonsingular change $y = Sx$, where S is a constant $n \times n$-matrix. It is clear that in the new basis condition (9.1) is also satisfied with the new matrix $Q_1(y)$:

$$Q_1(y) = Q(F(S^{-1}y))S.$$

Consider the important special case

$$Q(x) = p(x)S,$$

where S is a constant nondegenerate $n \times n$ matrix, $p(x)$ is the continuous function $\mathbb{R}^n \to \mathbb{R}^1$ for which

$$p_1 \le p(x) \le p_2, \quad \forall\, x \in K.$$

Here p_1 and p_2 are positive. In this case inequality (9.1) takes the form

$$\sup_K \omega_d \left(\frac{p(F(x))}{p(x)}\, S T_x F S^{-1} \right) < 1. \tag{9.3}$$

As will be shown below in condition (9.3) the multipliers of the type $p(F(x))/p(x)$ play the role of the Lyapunov type functions. This becomes especially clear in the case of the passage to the dynamical systems generated by differential equations.

Consider the system

$$\frac{dx}{dt} = f(t,x), \quad t \in \mathbb{R}^1,\ x \in \mathbb{R}^n, \tag{9.4}$$

where $f(t,x)$ is the continuously differentiable T-periodic vector-function $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t+T,x) = f(t,x)$. We assume that the solutions $x(t,x_0)$ of system (9.4) with the initial data $x(0,x_0) = x_0$ are defined on the interval $[0,T]$ and denote by G_T a shift operator along the solutions of system (9.4):

$$G_T q = x(T,q).$$

Suppose that the bounded set $K \subset \mathbb{R}^n$ is invariant with respect to the operator G_T, namely

$$G_T K = K.$$

Denote by $J(t,x)$ the Jacobian matrix of the vector-function $f(t,x)$:

$$J(t,x) = \frac{\partial f(t,x)}{\partial x}$$

and consider the nondegenerate $n \times n$ matrix S. Denote by $\lambda_1(t,x,S) \geq \cdots \geq \lambda_n(t,x,S)$ the eigenvalues of

$$\frac{1}{2}\left[SJ(t,x)S^{-1} + (SJ(t,x)S^{-1})^*\right].$$

Here the asterisk denotes transposition.

Theorem 9.2. *Suppose that for the integer $j \in [1,n]$ and $s \in [0,1]$ there exists a function $v(x)$, continuously differentiable on $\mathbb{R}^n$, and a nondegenerate $n \times n$ matrix S such that*

$$\sup_K \int_0^T [\lambda_1(t,x(t,q),S) + \ldots + \lambda_j(t,x(t,q),S) + \tag{9.5}$$
$$+ s\lambda_{j+1}(t,x(t,q),S) + \dot{v}(x(t,q))]\, dt < 0.$$

Then for sufficiently large i the inequality

$$\dim_L(G_T^i, K) \leq j + s. \tag{9.6}$$

holds.

Proof Denote the Jacobian matrix by

$$H(t,q) = \frac{\partial x(t,q)}{\partial q}.$$

Substituting $x(t,q)$ in (9.4) and differentiating both sides of (9.4) with respect to q, we obtain

$$\frac{dH(t,q)}{dt} = J(t,x(t,q))H(t,q).$$

Represent this relation as

$$\frac{d}{dt}[SH(t,q)S^{-1}] = [SJ(t,x(t,q))S^{-1}][SH(t,q)S^{-1}].$$

For the singular values $\sigma_1(t) \geq \cdots \geq \sigma_n(t)$ of the matrix $SH(t,q)S^{-1}$ we have the inequality [60]

$$\sigma_1 \cdots \sigma_k \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_k)\, d\tau\right)$$

for any $k = 1, \ldots, n$. From this and the relation

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s = (\sigma_1 \cdots \sigma_j)^{1-s}(\sigma_1 \cdots \sigma_{j+1})^s$$

we obtain the estimate

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1})\, d\tau\right). \tag{9.7}$$

Put

$$p(x) = \big(\exp v(x)\big)^{1/(j+s)}$$

and multiply both sides of (9.7) by the relation

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s} = \exp\big[v(x(t,q) - v(q)\big] = \exp\left(\int_0^t \dot{v}(x(\tau,q))\, d\tau\right).$$

As a result we obtain

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s}\sigma_1\dots\sigma_j\sigma_{j+1}^s$$
$$\le \exp\left(\int_0^t(\lambda_1+\dots+\lambda_j+s\lambda_{j+1}+\dot v(x(\tau,q))\,d\tau\right).$$

This implies the estimate

$$\begin{aligned}&\alpha_1(t,q)\dots\alpha_j(t,q)\alpha_{j+1}(t,q)^s\\&\quad\le \exp\left(\int_0^t\big(\lambda_1(\tau,x(\tau,q),S)+\dots+\lambda_j(\tau,x(\tau,q),S)\right.\\&\qquad\left.+\,s\lambda_{j+1}(\tau,x(\tau,q),S)+\dot v(x(\tau,q))\big)\,d\tau\right),\end{aligned}\tag{9.8}$$

where $\alpha_k(t,q)$ are the singular values of the matrix

$$\frac{p(x(t,q))}{p(q)}\,SH(t,q)S^{-1}.$$

From estimate (9.8) and condition (9.5) of Theorem 9.2 it follows that there exists $\varepsilon>0$ such that

$$\alpha_1(T,q)\cdots\alpha_j(T,q)\alpha_{j+1}(T,q)^s\le\exp(-\varepsilon)$$

for all $q\in K$. Thus, in this case condition (9.3) with $F=G_T$

$$T_qF=T_qG_T=H(T,q)$$

is satisfied and, therefore, estimate (9.6) is valid.

The following simple assertions will be useful in the sequel.

Lemma 9.1. *Suppose that the real matrix A can be reduced to the diagonal form*

$$SAS^{-1}=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix},$$

where S is a real nonsingular matrix. Then there exist positive numbers c_1 and c_2 such that

$$c_1|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i\le\omega_d(A^i)\le c_2|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i.$$

Proof It is sufficient to note that the singular values of the matrix

$$\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

are the numbers $|\lambda_j|$, and for the singular values $\alpha_1\ge\dots\ge\alpha_n$ the inequalities

$$\alpha_n(C)\alpha_j(B)\le\alpha_j(CB)\le\alpha_1(C)\alpha_j(B)$$

are satisfied.

Lemma 9.2. *Let $F(x) = x$ and the Jacobian matrix T_xF of the map F have the simple real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then the local Lyapunov dimension of the sequence of maps F^i at the point x is equal to $j+s$, where j and s are determined from*

$$|\lambda_1 \cdots \lambda_j \lambda_{j+1}^s| = 1.$$

Lemma 9.2 is a direct corollary of Lemma 9.1. A similar result holds for the map F_t.

Lemma 9.3. *Let $T_xF_t = e^{At}$ and the matrix A satisfy the condition of Lemma 9.1. Then the local Lyapunov dimension of the map F_t at the point x is equal to $j+s$, where j and s are determined from*

$$\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1} = 0.$$

Lemma 9.3 is also a corollary of Lemma 9.1.

Now we apply Theorems 9.1 and 9.2 to the Henon and Lorenz systems in order to construct Lyapunov functions $p(x)$ (for the Henon system) and $v(x)$ (for the Lorenz system). Consider the Henon map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} &x \to a + by - x^2, \\ &y \to x, \end{aligned} \tag{9.9}$$

where $a > 0$, $b \in (0,1)$ are the parameters of mapping. Consider the bounded invariant set K of map (9.9), $FK = K$, involving stationary points of this map:

$$\begin{aligned} x_+ &= \frac{1}{2}\left[b - 1 + \sqrt{(b-1)^2 + 4a}\,\right], \\ x_- &= \frac{1}{2}\left[b - 1 - \sqrt{(b-1)^2 + 4a}\,\right]. \end{aligned}$$

Theorem 9.3. *For the map F we have*

$$\dim_L K = 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)},$$

where

$$\alpha_1(x_-) = \sqrt{x_-^2 + b} - x_-.$$

Proof Denote $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$. The Jacobian matrix $T_\xi F$ of the map F takes the form

$$\begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}.$$

We introduce the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

In this case

$$ST_\xi FS^{-1} = \begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}. \tag{9.10}$$

We shall show that the singular values of (9.10) are

$$\begin{aligned} \alpha_1(x) &= \sqrt{x^2+b}+|x|, \\ \alpha_2(x) &= \sqrt{x^2+b}-|x| = \frac{b}{\alpha_1(x)}. \end{aligned} \tag{9.11}$$

It is obvious that

$$\begin{aligned} \alpha_1(x)^2 &= 2x^2+b+2|x|\sqrt{x^2+b}\,, \\ \alpha_2(x)^2 &= 2x^2+b-2|x|\sqrt{x^2+b}\,. \end{aligned}$$

It is clear that $\alpha_k(x)^2$ are zeros of the polynomial

$$\lambda^2-(4x^2+2b)\lambda+b^2,$$

which is the characteristic polynomial of the matrix

$$\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}.$$

Thus, formulas (9.11) are proved.

From Theorem 9.1 it follows that if there exist $s\in[0,1)$ and a continuously differentiable function $p(\xi)$, positive on K and such that

$$\sup_{\xi\in K}\alpha_1(x)\alpha_2(x)^s\left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s}<1, \tag{9.12}$$

then

$$\dim_L K \le 1+s.$$

Put

$$p(\xi)^{1+s}=e^{\gamma(1-s)(x+by)},$$

where γ is a positive parameter. It is not hard to prove that

$$\left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s}=e^{\gamma(1-s)(a+(b-1)x-x^2)}.$$

This implies that after taking the logarithm, condition (9.12) becomes

$$\begin{aligned} &\sup_K\big[\ln\alpha_1(x)+s\ln\alpha_2(x)+\gamma(1-s)(a+(b-1)x-x^2)\big] \\ &\quad=\sup_K\big[(1-s)\ln\alpha_1(x)+s\ln b+\gamma(1-s)(a+(b-1)x-x^2)\big]<0. \end{aligned}$$

This inequality is satisfied if

$$s\ln b+(1-s)\varphi(x)<0, \quad \forall\, x\in(-\infty,+\infty),$$

where

$$\varphi(x)=\ln\left[\sqrt{x^2+b}+|x|\right]+\gamma(a+(b-1)x-x^2).$$

The inequalities $\gamma>0$, $b-1<0$ result in the estimate

$$\varphi(-|x|)\ge\varphi(|x|).$$

Therefore it suffices to consider the extremum point of the functions $\varphi(x)$ for $x \in (-\infty, 0]$. It is clear that on this set we have

$$\varphi'(x) = \frac{-1}{\sqrt{x^2+b}} + \gamma\big[(b-1) - 2x\big], \quad \varphi''(x) < 0.$$

Letting

$$\gamma = \frac{1}{(b-1-2x_-)\sqrt{x_-^2+b}}\,,$$

we find that $\varphi'(x_-) = 0$ and therefore, for such a choice of γ,

$$\varphi(x) \le \ln\left(\sqrt{x_-^2+b} + |x_-|\right) = \ln \alpha_1(x_-).$$

Thus, inequality (9.12) holds for all s satisfying

$$s > \frac{\ln \alpha_1(x_-)}{\ln \alpha_1(x_-) - \ln b}. \tag{9.13}$$

Hence the estimate

$$\dim_L K \le 1 + s$$

is valid for all s satisfying (9.13). Passing to the limit, we obtain

$$\dim_L K \le 1 + \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}. \tag{9.14}$$

Note that the point $x = x_-$, $y = x_-$ is stationary for the map F. Then

$$\alpha_1(x_-)\alpha_2(x_-)^s = 1, \tag{9.15}$$

where

$$s = \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}.$$

It is easily shown that $\alpha_1(x_-)$ and $\alpha_2(x_-)$ are the eigenvalues of the Jacobian matrix $T_\xi F$ of the map F at the fixed point $y = x = x_-$:

$$T_\xi F = \begin{pmatrix} -2x_- & b \\ 1 & 0 \end{pmatrix}.$$

From relation (9.15) by Lemma 9.2 we conclude that the local Lyapunov dimension of the sequence of maps F^i at this stationary point is equal to

$$1 + \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}. \tag{9.16}$$

By inequality (9.14) we obtain the assertion of Theorem 9.3. □

Note that for $a = 1.4$, $b = 0.3$ from Theorem 9.3 we have

$$\dim_L K = 1.49532\ldots$$

Consider a Lorenz system

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned} \tag{9.17}$$

where r, b, σ are positive. Suppose that the inequalities $r > 1$,

$$\sigma + 1 \geq b \geq 2, \tag{9.18}$$

are valid. Consider the shift operator along the trajectory of system (9.17) G_T, where T is an arbitrary positive number. Let K be an invariant set with respect to this operator G_T. Suppose that K involves the stationary point $x = y = z = 0$. Such a set is represented in Fig. 3.1. We provide a formula for the Lyapunov dimension $\dim_L K$ of the set K with respect to the sequence of maps $(G_T)^i$.

Theorem 9.4. *Suppose the inequalities* (9.18) *and*

$$r\sigma^2(4-b) + 2\sigma(b-1)(2\sigma - 3b) > b(b-1)^2 \tag{9.19}$$

are valid. Then

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{9.20}$$

Proof The Jacobian matrix of the right-hand side of system (9.17) has the form

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

Introduce the matrix

$$S = \begin{pmatrix} -a^{-1} & 0 & 0 \\ -\sigma^{-1}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = \dfrac{\sigma}{\sqrt{r\sigma + (b-1)(\sigma - b)}}$. In this case we obtain

$$SJS^{-1} = \begin{pmatrix} b - \sigma - 1 & \sigma/a & 0 \\ \dfrac{\sigma}{a} - az & -b & -x \\ ay + \dfrac{a(b-1)}{\sigma}x & x & -b \end{pmatrix}.$$

Therefore the characteristic polynomial of the matrix

$$\frac{1}{2}((SJS^{-1})^* + (SJS^{-1})) =$$

$$\begin{pmatrix} b - \sigma - 1 & \dfrac{\sigma}{a} - \dfrac{az}{2} & \dfrac{1}{2}\left(ay + \dfrac{a(b-1)}{\sigma}x\right) \\ \dfrac{\sigma}{a} - \dfrac{az}{2} & -b & 0 \\ \dfrac{1}{2}\left(ay + \dfrac{a(b-1)}{\sigma}x\right) & 0 & -b \end{pmatrix}$$

takes the form

$$(\lambda+b)\left\{\left[\lambda^2+(\sigma+1)\lambda+b(\sigma+1-b)-\left(\frac{\sigma}{a}-\frac{az}{2}\right)^2\right]-\left[\frac{a(b-1)}{2\sigma}x+\frac{ay}{2}\right]^2\right\}.$$

This implies that eigenvalues of the matrix

$$\frac{1}{2}[(SJS^{-1})^*+(SJS^{-1})]$$

are the values

$$\lambda_2=-b,$$

and

$$\lambda_{1,3}=-\frac{\sigma+1}{2}\pm\frac{1}{2}\left[(\sigma+1-2b)^2+\left(\frac{2\sigma}{a}-az\right)^2 + \left(\frac{a(b-1)}{\sigma}x+ay\right)^2\right]^{1/2}.$$

From relations (9.18) it follows easily that $\lambda_1\ge\lambda_2\ge\lambda_3$.

Consider the Lyapunov type function

$$v(x,y,z)=\frac{1}{2}a\theta^2(1-s)\left(\gamma_1x^2+\gamma_2\left(y^2+z^2-\frac{(b-1)^2}{\sigma^2}x^2\right)+\gamma_3z\right),$$

where $s\in(0,1)$,

$$\theta^2=\left(2\sqrt{(\sigma+1-2b)^2+\left(\frac{2\sigma}{a}\right)^2}\right)^{-1},$$

$$\gamma_3=-\frac{4\sigma}{ab},\quad \gamma_2=\frac{a}{2},$$

$$\gamma_1=-\frac{1}{2\sigma}\left[2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma}+\gamma_3+2\frac{a(b-1)}{\sigma}\right].$$

Consider the relation

$$2\big[\lambda_1+\lambda_2+s\lambda_3+\dot v\big]=-(\sigma+1+2b)-s(\sigma+1)+(1-s)\varphi(x,y,z),$$

where

$$\begin{aligned}\varphi(x,y,z)=&\left((\sigma+1-2b)^2+\left(\frac{2\sigma}{a}-az\right)^2\right.\\&\left.+\left(\frac{a(b-1)}{\sigma}x+ay\right)^2\right)^{1/2}\\&+\theta^2\left\{\left(-2a\gamma_1\sigma+2\gamma_2\frac{a(b-1)^2}{\sigma}\right)x^2-2a\gamma_2y^2\right.\\&\left.-2a\gamma_2bz^2+a\left(2\sigma\gamma_1+2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma}+\gamma_3\right)xy-\gamma_3abz\right\}.\end{aligned}$$

By using the obvious inequality

$$\sqrt{u} \le \frac{1}{4\theta^2} + \theta^2 u,$$

we obtain the estimate

$$\begin{aligned}\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2 \Bigg\{ & (\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 \\ & + \left[-2a\gamma_1\sigma + 2\gamma_2\frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2}\right]x^2 \\ & + [a^2 - 2a\gamma_2]y^2 + [a^2 - 2\gamma_2 ab]z^2 \\ & + \left[a\left(2\sigma\gamma_1 + 2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma} + \gamma_3\right)\right. \\ & \left. + 2a^2\frac{b-1}{\sigma}\right]xy - [\gamma_3 ab + 4\sigma]z\Bigg\}.\end{aligned}$$

Note that the parameters γ_1, γ_2, and γ_3 are chosen in such a way that

$$\begin{aligned}\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2 \Bigg\{ & (\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 \\ & + \left[-2a\gamma_1\sigma + 2\gamma_2\frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2}\right]x^2\Bigg\}.\end{aligned}$$

It is not hard to prove that for the above parameters $\gamma_1, \gamma_2, \gamma_3$ under condition (9.19) we have

$$-2a\gamma_1\sigma + 2\gamma_2\frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2} \le 0.$$

Thus, for all x, y, z we have

$$\varphi(x,y,z) \le \sqrt{4r\sigma + (\sigma-1)^2}\ .$$

This implies that for any number

$$s < s_0 = \frac{\sqrt{4r\sigma+(\sigma-1)^2} - 2b - \sigma - 1}{\sqrt{4r\sigma+(\sigma-1)^2} + \sigma + 1}$$

there exists $\varepsilon > 0$ such that for all x, y, z the estimate

$$\lambda_1(x,y,z) + \lambda_2(x,y,z) + s\lambda_3(x,y,z) + \dot{v}(x,y,z) < -\varepsilon$$

is satisfied. Letting $s \to s_0$ on the right, by Theorem 9.2 we obtain

$$\dim_L K \le 3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}}. \tag{9.21}$$

By Lemma 9.3 we see that the local Lyapunov dimension of the stationary point $x = y = z = 0$ of system (9.17) is equal to

$$3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}}. \tag{9.22}$$

Relations (9.21) and (9.22) yield the formula (9.20).

By using a similar approach to the construction of the Lyapunov functions, we can obtain formulas for the Lyapunov dimension of the attractors of the dissipative Chirikov map [56].

10 Homoclinic Bifurcation

When the parameters of a dynamical system are varied, the structure of the minimal global attractor can vary as well. Such changes are the subject of bifurcation theory. Here we describe one particular phenomenon: the homoclinic bifurcation.

The first important results, concerning homoclinic bifurcations in dissipative dynamical systems, were obtained in 1933 by the outstanding Italian mathematician Franchesko Tricomi [15]. Here we give Tricomi's results and similar theorems for the Lorenz systems.

Consider the second-order differential equation

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = \gamma, \tag{10.1}$$

where α and γ are positive. This describes the motion of a pendulum with a constant moment of force, the operation of a synchronous electrical machine, and the phase-locked loop [61, 62]. For $\gamma < 1$ the equivalent system

$$\begin{aligned} &\dot{\theta} = z, \\ &\dot{z} = -\alpha z - \sin\theta + \gamma, \end{aligned} \tag{10.2}$$

has the saddle equilibria $z = 0,\ \theta = \theta_0 + 2k\pi$. Here θ_0 is a number for which $\sin\theta_0 = \gamma$ and $\cos\theta_0 < 0$.

Consider the trajectory $z(t), \theta(t)$ of (10.2) for which

$$\lim_{t\to+\infty} z(t) = 0, \qquad \lim_{t\to+\infty} \theta(t) = \theta_0, z(t) > 0, \qquad \forall\, t \geq T.$$

Here T is a certain number. In Fig. 1, such a trajectory is denoted by S. It is often called a separatrix of the saddle.

Fix $\gamma > 0$ and vary the parameter α. For $\alpha = 0$ the system (10.2) is integrable. It is easily shown that in this case, for the trajectory $S = \{z(t), \theta(t)\}$ there exists τ such that

$$\begin{aligned} &z(\tau) = 0, \quad \theta(\tau) \in (\theta_0 - 2\pi, \theta_0) \\ &z(t) > 0, \quad \forall\, t > \tau. \end{aligned} \tag{10.3}$$

Consider now the line segment $z = -K(\theta - \theta_0),\ \theta \in [\theta_0 - 2\pi, \theta_0]$. It is not hard to prove that on this segment for system (10.2) the relations

$$\begin{aligned} (z + K(\theta - \theta_0))^{\bullet} &= -\alpha z + Kz - \sin\theta + \gamma \\ &= (\theta - \theta_0)\left(-K(K - \alpha) + \frac{\gamma - \sin\theta}{\theta - \theta_0}\right) \end{aligned}$$

are valid. We make use of the obvious inequality

$$\left|\frac{\gamma - \sin\theta}{\theta - \theta_0}\right| \leq 1, \quad \forall\, \theta \neq \theta_0.$$

If the conditions

$$\alpha > 2, \qquad \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - 1} < K < \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - 1},$$

are satisfied, we obtain the estimate

$$(z + K(\theta - \theta_0))^{\bullet} < 0$$

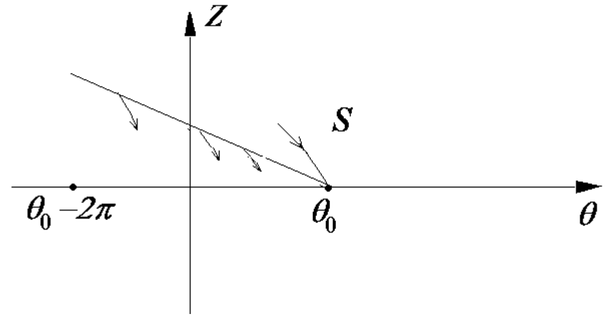

Figure 10.1: Estimate of separatrix.

for $z = -K(\theta - \theta_0),\ \theta \in (\theta_0 - 2\pi, \theta_0)$. See Fig. 10.1.

The figure shows that there does not exist τ such that conditions (10.3) are satisfied (Fig. 2).

It is well known that the piece of the trajectory $S : \{z(t), \theta(t) \mid t \geq \tau\}$ is continuously dependent on the parameter α. Here τ satisfies (10.3).

Then from the disposition of the trajectory S for $\alpha > 2$ (Fig. 10.2) it follows that there exists $\alpha_0 \in (0, 2)$ such that the trajectory S of system (10.2) with $\alpha = \alpha_0$ satisfies the relation

$$\lim_{t \to -\infty} z(t) = 0, \quad \lim_{t \to -\infty} \theta(t) = \theta_0 - 2\pi. \tag{10.4}$$

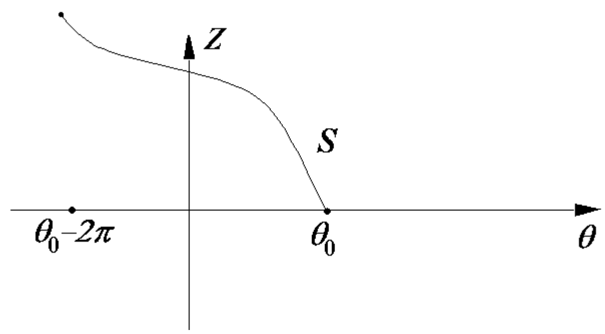

Figure 10.2: Behavior of separatrix.

Thus, $\alpha = \alpha_0$ is a bifurcational parameter. To this parameter corresponds the heteroclinic trajectory $S = \{z(t), \theta(t) \mid t \in \mathbb{R}^1\}$. Recall that the trajectory $x(t)$ of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{10.5}$$

is said to be *heteroclinic* if

$$\lim_{t \to +\infty} x(t) = c_1, \quad \lim_{t \to -\infty} x(t) = c_2, \quad c_1 \neq c_2.$$

In the case $c_1 = c_2$, the trajectory $x(t)$ is called *homoclinic*.

Sometimes for systems involving angular coordinates, the cylindrical phase space is introduced. We do this for system (10.2).

It is obvious that the properties of system (10.2) are invariant with respect to the shift $x + d$. Here

$$x = \begin{pmatrix} \theta \\ z \end{pmatrix}, \quad d = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}.$$

In other words, if $x(t)$ is a solution of system (10.2), then so is $x(t) + d$.

Consider a discrete group

$$\Gamma = \{x = kd \mid k \in Z\}.$$

We consider the factor group R^2/Γ, the elements of which are the classes of the residues $[x] \in R^2/\Gamma$. They are defined as

$$[x] = \{x + u \mid u \in \Gamma\}.$$

We introduce the so-called plane metric

$$\rho([x], [y]) = \inf_{\substack{u \in [x] \\ v \in [y]}} |u - v|.$$

Here, as above, $|\cdot|$ is a Euclidean norm in $\mathbb{R}^2$.

It is obvious that $[x(t)]$ is a solution and the metric space R^2/Γ is a phase space of system (10.2). This space is partitioned into the nonintersecting trajectories $[x(t)]$, $t \in \mathbb{R}^1$.

It is easy to establish the following diffeomorphism between R^2/Γ and a surface of the cylinder $R^1 \times C$. Here C is a circle of unit radius.

Consider the set $\Omega = \{x \mid \theta \in (0, 2\pi],\ z \in \mathbb{R}^1\}$, in which exactly one representer of each class $[x] \in \mathbb{R}^2/\Gamma$ is situated. Cover the surface of cylinder by the set Ω, winding Ω round this surface (Fig. 3)

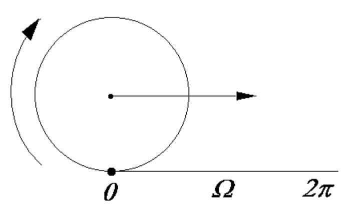

Figure 10.3: Cylindrical space.

It is obvious that the map constructed is a diffeomorphism. Therefore, the surface of the cylinder is also partitioned into nonintersecting trajectories. Such a phase space is called *cylindrical*.

Note that heteroclinic trajectory (10.4) in the phase space $\mathbb{R}^2$ becomes homoclinic in the cylindrical phase space and in the phase space R^2/Γ since we have

$$\lim_{t\to+\infty}[x(t)] = \lim_{t\to-\infty}[x(t)] = \begin{bmatrix} \theta_0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \theta_0 \\ 0 \end{bmatrix} = \left\{ \begin{pmatrix} \theta_0 + 2k\pi \\ 0 \end{pmatrix} \mid k \in Z \right\}.$$

Now the assertion obtained is formulated in the following way. Consider the smooth path $\alpha(s)$ $(s \in [0,1])$ such that $\alpha(0) = 0,\ \alpha(s) > 0, \forall\, s \in (0,1),\ \alpha(1) > 2$.

Theorem 10.1 (Tricomi). *For any $\gamma > 0$ there exists $s_0 \in (0,1)$ such that system (10.2) with the parameters $\gamma, \alpha(s_0)$ has a homoclinic trajectory in the phase space R^2/Γ.*

We proceed to obtain a similar assertion for the Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(x-y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{10.6}$$

where σ, b, r are positive. The function

$$V(x,y,z) = y^2 + z^2 + \frac{1}{\sigma}x^2$$

satisfies

$$\dot{V}(x(t),y(t),z(t)) = -2bz(t)^2 - 2y(t)^2 - 2x(t)^2 + 2(r+1)x(t)y(t).$$

From this we easily find that for $r \le 1$, all the solutions of system (10.6) tend to zero as $t \to +\infty$. Therefore we consider further the case $r > 1$.

Using the transformation

$$\begin{aligned} &\theta = \tfrac{\varepsilon x}{\sqrt{2\sigma}},\ \ \eta = \varepsilon^2\sqrt{2}(y-x),\ \ \xi = \varepsilon^2(z - \tfrac{x^2}{b}), \\ &t = t_1\tfrac{\sqrt{\sigma}}{\varepsilon},\ \ \varepsilon = \tfrac{1}{\sqrt{r-1}}, \end{aligned} \tag{10.7}$$

we reduce system (10.6) to the form

$$\begin{aligned} \dot{\theta} &= \eta, \\ \dot{\eta} &= -\mu\eta - \xi\theta - \varphi(\theta), \\ \dot{\xi} &= -\alpha\xi - \beta\theta\eta. \end{aligned} \tag{10.8}$$

Here

$$\varphi(\theta) = -\theta + \gamma\theta^3, \quad \mu = \frac{\varepsilon(\sigma+1)}{\sqrt{\sigma}}, \quad \alpha = \frac{\varepsilon b}{\sqrt{\sigma}}, \quad \beta = 2\left(\frac{2\sigma}{b} - 1\right), \quad \gamma = \frac{2\sigma}{b}.$$

It follows easily that if the conditions

$$\begin{aligned} &\lim_{t\to+\infty}\theta(t) = \lim_{t\to-\infty}\theta(t) = \lim_{t\to+\infty}\eta(t) = \\ &= \lim_{t\to-\infty}\eta(t) = \lim_{t\to+\infty}\xi(t) = \lim_{t\to-\infty}\xi(t) = 0 \end{aligned}$$

are satisfied, then

$$\lim_{t\to+\infty} x(t) = \lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} y(t) =$$
$$= \lim_{t\to-\infty} y(t) = \lim_{t\to+\infty} z(t) = \lim_{t\to-\infty} z(t) = 0.$$

Thus, a homoclinic trajectory of system (10.8) corresponds to a homoclinic trajectory of system (10.6).

Denote by $\theta^+(t), \eta(t)^+, \xi(t)^+$ a separatrix of the saddle $\theta = \eta = \xi = 0$, outgoing in the half-plane $\{\theta > 0\}$. See Fig. 4.

In other words, we consider a solution of system (10.6) such that

$$\lim_{t\to-\infty} \theta(t)^+ = \lim_{t\to-\infty} \eta(t)^+ = \lim_{t\to-\infty} \xi(t)^+ = 0$$

and $\theta(t)^+ > 0$ for $t \in (-\infty, T)$. Here T is a certain number or $+\infty$.

Consider the smooth path $b(s), \sigma(s), r(s)$ $(s \in [0,1])$ in a space of the parameters $\{b, \sigma, r\}$. It is clear that in this case the parameters $\alpha, \beta, \gamma, \mu$ are also smooth functions of $s \in [0,1]$.

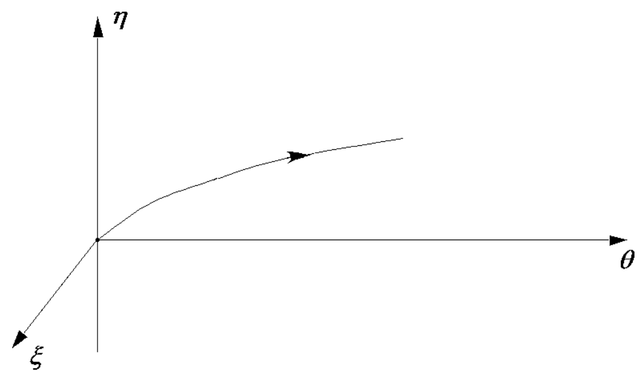

Figure 10.4: Separatrix of the Lorenz system.

Theorem 10.2. *Let $\beta(s) > 0$, $\forall s \in [0,1]$ and for $s \in [0, s_0)$ suppose there exist $T(s) > \tau(s)$ such that the relations*

$$\theta(T)^+ = \eta(\tau)^+ = 0 \tag{10.9}$$

$$\theta(t)^+ > 0, \ \forall t < T, \tag{10.10}$$

$$\eta(t)^+ \neq 0, \quad \forall t < T, \ t \neq \tau \tag{10.11}$$

are satisfied. Suppose also that for $s = s_0$ there does not exist the pair $T(s_0) > \tau(s_0)$ such that relations (10.9)–(10.11) are satisfied. Then system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$:

$$\lim_{t\to+\infty} \theta(t)^+ = \lim_{t\to+\infty} \eta(t)^+ = \lim_{t\to+\infty} \xi(t)^+ = 0.$$

To prove this theorem we need the following

Lemma 10.1. *If for system (10.8) the conditions*

$$\eta(\tau)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall\, t \in (-\infty, \tau)$$

are valid, then $\dot\eta(\tau)^+ < 0$.

Proof Suppose to the contrary that $\dot\eta(\tau)^+ = 0$. In this case from the last two equations of system (10.8) we obtain

$$\ddot\eta(\tau)^+ = \alpha\xi(\tau)^+\theta(\tau)^+. \tag{10.12}$$

From the relations $\eta(t)^+ > 0$, $\theta(t)^+ > 0$, $\forall\, t \in (-\infty, \tau)$ and the last equation of system (10.8) we obtain $\xi(t)^+ < 0$ $\forall\, t \in (-\infty, \tau]$. Then (10.12) yields the inequality $\ddot\eta(\tau)^+ < 0$, which contradicts the assumption $\dot\eta(\tau)^+ = 0$ and the hypotheses of the lemma.

Lemma 10.2. *Let* $\beta(s) > 0$, $\forall\, s \in [0,1]$. *Suppose that for system (10.8), relations (10.9), (10.10) and the inequalities*

$$\begin{array}{ll} \eta(t)^+ > 0, & \forall\, t \in (-\infty, \tau) \\ \eta(t)^+ \le 0, & \forall\, t \in (\tau, T) \end{array} \tag{10.13}$$

are valid. Then inequality (10.11) is also valid.

Proof Assuming the contrary, we see that there exists $\rho \in (\tau, T)$ such that the relations

$$\begin{array}{l} \eta(\rho)^+ = \dot\eta(\rho)^+ = 0, \\ \ddot\eta(\rho)^+ = \alpha\theta(\rho)^+\xi(\rho)^+ < 0, \\ \eta(t)^+ < 0, \quad \forall\, t \in (\rho, T), \end{array}$$

are satisfied. Then from conditions (10.9), (10.10) and from the fact that the trajectories $\theta(t) = \eta(t) = 0$, $\xi(t) = \xi(0)\exp(-\alpha t)$ belong to a stable manifold of the saddle $\theta = \eta = \xi = 0$ we obtain the crossing of the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ and this stable manifold. Therefore the separatrix belongs completely to the stable manifold of the saddle. In addition, the condition $\theta(t)^+ > 0$, $\forall\, t \ge \rho$ holds. The latter is in the contrast to condition (10.9). This contradiction proves Lemma 10.2.

It is possible to give the following geometric interpretation of the proof of Lemma 10.2 in the phase space with coordinates θ, η, ξ. "Under" the set $\{\theta > 0, \eta = 0, \xi \le 1 - \gamma\theta^2\}$ is situated the piece of stable two-dimensional manifold of the saddle $\theta = \eta = \xi = 0$. This does not allow the trajectories with the initial data from this set to attain the plane $\{\theta = 0\}$ if they remain in the quadrant $\{\theta \ge 0, \eta \le 0\}$.

Consider the polynomial

$$p^3 + ap^2 + bp + c, \tag{10.14}$$

where a, b, c are positive.

Lemma 10.3. *Either all zeros of (10.14) have negative real parts, or two zeros of (10.14) have nonzero imaginary parts.*

Proof It is well known [62] that all zeros of (10.14) have negative real parts if and only if $ab > 0$. For $ab = c$, polynomial (10.14) has two pure imaginary zeros.

Suppose now that for the certain a, b, c such that $ab < c$, polynomial (10.14) has real zeros only. From the positiveness of coefficients it follows that these zeros are negative. The latter yields $ab > c$, which contradicts the assumption.

Proof of Theorem 10.2. We shall show that to the values of the parameters $b(s_0), \sigma(s_0), r(s_0)$ there corresponds a homoclinic trajectory.

First note that for these parameters for the certain τ the relations

$$\begin{array}{l} \eta(t)^+ > 0, \quad \forall\, t < \tau, \quad \eta(t)^+ \le 0, \quad \forall\, t \ge \tau \\ \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, +\infty), \end{array} \tag{10.15}$$

hold. Actually, if there exist $T_2 > T_1 > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, T_2); \;\; \theta(T_2)^+ = 0, \;\; \eta(T_1)^+ > 0 \\ \eta(t)^+ > 0, \;\; \forall\, t < \tau; \;\; \eta(\tau)^+ = 0, \;\; \dot\eta(\tau)^+ < 0, \end{gathered}$$

then for the values $s < s_0$ and for the values s sufficiently close to s_0 the inequality $\eta(T_1)^+ > 0$ holds true. This is in the contrast to the definition of s_0. If there exist $T_1 > \tau$ such that

$$\begin{gathered} \eta(T_1)^+ > 0, \;\; \eta(t)^+ > 0, \;\; \forall\, t < \tau \\ \eta(\tau)^+ = 0, \;\; \dot\eta(\tau)^+ < 0, \;\; \theta(t)^+ > 0, \;\; \forall\, t \in (-\infty, +\infty) \end{gathered}$$

then for $s < s_0$, which is sufficiently closed to s_0, the inequality $\eta(T_1)^+ > 0$ holds true, which is in contrast to the definition of s_0. If there exist $T > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \; \forall\, t < T, \;\; \theta(T)^+ = 0, \;\; \eta(t)^+ > 0, \; \forall\, t < \tau \\ \eta(t)^+ \le 0, \quad \forall\, t \in [\tau, T], \end{gathered}$$

then by Lemma 10.2 inequality (10.11) holds. Therefore for $s = s_0$ relations (10.9)–(10.11) are valid, which is in the contrast to the hypotheses of the theorem. This contradiction proves inequality (10.15).

From (10.15) it follows that only one of the equilibria can be the ω-limit set of the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ for $s = s_0$. We shall show that the equilibrium $\theta = 1/\sqrt{\gamma}$, $\eta = \xi = 0$ cannot be the ω-limit point of the considered trajectory.

Having performed the linearization in the neighborhood of this equilibrium, we obtain the characteristic polynomial

$$p^3 + (\alpha + \mu)p^2 + (\alpha\mu + 2/\gamma)p + 2\alpha.$$

Suppose, for $s = s_0$ the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ has in its ω-limit set the point $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$. By Lemma 10.3 and from a continuous dependence of the semi-trajectories $\{\theta(t)^+, \eta(t)^+, \xi(t)^+ | \, t \in (-\infty, t_0)\}$ on the parameter s we obtain that for the values s sufficiently close to s_0, the separatrices $\theta(t)^+, \eta(t)^+, \xi(t)^+$ either tend to the equilibrium $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$ as $t \to +\infty$ or oscillate on the certain time interval with the sign reversal of the coordinate η. Both possibilities are in contrast to properties (10.9)–(10.11).

Thus, for system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ tends to zero equilibrium as $t \to +\infty$. □

Remark 10.1. It is well known that the semitrajectory

$$\{\theta(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$$

depends continuously on the parameter s. Here t_0 is a certain fixed number. Then Lemma 10.1 implies that if for system (10.8) with the parameters $b(s_1), \sigma(s_1), r(s_1)$ relations

(10.9)–(10.11) are satisfied, then these relations are also satisfied for $b(s), \sigma(s), r(s)$. Here $s \in (s_1 - \delta, s_1 + \delta)$ where δ is sufficiently small. □

Theorem 10.2 and Remark 10.1 result in the following

Theorem 10.3. *Let be $\beta(s) > 0$, $\forall s \in [0,1]$. Suppose, for system (10.8) with the parameters $b(0), \sigma(0).r(0)$ there exist $T > \tau$ such that relations (10.9)–(10.11) are valid. Suppose also that for system (10.8) with the parameters $b(1), \sigma(1), r(1)$ the inequality*

$$\theta(t)^+ > 0, \quad \forall t \in (-\infty, +\infty)$$

holds. Then there exists $s_0 \in [0,1]$ such that system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$.

We shall show that if

$$3\sigma - 2b > 1, \tag{10.16}$$

then for sufficiently large r the relations (10.9)–(10.11) are valid. Consider the system

$$\begin{aligned} Q\frac{dQ}{d\theta} &= -\mu Q - P\theta - \varphi(\theta), \\ Q\frac{dP}{d\theta} &= -\alpha P - \beta Q\theta, \end{aligned} \tag{10.17}$$

which is equivalent to (10.8) in the sets $\{\theta \geq 0, \eta > 0\}$ and $\{\theta \geq 0, \eta < 0\}$. Here P and Q are the solutions of system (10.17). It is clear that P and Q are functions of $\theta : P(\theta), Q(\theta)$.

We perform the asymptotic integration of the solutions of system (10.17) with the small parameter ε, which corresponds to the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$. For this purpose we transform (10.17) as follows:

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\theta}(Q(\theta))^2 = -\mu Q(\theta) - P(\theta)\theta - \varphi(\theta), \\ &\frac{dP(\theta)}{d\theta} = -\alpha\frac{P(\theta)}{Q(\theta)} - \beta\theta. \end{aligned}$$

Here α and μ are small parameters. In the first approximation the solutions considered can be represented in the form

$$\begin{aligned} &Q_1(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_0^\theta \theta\sqrt{1 - \tfrac{\theta^2}{2}}d\theta - \\ &-2\alpha\beta\int_0^\theta \theta\left(1 - \sqrt{1 - \tfrac{\theta^2}{2}}\right)d\theta, \\ &Q_1(\theta) \geq 0, \;\; P_1(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1 - \sqrt{1 - \tfrac{\theta^2}{2}}\right), \\ &Q_2(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_\theta^{\sqrt{2}} \theta\sqrt{1 - \tfrac{\theta^2}{2}}\,d\theta - \tfrac{4}{3}\mu + \\ &+2\alpha\beta\int_\theta^{\sqrt{2}} \theta\left(1 + \sqrt{1 - \tfrac{\theta^2}{2}}\right)d\theta - \tfrac{2}{3}\alpha\beta \\ &Q_2(\theta) \leq 0, \;\; P_2(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1 + \sqrt{1 - \tfrac{\theta^2}{2}}\right). \end{aligned}$$

This implies that if inequality (10.16) is satisfied, then for the certain $T > \tau$ relations (10.9)–(10.11) are valid. In addition we have

$$\begin{aligned} &\xi(T)^+ = P_2(0) = 2\alpha\beta, \\ &\eta(T)^+ = Q_2(0) = -\sqrt{8(\alpha\beta - \mu)/3} = -\sqrt{8\varepsilon(3\sigma - 2b - 1)/3\sqrt{\sigma}}. \end{aligned}$$

Thus, if inequality (10.16) is satisfied, then for sufficiently large r relations (10.9)–(10.11) are valid . $\square$

We now obtain conditions such that relations (10.9)–(10.11) do not hold and

$$\theta(t)^+ > 0, \ \forall\, t \in (-\infty, +\infty). \tag{10.18}$$

Consider first the case $\beta < 0$. Here for the function

$$V(\theta, \eta, \xi) = \eta^2 - \frac{1}{\beta}\xi^2 + \int_0^\theta \varphi(\theta) d\theta$$

we have

$$\dot V(\theta(t), \eta(t), \xi(t)) = -2\left(\mu\eta(t)^2 - \frac{\alpha}{\beta}\xi(t)^2\right). \tag{10.19}$$

Thus, for $\beta < 0$ the function V is the Lyapunov function for system (10.8). From the conditions (10.19) and $\beta < 0$ we obtain

$$V(\theta(t)^+, \eta(t)^+, \xi(t)^+) < V(\theta(-\infty)^+, \eta(-\infty)^+, \xi(-\infty)^+) = V(0,0,0) = 0, \\ \forall\, t \in (-\infty, +\infty).$$

This implies (10.18). In this case the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ does not tend to zero as $t \to +\infty$. For $\beta = 0$ we have $\xi(t)^+ \equiv 0$ and from the first two equations of system (10.8) we obtain at once (10.18). In this case the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ does not tend to zero as $t \to +\infty$. $\square$

Consider the case

$$\beta = \frac{2}{b}(2\sigma - b) > 0.$$

In this case by using the change of variables

$$\eta = \sigma(x - y), \quad Q = z - x^2/(2\sigma),$$

we can reduce system (10.6) to the form

$$\begin{aligned} &\dot x = \eta, \\ &\dot\eta = -(\sigma+1)\eta + \sigma\{(r-1) - Q - \tfrac{x^2}{2\sigma}\}x, \\ &\dot Q = -bQ + (1 - \tfrac{b}{2\sigma})x^2. \end{aligned} \tag{10.20}$$

Consider the separatrix $x(t)^+, \eta(t)^+, Q(t)^+$ of zero saddle equilibrium such that

$$\begin{aligned} &\lim_{t\to-\infty} x(t)^+ = \lim_{t\to-\infty} \eta(t)^+ = \lim_{t\to-\infty} Q(t)^+ = 0, \\ &x(t)^+ > 0, \quad \forall\, t \in (-\infty, T). \end{aligned} \tag{10.21}$$

Find the estimates of this separatrix.

Lemma 10.3. *The estimate*

$$Q(t)^+ \geq 0, \quad \forall\, t \in (-\infty, +\infty) \tag{10.22}$$

is valid.

Proof From the inequality $2\sigma > b$ and from the last equation of (10.20) we have

$$\dot Q(t) \geq -bQ(t).$$

This implies that

$$Q(t) \geq \exp(-bt)Q(0).$$

Therefore (10.22) holds.

Lemma 10.4. *From condition (10.21) follows the inequality*

$$\eta(t)^+ \leq Lx(t)^+, \quad \forall\, t \in (-\infty, T), \tag{10.23}$$

where

$$L = -\frac{\sigma+1}{2} + \sqrt{\frac{(\sigma+1)^2}{4} + \sigma(r-1)}.$$

Proof Relation (10.22) and the first two equations of system (10.20) give

$$\eta(t)^+ \leq \widetilde{\eta}(t)^+, \quad \forall\, t \in (-\infty, T). \tag{10.24}$$

Here $\widetilde{\eta}(t)^+, \widetilde{x}(t)^+$ is the separatrix of zero saddle of the system

$$\begin{aligned} &\dot{x} = \eta, \\ &\dot{\eta} = -(\sigma+1)\eta + \sigma(r-1)x. \end{aligned}$$

Obviously, $\widetilde{\eta}(t)^+ = L\widetilde{x}(t)^+$. The lemma follows from (10.24).

Lemma 10.5. *From condition (10.21) follows the estimate*

$$Q(t)^+ \geq a(x(t)^+)^2, \quad \forall\, t \in (-\infty, T), \tag{10.25}$$

where

$$a = \frac{(2\sigma - b)}{(2\sigma(2L+b))}.$$

Proof Estimate (10.23) gives the differential inequality

$$\begin{aligned} &\left(Q(t)^+ - a(x(t)^+)^2\right)^{\bullet} + b(Q(t)^+ - a(x(t)^+)^2) \geq \\ &\geq \left[(1 - \tfrac{b}{2\sigma}) - 2aL - ab\right](x(t)^+)^2 = 0. \end{aligned}$$

This implies (10.25).

Consider now the Lyapunov-type function introduced in [63]:

$$V(x, \eta, Q) = \eta^2 + \sigma x^2 \left(\frac{x^2}{4\sigma} + Q - (r-1)\right) + (\sigma+1)x\eta. \tag{10.26}$$

It can easily be checked that for the solutions $x(t), \eta(t), Q(t)$ of system (10.20) we have

$$\begin{aligned} &\dot{V}(x(t), \eta(t), Q(t)) = -(\sigma+1)V(x(t), \eta(t), Q(t)) + \\ &+ \tfrac{3}{4}\left(\sigma - \tfrac{2b+1}{3}\right)x(t)^4 - b\sigma Q(t)x(t)^2. \end{aligned} \tag{10.27}$$

Lemma 10.6. *Let the inequality*

$$3\sigma - (2b+1) < \frac{2b(2\sigma - b)}{2L+b} \tag{10.28}$$

be valid. Then condition (10.21) results in the estimate

$$\begin{aligned}&\dot V(x(t)^+,\eta(t)^+,Q(t)^+)+(\sigma+1)V(x(t)^+,\eta(t)^+,Q(t)^+)<0,\\&\forall\, t\in(-\infty,T).\end{aligned}\tag{10.29}$$

Proof From (10.28) and (10.25) we have

$$\frac{3}{4}\left(\sigma+\frac{2b+1}{3}\right)(x(t)^+)^4-b\sigma Q(t)(x(t)^+)^2<0,\quad \forall\, t\in(-\infty,T).$$

Then (10.27) yields estimate (10.29). Note that relation (10.29) results in the inequality

$$V(x(T)^+,\eta(T)^+,Q(T)^+)<0.$$

It is easy to see that

$$V(0,\eta,Q)\geq 0,\quad \forall\eta\in R^1,\ \forall Q\in R^1.$$

Therefore, if (10.28) is satisfied, then (10.29) is satisfied for all $T\in\mathbb{R}^1$.

Thus, we can formulate

Theorem 10.4 [63]. *If inequality (10.28) holds, then so does (10.18) and the separatrix $x(t)^+,\eta(t)^+,Q(t)^+$ does not tend to zero as $t\to+\infty$.*

This implies

Theorem 10.5. *If*

$$2b+1\geq 3\sigma,$$

then for any $r>1$ the homoclinic trajectory of system (10.6) does not exist.

Theorem 10.6. *If*

$$2b+1<3\sigma,$$

then for the values $r>1$ and sufficiently close to 1 the conditions (10.9)–(10.11) are not valid.

Theorems 10.3, 10.5, 10.6 imply the following

Theorem 10.7. *Given b and σ fixed, for the existence of $r\in(1,+\infty)$, corresponding to the homoclinic trajectory of the saddle $x=y=z=0$, it is necessary and sufficient that*

$$2b+1<3\sigma.\tag{10.30}$$

The sufficiency of condition (10.30) was first proved in [64, 65]. It was proved by another method (the shooting method [66]–[68]) in [69]. The papers [68, 69] involve the notes, added in the proof, about a priority of the assertion from [64].

In the papers [64, 65] the conjecture was asserted that (10.30) is a necessary condition for the existence of a homoclinic trajectory. This conjecture is proved in [69] on the basis of constructing the Lyapunov-type function (10.26).

We remark that the consideration of the smooth paths, in the space of parameters of nonlinear dynamic systems, on which there exist the points of homoclinic bifurcation, is a fruitful direction in the development of the analytic theory of global bifurcations.

We formulate now one more assertion of the same type, obtained for the Lorenz system in the paper [55].

Theorem. *Let be $\sigma=10, r=28$. Then there exists $b\in(0,+\infty)$ such that (10.6) has a homoclinic trajectory.*

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