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PERSONAGE IN SCIENCE

Professor Myron K. Grammatikopoulos

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We dedicate this article to the memory of our friend and colleague Professor M. K. Grammatikopoulos, in recognition of his outstanding contributions to mathematics. Myron's research and teaching contributions spanned qualitative theory of ordinary, functional and partial differential equations with deviating arguments.

M.K. GRAMMATIKOPOULOS was born on November 14, 1938 (after World War II his official documents passport gives his birthday as November 8, 1935) in the village of Grebeshok of the Gagra District of Abkhasian ASSR of the Georgian SSSR. He was a son of the farmhand Kyriakos Grammatikopoulos, a Greek citizen who had fled his native Pontus sometime between 1918 and 1923.

In 1946, Myron entered elementary school in the city of Gudauta in Abkhasia, and in 1949, his family, as well as all refugees from Pontus, were forcibly relocated in Central Asia. There, under strict police surveillance, he continued his education in the village of Tamerlanovka in the Arys' area of the District of Chimkent in Southern Kazakhstan.

In 1956, Myron graduated from the 10-grade Amangeldy Imanov High School with a Silver medal. He received a scholarship and matriculated in the Physics Department of the N. Krupskaya Pedagogical Institute of Chimkent. In 1959, the Administration of this Institution discovered that due to his Greek citizenship, he was barred from entering higher education, and he was expelled. As a result of his protest letters that he addressed to the Greek Embassy in Moscow and the Ministers of Education and Exterior Affairs of the former USSR, he was allowed to continue his studies only in the Literature Department or the Mathematics Department of Chimkent Pedagogical Institute. Myron opted for Mathematics, and after being successfully examined in seventeen mathematics courses in forty days (these courses were not included in the first three years of the Physics Department's curriculum), he continued his studies in mathematics graduating finally "with distinction" in 1961. Despite the fact that he had no Graduate Studies, Mr. Jangildin, Chairman of the State Committee and Undersecretary of Education of Kazakhstan, made the unusual proposal for Myron to stay on at the Institute as an Assistant under the condition that he relinquish his Greek citizenship and become a Soviet citizen. He refused the offer and accepted a position as a teacher of Physics, Mathematics and Design at his *alma mater*, the Amangeldy Imanov High School of Tamerlanovka. In 1966, after working there for six years, Myron left and took up permanent residence in Greece.

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Myron received his Diploma in Mathematics from the Pedagogical Institute of Chimkent (a branch of the University of Kazakhstan) in 1961, and the University Thessaloniki in 1967. The Ph.D. in Mathematics (first dissertation) was awarded from the University of Ioannina 1975 and the Docent in Mathematics (second dissertation) from the University of Ioannina in 1981. In 1986, Myron was made a full Professor of Mathematics at the University of Ioannina.

In 1978, Myron was awarded a NATO Research Grant and he contacted J. R. Graef at Mississippi State University (MSU) to arrange a three month visit. In the afternoon of Wednesday, November 1, of that year, Myron arrived at MSU - small in stature, big in heart, and giant in intellect. The collaboration with Graef and P. W. Spikes at Mississippi State was fruitful and lasted long after that initial visit. Shortly before he left Ioannina for Mississippi, Myron's last son, Petros, was born. Unfortunately, the child had a serious birth defect and eventually only lived to be ten years old. Over the Christmas holidays in 1978, Myron returned to Greece only to immediately bring his son to New York in hopes of learning of some new medical technique that might have reversed the inevitable. But it was not to be. Because Myron essentially spent a month of his scheduled time in the US trying to help his son, he requested and immediately received a month extension to his visit.

Myron returned to Greece on April 15, 1979. In those four months, results for several different papers were outlined, some with more details than others. The collaboration continued over the next several years, and in 1985, Myron again visited MSU this time for only one week. The collaboration continued with the last of the more than twenty joint publications finally appearing in 1993.

Myron also held a Visiting Assistant Professor position in the Department of Mathematics at the University of Rhode Island in 1984-1985, and he was a Visiting Professor at the Center of Mathematics of the University of Rousse, Bulgaria, in 1990, 1991, and 1996.

Myron was the author or coauthor of more than 70 research papers. Monographs in which some of his research is described will be listed below. He received Distinguished Teaching Awards, Silver Medal in Secondary School USSR in 1956, and the Award in Higher Education USSR in 1961. He was awarded a Doctor Honoris Causa from the University of Rousse, Bulgaria, in 1995.

We will now give a brief outline of his results in the area of oscillation theory of ordinary and functional differential equations with deviating arguments, that is to say, equations of the form

$$x^{(n)}(t) = F(t; x^{(m_1)}(t-\tau_1), \dots, x^{(m_s)}(t-\tau_s)), \quad t \geq t_0, \tag{1}$$

where n and m_i are nonnegative integers, $\tau_i \in R$, $i = 1, 2, \dots, s$, and x, F may be vectors. Set $m = \max\{m_i\colon\ i = 1, 2, \dots, s\}$. Then equation (1) is said to be of the:

— retarded (delay) type, if $m < n$;

— advanced type, if $m > n$, and

— neutral type, if $m = n$.

Moreover, equation (1) has:

— retarded (delay) arguments, if $\tau_i \geq 0$, $i = 1, 2, \dots, s$;

— advanced arguments, if $\tau_i \leq 0$, $i = 1, 2, \dots, s$, and

— mixed arguments, if there is $s_1 \in \{1, 2, \dots, s-1\}$ such that $\tau_i \geq 0$ for $i = 1, 2, \dots, s_1$ and $\tau_i \leq 0$ for $i = s_1 + 1, s_1 + 2, \dots, s$.

The deviating arguments may depend on time t or may even depend on the solution of the equation. The type of deviation of the arguments may change depending on time or on the solution itself. It should be pointed out that deviating arguments, in some cases, do not affect the oscillatory character of the solutions of differential equations under consideration, while in some other cases, they can either generate oscillations or stop them. Consequently, it is interesting to investigate phenomena of this kind in order to choose the appropriate mathematical model of real world systems whose oscillatory character depends on differential equations with deviating arguments.

In a number of papers, Myron studied retarded type differential equations, that is, equations in which the unknown function depends on a continuous function r defined on an interval $[t_r, \infty)$. The presence of the function r is justified by the fact that these equations constitute generalizations of the well known Emden-Fowler and Thomas-Fermi equations that often arise in applications and contain functions of the type r. Such an occurrence of the function r, for example, could be a cause for the existence of oscillatory behavior of the solutions of the differential equations under consideration. The results obtained are interesting not only from the theoretical aspect, but also from the point of view of applications. Indeed, the role of these equations, for example, in relativistic electrodynamics and other natural sciences, is very important.

Another important topic is the problem of oscillatory and asymptotic behavior of the solutions of neutral differential equations with deviating arguments. This problem is interesting both from the theoretical and practical aspect. In fact, neutral differential equations have applications in electric networks containing lossless transmission lines. Such networks arise, for example, in high speed computers, where the lossless transmission lines are used to interconnect switching circuits. Second order neutral differential equations appear in the study of vibrating masses attached to an elastic bar, and they also appear as the Euler equations for the minimization of functionals involving a time delay.

In general, the study of neutral differential equations presents complications that are not present in non-neutral type equations. Indeed, it has been proved that even though the characteristic roots of a neutral differential equation may all have negative real parts, it is still possible for this equation to have unbounded solutions. Furthermore, the oscillatory character of the solutions of a neutral differential equation is determined by the roots of the corresponding characteristic equation, which is in contrast to the fact that the stability character is not determined by those characteristic roots.

Myron developed techniques and methods that have been adopted by a number of other researchers in this area. Also, it should be pointed out that results obtained for neutral differential equations with constant coefficients and constant deviations are crucial with respect to drawing conclusions concerning neutral differential equations of the same form where the coefficients and deviations are not constants but are functions of time. For this reason, the large number of important references to his work is not a surprise. Beyond the above areas, Professor Grammatikopoulos was interested in applications of partial differential equations. The results he obtained in this direction concern boundary value problems for some special types of partial differential equations (for example, wave equations); he was interested in the problem of existence and uniqueness of solutions of this type of equation and the possibility to treat practical problems appearing in technology, etc.

As a result of an analysis of Myron's research work, we see that some of the main topics treated are the following:

— Establishing criteria (necessary and sufficient conditions) for oscillation (non-oscillations) of all solutions of differential equations under consideration.

— Establishing criteria for existence of oscillatory (non-oscillatory) solutions of differential equations with some asymptotic property.

— Obtaining sufficient conditions for oscillation (non-oscillation) of all solutions of equations in question.

— Finding the relation between oscillation and other qualitative properties such as boundedness, convergence to zero, etc.

— Investigating the oscillatory and asymptotic properties of the non-oscillatory solutions of differential equations with forcing or discontinuous terms.

— Investigating the oscillatory phenomena caused by deviating arguments.

— Classification of all solutions of differential equations under consideration with respect to their behavior at infinity.

— Other topics: boundary value problems for partial differential equations, etc.

Myron made many significant contributions in all of these areas.

MONOGRAPHS in which of M.K. Grammatikopoulos research work is cited:

[1] Koplatadze, R.G. and Chanturia, T.A. *On The Oscillatory Properties of Differential Equations with a Deviating Argument.* State University of Tbilisi, Tbilisi, 1977. [Russian].

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Besides doing an incredible amount of mathematical work, Myron took time to enjoy being with his family, his wife Alla, his children (Andreas, Dimitrios, Kyriakos), and his grandchildren (Eugenia, Ioanna-Hypatia, Philarete). He also enjoyed classical ballet and history. In particular, he was a connoisseur of dramatic and humorous facts in Greek and World History. He was like his garden on Corfu, where according to description of "the island of the Phaeacians" in the Odyssey, Odysseus' last stop before arriving in his beloved Ithaca. Myron was always available for discussions concerning teaching or research problems with anyone who wished to meet with him.

The mathematics community is saddened to announce the unexpected passing of Myron Grammatikopoulos in Bulgaria on June 21, 2007. He was interred at Kastanoussa, Serres.

Sto kalo — we will miss you our friend.

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Synchronization of Different Hyperchaotic Maps for Encryption

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Abstract: In this paper, the synchronization problem of different hyperchaotic maps is presented. In particular, we appeal to model-matching approach from nonlinear control theory to synchronize the otputs of the coupled Rössler and Hénon hyperchaotic maps. An application to secure communication of confidential information is also given. By using a hyperchaotic encryption scheme, we show that output synchronization of different hyperchaotic maps is indeed suitable for encryption, transmission, and decryption of confidential information which can be implemented for use in computer communication.

Keywords: *Synchronization; hyperchaotic maps; model-matching problem; secure communication.*

Mathematics Subject Classification (2000): 37N35, 65P20, 68P25, 70K99, 93D20, 94A99.

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1 Introduction

In modern communication systems, data secury is a requirement of central importance. As a result, with the rapid development of the different communication systems, there exits a great demand of cryptography algorithms to protect the confidential information, see e.g. [1, 2]. In particular, nowadays most communication is through computers and even real-time communication systems are digital. Recently, by using chaotic dynamics to address the secure communication problem has received a great interest. In several articles is reported the extreme relationship between chaotic dynamics and conventional cryptography, some common properties are:

- a small variation in the input originates a large change at the output,
- the output preserves the same distribution for any input,
- a small variation in the local area originates a large variation in the whole space,
- a simple process has a very high complexity, and
- a deterministic system originates a pseudorandom dynamics.

During last decades, the problem of chaos synchronization has received a lot attention, see e.g. [3, 4, 5, 6, 7, 8, 9, 37] and references therein. This interest increases by practical reasons, mainly to design secure communication systems. Chaos synchronization can be used in different ways for encryption of confidential information in secure communication systems, see e.g. [7, 11, 12, 13, 14, 15, 26, 28, 29, 33, 35, 36, 37]. However, in subsequent works, see e.g. [16, 17] it has been shown that encrypted information by means of comparatively "simple" chaos with only one positive Lyapunov exponent, does not ensure a sufficient security level. For higher security purpose, *hyperchaotic dynamics* characterized by more than one positive Lyapunov exponents are advantageous over simple chaotic dynamics.

On the basis of these considerations, one way to enhance the level of encryption security is by applying conventional *cryptographic techniques* to the information in *combination with chaotic encryption schemes* [18, 19]. Another way is to encode information by using systems generated of *high dimensional chaotic attractors,* or *hyperchaotic attractors*. In this case, one generally encounters *multiple positive Lyapunov exponents*. However, hyperchaos synchronization is a much more difficult problem, see e.g. [9, 21, 22, 23, 24, 25, 27, 34, 37]. The level of security is also enhanced by using *chaos modeled by delay differential equations*, such systems have an infinite-dimensional state space, and produce hyperchaotic dynamics with an *arbitrarily large number of positive Lyapunov exponents* [26, 27, 28, 29].

The aim of this paper is to present a communication scheme to transmit encrypted audio and image information, which is based on synchronized different hyperchaotic discrete-time systems; in particular, we use the generalized Rössler and Hénon maps. This objective is achieved by appealing to nonlinear control theory, in particular, we use the model-matching approach given in [37]. We enumerate several advantages over the existing synchronization methods reported in the current literature:

- It enables synchronization be achieved in a systematic way and clarifies the issue of deciding on the nature of the coupling signal to be transmitted.
- It can be successfully applied to many chaotic and hyperchaotic systems (in continuous-time, or discrete-time).
- It can be applied to identical and nonidentical systems in continuous-time [7] and in discrete-time [37].
- It does not require the computation of any Lyapunov exponent.
- It does not require initial conditions belonging to the same basin of attraction.

In addition, we use output synchronization for encoding, transmission, and decoding of confidential information.

The organization of the sections of this paper is as follows: In Section 2, the proposed hyperchaotic encryption scheme is described. In Section 3, a review on output synchronization of hyperchaotic maps via model matching is provided. By using computer simulations, the approach used is explained by means of the hyperchaotic generalized Rössler and Hénon maps in Section 4. An application of output synchronization to secure communication systems is illustrated in Section 5. The paper is concluded with some remarks in Section 6.

2 Hyperchaotic Encryption Scheme

In this section, a cryptosystem based on synchronized hyperchaotic (three-dimensional) maps is described. The aim is to transmit encrypted information from side A to side B (the so-called *authorized* communicating remote parts) as is illustrated in Figure 2.1. A confidential *information* m is to be transmitted over an *insecure* communication channel. To avoid any *unauthorized* part (intruder) located at the mentioned channel; m is encrypted prior to transmission to generate an *encrypted* information s,

$$s = f(m, k),$$

by using hyperchaotic dynamics generated by the map f on side A.

The encrypted information s is sent to remote side B, where m is *recovered* as $\hat{m}$ from the hyperchaotic decryption. g, as

$$\hat{m} = g(s, k).$$

If f and g have used the same *key* k, then at remote side B it is possible to obtain the recovered information $\hat{m} = m$. A *secure* channel (dashed line) is used for transmission of the keys. Generally, this secure communication channel is a courier and is too slow for the transmission of the confidential information m. Our hyperchaotic cryptosystem is reliable, if it preserves the security of m, i.e. if $m\prime \neq m$ for even the best *cryptanalytic* function h, given by

$$m\prime = h(s).$$

To achieve the proposed hyperchaotic encryption scheme, we appeal to three-dimensional hyperchaotic *generalized Rössler and Hénon maps* for encryption/decryption

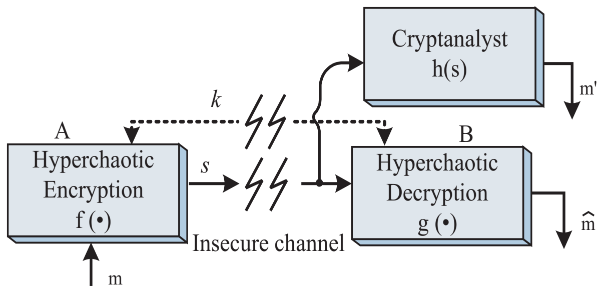

Figure 2.1: Secure hyperchaotic cryptosystem.

purposes (f and g, respectively), as will be shown in Section 5. The hyperchaotic Rössler and Hénon maps have a number of parameters determining their dynamics; such parameters and initial conditions are the coding "keys", k. We expect that it can perform the objective of the secure communication and the transmitting information can be recovered at the receiver. In order to guarantee the encryption and decryption, the generalized hyperchaotic Rössler and Hénon maps have to achieve the so-called *synchronization* on both remote sides A and B. For such reason, our first problem to solve is to design a control law u for hyperchaotic synchronization, which will be shown in next sections.

3 Output Synchronization of Different Hyperchaotic Maps

Consider a nonlinear discrete-time system, defined by

$$P : \begin{cases} x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k)), \end{cases} \tag{1}$$

where the state vector $x \in X$ (an open set in $\mathbb{R}^n$), the input u is inside an open set U in $\mathbb{R}$, and the output y belongs to an open set Y in $\mathbb{R}$. The mappings $f : X \times U \to X$ and $h : X \to Y$ are analytic. In addition, consider the following nonlinear discrete-time system, described by

$$M : \begin{cases} x_M(k+1) &= f_M(x_M(k), u_M(k)), \\ y_M(k) &= h_M(x_M(k)), \end{cases} \tag{2}$$

where the state vector $x_M \in X_M$ (an open set in $\mathbb{R}^{n_M}$), the input $u_M \in U_M$ (an open set in $\mathbb{R}$), and the output y_M belongs to an open set Y_M in $\mathbb{R}$. Also, the mappings $f_M : X_M \times U_M \to X_M$ and $h_M : X_M \to Y_M$ are analytic. Assume that for certain parameter values, the uncontrolled discrete-time dynamical systems (1) and (2), i.e. for $u(k) = u_M(k) = 0$, exhibit *hyperchaotic behavior*; that is, the dynamical systems have *multiple positive Lyapunov exponents*. The synchronization problem addressed here is defined as follows.

Definition 3.1 (Output Synchronization Problem, OSP) [30] The output $y(k)$ of the hyperchaotic discrete-time system (1) **synchronizes** with the output $y_M(k)$ of the hyperchaotic discrete-time system (2), if

$$\lim_{k\to\infty} |y(k) - y_M(k)| = 0, \tag{3}$$

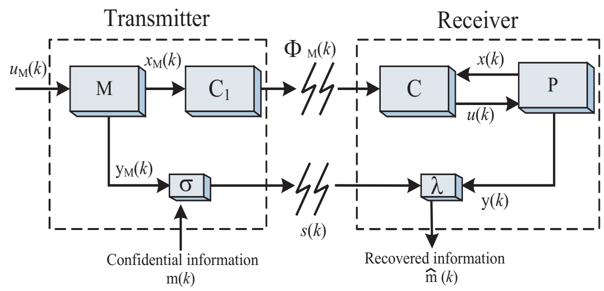

Figure 3.1: Output synchronization scheme based on model matching approach.

no matter which initial conditions $x(0)$ and $x_M(0)$ have, and for suitable input signals $u(k)$ and $u_M(k)$.

Notice that, we are considering *partial synchronization* between hyperchaotic maps (1) and (2), which is a substantial difference with other approaches based on complete synchronization.

Figure 3.1 shows the *output synchronization scheme* by using model-matching approach: the **master** system is the hyperchaotic map M with state x_M, input u_M, and output y_M. The nonlinear function $\phi_M(k) = \phi_M(x_M, u_M)$ is the coupling signal, which is transmitted through a public channel to the slave system, and is used to synchronize the master and slave systems to satisfy the condition (3). The **slave** consists of the hyperchaotic map P and a compensator C. The **compensator** C is utilized to control P with inputs ϕ_M and x, and output u. If the compensator C yields properly the control signal u, then the *output error synchronization* $e(k) = y_E(k) = y(k) - y_M(k)$ *asymptotically converges to zero*.

For secure communications based on previous output synchronization scheme between maps (1) and (2): at the hyperchaotic transmitter, the confidential information is encrypted (by direct modulation, additive masking, or another technique) and sent to the hyperchaotic receiver via a insecure channel. Finally, the original information is decrypted at the receiver end by using output synchronization $e(k) = y_E(k)$. For this purpose, we will use a communication scheme based on *hyperchaotic encryption*, to send encrypted audio and image information.

3.1 Model-matching problem

Considering the hyperchaotic maps (1) and (2), we assume that P evolves in a neighborhood of an equilibrium point x^0; that is, around $\left(x^0, u^0\right) \in X \times U$ such that $f\left(x^0, u^0\right) = x^0$, with $\left\{u(k) = u^0 : k \geq 0\right\}$ being a (constant) input sequence. For this sequence there exists another (constant) output sequence $\left\{y(k) = h\left(x^0\right) = y^0 : k \geq 0\right\}$. In the same way, let the equilibrium point of model M be denoted by x_M^0 around $\left(x_M^0, u_M^0\right) \in X_M \times U_M$. According to Figure 3.1 we are interested in to design a control u for P which, irrespectively of the initial conditions of P and M, makes the output $y(k)$ of P asymptotically converges to the output $y_M(k)$ produced by M under an arbitrary input $u_M(k)$. This problem is the so-called *discrete-time asymptotic model-matching problem (DAMMP)* from nonlinear control theory which coincides with the OSP, see [7, 37]. Similar to [37] we adopt the following approach: where the DAMMP is reduced into a problem of decoupling the output of a suitable auxiliary system from the input u_M to the model M. In this way, we define an *output error* $y_E(k) = y(k) - y_M(k)$, and we choose the control law $u(k)$ such that the output $y_E(k)$ is decoupled from $u_M(k)$ for all $k \geq 0$, and converges asymptotically to zero. The *auxiliary system* is defined as follows

$$E : \begin{cases} x_E(k+1) &= f_E(x_E(k), u_E(k), w_E(k)), \\ y_E(k) &= h_E(x_E(k)), \end{cases} \tag{4}$$

with *auxiliary state* $x_E = (x, x_M)^T \in \mathbb{R}^{n+n_M}$, and *auxiliary inputs* $u_E = u$ and $w_E = u_M$, where

$$f_E(x_E, u_E, w_E) = \begin{pmatrix} f(x,u) \\ f_M(x_M, u_M) \end{pmatrix}, \quad h_E(x_E) = h(x) - h_M(x_M).$$

Given this system, together with an equilibrium point $x_E^0 = (x^0, x_M^0)$ it is known that, if the disturbance-decoupling problem with measurement disturbance w_E associated with the system E has a solution on Ω_0^E, an open and dense subset of $X \times X_M \times U \times U_M$, defined around the equilibrium point (x^0, x_M^0, u^0, u_M^0), then there exists an analytic mapping γ^E defined on Ω_0^E with the property that the control law

$$u(k) = \gamma^E(x_E(k), w_E(k)) = \gamma^E(x_E(k), u_M(k)) \tag{5}$$

decouples the output y_E of the closed-loop system (4)-(5) from the disturbance w_E for every initial state of x_E in an open and dense subset of $X \times X_M$ contained in Ω_0^E.

The DAMMP is treated in terms of a *relative degree* associated with the outputs y and y_M. Thus, the following definitions are introduced. Let f_0, f_{M_0}, and f_{E_0} be the undriven state dynamics $f(\cdot, 0)$, $f_M(\cdot, 0)$, and $f_E(\cdot, 0, 0)$, respectively, and f_0^j, $f_{M_0}^j$, and $f_{E_0}^j$ the j-times iterated compositions of f_0, f_{M_0}, and f_{E_0} with $f_0^0(x) = x$, $f_{M_0}^0(x_M) = x_M$, and $f_{E_0}^0(x_E) = x_E$.

Definition 3.2 (Relative degree) [31] The output y of the plant Eq. (1) is said to have a relative degree d in an open and dense subset O of $X \times U$ containing the equilibrium point (x^0, u^0), if

$$\frac{\partial}{\partial u}\left[h \circ f_0^l(f(x,u))\right] \equiv 0$$

for all $0 \leq l \leq d-1$, for all $(x,u) \in O$, and

$$\frac{\partial}{\partial u}\left[h \circ f_0^d(f(x,u))\right] \neq 0$$

for all $(x,u) \in O$.

A similar definition can be given for the relative degree of the model M Eq. (2), d_M, in an open and dense subset O_M, of $X_M \times U_M$ containing the equilibrium point (x_M^0, u_M^0).

The following theorem gives necessary and sufficient conditions for the local solvability of the OSP for hyperchaotic maps.

Theorem 3.1 *[37] Consider the hyperchaotic maps P Eq. (1) and M Eq. (2) around, respectively, their equilibria* $\left(x^0, u^0\right)$ *and* $\left(x_M^0, u_M^0\right)$*. Suppose that the outputs* y *of* P *and* y_M *of* M *have finite relative degree* d *and* d_M*, respectively defined on* O *and* O_M*. Assume that for all* $x_E = (x, x_M)^T \in X \times X_M$ *and* $u_M \in U_M$*,*

$$0 \in Im\left\{h_E \circ f_{E_0}^d\left(f_E\left(x_E, \cdot, u_M\right)\right)\right\},$$

holds, where $Im\{\varphi\}$ *denotes the image of* φ*. Then the OSP is locally solvable on* Ω_0^E *if and only if*

$$d \leq d_M. \tag{6}$$

If the condition (6) holds, then from definition of relative degrees d and d_M we have that there exists an analytic mapping $\gamma^E : \mathbb{R}^{n+n_M} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$y_E\left(k+d+1\right) = h_E \circ f_{E_0}^d \circ f_E\left(x_E\left(k\right), \gamma^E\left(x_E\left(k\right), u_M\left(k\right), v\left(k\right)\right)\right) = v\left(k\right),$$

with $v \in \mathbb{R}$ an external control, or equivalently,

$$S\left(x\left(k\right), \gamma^E\left(x_E\left(k\right), u_M\left(k\right), v\left(k\right)\right)\right) =$$

$$v\left(k\right) - h \circ f_0^d \circ f\left(x\left(k\right)\right) + h_M \circ f_{M_0}^l \circ f_M\left(x_M\left(k\right), u_M\left(k\right)\right).$$

The analytic mapping $\gamma^E\left(x_E, u_M, v\right)$ is the inverse of $S\left(x, \cdot\right)$, that is

$$\gamma^E\left(x_E\left(k\right), u_M\left(k\right), v\left(k\right)\right) =$$

$$S^{-1}\left(x\left(k\right),\ v\left(k\right) - h \circ f_0^d \circ f\left(x\left(k\right)\right) + h_M \circ f_{M_0}^l \circ f_M\left(x_M\left(k\right), u_M\left(k\right)\right)\right), \tag{7}$$

where the external control is given by

$$v\left(k\right) = -\sum_{l=0}^{d} \alpha_l\left[h \circ f_0^l\left(x\left(k\right)\right) - h_M \circ f_{M_0}^l\left(x_M\left(k\right)\right)\right]. \tag{8}$$

Under the new coordinates

$$\left(\zeta\left(x_E\right), x_M\right) = \phi\left(x_E\right) = \phi\left(x, x_M\right),$$

where $\zeta\left(x_E\right) = \left(\zeta_1\left(x_E\right), ..., \zeta_{d+1}\left(x_E\right)\right)^T$ and $\zeta_i\left(x_E\right) = h_{E_i} \circ f_{E_0}^{i-1}\left(x_E\right) = \xi_i\left(x\right) - h_{M_i} \circ f_{M_0}^{i-1}\left(x_M\right)$ for all $i = 1, 2, \ldots, d+1$. The closed-loop auxiliary system E, by using the control law $u = \gamma^E\left(x_E, u_M\right)$ Eqs. (7)-(8), takes the form

$$\begin{aligned} \zeta_i\left(k+1\right) &= \zeta_{i+1}\left(k\right), \qquad i = 1, ..., d, \\ \zeta_{d+1}\left(k+1\right) &= -\alpha_0\zeta_1\left(k\right) - \ ... \ - \alpha_d\ \zeta_{d+1}\left(k\right) = v\left(k\right), \\ x_M\left(k+1\right) &= f_M\left(x_M\left(k\right), u_M\left(k\right)\right), \\ y_E\left(k\right) &= \zeta_1\left(k\right). \end{aligned} \tag{9}$$

From Eq. (9) we see that the output $y\left(k\right)$ of the closed-loop slave system P differs from the output y_M of the model M by a signal $y_E\left(k\right)$ obeying the linear difference equation

$$y_E\left(k+d+1\right) + \alpha_d\ y_E\left(k+d\right) + \ldots + \alpha_1 y_E(k+1) + \alpha_0\ y_E\left(k\right) = 0,$$

where $\alpha_0, \ldots, \alpha_d$ are constant real coefficients. A proper location of the roots of the polynomial

$$\lambda^{d+1} + \alpha_d \lambda^d + \ldots + \alpha_1 \lambda + \alpha_0$$

entails the desired asymptotic behavior $y_E(k) = 0$, i.e. $y(k)$ converges to $y_M(k)$ as $k \to \infty$, and therefore the output synchronization condition (3) holds. We can identify two subsystems in the closed-loop system (9), as follows:

1. The subsystem is described by

$$x_M(k+1) = f_M(x_M(k), u_M(k)),$$

which represents the dynamics of the model M, and

2. The subsystem is described by

$$\zeta(k+1) = A\zeta(k),$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_d \end{pmatrix},$$

which represents the dynamics of the signal $y_E(k)$.

The dynamics of model M is stable by assumption and, if we choose u Eqs. (7)–(8) such that the eigenvalues of matrix A have magnitude strictly less than one, then the closed-loop system (9) will be exponentially stable, and the output synchronization condition (3) holds.

3.2 Output synchronization procedure

From Eq. (5) we can express the control law u in the following form

$$u(k) = \gamma^E(x(k), x_M(k), u_M(k)) = \gamma^E(x(k), \phi_M(x_M(k), u_M(k))), \tag{10}$$

where the nonlinear function $\phi_M(x_M, u_M)$ is the coupling signal to be transmitted from the master M to construct the control law u in C, which solves the OSP, see Figure 3.1. In the context of synchronization, a key observation, provided by the special form of the control law u in (10), is that the nonlinear function $\phi_M(x_M, u_M)$ fixes the coupling signal to be transmitted to the slave system. We rewrite the following procedure to achieve output synchronization between hyperchaotic maps P and M proposed in [37]:

Step 1. Given two hyperchaotic maps $x(k+1) = f(x(k))$ and $x_M(k+1) = f_M(x(k))$ we write it in the forms P Eq. (1) and M Eq. (2) by adding the control inputs $u(k)$ and $u_M(k)$, respectively.

Step 2. We define properly the outputs y and y_M for maps P and M, respectively; such that the OSP has a solution, that is the condition $d \leq d_M$ holds.

Step 3. We obtain the control law u according to Eqs. (7)-(8).

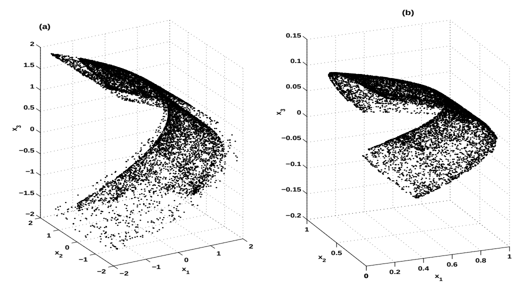

Figure 4.1: Hyperchaotic attractors generated by the uncontrolled: (a) Hénon and (b) Rössler maps.

Step 4. From $u = \gamma^E(x_E, u_M)$, we proceed to identify the nonlinear coupling signal $\phi_M(x_M, u_M)$.

Step 5. Once the coupling signal $\phi_M = \phi_M(x_M, u_M)$ has been decided, then the output y of slave P can track arbitrary the reference signal y_M of model M in the sense of condition (3).

In next section, we will appeal to the above procedure to synchronize the outputs of the hyperchaotic generalized Hénon and Rössler maps, which is a necessary condition in secure communications for encryption and decryption of confidential information.

4 Output Synchronization of Hyperchaotic Hénon and Rössler Maps

Consider the following hyperchaotic generalized **Hénon map** described by the third order difference equations [38]:

$$\left\{ \begin{array}{lcl} x_1(k+1) & = & 1.76 - x_2^2(k) - 0.1x_3(k), \\ x_2(k+1) & = & x_1(k), \\ x_3(k+1) & = & x_2(k), \end{array} \right. \tag{11}$$

In addition, consider the **Rössler map** defined by [34]:

$$\left\{ \begin{array}{lcl} x_1(k+1) & = & \alpha x_1(k)(1 - x_1(k)) - \beta(x_3(k) + \gamma)(1 - 2x_2(k)), \\ x_2(k+1) & = & \delta x_2(k)(1 - x_2(k)) + \varsigma x_3(k), \\ x_3(k+1) & = & \eta((x_3(k) + \gamma)(1 - 2x_2(k)) - 1)(1 - \theta x_1(k)), \end{array} \right. \tag{12}$$

for parameter set: $\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\varsigma = 0.2$, $\eta = 0.1$, and $\theta = 1.9$; the uncontrolled generalized Hénon and Rössler maps exhibit hyperchaotic dynamics. Figures 4.1(a) and 4.1(b) show the hyperchaotic attractors projected onto three-dimensional space generated by the generalized Hénon and Rössler maps, respectively (when we have used 10 000 iterations). Following the Step 1, we add the control inputs $u(k)$ and $u_M(k)$ to Hénon and Rössler maps, respectively. In addition for Step 2, we define the outputs $y(k) = x_2(k)$ and $y_M(k) = x_{M2}(k)$ in (11) and (12), respectively. In this way, we have the generalized Hénon map P for the *slave system* (in the form of Eq. (1)), as follows

$$P:\begin{cases} x_1(k+1) &= 1.76 - x_2^2(k) - 0.1x_3(k), \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= x_2(k) + u(k), \\ y(k) &= x_2(k) \end{cases} \tag{13}$$

and the Rössler map M for the *master system* (in the form of Eq. (2)), described by

$$M:\begin{cases} x_{M1}(k+1) &= \alpha x_{M1}(k)(1-x_{M1}(k)) - \beta(x_{M3}(k)+\gamma)(1-2x_{M2}(k)) + u_M(k), \\ x_{M2}(k+1) &= \delta x_{M2}(k)(1-x_{M2}(k)) + \varsigma x_{M3}(k), \\ x_{M3}(k+1) &= \eta((x_{M3}(k)+\gamma)(1-2x_{M2}(k))-1)(1-\theta x_{M1}(k)), \\ y_M(k) &= x_{M2}(k), \end{cases} \tag{14}$$

the relative degrees are $d = d_M = 2$, with this the OSP has a solution. In order to obtain the particular solution u (Step 3) to control to P, we define $\zeta_1 = y_E = x_2 - x_{M2}$, in this way, the auxiliary system in new coordinates is described by

$$\begin{aligned} \zeta_1(k+1) &= \zeta_2(k), \\ \zeta_2(k+1) &= \zeta_3(k), \\ \zeta_3(k+1) &= -\alpha_2\zeta_3(k) - \alpha_1\zeta_2(k) - \alpha_0\zeta_1(k) = v\,(k)\,. \end{aligned}$$

The control law u is given by

$$u\,(k) = 10\left(1.76 - x_1^2\,(k) - 0.1x_2\,(k)\right) - a - \phi_M\left(x_M\,(k)\,, u_M\,(k)\right)), \tag{15}$$

where

$$a = -\alpha_2\left(1.76 - x_2^2\,(k) - 0.1x_3\,(k)\right) - \alpha_1 x_1\,(k) - \alpha_0 x_2\,(k)\,.$$

Step 4, from Eq. (15) the coupling function $\phi_M\left(x_M, u_M\right)$ is given by

$$\phi_M\left(x_M\,(k)\,, u_M\,(k)\right) = -\left(-\alpha_2\rho_1 - \alpha_1\rho_2 - \alpha_0 x_{M2}\,(k)\right) + \rho_4, \tag{16}$$

where

$$\begin{aligned} \rho_1 &= \delta\rho_2\,(1-\rho_2) + \varsigma\rho_3, \\ \rho_2 &= \delta x_{M2}\,(k)\,(1-x_{M2}\,(k)) + \varsigma x_{M3}\,(k)\,, \\ \rho_3 &= \eta\left(\left(\left(x_{M3}\,(k)+\gamma\right)(1-2x_{M2}\,(k))\right)-1\right)(1-\theta x_{M1}\,(k))\,, \\ \rho_4 &= \delta\left(\delta\rho_2\,(1-\rho_2)+\varsigma\rho_3\right)\left(1-(\delta\rho_2(1-\rho_2)+\varsigma\rho_3\right)+\rho_5, \\ \rho_5 &= \varsigma(\eta(((\rho_3+\gamma)(1-2\rho_2))-1)(1-\theta\rho_6), \\ \rho_6 &= \alpha x_{M1}\,(k)\,(1-x_{M1}\,(k)) - \beta\left(x_{M3}\,(k)+\gamma\right)(1-2x_{M2}\,(k)) + u_M\,(k)\,. \end{aligned}$$

In the following, we carry out some numerical simulations by using the initial conditions $x\,(0) = (0.3, 0, 0.05)$ and $x_M\,(0) = (0.1, 0.2, -0.1)$ with the selection $\alpha_i = 0.1$, $i = 0, 1, 2$. In this case, we use $u_M\,(k) = 0$ to keep the master system M Eq. (14) with hyperchaotic dynamics. With the above selection, Step 5 is achieved.

Figure 4.2 shows the matching between the output signals $y\,(k) = x_2\,(k)$ and $y_M\,(k) = x_{M_2}\,(k)$ (top of figure); in addition, the output synchronization error $e_2\,(k) = x_2\,(k) - x_{M2}\,(k)$ is shown (top of figure). Meanwhile, Figure 4.3 illustrates the synchronization errors $e_1\,(k) = x_1\,(k) - x_{M1}\,(k)$ and $e_3\,(k) = x_3\,(k) - x_{M3}\,(k)$. In this case, notice that

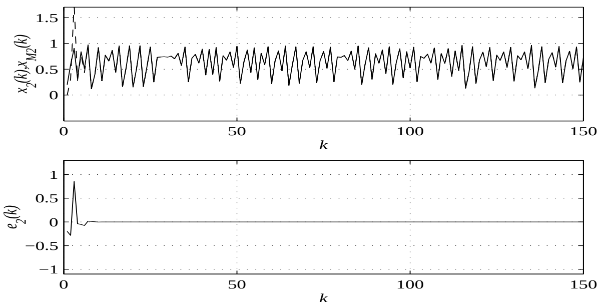

Figure 4.2: Matching between output signals $y(k) = x_2(k)$ and $y_M(k) = x_{M_2}(k)$ (top of figure). Output synchronization error $e_2(k) = x_2(k) - x_{M2}(k)$ (bottom of figure).

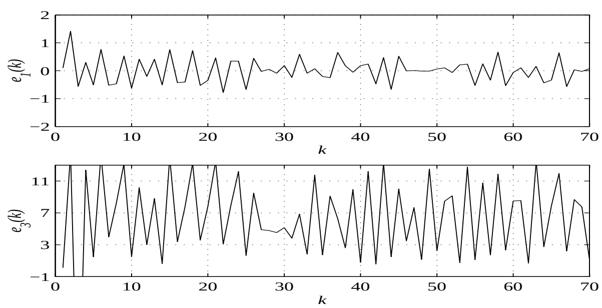

Figure 4.3: Output synchronization errors $e_1(k) = x_1(k) - x_{M1}(k)$ and $e_3(k) = x_3(k) - x_{M3}(k)$.

remaining output synchronization errors $e_1(k)$ and $e_3(k)$ are different from zero; also, notice that there exist big magnitudes of the synchronization errors $e_1(k)$ and $e_3(k)$ which can be estimated by the enormous difference between the hyperchaotic attractors generated by the hyperchaotic Rössler and Hénon maps, which is depicted in Figure 4.4: hyperchaotic attractors generated by the controlled hyperchaotic Hénon and Rössler maps, i.e. after we have achieved output synchronization, when we have used 50 000 iterations.

5 Secure Hyperchaotic Encryption

In this section, we show how output synchronization of the hyperchaotic Hénon and Rössler maps is used in a secure communication scheme to send confidential information. In particular, we propose a communication scheme to transmit encrypted audio and image information.

The communication scheme to send confidential information is shown in Figure 5.1. This cryptosystem uses two transmission channels, in one the complex coupling sequence $\phi_M(k) = \phi_M(x_M(k), u_M(k))$ is transmitted to achieve output synchronization between hyperchaotic transmitter and receiver. The signal $\phi_M(k)$ is only used for fast synchronization and does not contain any information of the confidential information $m(k)$.

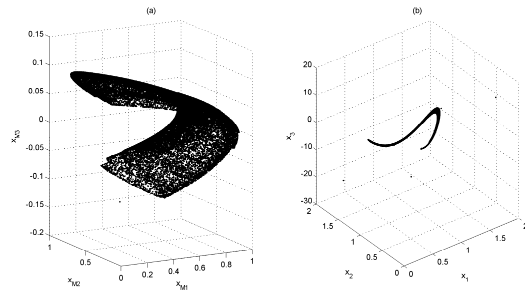

Figure 4.4: Hyperchaotic attractors generated by the controlled (after output synchronization): (a) Rössler and (b) Hénon maps.

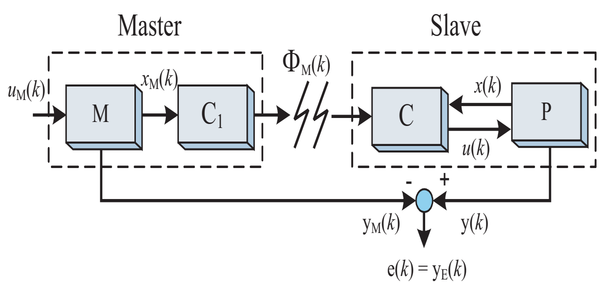

Figure 5.1: Secure communication scheme for transmission of encrypted audio and image information.

While, in the second channel, we send the encrypted confidential information $m(k)$, here the nonlinear function $\sigma(\cdot,\cdot)$ encrypts both the information $m(k)$ and chaotic output $y_M(k)$ in the transmitter. The encrypted message $s(k)$ is transmitted to the receiver end. The nonlinear function for encryption is proposed as follows

$$\sigma(y_M, m) = s = g_1(y_M) + g_2(y_M)\, m,$$

and the nonlinear function for decryption is given by

$$\lambda(y, s) = \frac{-g_1(y)}{g_2(y)} + \frac{s}{g_2(y)}.$$

In particular, the *encryption function* installed in the transmitter computer is given by

$$\sigma(y_M, m) = y_M^3 + \left(1 + y_M^3\right) m = s, \tag{17}$$

and the decryption function installed into the remote receiver computer is defined by

$$\lambda(y, s) = \frac{-y^3}{1 + y^3} + \frac{s}{1 + y^3} \ . \tag{18}$$

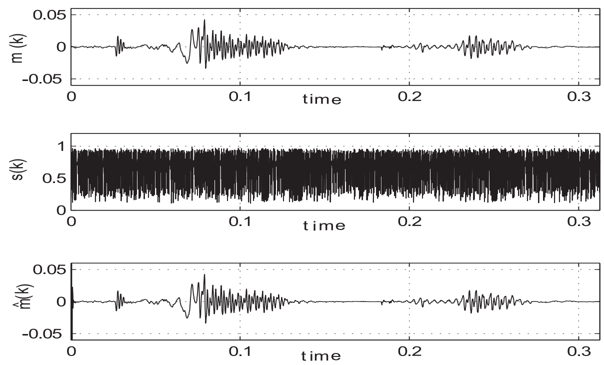

Figure 5.2: Original voice information to be encrypted, m (middle of figure). The transmitted signal with the hidden information. The recovered information $\hat{m}$, at the reciver end (bottom of figure).

5.1 Communicating encrypted audio information signals

Firstly, we use like confidential information $m\,(k)$ a *voice message*, the transmitted signal with the hidden the information is $s\,(k)$, and at the receiver end, the recovered information $\hat{m}\,(k)$ is given by Figure 5.2 shows the encrypted transmission and recovery when the confidential information $m\,(k)$ (top of figure) is a voice signal, in this case the word "*cuatro*" that means *four* in Spanish. The transmitted hyperchaotic signal $s(k)$ (middle of figure), and recovered information signal $\hat{m}$ (bottom of figure). We can see after brief transient time that information is recovered faithfully.

5.2 Communicating encrypted images

Figure 5.3 shows the transmission and recovering of an image message by using hyperchaotic encryption, which is based on output synchronization of Hénon and Rössler maps. The original image to be encrypted and transmitted is shown in Figure 5.3(a). While Figure 5.3(b) shows the transmitted encrypted image to the remote receiver via an insecure channel. Finally, the recovered image at the receiver end is depicted in Figure 5.3(c).

Remark 5.1 In our cryptosystem, the processes of encryption and synchronization are completely separated with no interference between them. So, encrypted information does not interfere with synchronization, therefore not increasing the sensitivity of synchronization to external errors. As a result, the hyperchaotic communication scheme with two transmission channels gives faster synchronization and high security, see [35].

6 Conclusions

In this paper, we have presented a scheme to achieve output synchronization of different discrete-time hyperchaotic maps via model-matching approach. This method is inspired from nonlinear control theory. We have showed by computer simulations, that this

(a) (b) (c)

Figure 5.3: (a) Original jpg image information to be send for thransmitter, (b) hyperchaotic encrypted image through insecure channel, and (c) recovered jpg information at the receiver end.

approach is indeed suitable to synchronize hyperchaotic generalized Hénon and Rössler maps, in a master-slave coupled configuration.

We have applied output synchronization in secure communication based on hyperchaos. In particular, we have presented a hyperchaotic communication scheme to transmit encrypted confidential (audio and image) information. As well as, the intrinsic advantages for the encryption presented by the mentioned schemes (σ and λ function for exception/decryption, respectively), we have increased the security by using complex hyperchaotic transmitted signals.

Acknowledgments

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Approximation of Solutions to a Class of Second Order History-valued Delay Differential Equations

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Abstract: In this paper we shall study the approximations of solutions to a class of second order history-valued delay differential equations in a separable Hilbert space. Using a pair of associated nonlinear integral equations and projection operators we consider a pair of approximate nonlinear integral equations. We first show the existence and uniqueness of solutions to this pair of approximate integral equations and then establish the convergence of the sequences of the approximate solutions to the solution and the pair of associated integral equations, respectively. Also, we consider the Faedo–Galerkin approximations of the solution and prove some convergence results. Finally, we give an example.

Keywords: *Second order history-valued delay differential equations; analytic semigroup; Banach fixed point theorem; Faedo-Galerkin approximation.*

Mathematics Subject Classification (2000): 34K30, 35R10, 47D06.

1 Introduction

We consider the following second order history-valued abstract delay differential equation in a separable Hilbert space $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$:

$$\begin{aligned} u''(t) + Av(t) &= f(t, u(t), v(t), u(t-\tau), v(t-\tau)), \quad t \in (0, T], \\ u(t) &= h(t), \quad v(t) = g(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

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where A is a closed linear operator defined on a dense subset of H and $v(t) = u'(t)$ for all $t \in [-\tau, T]$. We assume that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\colon\ t \geq 0\}$ in H and the nonlinear map f is defined from $[0,T] \times H^4$ into H satisfying certain conditions to be specified later.

Regarding the earlier works on existence, uniqueness, regularity and stability of various types of solutions to evolutions equations, delay differential equations and neutral functional differential equations under different conditions, we refer to Bahuguna and Muslim [1, 2, 3], Bahuguna *et al* [4], Wei *et al* [5], Balachandran and Chandrasekaran [6], Lin and Liu [7], Alaoui [8], Adimy [9], Hernandez and Henriquez [10, 11], Blasio and Sinestrari [12], Jeong [13], Rhandi [14] and the references cited in these papers.

The related results for the approximation of solutions to the first order evolution equations with and without delay can be found in Bahuguna and Muslim [1, 2], Henriquez [15] and Muslim [16].

Initial studies concerning existence, uniqueness and finite-time blow-up of solutions for the following equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0,\\ u(0) &= \phi, \end{aligned}$$

have been considered by Segal [17], Murakami [18] and Heinz and Von Wahl [19]. Bazley [20, 21] has considered the following semilinear wave equation

$$\begin{aligned} u''(t) + Au(t) &= g(u(t)), \quad t \geq 0,\\ u(0) &= \phi, \quad u'(0) = \psi, \end{aligned} \tag{2}$$

and has established the uniform convergence of approximations of solutions to (2) using the results of Heinz and von Wahl [19]. Goethel [22] has proved the convergence of approximations of solutions to (2) but assumed g to be defined on the whole of H. Based on the ideas of Bazley [20, 21], Miletta [23] has proved the existence and convergence of approximate solutions to (2).

The authors Bahuguna and Muslim [2] have considered the following first order retarded integro-differential equation

$$\begin{aligned} u'(t) + Au(t) &= Bu(t) + Cu(t-\tau) + \int_{-\tau}^{0} a(\theta) Lu(t+\theta)\, d\theta, \;\; 0 < t \leq T < \infty, \;\; \tau > 0,\\ u(t) &= h(t), \quad t \in [-\tau, 0] \end{aligned} \tag{3}$$

in a separable Hilbert space and studied the approximation of solution of the above problem under the conditions when $-A$ is the infinitesimal generator of an analytic semigroup, $B,\ C$ and L are nonlinear continuous operators suitably defined on H.

In [23], Miletta has established the convergence of Faedo-Galerkin approximation of the solution to

$$u'(t) + Au(t) = M(u(t)), \quad u(0) = \phi,$$

in a separable Hilbert space where A satisfies the same condition as in this paper and M is a nonlinear map defined on $D(A^{\alpha})$, for some $\alpha,\ 0 < \alpha < 1,$ which satisfies a Lipschitz condition in a ball in $D(A^{\alpha})$.

Despite the widespread use of the Faedo-Galerkin method (in many applications it is referred to as the method of harmonic balance), the convergence behaviour in many cases is not known. Bazely [20, 21] has proved the uniform convergence of the approximation solution of the nonlinear wave equation

$$u''(t) + Au(t) + M(u(t)) = 0, \quad u(0) = \phi, \quad u'(0) = \psi,$$

on any closed subinterval $[0, T]$ of the existence of the solution.

2 Preliminaries

We note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality we suppose that

$$\|e^{-tA}\| \leq M \quad \text{for} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in X.

In view of the facts mentioned above we have the following Lemma for an analytic semigroup $\{e^{-tA},\ t \geq 0\}$ (cf. Pazy [24], pp. 195–196).

Lemma 2.1 *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}, t \geq 0\}$ with $\|e^{-tA}\| \leq M$, for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following*

(i) *$D(A^\alpha)$ for $0 \leq \alpha \leq 1$ is a Banach space endowed with the norm $\| \cdot \|_\alpha$,*

(ii) *For $0 < \beta \leq \alpha$, the embedding $H_\alpha \hookrightarrow H_\beta$ is continuous,*

(iii) *A^α commutes with e^{-tA} and there exists a constant $C_\alpha > 0$ depending on α such that*

$$\|A^\alpha e^{-tA}\| \leq C_\alpha t^{-\alpha}, \quad t > 0,$$

(iv) *There exists a constant C such that*

$$\|A^{-\alpha}\| \leq C, \quad \textit{for} \quad 0 \leq \alpha \leq 1.$$

We assume that the linear operator A satisfies the following assumption.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_2 \ldots$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then $-A$ generates an analytic semigroup $\{e^{-tA}: \ t \geq 0\}$ in H. Further assume that the maps h, g and f satisfy the following hypotheses.

(H2) The maps $h, g \in \mathcal{C}_0^1$ are locally Hölder continuous on $[-\tau, 0]$.

We define the two new functions $\tilde{h}$ and $\tilde{g}$ given by

$$\tilde{h}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0), & t \in [0, T] \end{cases} \tag{4}$$

and

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ g(0), & t \in [0, T]. \end{cases} \tag{5}$$

(H3) The nonlinear map f is defined from $[0,T] \times D(A) \times D(A^\alpha) \times D(A) \times D(A^\alpha)$ into H and there exists a nondecreasing function L_f from $[0, \infty)$ into $[0, \infty)$ depending on some $r_1 > 0$ such that

$$\begin{aligned} \|f(t, u_1, &v_1, w_1, z_1) - f(s, u_2, v_2, w_2, z_2)\| \\ &\leq L_f(r_1)\{|t-s|^\theta + \|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha + \|w_1 - w_2\|_1 + \|z_1 - z_2\|_\alpha\}, \end{aligned}$$

for all $t, s \in [0,T]$, $\theta \in (0,1]$, and (u_1, v_1), (u_2, v_2), (w_1, z_1),$(w_2, z_2) \in B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t)))$ where $B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t))) = \{(x_1, y_1) \in D(A) \times D(A^\alpha): \|x_1 - \tilde{h}(t)\|_1 + \|y_1 - \tilde{g}(t)\|_\alpha \leq r_1\}$.

3 Approximate Integral Equations

The existence of solutions to equation (1) is closely associated with the following pair of integral equations

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - & \\ \int\limits_0^t (e^{-(t-s)A} - I)A^{-1} f(s, u(s), v(s), u(s-\tau), v(s-\tau))\, ds, & t \in [0, T], \end{cases} \tag{6}$$

$$v(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA} g(0) + & \\ \int\limits_0^t e^{-(t-s)A} f(s, u(s), v(s), u(s-\tau), v(s-\tau))\, ds, & t \in [0, T]. \end{cases} \tag{7}$$

By a solution (u, v) to equations (6)–(7) on $[-\tau, T]$, we mean a pair of functions $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ for some $0 < \alpha < 1$ satisfying (6)–(7), where $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ is the Banach space $C([-\tau, T], D(A) \times D(A^\alpha))$ of all continuous functions from $[-\tau, T]$ into $D(A) \times D(A^\alpha)$ endowed with the norm

$$\|(u, v)\|_{C_{T,1} \times C_{T,\alpha}} = \|u\|_{T,1} + \|v\|_{T,\alpha},$$

where

$$\|u\|_{T,1} = \sup_{-\tau \leq t \leq T} \|Au(t)\| = \sup_{-\tau \leq t \leq T} \|u(t)\|_1$$

and

$$\|v\|_{T,\alpha} = \sup_{-\tau \le t \le T} \|A^\alpha v(t)\| = \sup_{-\tau \le t \le T} \|v(t)\|_\alpha.$$

Let $0 < T_0 < \infty$ be an arbitrary fixed real number and

$$L(R) = (1+R)2F_R(T_0), \tag{8}$$

where

$$F_R(T_0) = 2L_f(R)[T_0^\theta + R + \|\tilde{h}\|_{T,1} + \|\tilde{g}\|_{T,\alpha}] + \|f_n(0,0,0,0,0)\|. \tag{9}$$

Let $0 < T \le T_0$ be such that

$$\sup_{0 \le t \le T} \big\{ \|(e^{-tA} - I)g(0)\| + \|(e^{-tA} - I)A^\alpha g(0)\| \big\} < \frac{R}{3}$$

and

$$T < \min\left\{ T_0,\ \frac{R}{3}\big[(M+1)L(R)\big]^{-1},\ \left[\frac{R}{3}(1-\alpha)[L(R)C_\alpha]^{-1}\right]^{\frac{1}{1-\alpha}} \right\}.$$

Let H_n denote the finite dimensional subspace of H spanned by $\{\phi_0, \phi_1, \ldots, \phi_n\}$ and for each $n = 0, 1, 2, \ldots,$ $P^n : H \to H_n$ be the corresponding projection operators. For each n we define $f_n : [0, T_0] \times D(A) \times D(A^\alpha) \times D(A) \times D(A^\alpha) \to H$ such that $f_n(t, u, v, w, z) = f(t, P^n u, P^n v, P^n w, P^n z)$, where $(u, v), (w, z) \in D(A) \times D(A^\alpha)$ and $t \in [0, T_0]$.

Let $W_R = B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g}))$, where

$$B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g})) = \{(y_1, y_2) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha \colon \|y_1 - \tilde{h}\|_{T,1} + \|y_2 - \tilde{g}\|_{T,\alpha} \le R\}.$$

Define a map S_n on W_R such that $S_n(u, v) = (\hat{u}, \hat{v})$ with

$$\hat{u}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - \\ \int_0^t (e^{-(t-s)A} - I)A^{-1} f_n(s, u(s), v(s), u(s-\tau), v(s-\tau))ds, & t \in [0, T], \end{cases} \tag{10}$$

$$\hat{v}(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA}g(0) + \int_0^t e^{-(t-s)A} f_n(s, u(s), v(s), u(s-\tau), v(s-\tau))ds, & t \in [0, T]. \end{cases} \tag{11}$$

Theorem 3.1 *If all the assumptions (H1)–(H3) are satisfied then there exists a unique* $(u_n, v_n) \in W_R$ *such that* $S_n(u_n, v_n) = (u_n, v_n)$ *for each* $n = 0, 1, 2, \ldots.$

Proof We claim that $S_n : W_R \to W_R$. For this we need to show that the map $t \mapsto (S_n(u, v))(t)$ is continuous from $[-\tau, T]$ into $D(A) \times D(A^\alpha)$ with respect to the norm $\|\cdot\|_1 + \|\cdot\|_\alpha$. For $t \in [-\tau, 0]$ we have

$$\|\hat{u}(t_2) - \hat{u}(t_1)\|_1 + \|\hat{v}(t_2) - \hat{v}(t_1)\|_\alpha = \|h(t_2) - h(t_1)\|_1 + \|g(t_2) - g(t_1)\|_\alpha. \tag{12}$$

For $t_1, t_2 \in (0,T]$ with $t_1 < t_2$, we have

$$[\hat{u}(t_2)-\hat{u}(t_1)]+[\hat{v}(t_2)-\hat{v}(t_1)] = [(e^{-t_2A}-e^{-t_1A})(-A)^{-1}g(0)]+[(e^{-t_2A}-e^{-t_1A})g(0)]$$
$$+\int_{t_1}^{t_2}[e^{-(t_2-s)A}-I](-A)^{-1}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds$$
$$+\int_0^{t_1}\left[e^{-(t_2-s)A}-e^{-(t_1-s)A}\right]$$
$$\times(-A)^{-1}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds$$
$$+\int_{t_1}^{t_2}e^{-(t_2-s)A}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds$$
$$+\int_0^{t_1}[(e^{-(t_2-s)A}-e^{-(t_1-s)A})]f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds.$$

Hence from the above equation we get

$$\|\hat{u}(t_2)-\hat{u}(t_1)\|_1+\|\hat{v}(t_2)-\hat{v}(t_1)\|_\alpha \le \|(e^{-t_2A}-e^{-t_1A})g(0)\|+\|(e^{-t_2A}-e^{-t_1A})g(0)\|_\alpha$$
$$+\int_{t_1}^{t_2}\|e^{-(t_2-s)A}-I\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds$$
$$+\int_0^{t_1}\|e^{-(t_2-s)A}-e^{-(t_1-s)A}\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds$$
$$+\int_{t_1}^{t_2}\|A^\alpha e^{-(t_2-s)A}\|\,\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds$$
$$+\int_0^{t_1}\|A^\alpha(e^{-(t_2-s)A}-e^{-(t_1-s)A})\|$$
$$\times\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds.$$

We calculate the above inequality as follows

$$\begin{aligned}\int_{t_1}^{t_2}\|e^{-(t_2-s)A}-I\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds&\\ \le (M+1)L(R)(t_2-t_1)&\end{aligned} \tag{13}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\|e^{-(t_2-s)A}A^\alpha\|\,\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds&\\ \le L(R)C_\alpha\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds = L(R)C_\alpha\frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}\,.&\end{aligned} \tag{14}$$

Part (d) of Theorem 2.6.13 in Pazy [24] implies that for $0<\vartheta\le 1$ and $x\in D(A^\vartheta)$, we have

$$\|(e^{-tA}-I)x\| \le C'_\vartheta t^\vartheta\|x\|_\vartheta. \tag{15}$$

If $0<\vartheta<1$ and $0<\alpha+\vartheta<1$, then $A^\alpha y\in D(A^\vartheta)$ for any $y\in D(A^{\alpha+\vartheta})$. Therefore, for $t\in[0,T]$ and $s\in(0,T]$, we have

$$\begin{aligned}\|(e^{-tA}-I)A^\alpha e^{-sA}x\| &\le C'_\vartheta t^\vartheta\|A^\alpha e^{-sA}x\|_\vartheta = C'_\vartheta t^\vartheta\|A^{\alpha+\vartheta}e^{-sA}x\|\\ &\le C'_\vartheta C_{\alpha+\vartheta}t^\vartheta s^{-(\alpha+\vartheta)}\|x\|.\end{aligned} \tag{16}$$

Hence from (16) we get

$$\|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| = \|(e^{-(t_2-t_1)A} - I)A^\alpha e^{-(t_1-s)A}\| \\ \leq C'_\vartheta C_{\alpha+\vartheta}(t_2-t_1)^\vartheta (t_1-s)^{-(\alpha+\vartheta)}.$$

Hence

$$\begin{aligned} &\int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| \, \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &\quad \leq C'_\vartheta C_{\alpha+\vartheta} L(R)(t_2-t_1)^\vartheta \int_0^{t_1} (t_1-s)^{-(\alpha+\vartheta)} ds \\ &\quad \leq C'_\vartheta C_{\alpha+\vartheta} L(R) \, \frac{T_0^{1-(\alpha+\vartheta)}}{1-(\alpha+\vartheta)} \, (t_2-t_1)^\vartheta. \end{aligned} \tag{17}$$

Also, from (16), we have

$$\|(e^{-tA} - I)e^{-sA}x\| \leq C'_\vartheta t^\vartheta \|e^{-sA}x\|_\vartheta = C'_\vartheta t^\vartheta \|A^\vartheta e^{-sA}x\| \\ \leq C'_\vartheta C_\vartheta t^\vartheta s^{-\vartheta}\|x\|.$$

Therefore

$$\|e^{-(t_2-s)A} - e^{-(t_1-s)A}\| = \|(e^{-(t_2-t_1)A} - I)e^{-(t_1-s)A}\| \\ \leq C'_\vartheta C_\vartheta (t_2-t_1)^\vartheta (t_1-s)^{-\vartheta}.$$

Hence

$$\begin{aligned} &\int_0^{t_1} \|e^{-(t_2-s)A} - e^{-(t_1-s)}\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &\quad \leq C'_\vartheta C_\vartheta L(R)(t_2-t_1)^\vartheta \int_0^{t_1} (t_1-s)^{-\vartheta} ds \\ &\quad \leq C'_\vartheta C_\vartheta L(R) \, \frac{T_0^{1-\vartheta}}{1-\vartheta} \, (t_2-t_1)^\vartheta. \end{aligned} \tag{18}$$

From inequalities (13), (14), (17) and (18), it follows that $S_n(u,v)(t)$ is continuous from $[-\tau, T]$ into $D(A) \times D(A^\alpha)$ with respect to the norm $\|\cdot\|_1 + \|\cdot\|_\alpha$. Next we want to show that $S_n(u,v) \in W_R$ i.e., $(\hat{u}, \hat{v}) \in W_R$. Now if $t \in [-\tau, 0]$ then we have

$$\|\hat{u}(t) - \tilde{h}(t)\|_1 + \|\hat{v}(t) - \tilde{g}(t)\|_\alpha = 0.$$

Now, if $t \in (0, T]$, then we have

$$\begin{aligned} \|\hat{u}(t) - \tilde{h}(t)\|_1 + \|\hat{v}(t) - \tilde{g}(t)\|_\alpha &\leq \|(e^{-tA} - I)g(0)\| + \|(e^{-tA} - I)A^\alpha g(0)\| \\ &+ \int_0^t \|e^{-(t-s)A} - I\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &+ \int_0^t \|e^{-(t-s)A} A^\alpha\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &\leq \frac{R}{3} + (M+1)L(R))(T_0)T + C_\alpha L(R)(T_0) \int_0^t (t-s)^{-\alpha} ds \\ &\leq \frac{R}{3} + (M+1)L(R)T + C_\alpha L(R) \, \frac{T^{1-\alpha}}{1-\alpha} \leq R. \end{aligned}$$

Taking the supremum over $[-\tau, T]$, we get

$$\|\hat{u} - \tilde{h}\|_{T,1} + \|\hat{v} - \tilde{x}_1\|_{T,\alpha} \leq R,$$

which implies that $S_n(u,v) \in W_R$. Hence, S_n maps W_R into W_R. Now to complete the proof of this theorem it only remains to show that S_n is a strict contraction mapping on W_R.

If $t \in [-\tau, 0]$, and $(u_1, v_1), (u_2, v_2) \in W_R$, then we have

$$\begin{aligned}
&\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 + \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\
&\qquad \leq \int_0^t \|e^{-(t-s)A} - I\| \, \|f(s, P^n u_1(s), P^n v_1(s), P^n u_1(s-\tau), P^n v_1(s-\tau)) \\
&\qquad\quad - f(s, P^n u_2(s), P^n v_2(s), P^n u_2(s-\tau), P^n v_2(s-\tau))\| ds \\
&\qquad + \int_0^t \|e^{-(t-s)A} A^\alpha\| \, \|f(s, P^n u_1(s), P^n v_1(s), P^n u_1(s-\tau), P^n v_1(s-\tau)) \\
&\qquad\quad - f(s, P^n u_2(s), P^n v_2(s), P^n u_2(s-\tau), P^n v_2(s-\tau))\| ds.
\end{aligned}$$

From assumption (H3), we get

$$\begin{aligned}
\|f(t, &P^n u_1(t), P^n v_1(t), P^n u_1(t-\tau), P^n v_1(t-\tau)) \\
&- f(t, P^n u_2(t), P^n v_2(t), P^n u_2(t-\tau), P^n v_2(t-\tau))\| \\
&\leq F_R(T_0)[\|u_1(s) - u_2(s)\|_1 + \|v_1(s) - v_2(s)\|_\alpha \\
&\quad + \|u_1(s-\tau) - u_2(s-\tau)\|_1 + \|v_1(s-\tau) - v_2(s-\tau)\|_\alpha] \\
&\leq \frac{2RF_R(T_0)}{R} (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha)}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f(t, &P^n u_1(t), P^n v_1(t), P^n u_1(t-\tau), P^n v_1(t-\tau)) \\
&- f(t, P^n u_2(t), P^n v_2(t), P^n u_2(t-\tau), P^n v_2(t-\tau))\| \\
&\leq 2F_R(T_0)(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}) \\
&\leq \frac{L(R)}{R} (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 + \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\
&\qquad \leq [(M+1)2F_R(T_0)T + 2C_\alpha F_R(T_0)] \int_0^t (t-s)^{-\alpha} ds \, (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T\alpha}) \\
&\qquad \leq \frac{1}{R} \left[(M+1)L(R)T + C_\alpha L(R) \, \frac{T^{1-\alpha}}{1-\alpha} \right] (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}) \\
&\qquad \leq \frac{2}{3} \, (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).
\end{aligned}$$

Taking the supremum over $[-\tau, T]$, we get

$$\|\hat{u}_1 - \hat{u}_2\|_{T,1} + \|\hat{v}_1 - \hat{v}_2\|_{T,\alpha} \leq \frac{2}{3} \, (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).$$

Thus S_n is a strict contraction mapping on W_R. Hence, there exists a unique pair $(u_n, v_n) \in W_R$ such that

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - \\ \int_0^t (e^{-(t-s)A} - I)A^{-1} f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))ds, & t \in [0, T], \end{cases} \tag{19}$$

and

$$v_n(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA}g(0) + \int_0^t e^{-(t-s)A} f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))ds, & t \in [0, T]. \end{cases} \tag{20}$$

The equations (19)–(20) are known as a pair of approximate solutions related to the given problem (1). □

Corollary 3.1 Let all the assumptions (H1)–(H3) hold. If $(h(t), g(t)) \in D(A) \times D(A)$ for all $t \in [-\tau, 0]$ then $(u_n(t), v_n(t)) \in D(A) \times D(A^\vartheta)$ for all $t \in [-\tau, T]$, where $0 \le \vartheta < 1$.

Proof From Theorem 3.1, we have the existence of a unique pair $(u_n, v_n) \in B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g}))$ satisfying (19)–(20). By Theorem (1.2.4) in Pazy [24], we have for $x \in H$, $\int_0^t e^{-tA}x ds \in D(A)$ and if $x \in D(A)$ then $e^{-tA}x \in D(A)$. Thus the result follows from these facts and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$. □

Corollary 3.2 If all the conditions (H1)–(H3) hold then for $g(0) \in D(A)$ there exists a constant V_0 independent of n such that

$$\|v_n(t)\|_\vartheta \le V_0, \quad \text{where} \quad 0 \le \vartheta < 1, \quad -\tau \le t \le T.$$

Proof If $t \in [-\tau, 0]$, then from equation (20), we get the following

$$\|v_n(t)\|_\vartheta \le \|A^\vartheta g(0)\|.$$

If $t \in (0, T]$, then we have

$$\begin{aligned} \|v_n(t)\|_\vartheta &\le \|e^{-tA}A^\vartheta g(0)\| \\ &\quad + \int_o^t \|e^{-(t-s)A}A^\vartheta\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &\le M\|g(0)\|_\vartheta + C_\vartheta L(R)\frac{T^{1-\vartheta}}{1-\vartheta} \le V_0'. \end{aligned}$$

This completes the proof of the Corollary. □

Corollary 3.3 If all the conditions (H1)–(H3) are hold then for $h(0) \in D(A)$ there exist a constant V_1 independent of n such that

$$\|u_n(t)\|_1 \le V_1, \quad \text{for all} \quad -\tau \le t \le T.$$

Proof If $t \in [-\tau, 0]$, then from equation (19) $\|v_n(t)\|_1 \le \|Ag(0)\|$.

If $t \in (0,T]$, then we have

$$\begin{aligned}\|u_n(t)\|_1 &\leq \|h(0)\|_1 + \|(e^{-tA}-I)g(0)\| \\ &\quad + \int_o^t \|(e^{-(t-s)A}-I)\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|\,ds \\ &\leq \|h(0)\|_1 + (M+1)\|g(0)\| + (M+1)L(R)T \leq V_1'.\end{aligned}$$

This completes the proof of the Theorem. □

4 Convergence of Approximate Solutions

In this section we will establish the convergence of the solution $(u_n,v_n) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ of approximate integral equations to a unique solution (u,v) of equation (1).

For proving the convergence, we need the following stronger assumption on the nonlinear map f than (H3).

(H3′) The nonlinear map f is defined from $[0,T]\times D(A)\times D(A^\alpha)\times D(A)\times D(A^\alpha)$ into $D(A^\beta)$ for $0<\alpha<\beta<1$ and there exists a nondecreasing function $\tilde{L}_f$ from $[0,\infty)$ into $[0,\infty)$ depending on some $r_1>0$ such that

$$\begin{aligned}&\|f(t,u_1,v_1,w_1,z_1)-f(s,u_2,v_2,w_2,z_2)\|_\beta \\ &\quad \leq \tilde{L}_f(r_1)\{|t-s|^\theta + \|u_1-u_2\|_1 + \|v_1-v_2\|_\alpha + \|u_1-u_2\|_1 + \|v_1-v_2\|_\alpha\}\end{aligned}$$

for all $t,s \in [0,T]$, $\theta \in (0,1]$ and $(u_1,v_1),(u_2,v_2),(w_1,z_1),(w_2,z_2) \in B_{r_1}(D(A)\times D(A^\alpha),(\tilde{h}(t),\tilde{g}(t)))$, where $B_{r_1}(D(A)\times D(A^\alpha),(\tilde{h}(t),\tilde{g}(t))) = \{(x_1,y_1)\in D(A)\times D(A^\alpha)$: $\|x_1-\tilde{h}(t)\|_1 + \|y_1-\tilde{g}(t)\|_\alpha \leq r_1\}$.

We can easily observe that the conditions (H3′) is stronger than (H3) because the same condition is satisfied in $D(A^\beta)$ rather than in H. Now, we are in a position to state a theorem.

Theorem 4.1 *Let (H1), (H2) and (H3′) be satisfied and* $(h(0),g(0)) \in D(A)\times D(A)$. *Then,*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ -\tau\leq t\leq T\}} \{\|u_n-u_m\|_{T,1} + \|v_n-v_m\|_{T,\alpha}\} = 0,$$

where u_n *and* v_n *are given by (19) and (20) respectively.*

Proof For $n \geq m$, we have

$$\begin{aligned}&\|f_n(t,u_n(t),v_n(t),u_n(t-\tau),v_n(t-\tau)) - f_m(t,u_m(t),v_m(t),u_m(t-\tau),v_m(t-\tau))\| \\ &\leq \|f_n(t,u_n(t),v_n(t),u_n(t-\tau),v_n(t-\tau)) - f_n(t,u_m(t),v_m(t),u_m(t-\tau),v_m(t-\tau))\| \\ &+ \|f_n(t,u_m(t),v_m(t),u_m(t-\tau),v_m(t-\tau)) - f_m(t,u_m(t),v_m(t),u_m(t-\tau),v_m(t-\tau))\| \\ &\leq L_f(R)[\|P^nu_n(t)-P^nu_m(t)\|_1 + \|P^nv_n(t)-P^nv_m(t)\|_\alpha \\ &\quad + \|P^nu_n(t-\tau)-P^nu_m(t-\tau)\|_1 + \|P^nv_n(t-\tau)-P^nv_m(t-\tau)\|_\alpha \\ &\quad + \|(P^n-P^m)u_m(t)\|_1 + \|(P^n-P^m)v_m(t)\|_\alpha \\ &\quad + \|(P^n-P^m)u_m(t-\tau)\|_1 + \|(P^n-P^m)v_m(t-\tau)\|_\alpha].\end{aligned}$$

Also, we can see that

$$\begin{aligned}\|(P^n - P^m)v_m(t)\|_\alpha &= \|A^\alpha (P^n - P^m)v_m(t)\| = \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta v_m(t)\| \\ &\le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|(P^n - P^m)A^\vartheta v_m(t)\| \le \frac{\|A^\vartheta v_m(t)\|}{\lambda_m^{\vartheta-\alpha}}\end{aligned}$$

and

$$\begin{aligned}\|(P^n - P^m)v_m(t-\tau)\|_\alpha &= \|A^\alpha (P^n - P^m)v_m(t-\tau)\| = \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta v_m(t-\tau)\| \\ &\le \frac{1}{\lambda_m^{\vartheta-\alpha}}\,\|(P^n - P^m)A^\vartheta v_m(t-\tau)\| \le \frac{\|A^\vartheta v_m(t-\tau)\|}{\lambda_m^{\vartheta-\alpha}}\,.\end{aligned}$$

For convenience, we denote

$$\xi_{m,n}(t) = \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha$$

and

$$\xi_{m,n}(t-\tau) = \|u_n(t-\tau) - u_m(t-\tau)\|_1 + \|v_n(t-\tau) - v_m(t-\tau)\|_\alpha.$$

Thus, we have

$$\begin{aligned}&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f_m(t, u_m(t), v_m(t), u_m(t-\tau), v_m(t-\tau))\| \\ &\quad \le L_f(R)[\xi_{m,n}(t) + \xi_{m,n}(t-\tau) + \|(P^n - P^m)u_m(t)\|_1 \\ &\qquad + \frac{\|v_m(t)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}} + \|(P^n - P^m)u_m(t-\tau)\|_1 + \frac{\|v_m(t-\tau)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}] \\ &\quad \le 2L_f(R)\left[\{\|u_n - u_m\|_{t,1} + \|v_n - v_m\|_{t,\alpha}\} + \|(P^n - P^m)u_m\|_{t,1} + \frac{\|v_m\|_{t,\vartheta}}{\lambda_m^{\vartheta-\alpha}}\right].\end{aligned} \tag{21}$$

Now, from the pair of integral equations (19)–(20), for any $0 < t_0' < t < T_0$, we have

$$\begin{aligned}&\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\ &\quad \le \left\{ \int_0^{t_0'} \|e^{-(t_0'-s)A} - I\| + \int_{t_0'}^t \|e^{-(t-s)A} - I\| \right\} \\ &\quad \times \big[\|f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau)) \\ &\qquad - f_m(s, u_m(s), v_m(s), u_m(s-\tau), v_m(s-\tau))\|\big]\,ds \\ &\quad + \left\{ \int_0^{t_0'} \|e^{-(t_0'-s)A}A^\alpha\| + \int_{t_0'}^t \|e^{-(t-s)A}A^\alpha\| \right\} \\ &\quad \times \big[\|f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau)) \\ &\qquad - f_m(s, u_m(s), v_m(s), u_m(s-\tau), v_m(s-\tau))\|\big]\,ds.\end{aligned} \tag{22}$$

By using the estimate of the inequality (21) in the inequality (22), we get

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\
&\quad \le A_1 t_0' + L(R) \int_{t_0'}^{t} \left((M+1) + \frac{C_\alpha}{(t-s)^\alpha} \right) ds \left(\|(P^n - P^m)u_m\|_{T,1} + \frac{V_0}{\lambda_m^{\vartheta-\alpha}} \right) \\
&\qquad + L(R) \int_{t_0'}^{t} \left((M+1) + \frac{C_\alpha}{(t-s)^\alpha} \right) \{\|u_n - u_m\|_{s,1} + \|v_n - v_m\|_{s,\alpha}\} ds \\
&\quad \le A_1 t_0' + C(R,T) B_{mn} + N_1 \int_{t_0'}^{t} \frac{1}{(t-s)^\alpha} \{\|u_n - u_m\|_{s,1} + \|v_n - v_m\|_{s,\alpha}\} ds,
\end{aligned} \tag{23}$$

where

$$\begin{gathered}
B_{mn} = B_{mn}^1 + B_{mn}^2, \quad B_{mn}^1 = \|(P^n - P^m)u_m\|_{T,1}, \quad B_{mn}^2 = \frac{V_0}{\lambda_m^{\vartheta-\alpha}}, \\
C(R,T) = L(R) \left((M+1)T + \frac{C_\alpha T^{1-\alpha}}{1-\alpha} \right), \\
N_1 = L(R)(T^\alpha + 1) \max\{(M+1), C_\alpha\}
\end{gathered}$$

and

$$\begin{aligned}
A_1 = \{(M+1) + C_\alpha (t_0 - t_0')^{-\alpha}\} 2L_f(R) \Big[&\{\|u_n - u_m\|_{t_0',1} + \|v_n - v_m\|_{t_0',\alpha}\} \\
&+ \|(P^n - P^m)u_m\|_{t_0',1} + \frac{V_0}{\lambda_m^{\vartheta-\alpha}} \Big] t_0'.
\end{aligned} \tag{24}$$

Now we replace t by $t+\theta$ in the inequality (23), where $\theta \in [t_0' - t, 0]$, we get

$$\begin{aligned}
\|u_n(t+\theta) - u_m(t+\theta)\|_1 + \|v_n(t+\theta) - v_m(t+\theta)\|_\alpha \le A_1 t_0' + C(R,T) B_{mn} \\
+ N_1 \int_{t_0'}^{t+\theta} (t+\theta-s)^{-\alpha} \{\|u_n - u_m\|_{s,1} + \|v_n - v_m\|_{s,\alpha}\}\, ds.
\end{aligned} \tag{25}$$

We put $s - \theta = \gamma$ in inequality (25) and get

$$\begin{aligned}
&\|u_n(t+\theta) - u_m(t+\theta)\|_1 + \|v_n(t+\theta) - v_m(t+\theta)\|_\alpha \\
&\quad \le A_1 t_0' + C(R,T) B_{mn} + N_1 \int_{t_0'-\theta}^{t} (t-\gamma)^{-\alpha} \{\|u_n - u_m\|_{\gamma,1} + \|v_n - v_m\|_{\gamma,\alpha}\} d\gamma \\
&\quad \le A_1 t_0' + C(R,T) B_{mn} + N_1 \int_{t_0'}^{t} (t-\gamma)^{-\alpha} \{\|u_n - u_m\|_{\gamma,1} + \|v_n - v_m\|_{\gamma,\alpha}\} d\gamma.
\end{aligned} \tag{26}$$

Thus

$$\begin{aligned}
&\sup_{t_0'-t \le \theta \le 0} \{\|u_n(t+\theta) - u_m(t+\theta)\|_1 + \|v_n(t+\theta) - v_m(t+\theta)\|_\alpha\} \\
&\le A_1 t_0' + C(R,T) B_{mn} + N_1 \int_0^t (t-\gamma)^{-\alpha} \{\|u_n - u_m\|_{\gamma,1} + \|v_n - v_m\|_{\gamma,\alpha}\} d\gamma.
\end{aligned} \tag{27}$$

We have

$$\begin{aligned} &\sup_{-\tau-t\le\theta\le 0}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\} \\ &\quad\le \sup_{0\le\theta+t\le t_0'}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\} \\ &\quad+ \sup_{t_0'-t\le\theta\le 0}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}. \end{aligned} \tag{28}$$

By using the inequalities (26) and (27) in the inequality (28), we get

$$\begin{aligned} &\sup_{-\tau\le t+\theta\le t}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\} \\ &\le 2A_1t_0'+C(R,T)B_{mn}+N_1\int_0^t(t-\gamma)^{-\alpha}\{\|u_n-u_m\|_{\gamma,1}+\|v_n-v_m\|_{\gamma,\alpha}\}d\gamma. \end{aligned} \tag{29}$$

Hence, from Gronwall's Lemma and taking the limit as $m\to\infty$ on both sides, we get the required result, since $B_{mn}\to 0$ as $m\to\infty$ provided $\|(P^n-P^m)u_m\|_{T,1}\to 0$ as $m\to\infty$ for $-\tau\le t\le T$. Since $B_{mn}^2\to 0$ as $m\to\infty$, hence to prove that $B_{mn}\to 0$, we only need to prove that for $-\tau\le t\le T$, $\|(P^n-P^m)u_m(t)\|_1\to 0$ as $m\to\infty$. We can easily check that for every $x\in H$ and $\eta<0$

$$\|A^\eta(P^n-P^m)x\|\le\lambda_m^\eta\|(P^n-P^m)x\|\le\lambda_m^\eta\|x\|. \tag{30}$$

From the equation (19), for any $t\in[-\tau,0]$ we have

$$\|A(P^n-P^m)u_m(t)\|=\|(P^n-P^m)Ah(0)\|. \tag{31}$$

For $t\in(0,T]$, we have

$$\begin{aligned} &\|A(P^n-P^m)u_m(t)\|\le\|(P^n-P^m)Ah(0)\|+(M+1)\|(P^n-P^m)g(0)\| \\ &+(M+1)\int_0^t\|(P^n-P^m)f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\,ds. \end{aligned} \tag{32}$$

Since $A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(t-\tau))\in H,$ hence from inequality (30), we have

$$\begin{aligned} &\|(P^n-P^m)f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\| \\ &\qquad\le\|A^{-\beta}(P^n-P^m)A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\| \\ &\qquad\le\frac{1}{\lambda_m^\beta}\|A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\| \\ &\qquad\le\frac{1}{\lambda_m^\beta}\tilde F_R(T_0), \end{aligned} \tag{33}$$

where

$$\tilde F_R(T_0)=2\tilde L_f(R)[T_0^\theta+R+\|\tilde h\|_{T,1}+\|\tilde g\|_{T,\alpha}]+\|f_n(0,0,0,0,0)\|. \tag{34}$$

Using the inequality (33) in the inequality (32), we get

$$\|(P^n-P^m)u_m(t)\|_1\le\|(P^n-P^m)Ax_0\|+(M+1)\{\|(P^n-P^m)x_1\|+\frac{1}{\lambda_m^\beta}T(\tilde F_R(T_0), \tag{35}$$

which tend to zero as $m\to\infty$ for $0\le t\le T$. Hence from (32) and (35) we get the required result. This completes the proof of the theorem. □

Theorem 4.2 *If (H1)–(H2) and (H3′) are satisfied and* $(h(0), g(0)) \in D(A) \times D(A)$ *then there exists a pair of functions* $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ *such that* $(u_n, v_n) \to (u, v)$ *as* $n \to \infty$ *in* $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ *and* (u, v) *satisfies (6)–(7) on* $[-\tau, T]$.

Proof Theorem 4.1 implies that there exists $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ such that (u_n, v_n) converges to (u, v) in $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$. Since $(u_n, v_n) \in W_R$ for each n, (u, v) is also in W_R. Further, we have

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le \|f(t, P^n u_n(t), P^n v_n(t), P^n u_n(t-\tau)), P^n v_n(t-\tau)) \\
&\qquad - f(t, P^n u(t), P^n v(t), P^n u(t-\tau), P^n v(t-\tau))\| \\
&\qquad + \|f(t, P^n u(t), P^n v(t), P^n u(t-\tau), P^n v(t-\tau)) \\
&\qquad - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\|.
\end{aligned}$$

Hence from the above inequality, we have

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le L_f(R)[\|P^n u_n(t) - P^n u(t)\|_1 + \|P^n v_n(t) - P^n v(t)\|_\alpha \\
&\qquad + \|P^n u_n(t-\tau) - P^n u(t-\tau)\|_1 + \|P^n v_n(t-\tau) - P^n v(t-\tau)\|_\alpha \\
&\qquad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha \\
&\qquad + \|(P^n - I)u(t-\tau)\|_1 + \|(P^n - I)v(t-\tau)\|_\alpha] \\
&\quad \le L_f(R)[\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha \\
&\qquad + \|u_n(t-\tau) - u(t-\tau)\|_1 + \|v_n(t-\tau) - v(t-\tau)\|_\alpha \\
&\qquad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha \\
&\qquad + \|(P^n - I)u(t-\tau)\|_1 + \|(P^n - I)v(t-\tau)\|_\alpha].
\end{aligned}$$

Thus finally we get

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le 2L_f(R)[\|u_n - u\|_{T,1} + \|v_n - v\|_{T,\alpha} | (P^n - I)u\|_{T,1} + \|(P^n - I)v\|_{T,\alpha}].
\end{aligned} \tag{36}$$

Hence, by using the inequality (36) and the bounded convergence theorem we can see easily that the pair of functions (u, v) must be given by equations (6)–(7). □

5 Faedo-Galerkin Approximations

From the previous sections we know that for any $-\tau \le T < \infty$ we have a unique pair $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ satisfying the integral equations (6)–(7).

Also we have a unique pair $(u_n, v_n) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ which is the solution of the approximate integral equations (19)–(20).

If we project the equations (19)–(20) onto H_n, we get the Faedo-Galerkin approximation $(\hat{u}_n(t), \hat{v}_n(t)) = (P^n u_n(t), P^n v_n(t))$ satisfying

$$\hat{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ P^n h(0) - (e^{-tA} - I)A^{-1}P^n g(0) - \int_0^t (e^{-(t-s)A} - I)A^{-1} \times & \\ P^n f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))\, ds, & t \in [0, T], \end{cases} \tag{37}$$

$$\hat{v}_n(t) = \begin{cases} P^n g(t), & t \in [-\tau, 0], \\ e^{-tA} P^n g(0) + & \\ \int_0^t e^{-(t-s)A} P^n f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))\, ds, & t \in [0, T]. \end{cases} \tag{38}$$

The solution (u, v) of (6)–(7) and $(\hat{u}_n, \hat{v}_n)$ of (37)–(38), have the representations

$$\begin{aligned} u(t) &= \sum_{i=0}^{\infty} \alpha_i(t)\phi_i, \quad \alpha_i(t) = \langle u(t), \phi_i \rangle, \quad i = 0, 1, \ldots, \\ v(t) &= \sum_{i=0}^{\infty} \beta_i(t)\phi_i, \quad \beta_i(t) = \langle v(t), \phi_i \rangle, \quad i = 0, 1, \ldots, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \hat{u}_n(t) &= \sum_{i=0}^{n} \alpha_i^n(t)\phi_i, \quad \alpha_i^n(t) = \langle \hat{u}_n(t), \phi_i \rangle, \quad i = 0, 1, \ldots, n, \\ \hat{v}_n(t) &= \sum_{i=0}^{n} \beta_i^n(t)\phi_i, \quad \beta_i^n(t) = \langle \hat{v}_n(t), \phi_i \rangle, \quad i = 0, 1, \ldots, n. \end{aligned} \tag{40}$$

Now, we shall show the convergence of (α_i^n, β_i^n) to (α_i, β_i). It can be easily checked that

$$A[u(t) - \hat{u}(t)] = \sum_{i=0}^{\infty} \lambda_i(\alpha_i(t) - \alpha_i^n(t))\phi_i$$

and

$$A^\alpha[v(t) - \hat{v}(t)] = \sum_{i=0}^{\infty} \lambda_i^\alpha(\beta_i(t) - \beta_i^n(t))\phi_i.$$

Thus, we have

$$\|A[u(t) - \hat{u}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^2 |\alpha_i(t) - \alpha_i^n(t)|^2$$

and

$$\|A^\alpha[v(t) - \hat{v}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} |\beta_i(t) - \beta_i^n(t)|^2.$$

Hence, we have the following convergence theorem.

Theorem 5.1 *Let (H1), (H2) and (H3') be satisfied and* $(h(0), g(0)) \in D(A) \times D(A)$. *Then,*

$$\lim_{n\to\infty} \sup_{-\tau \leq t \leq T} \left\{ \sum_{i=0}^{n} \lambda_i^2 |\alpha_i(t) - \alpha_i^n(t)|^2 + \sum_{i=0}^{n} \lambda_i^{2\alpha} |\beta_i(t) - \beta_i^n(t)|^2 \right\} = 0.$$

The assertion of Theorem 5.1 follows from the facts mentioned above and from the following proposition.

Theorem 5.2 *Let (H1), (H2) and (H3') be satisfied and let T be any number such that $0 < T < \infty$, and $(h(0), g(0)) \in D(A) \times D(A)$. Then,*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ -\tau\leq t\leq T\}} \{\|A[\hat{u}_n(t) - \hat{u}_m(t)]\| + \|A^\alpha[\hat{v}_n(t) - \hat{v}_m(t)]\|\} = 0.$$

Proof For $n \geq m$, we have

$$\begin{aligned}
&\|A(\hat{u}_n(t) - \hat{u}_m(t))\| + \|A^\alpha(\hat{v}_n(t) - \hat{v}_m(t))\| \\
&\quad = \|A(P^n u_n(t) - P^m u_m(t))\| + \|A^\alpha(P^n v_n(t) - P^m v_m(t))\| \\
&\quad \leq \|AP^n(u_n(t) - u_m(t))\| + \|A(P^n - P^m)u_m(t)\| \\
&\qquad + \|A^\alpha P^n(v_n(t) - v_m(t))\| + \|A^\alpha(P^n - P^m)v_m(t)\| \\
&\quad \leq \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_1 + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\beta v_m\|.
\end{aligned}$$

Hence, the result follows directly from Theorem 4.1. □

6 Example

Let $H = L^2((0,1);\mathbb{R})$. Consider the following partial delay differential equations

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}(t,x) - \frac{\partial^2 w}{\partial x^2}(t,x) \\
&\qquad = F(t, x, \frac{\partial^2 w}{\partial x^2}(t,x), \frac{\partial^2 w}{\partial x \partial t}(t,x), \frac{\partial^2 w}{\partial x^2}(t-\tau,x), \frac{\partial^2 w}{\partial x \partial t}(t-\tau,x)), \\
&\qquad\qquad x \in (0,1), \quad t > 0, \\
&w(\xi,x) = h_1(\xi,x), \quad \frac{\partial w}{\partial t}(\xi,x) = g_1(\xi,x) \quad \text{for all} \quad \xi \in [-\tau,0], \quad x \in (0,1) \\
&\qquad \text{and} \quad w(t,0) = w(t,1) = 0, \quad t \in [0,T], \quad 0 < T < \infty,
\end{aligned} \tag{41}$$

where F is a sufficiently smooth nonlinear function, h_1 and g_1 are given locally Hölder continuous functions on $[-\tau, 0]$.

We define an operator A as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = H_0^1(0,1). \tag{42}$$

Here clearly the operator A is self-adjoint with the compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm

$$\|x\|_{1/2} = \|A^{1/2}x\|, \quad x \in D(A^{1/2}),$$

and we denote this space by $H_{1/2}$.

The equation (41) can be reformulated as the following abstract equation in H:

$$\begin{aligned}
&\frac{d^2u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) = f\left(t, u(t), \frac{du}{dt}(t), u(t-\tau), \frac{du}{dt}(t-\tau)\right), \quad t > 0, \\
&\qquad u(t) = h(t), \quad u'(t) = g(t) \quad \text{for all} \quad t \in [-\tau, 0],
\end{aligned} \tag{43}$$

where $u(t)(x) = w(t,x)$, $h(t)(x) = h_1(t,x)$, $g(t)(x) = g_1(t,x)$, the linear operator A is given by equation (42) and the function f is defined from $[0,T] \times D(A) \times D(A^{1/2}) \times D(A) \times D(A^{1/2})$ into H such that

$$f\left(t,\, u(t),\, \frac{du}{dt}(t),\, u(t-\tau),\, \frac{du}{dt}(t-\tau)\right)(x) = F\left(t,\, x,\, \frac{\partial^2 w}{\partial x^2}(t,x),\, \frac{\partial^2 w}{\partial x \partial t}(t,x),\, \frac{\partial^2 w}{\partial x^2}(t-\tau,x),\, \frac{\partial^2 w}{\partial x \partial t}(t-\tau,x)\right).$$

It can be verified that the assumptions of Theorem 3.1 for (43) are satisfied and hence the existence of a unique solution of (43) is guaranteed which in turn ensures the existence of a unique solution to (41).

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An Extension of Barbashin–Krasovskii–LaSalle Theorem to a Class of Nonautonomous Systems

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Abstract: In this paper we give an extension of the Barbashin-Krasovskii-LaSalle theorem to a class of time-varying dynamical systems, namely the class of systems for which the restricted vector field to the zero-set of the time derivative of the Liapunov function is time invariant and this set includes some trajectories. Our goal is to improve the sufficient conditions for the case of uniform asymptotic stability of the equilibrium. We obtain an extension of a well-known result of linear zero-state detectability to nonlinear systems, as well as a robust stabilizability result of nonlinear affine control systems.

Keywords: *Invariance Principle; Liapunov functions; detectability; robust stabilizability.*

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1 Introduction and Main Results

Let us consider the following time-varying dynamical system:

$$\dot{x} = f(t, x), \quad x \in D, \ t \in R, \tag{1}$$

where D is a domain in R^n containing the origin ($0 \in D \subset R^n$). About f we suppose the following:

1) $f(t, 0) = 0$, for any $t \in R$;

2) uniformly continuous in t, uniformly in $x \in D$, i.e. $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ such that $\forall t_1, t_2 \in R, |t_1 - t_2| < \delta_\varepsilon$ and $\forall x \in D$, $\| f(t_1, x) - f(t_2, x) \| < \varepsilon$;

3) uniformly local Lipschitz continuous in x for any $t \in R$, i.e. for any compact set $K \subset D$, there exists a positive constant $L_K > 0$ such that:

$$\|f(t, x) - f(t, y)\| \le L_K \|x - y\| \quad \text{for any } x, y \in K \text{ and } t \in R.$$

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4) bounded in time, that means there exists a continuous function $M : D \to R$ such that:

$$\|f(t,x)\| \leq M(x) \quad \text{for any } t \in R.$$

With these hypotheses we know that for any $(t_0, x_0) \in R \times D$ there exists a unique solution of the Cauchy problem:

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{2}$$

with the initial data (t_0, x_0). We denote by $x(t; t_0, x_0)$ this solution. One can define this solution for $t \in (t_0 - T, t_0 + T)$ where $T = \sup_{r>0, B_r(x_0) \subset D} \frac{r}{\|f\|_{B_r(x_0)}}$, the supremum is taken over all positive radius such that the ball centered around x_0, $B_r(x_0) = \{x \in R^n | \|x - x_0\| < r\}$, is completely included in D and $\|f\|_{B_r(x_0)} = \sup_{(t,x) \in R \times \bar{B}_r(x_0)} \|f(t,x)\|$ is a supremum norm of f with respect to $B_r(x_0)$ (where is no confusion we denote $B_r = B_r(0)$). The function $\gamma_{t,t_0}(x_0) = x(t; t_0, x_0)$ is well defined for some bounded open set S, $\gamma_{t,t_0} : S \to U \subset D$ (with U open and bounded) and it is Lipschitz continuous with a Lipschitz constant given by $L = exp(L_U|t - t_0|)$ (L_U being the Lipschitz constant associated to f, as above, on the compact set $\bar{U}$). All these results can be found in any textbook of differential equations (for instance see [6]).

Our concern regards the stability behaviour of the equilibrium point $\bar{x} = 0$. First we recall some definitions about stability (in Liapunov sense).

Definition 1.1 We say the equilibrium point $\bar{x} = 0$ for (1) is *uniformly stable*, if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $t_0 \in R$ and $x_0 \in R$ with $\|x_0\| < \delta_\varepsilon$ the solution $x(t; t_0, x_0)$ is defined for all $t \geq t_0$ and furthermore $\|x(t; t_0, x_0)\| < \varepsilon$, for every $t > t_0$.

Definition 1.2 We say that the equilibrium point $\bar{x} = 0$ for (1) is *uniformly asymptotic stable*, if it is uniformly stable and there exists a $\delta > 0$ such that for any $t_0 \in R$ and $x_0 \in D$ with $\|x_0\| < \delta$ the solution $x(t; t_0, x_0)$ is defined for every $t \geq t_0$ and $\lim_{t\to\infty} x(t; t_0, x_0) = 0$.

If in the definition of uniform stability we interchange "there exists $\delta_\varepsilon > 0$" with "for any $t_0 \in R$" (thus δ will depend on ε and t_0, δ_{ε,t_0}) then the equilibrium is said (just) *stable*. If we proceed the same in the second definition we obtain that the equilibrium is *asymptotic stable*. For time-invariant systems there is no distinction between uniform stability and stability, or uniform asymptotic stability and asymptotic stability. In general case, the uniform (asymptotic) stability implies (asymptotic) stability, but the converse is not true (see for instance [7]).

We say that the dynamics (1) has a *positive invariant set* N if for any $t_0 \in R$ and $x_0 \in N$ the solution $x(t; t_0, x_0) \in N$ for all $t \geq t_0$ for which it is well-defined. Then it makes sense to consider the *dynamics restricted to* N, i.e. the function:

$$X : R^+ \times R \times N \to N, \quad X(\tau; t_0, x_0) = x(\tau + t_0; t_0, x_0),$$

where τ runs up to a maximal value depending on (t_0, x_0). Moreover, by considering the case of f from (1) we obtain that $X(\tau; t_0, 0) = 0$, for any $\tau > 0$, $t_0 \in R$. Therefore we may define the corresponding stability properties of the restricted dynamics as above, where we replace D by N.

The main result of this paper is given by the following theorem:

Theorem 1.1 *Consider the time-varying dynamical system* (1) *for which* f *has the properties* $1) - 4)$. *Suppose there exists a function* $V : D \to R$ *of class* $\mathcal{C}^1$ *such that:*

$H1)$ $V(x) \geq 0$ *for every* $x \in D$ *and* $V(0) = 0$*;*

$H2)$ *There exists a continuous function* $W : D \to R$ *such that*

$$\frac{dV}{dt}(t, x) = \nabla V(x) \cdot f(t, x) \leq W(x) \leq 0.$$

$H3)$ *Let* $E = \{x \in D | W(x) = 0\}$ *denote the zero-set (or kernel) of* W*; suppose that* f *restricted to* E *is time-invariant (i.e.* $f(t, x) = f(t_0, x)$*, for every* $t \in R$ *and* $x \in E$*). Let us denote by* N *the maximal positive invariant set in* E*, i.e. for any* $x_0 \in N$ *and* $t_0 \in R$*,* $x(t; t_0, x_0) \in N$*, for every* $t \in [t_0, t_0 + T_{x_0})$ *in the maximal interval of definition of the solution.*

Then the dynamics (1) *has at* $\bar{x} = 0$ *an uniformly asymptotic stable equilibrium point if and only if the dynamics restricted to* N *has an asymptotic stable equilibrium at* $\bar{x} = 0$.

Even if it has appeared in the literature in a more general setting (I refer to [23]), it is worth mentioning the form the invariance principle takes in this context:

Theorem 1.2 (Invariance principle) *Consider the time-varying dynamical system* (1) *for which* f *has the properties* $1) - 4)$. *Suppose there exists a function* $V : D \to R$ *of class* $\mathcal{C}^1$ *such that:*

$H1)$ *It is bounded below, i.e.* $V(x) \geq V_0$ *for any* $x \in D$ *for some* $V_0 \in R$*;*

$H2)$ *There exists a continuous function* $W : D \to R$ *such that*

$$\frac{dV}{dt}(t, x) = \nabla V(x) \cdot f(t, x) \leq W(x) \leq 0.$$

$H3)$ *Let* $E = \{x \in D | W(x) = 0\}$ *denote the zero-set (or kernel) of* W*; suppose that* f *resticted to* E *is time-invariant (i.e.* $f(t, x) = f(t_0, x)$*, for any* $t \in R$ *and* $x \in E$*). Let us denote by* N *the maximal positive invariant set included in* E*, i.e. for any* $x_0 \in N$ *and* $t_0 \in R$*,* $x(t; t_0, x_0) \in N$*, for any* $t \in [t_0, t_0 + T_{x_0})$ *in the maximal interval of definition of the solution.*

Then any bounded trajectory of (1) *tends to* N*, i.e. if* (t_0, x_0) *is the initial data for a bounded solution included in* D *then:*

$$\lim_{t \to \infty} d(x(t; t_0, x_0), N) = 0. \tag{3}$$

Remark 1.1 There are two directions in which Theorem 1.1 generalizes the well-known Barbashin–Krasovskii–LaSalle's theorem (see [15], [16] or [14]); firstly it requires V to be only nonnegative and not strictly positive, secondly it applies to the case of time-varying dynamical systems. Several extensions were presented in literature dealing with the stability result.

The first result that I am referring to is Lemma 5 from [5]. In that lemma only autonomous systems are considered and the restricted dynamics is required to be attractive in the sense that all trajectories should tend to the origin. I point out that only the requirement of attractivity is not enough; this can be seen in a trivial case, namely the 2 dimensional system given by Vinograd (conform [7]), for which the origin is an attractive equilibrium but not stable, and take $V \equiv 0$. I need to point out also that, for the purposes of their paper [5], their Lemma 5 can be replaced by Theorem 3.1 of this paper without affecting the other results from their paper.

A second result has appeared in [22] but not in a general and explicit form as here. In fact in [22] the author is concerned with the stability of the large-scale systems which are already decomposed in triangular form. Thus, this result solves the problem only in the case when we can perform the observability decomposition of the dynamics (1) with respect to the output $W(x)$. This case requires a supplementary condition, namely the codistribution span by $dW, dL_f W, \ldots, dL_f^n W$ to be of constant rank on D (see [12]). Among other requirements, this geometric condition implies also that N is a manifold, whereas we do not assume here this rather strong assumption.

I acknowledge the existence of a recently published paper that deals with a similar extension of the Liapunov theorem, yet only for autonomous systems ([10]). However, we were unaware of this result at the time we were working in this field (i.e. 1993–1995). More recently, in [11], the authors extended to time-varying systems these previous results. It is interesting to note, based on this last paper and historical references therein, the autonomous version of these results were first stated and proved by Boulgakov and Kalitine in [3]. Compared to [11], here we present a stabilizability result (Theorem 4.1) tailored specifically for affine nonlinear control systems.

Remark 1.2 Some other papers deal with extensions of the invariance principle for nonautonomous systems. In two special cases, when the system is either asymptotically autonomous (in [23]) or asymptotically almost periodic (in [19]), the bounded solution tends to the largest pseudo-invariant set in E. However they use the classical Liapunov theorems to obtain the uniform boundedness of the solutions. Thus they require the existence of a strictly positive definite function playing the rôle of Liapunov function, while here we require only nonnegativeness of the Liapunov-like function. In other approaches an additional auxiliary function is assumed and by means of extra conditions the time in E is controlled (see the results of Salvadori or Matrosov, e.g. in [20]). In a third approach an extra condition on $\dot{V}$ is considered without any additional condition on the vector field; such an approach is considered in [1].

Remark 1.3 The condition that the restricted dynamics to be uniformly asymptotically stable is necessary and sufficient. Thus it is a center-manifold-type result where a knowledge about a restricted dynamics to some invariant set implies the same property of the whole dynamics. We point out here that the set N does not need to be a manifold.

Remark 1.4 One could expect that simple stability of the restricted dynamics would imply uniform stability of the restricted dynamics. But this is not true as we can see from the following example:

Example 1.1 Consider the following autonomous planar system:

$$\left\{ \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = -y^3 \end{array} \right. , \quad (x,y) \in R^2, \tag{4}$$

The solution of the system is given by $(x,y) \to \left(x + ln(1+y^2 t), \frac{y}{\sqrt{1+y^2 t}} \right)$. It is obvious that the equilibrium is not stable but if we take $V = y^2$ we have $\frac{dV}{dt} = -2y^4$ and on the set $E = N = \{(x,0),\ x \in R\}$ the dynamics is trivial stable $\dot{x} = 0$.

The problem is not the nonisolation of the equilibrium, but the existence of some invariant sets in any neighborhood of the equilibrium;

Remark 1.5 Theorem 1.2 is the natural generalization of the invariance principle to the class of systems considered in this paper. The conclusion of this theorem applies only to bounded trajectories. Thus we have to know apriori which solutions are bounded. Since they are bounded we can extend them indefinitely in positive time. Thus it makes sense to take the limit $t \to \infty$ in (3). We mention that a more general invariance principle can be obtained even under weaker conditions than those from here (see [23]).

The organization of the paper is the following: in the next section we give the proof of these results. In the third section we consider the autonomous case and we present the systemic consequences related to the nonlinear Liapunov equation and a special type of zero-state detectability. In the fourth section we consider a nonlinear Riccati equation (or Hamilton-Jacobi equation) and we present a result of robust stabilizability by output feedback. The last section contains the conclusions and is followed by the bibliography.

2 Proof of the Main Results

We prove by contradiction the uniform stability of the equilibrium. For this, we construct a $\mathcal{C}^1$-convergent sequence of solutions that are going away from the origin and whose limit is a trajectory, thus contradicting the hypothesis.

For the uniform asymptotic stability, we prove first that the ω-limit set of bounded trajectories is included in N (implicitly proving the invariance principle — Theorem 1.2) and then we adapt a classical trick (used for instance in Theorem 34.2 from [7]) that the convergence of trajectories in ω-limit set will attract the convergence of the bounded trajectory itself. In both steps we use essentially the time-invariant property of f restricted to E. In proving the uniform stability we also obtain that the solution can be defined on the whole positive real set (can be completely extended in future).

Theorem 1.2 (the invariance principle) will follow simply from a lemma that we state during the proof of uniform attractivity.

First we need a lemma.

Lemma 2.1 *Let f be a vector field defined on a domain D and having the properties 1-4 as above. Let $(t_i)_i$ be a sequence of real numbers and $(w_i)_i$, $w_i : [a,b] \to D$ be a sequence of trajectories for the time-translated vector field f with t_i, i.e. $\dot{w}_i(t) = f(t+t_i, w_i(t))$.*

If the trajectories are uniformly bounded, i.e. there exists $M > 0$ such that $\|w_i\|_\infty < M$, for any i, then we can extract a subsequence, denoted also by $(w_i)_i$, uniformly convergent to a function w in $\mathcal{C}^1([a,b];D)$, i.e. $w_i \to w$ and $\dot{w}_i \to \dot{w}$ both uniformly in $\mathcal{C}^0([a,b];D)$.

Proof We apply the Ascoli-Arzelà lemma twice: first to extract a subsequence such that $(w_i)_i$ is uniformly convergent and second to extract further another subsequence such that $(\dot{w}_i)_i$ is uniformly convergent. Then we obtain that $\lim_i \frac{d}{dt} w_i = \frac{d}{dt} \lim_i w_i$.

1. We verify that $(w_i)_i$ are uniformly bounded and equicontinuous. The uniformly boundedness comes from $\|w_i\|_\infty < M$. The equicontinuity comes from the uniformly boundedness of the first derivative. Indeed, since $\|w_i\| \le M$, the closed ball $\bar{B}_M$ is compact and $f(t,\cdot)$ is continuous on $\bar{B}_M$, there exists a constant A such that $\|f(t,x)\| \le A$, for any $(t,x) \in R \times \bar{B}_M$. Then

$$\|\dot{w}_i(t)\| = \|f(t+t_i, w_i(t))\| \le A, \text{ for any } i \text{ and } t \in [a,b].$$

Thus $(w_i)_i$ is relatively compact and we can extract a subsequence, that we denote also by $(w_i)_i$, which is uniformly convergent to a function $w \in \mathcal{C}^0([a,b];D)$.

2. We prove that $(\dot{w}_i)_i$ is relatively compact. We have already proved the uniform boundedness $\|\dot{w}_i\|_\infty \le A$. For the equicontinuity we use both the uniform continuity in t and uniform local Lipschitz continuity in x, of f. Let L_M be the uniform Lipschitz constant corresponding to the compact set $\bar{B}_M$. Then

$$\|\dot{w}_i(t_1) - \dot{w}_i(t_2)\| = \|f(t_i + t_1, w_i(t_1)) - f(t_i + t_2, w_i(t_2))\| \le$$

$$\|f(t_i + t_1, w_i(t_1)) - f(t_i + t_2, w_i(t_1))\| + \|f(t_i + t_2, w_i(t_1)) - f(t_i + t_2, w_i(t_2))\|.$$

Let $\varepsilon > 0$ be arbitrarily. Then we choose δ_1 such that $\|f(s_1,x) - f(s_2,x)\| < \frac{\varepsilon}{2}$, for any $|s_1 - s_2| < \delta_1$ and $x \in \bar{B}_M$. On the other hand: $\|f(t_i + t_2, w_i(t_1)) - t(t_i + t_2, w_i(t_2))\| \le L_M \|w_i(t_1) - w_i(t_2)\| \le L_M A |t_1 - t_2|$. Then we choose $\delta = \min(\delta_1, \frac{\varepsilon}{2L_M A})$. Then the left-hand side from the above inequality is also bounded by $\frac{\varepsilon}{2}$ for any t_1, t_2 with $|t_1 - t_2| < \delta$. Thus $\|\dot{w}_i(t_1) - \dot{w}_i(t_2)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for any i and $t_1, t_2 \in [a,b]$, $|t_1 - t_2| < \delta$.

We can now extract a second subsequence from $(w_i)_i$ such that $(\dot{w}_i)_i$ is also uniformly convergent and this ends the proof of lemma. □

Proof of uniform stability in Theorem 1.1.

Let us assume that the equilibrium is not uniformly stable. Then there exists $\varepsilon_0 > 0$ such that for any δ, $0 < \delta < \varepsilon_0$ there are x_0, t and $\Delta > 0$ such that $\|x_0\| < \delta$ and $\|x(t + \Delta; t, x_0)\| = \varepsilon_0$, $\|x(t + \tau; t, x_0)\| < \varepsilon_0$, for $0 \le \tau < \Delta$. We choose ε_0 (eventually by shrinking it) such that $\bar{B}_{\varepsilon_0} \cap N$ is included in the attraction domain of the origin (for the restricted dynamics).

By choosing a sequence $(\delta_i)_i$ converging to zero we obtain sequences $(x_{0i})_i$, $(t_i)_i$ and $(\Delta_i)_i$ such that: $\|x_{0i}\| \to 0$ and $\|x(t_i + \Delta_i; t_i, x_{0i})\| = \varepsilon_0$.

Let $\delta < \varepsilon_0$ be such that for any $z_0 \in B_\delta \cap N$ we have $\|x(t; 0, z_0)\| < \frac{\varepsilon_0}{2}$ for any $t > 0$ (such a choice for δ is possible since the dynamics restricted to N is stable). Let i_0 be such that $\delta_i < \delta$, for $i > i_0$. We denote by $(u_i)_{i > i_0}$ the time moments such that $\|x(t_i + u_i; t_i, x_{0i})\| = \delta$ and $\|x(t; t_i, x_{0i})\| > \delta$ for $t > t_i + u_i$. Since the spheres $\bar{S}_{\varepsilon_0}$ and $\bar{S}_\delta$ are compact we can extract a subsequence (indexed also by i) such that both $x_i = x(t_i + \Delta_i; t_i, x_{0i})$ and $y_i = x(t_i + u_i; t_i, x_{0i})$ are convergent to x^*, respectively to y^*; $x_i \to x^*$, $y_i \to y^*$, $\|x^*\| = \varepsilon_0$, $\|y^*\| = \delta$. Since V is continuously nonincreasing on trajectories and $\lim_i V(x_{0i}) = 0$, we get $V(x^*) = V(y^*) = 0$. Therefore $x^*, y^* \in N$.

Suppose $\|f(x,t)\| \le A$ on $\bar{B}_{\varepsilon_0}$, for some $A > 0$. Then one can easily prove that $\Delta_i - u_i \ge \frac{\varepsilon_0 - \delta}{A} = T_1$, for any $i > i_0$ (i.e. the flight time between two spheres of radius δ and ε_0 has a lower bound).

Define now the time-translated vector fields $f_i(t,x) = f(t + t_i + u_i, x)$ and denote by $w_i : [0, T_1] \to \bar{B}_{\varepsilon_0}$ the time-translated solutions $w_i(t) = x(t + t_i + u_i; t_i, x_{0i})$. Then: $\dot{w}_i(t) = f_i(t, w_i(t))$, $0 \le t \le T_1$. By applying Lemma 2.1 we get a subsequence uniformly convergent to a trajectory $w^1 : [0, T_1] \to \bar{B}_{\varepsilon_0} \cap N$, such that $w^1(0) = \lim_i w_i(0) = y^*$ and $\|w^1(t)\| > \delta$, for $0 < t \le T_1$. If $\|w^1(T_1)\| < \varepsilon_0$ we obtain that $\Delta_i - u_i - T_1 > \frac{\varepsilon_0 - \|w^1(T_1)\|}{A}$, for some $i \ge i_1 > i_0$. Then, we denote $T_2 = T_1 + \frac{\varepsilon_0 - \|w^1(T_1)\|}{A}$ and we repeat the scheme. We obtain another sequence which is uniformly convergent to a trajectory $w^2 : [0, T_2] \to \bar{B}_{\varepsilon_0} \cap N$ such that $w^2(0) = y^*$, $\|w^2(t)\| > \delta$, $0 < t \le T_2$ and $w^2(t) = w^1(t)$, for $0 \le t \le T_1$.

Thus we extend each trajectory $w^k : [0, T_k] \to \bar{B}_{\varepsilon_0} \cap N$ to a trajectory $w^{k+1} : [0, T_{k+1}] \to \bar{B}_{\varepsilon_0} \cap N$ such that $T_{k+1} \geq T_k$, $w^{k+1}(t) = w^k(t)$ for $0 \leq t \leq T_k$ and $\|w^{k+1}(t)\| > \delta$, for $0 < t \leq T_{k+1}$.

We end this sequence of extensions in two cases:

1) $\lim_k T_k = T^* < +\infty$ (the limit may be reached in a finite number of steps), in which case we have $\lim_k \|w^k(T_k)\| = \varepsilon_0$ and thus $\lim_k w^k(T_k) = x^*$; or:

2) $\lim_k T_k = +\infty$.

In the first case we obtain a trajectory $w^* : [0, T^*] \to \bar{B}_{\varepsilon_0} \cap N$ such that $w^*(0) = y^*$, $w^*(T^*) = x^*$ with $\|w^*(0)\| = \delta$ and $\|w^*(T^*)\| = \varepsilon_0$. But this is a contradiction with the choice of δ (and of stability of the restricted dynamics).

In the second case we obtain a trajectory $w^* : [0, \infty) \to \bar{B}_{\varepsilon_0} \cap N$ such that $\|w^*(0)\| = \delta < \varepsilon_0$ and $\|w^*(t)\| > \delta$ for $t > 0$. Thus $\lim_{t\to\infty} w^*(t) \neq 0$ contradicting the assumption that $\bar{B}_{\varepsilon_0} \cap N$ is included in the attraction domain of the origin. Now the proof is complete. □.

For the proof of uniformly attractivity we recall a few definitions and results.

Definition 2.1 A point x^* is called *ω-limit point* for the trajectory $x(t; t_0, x_0)$ if there exists a sequence $(t_k)_k$ such that $\lim_{k\to\infty} t_k = \infty$, $x(t; t_0, x_0)$ is defined for all $t > t_0$ and $\lim_k x(t_k; t_0, x_0) = x^*$. The set of all ω-limit points is called the ω-limit set and is denoted by $\Omega(t_0, x_0)$. It characterizes the trajectory $x(t; t_0, x_0)$ and it depends on the initial data (t_0, x_0).

Theorem 2.1 (Birkoff's limit set theorem, see [4]) *A bounded trajectory approaches its ω-limit set, i.e.* $\lim_{t\to\infty} d(x(t; t_0, x_0), \Omega(t_0, x_0)) = 0$, *where* $d(p, S) = \inf_{x\in S} \|p - x\|$ *is the distance between the point* p *and the set* S.

There is also a very useful result about uniformly continuous functions.

Lemma 2.2 (Barbălat's lemma, see [2]) *If* $g : [t_0, \infty) \to \infty$ *is a uniformly continuous function such that the following limit exists and is finite,* $\lim_{t\to\infty} \int_{t_0}^{t} g(\tau)d\tau$, *then* $\lim_{t\to\infty} g(t) = 0$.

Proof of uniform attractivity in Theorem 1.1

We already know that $\bar{x} = 0$ is uniformly stable. What we have to prove is the uniform attractivity.

Let $\varepsilon_0 > 0$ be chosen with the following properties: for any t_0 and $x_0 \in D \cap \bar{B}_{\varepsilon_0}$ the positive trajectory $x(t; t_0, x_0)$ is bounded by ε_1 (i.e. $x(t; t_0, x_0) \in B_{\varepsilon_1}$); for any t_1 and $x_1 \in D \cap B_{\varepsilon_1}$ the trajectory $x(t; t_1, x_1)$, $t > t_1$, is bounded by some M; and for any $x_2 \in N \cap B_{\varepsilon_1}$ the trajectory $x(t; t_0, x_2)$ tends to the origin $\lim_{t\to\infty} x(t; t_0, x_2) = 0$. We are going to prove that $\lim_{t\to\infty} x(t; t_0, x_0) = 0$.

Let us consider the ω-limit set $\Omega(t_0, x_0)$. It is enough to prove that $\Omega(t_0, x_0) = \{0\}$, because of Birkoff's limit set theorem.

Let $x^* \in \Omega(t_0, x_0)$ and suppose $x^* \neq 0$. Let us denote by $x(t) = x(t; t_0, x_0)$ and $g(t) = \nabla V(x(t)) \cdot f(t, x(t))$. Since the solution is continuous and bounded, so is $g(t)$. On the other hand

$$V(x(t)) = V(x_0) + \int_{t_0}^{t} g(\tau)d\tau.$$

Since $\dot{x}(t) = f(t, x(t))$ and $x(t)$ is bounded we obtain that it is also uniformly continuous. Thus $g(t)$ is also uniformly continuous (recall we have assumed $f(\cdot, x)$ is uniformly continuous in t). Let $(t_k)_k$ be a sequence that renders x^* a ω-limit point. Then $\lim_k V(x(t_k)) = V(\lim_k x(t_k)) = V(x^*)$. Since $V(x(t))$ is a decreasing function bounded below, there exists the limit: $\lim_{t\to\infty} V(x(t)) = V(x^*)$. Now, applying Barbălat's lemma we obtain $\lim_{t\to\infty} g(t) = 0$ or $W(x^*) = 0$. Thus $\Omega(t_0, x_0) \subset E$, the kernel of W.

In this point we need a result about the behaviour of solutions starting at x^*. We mention that the following lemma is a consequence of Theorem 3 from [23]. But, since we are under stronger conditions, we have found a simpler proof that we are going to present here (our conditions are stronger because we need to obtain uniform stability and consequently boundedness of the solutions when Liapunov function is only positive semidefinite, which overall means a weaker condition).

Lemma 2.3 *The positive trajectory starting at x^* is included in E and thus the Ω-limit set is a positive invariant set included in N.*

Proof Let $\tau > 0$ be an arbitrary time interval. Let $(t_k)_k$ be the sequence that renders x^* a ω-limit point for the trajectory $x(t) = x(t; t_0, x_0)$. Then, if we denote by $x_k = x(t_k)$ we have $\lim_k x_k = x^*$. Consider the following sequence of functions: $w_k : [0, \tau] \to D,\ w_k(t) = x(t + t_k; t_k, x^*)$. We have chosen x_0, t_0 such that all these functions are bounded by M, i.e. $\|w_k\|_\infty < M$. We have $w_k(0) = x^*$ and $V(w_k(t)) \leq V(x^*)$. Let us denote by $y_k^t = x(t+t_k)$, for any $0 \leq t \leq \tau$, and let L_M be the Lipschitz constant of f on the compact $\bar{B}_M$. Then: $\|y_k^t - w_k(t)\| \leq e^{L_m t}\|x_k - x^*\|$ and, since $\lim_k x_k = x^*$ we get $\lim_k \|y_k^t - w_k(t)\| = 0$. On a hand, since $V(x^*) = \lim_{t\to\infty} V(x(t))$ and V is nonincreasing on trajectories we have $V(y_k^t) > V(x^*)$ and also $\lim_k V(y_k^t) = V(x^*) = \lim_k V(w_k(t))$. On the other hand, since $(w_k)_k$ are uniformly bounded we apply Lemma 2.1 and we obtain a subsequence uniformly convergent to a function $w \in \mathcal{C}^1([0, \tau]; D \cup \bar{B}_M)$. Obviously $V(w(t)) = V(x^*)$ for any $0 \leq t \leq \tau$. Thus $W(w(t)) = 0$ and $w(t) \in E$. On the other hand, since f is continuous in (t, x) we obtain that w is an integral curve of f, i.e. $\dot{w}(t) = f(t_*, w(t))$, for $0 \leq t \leq \tau$ and any t_*. In particular, for $t_* = t_k$ we get $w(t)$ is a solution of the same equation as $w_k(t)$ and $w(0) = w_k(0) = x^*$. By the uniqueness of the solution they must coincide. Then $x(t + t_k; t_k, x^*) \in E$ for $0 \leq t \leq \tau$. But τ was arbitrarily; thus $x(t; t_0, x^*) \in E$ for any t and then $x^* \in N$. □

Since the trajectory starting at x^* is included in N, it should converge to the origin (the equilibrium point). Let us denote by $\varepsilon = \frac{\|x^*\|}{2}$. From uniform stability there exists a $\delta > 0$ such that for any $\tilde{x} \in D$, $\|\tilde{x}\| < \delta$ implies $\|x(t_2; t_1, \tilde{x})\| < \varepsilon$, for any $t_2 > t_1$. Let Δt be a time interval such that $\|x(t; 0, x^*)\| < \frac{\delta}{2}$ for any $t > \Delta t$. We consider the compact set C, the $\frac{\delta}{2}$-neighborhood of the compact curve $\Gamma = \{x(t; 0, x^*) | 0 \leq t \leq \Delta t\}$:

$$C = \{x \in D | d(x, \Gamma) \leq \frac{\delta}{2}\} = \bigcup_{t\in[0,\Delta t]} B_{\delta/2}(x(\bar{t}; 0, x^*))$$

which is the union of the closed balls centered at $x(t; 0, x^*)$ and of radius $\frac{\delta}{2}$. We set $\delta_1 = \frac{\delta}{2} exp(-L_C \Delta t)$ where L_C is the uniform Lipschitz constant of f on the compact set C. Since the solution is uniformly Lipschitz with respect to the initial point x_0 we have that for any $t_1 \in R$ and x_1 such that $\|x_1 - x^*\| < \delta_1$ we get: $\|x(t_1 + \Delta t; t_1, x_1) - x(\Delta t; 0, x^*)\| < \frac{\delta}{2}$ and then $\|x(t_1 + \Delta t; t_1, x_1)\| < \delta$. Furthermore, from the choice of δ we obtain that $\|x(t_1 + \tau; t_1, x_1)\| < \varepsilon$, for any $\tau > \Delta t$ or $\|x(t_1 + \tau; t_1, x_1) - x^*\| > \varepsilon$, for any $\tau > \Delta t$.

Now we pick a t_n such that $\|x(t_n;t_0,x_0) - x^*\| < \delta_1$. Then, from the previous discussion $\|x(t_n+\tau;t_0,x_0) - x^*\| > \varepsilon$, for any $\tau > \Delta t$ which contradicts the limit $\lim_k x(t_k;t_0,x_0) = x^*$. This contradiction comes from the hypothesis that $x^* \neq 0$. Thus $\Omega(t_0,x_0) = \{0\}$ and now the proof is complete. □

Proof of Theorem 1.2 (The invariance principle)
If $x(t;t_0,x_0)$ is a bounded trajectory then, from Birkoff's limit set theorem it approaches its ω-limit set. On the one hand we can use Barbălat's lemma and prove that W vanishes on ω-limit set of bounded trajectories. On the other hand, as we have proved in Lemma 2.3, the ω-limit set is invariant and included in N. Thus the bounded trajectory approaches the set N. □

3 The Autonomous Case: Consequences in Nonlinear Control Theory

Consider the following inputless nonlinear control system:

$$S\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}, \quad x \in D \subset R^n, \ y \in R^p, \tag{5}$$

such that $f(0) = 0$, $h(0) = 0$ and D a neighborhood of the origin. Suppose f is local Lipschitz continuous and h continuous on D. Then denote by $x(t,x_0)$ the flow generated by f on D (i.e. the solution of $\dot{x} = f(x)$, $x(0) = x_0$), by $E = ker\ h = \{x \in D | h(x) = 0\}$, the kernel of h and by N the maximal positive invariant set included in E, i.e. the set $N = \{\tilde{x} \in D | h(x(t,\tilde{x})) = 0$ for any $t \geq 0$ such that $x(t,\tilde{x})$ has sense$\}$.

We present two concepts of detectability for (5). The first one has been used by many authors (see for instance [13]).

Definition 3.1 The pair (h,f) is called *zero-state detectable* (or *z.s.d.*) if $\bar{x} = 0$ is an attractive point for the dynamics restricted to N, i.e. there exists an $\varepsilon_0 > 0$ such that for any $x_0 \in N$, $\|x_0\| < \varepsilon_0$, $\lim_{t\to\infty} x(t,x_0) = 0$.

Definition 3.2 The pair (h,f) is called *strong zero-state detectable* (or *strong z.s.d.*) if $\bar{x} = 0$ is an asymptotical stable equilibrium point for the dynamics restricted to N, i.e. it is zero-state detectable and for some ε_0 and for any $x_0 \in N$ with $\|x_0\| < \varepsilon_0$, $\lim_{t\to\infty} x(t,x_0) = 0$.

We see that strong z.s.d. implies z.s.d., but obviously the converse is not true.

In this framework, as a consequence of the main result we can state the following theorem.

Theorem 3.1 *For the inputless nonlinear control system* (5) *with f local Lipschitz continuous and h continuous, consider the following nonlinear Liapunov equation:*

$$\nabla V \cdot f + \|h\|^q = 0 \tag{6}$$

or the following nonlinear Liapunov inequality:

$$\nabla V \cdot f + \|h\|^q \leq 0 \tag{7}$$

for some $q > 0$. Suppose there exists a positive semidefinite solution of (6) *or* (7) *of class $\mathcal{C}^1$ defined on D such that $V(0) = 0$.*

Then the pair (h,f) is strong zero-state detectable if and only if $\bar{x} = 0$ is an asymptoticaly stable equilibrium for the dynamics (5).

Below we give an example.

Example 3.1 Consider the dynamics:

$$\begin{array}{l}\dot{x}_1 = -x_1^3 + \Psi(x_2) \\ \dot{x}_2 = -x_2^3\end{array} \quad , \quad (x_1, x_2) \in R^2, \tag{8}$$

where $\Psi : R \to R$ is local Lipschitz continuous, $\Psi(0) = 0$ and there exist constants $a > 0$, $b \geq 1$ such that:

$$|\Psi(x)| \leq a|x|^b, \quad \forall x_2.$$

If we choose as output function $h(x) = x_2^2$ we see that the pair (h, f) is strong zero-state detectable; indeed, the set $E = \{x \in R^2 | h(x) = 0\} = \{(x_1, 0) | x_1 \in R\}$ and the dynamics restricted to E is $\dot{x}_1 = -x_1^3$ which is asymptotically stable.

Now, if we choose $V(x) = \frac{x_2^2}{2}$ we have $\dot{V} = -x_2^4$ and thus V is a solution of the Liapunov equation (6) with $q = 2$. Then, the equilibrium is asymptotically stable, as a consequence of the Theorem 3.1.

On the other hand we can explicitely solve for x_2: $x_2(t) = \frac{x_{20}}{\sqrt{2(1+x_{20}^2 t)}}$ and then we have: $|\Psi(x_2(t))| \leq C(1 + Bt)^{-1/2}$ for some $B, C > 0$ and any $t \geq 0$. Now the asymptotic stability follows as a consequence of Theorem 68.2 from [7] (stability under perturbation).

4 An Application to Robust Stabilizability

We present here, as an application, a robust stabilizability result for a nonlinear affine control system. In fact it is an absolute stability result about a particular situation. More general results about absolute stability for nonlinear affine control system will appear in a forthcoming paper. We base our approach on the existence of a positive semidefinite solution of some Hamilton-Jacobi equation or inequality. Discussions about solutions of this type of equation may be found in [21].

Consider the following single input–single output control system:

$$\left\{\begin{array}{lcl}\dot{x} & = & f(x) + g(x)u \\ y & = & h(x)\end{array}\right. \quad , \quad x \in D \subset R^n, \; u, y \in R, \tag{9}$$

where f and g are local Lipschitz continuous vector fields on a domain D including the origin, h is a local Lipschitz real-valued function on D, and $f(0) = 0$, $h(0) = 0$. Consider also a local Lipschitz output feedback:

$$\varphi : R \to R, \quad \varphi(0) = 0. \tag{10}$$

We define now two classes of perturbations associated to this feedback. Let $a > 0$ be a positive real number. The first class contains time-invariant perturbations:

$$P_1 = \{p : R \to R \; , \; p \text{ is local Lipschitz, } p(0) = 0 \text{ and } |p(y)| < a|\varphi(y)| \; , \; \forall y \neq 0\}$$

while the second class is composed by time-varying perturbations:

$$P_2 = \{p : R \times R \to R \; , \; p(y,t) \text{ is local Lipschitz in } y \text{ for } t \text{ fixed and uniformly continuous in } t$$
$$\text{for any } y \text{ fixed} \; , \; p(0,t) \equiv 0 \text{ and there exists } \varepsilon > 0 \text{ such that } |p(y,t)| < (a - \varepsilon)|\varphi(y)| \; , \; \forall y \neq 0, t\}$$

Now we can define more precisely the concept of robust stability.

Definition 4.1 We say the feedback (10) *robustly stabilizes* the system (9) *with respect to the class* $P_1 \cup P_2$ if for any perturbation $p \in P_1 \cup P_2$ the closed-loop with the perturbed feedback $\varphi + p$ has an asymptotically stable equilibrium at the origin.

In other words, we require that the origin to be asymptotically stable for the dynamics:

$$\dot{x} = f(x) + g(x)(\varphi(h(x)) + p(h(x), t)) \tag{11}$$

for any $p \in P_1 \cup P_2$. Since the null function belongs to P_1, the feedback φ itself must stabilize the closed-loop too.

With these preparations we can state the result.

Theorem 4.1 *Consider the nonlinear affine control system* (9) *and the feedback* (10)*. Suppose the pair* (h, f) *is strong zero-state detectable and suppose the following Hamilton-Jacobi equation:*

$$\nabla V \cdot f + (\frac{1}{2}\nabla V \cdot g + \varphi \circ h)^2 - (1 - a^2)(\varphi \circ h)^2 = 0, \quad V(0) = 0 \tag{12}$$

or inequality:

$$\nabla V \cdot f + (\frac{1}{2}\nabla V \cdot g + \varphi \circ h)^2 - (1 - a^2)(\varphi \circ h)^2 \leq 0, \quad V(0) = 0 \tag{13}$$

has a positive semidefinite solution V *of class* $\mathcal{C}^1$ *on* D*.*

Then the feedback φ *robustly stabilizes the system* (9) *with respect to the class* $P_1 \cup P_2$*.*

Proof Let us consider a perturbation $p \in P_1 \cup P_2$. Then, the closed-loop dynamics is given by (11). We compute the time derivative of the solution V of (12) with respect to this dynamics:

$$\frac{dV}{dt} = \nabla V \cdot f(x) + \nabla V \cdot g(x)(\varphi(h(x)) + p(h(x), t)).$$

After a few algebraic manipulations we get:

$$\frac{dV}{dt} \leq -(\frac{1}{2}\nabla V \cdot g - p \circ h)^2 + (p \circ h)^2 - a^2(\varphi \circ h)^2.$$

Now, for $p \in P_1$, $\frac{dV}{dt}$ is time-independent and we may take for instance:

$$W(x) = (p(h(x)))^2 - a^2(\varphi(h(x)))^2 \leq 0.$$

For $p \in P_2$, $\frac{dV}{dt}$ is time-dependent and we define:

$$W(x) = -(2a\varepsilon - \varepsilon^2)(\varphi(h(x)))^2 \leq 0.$$

Either a case or the other, we obtain (recall the definitions of P_1 and P_2):

$$\frac{dV}{dt} \leq W(x) \leq 0.$$

The kernel-set of W is given by:

$$E = \{x \in D |\ W(x) = 0\} = \{x \in D |\ h(x) = 0\}.$$

We see that the closed-loop dynamics (11) restricted to E is simply given by $\dot{x} = f(x)$ and is time-independent. Moreover, since we have supposed (h, f) is strong zero-state detectable, it follows that the restricted dynamics to the maximal positive invariant set in E has an asymptotically stable equilibrium at the origin. Now, applying Theorem 1.1, the result follows. □

Let us consider now an example.

Example 4.1 Consider the following planar nonlinear control system:

$$\begin{cases} \dot{x}_1 = -x_1^3 + u, \\ \dot{x}_2 = -x_2^3, \\ y = x_2^3. \end{cases} \tag{14}$$

We are interested to find how robust the feedback $\varphi(y) = y$ is, i.e. how large we can choose a such that φ robustly stabilizes the system (14) with respect to the class $P_1 \cup P_2$.

The Hamilton-Jacobi equation (12) takes the form:

$$-x_1^3 \frac{\partial V}{\partial x_1} - x_2^3 \frac{\partial V}{\partial x_2} + (\frac{1}{2}\frac{\partial V}{\partial x_1} + x_2^3)^2 - (1 - a^2)x_2^6 = 0$$

or:

$$-x_1^3 \frac{\partial V}{\partial x_1} - x_2^3 \frac{\partial V}{\partial x_2} + \frac{1}{4}(\frac{\partial V}{\partial x_1})^2 + x_2 \frac{\partial V}{\partial x_1} + a^2 x_2^6 = 0.$$

A solution of this equation is:

$$V(x_1, x_2) = \frac{a^2}{4} x_2^4.$$

For any $a > 0$ it is positive semidefinite and the system (14) is strong zero-state detectable. Thus, as a consequence of Theorem 4.1, we can choose a arbitrary large such that φ robustly stabilizes the system (14) with respect to the class $P_1 \cup P_2$.

On the other hand, for any feedback Φ, local Lipschitz and:

$$|\Phi(y)| \leq a|y| \quad \text{for some } a > 0,$$

we have seen in the previous example that the closed-loop has an asymptotically stable equilibrium at the origin.

5 Conclusions

In this paper we study an extension of Barbashin-Krasovskii-LaSalle and Invariance Principle to a class of time-varying dynamical systems. We impose two type of conditions on the vector field: one is regularity (we require uniformly continuity with respect to t and uniformly local Lipschitz continuoity and boundedness with respect to x); the other condition requires the vector field to be time-invariant on the zero-set E of an auxiliary function. In this setting we find that the asymptotic behaviour of the dynamics restricted to the largest positive invariant set in E determines the asymptotic stability character of the full dynamics.

Then we study two applications in control theory. The first application concerns the notion of detectability. We give another definition for this notion, called strong zero-state detectability and we show how the existence of a positive semidefinite solution of the Liapunov equation or inequation is related to the asymptotic stability of the equilibrium.

We obtain a nonlinear equivalent of the linear well-known result: if the pair (C, A) is detectable and there exists a positive solution $P \geq 0$ of the Liapunov algebraic equation $A^T P + PA + C^T C = 0$, then the matrix A has all eigenvalues with negative real part.

The second application is on the problem of robust stabilizability. We give sufficient conditions such that a given feedback robustly stabilizes the closed-loop with respect to two sector classes of perturbations (time-invariant and time-varying). The condition is formulated in term of the existence of a positive solution of some Hamilton-Jacobi equation or inequality.

Interesting open questions are to find extensions of the results presented here to the class of switched linear systems (see [8] for an excellent starting point), and to the class of large scale systems (see [18] for a novel approach).

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DTC based on Fuzzy Logic Control of a Double Star Synchronous Machine Drive

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Abstract: The paper discusses a direct torque control (DTC) strategy based on a fuzzy logic for double star synchronous machine (DSSM). The DSSM is built with two symmetrical 3-phase armature winding systems, electrically shifted by 30. A suitable transformation matrix is used to develop a simple dynamic model in view of control. The analysis of the torque in the stator flux linkage reference frame shows that the concept of DTC can be applied in DSSM. A set of voltage vectors corresponding to the switching mode are chosen to offer a maximum voltage and keep the harmonics at a minimum. Further, a switching table specific for DSSM is proposed. Simulations results are given to show the effectiveness and the robustness of our approach.

Keywords: *Double star synchronous machine (DSSM); direct torque control (DTC); fuzzy control; robustness; resistance stator estimator.*

Mathematics Subject Classification (2000): 34C60, 93D09, 93C42.

1 Introduction

AC machines with variable speed drives are widely employed in high power applications. In addition to the multilevel inverter fed electric machine drive systems ([4, 5]), one approach in achieving high power with rating limited power electronic devices is the multiphase inverter system. In a multiphase inverter fed machine, the windings of more than three phases are connected in the same stator of the machine, consequently the current per phase in machine is reduced [7, 19].

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Multiphase machine possess several advantages over conventional three phase machine. These include increasing the inverter output power, reducing the amplitude of torque ripple and lowering the dc link current harmonics. Multiphase drive system improves the reliability, the motor can start and run since the loss of one or many phase [9]. For high power, the use of the synchronous machine specially finds its application in the motorisation at variable speed of the embedded systems [16]. But when they are supplied by thyristor current source inverter, torque ripples of high amplitude appear [23, 15]. Increasing the number of triple armature windings, witch is supplied in relation to each other one, lowers the rate of the torque ripples. Especially, the first harmonic of double star synchronous machine is twelve times the operating frequency of the machine [22].

During the last years, the modeling and control of double star synchronous machine has been the subject of investigations [20, 17, 21, 18, 1]. However the difficulty to control the DSSM supplied by two voltage source inverters (VSIs) is related to the fact that the model in Park frame is high order, multivariable and non linear. In [20] a monovariable approach in view of control of DSSM is proposed. This approach needs precise information about the parameters and rotor position of DSSM. A vector control method has been proposed to achieve a decoupling of rotor flux linkage and torque of DSSM in [16]. The proposed scheme used a rotor position and the torque was controlled via a stator current. One possible alternative to the vector control is the use of direct torque control strategies with several advantages based on possible control directly the stator flux linkage and the torque by selecting appropriate switching voltage vectors of the inverter. The method has been developed for electrical machines and first applied to induction motor drives and now, due to the availability of high-performance DSP process has resulted in the wide application of this technique in AC motor drives. The principle of a DTC consists to select stator voltage vectors according to the differences between the references of stator flux linkage and torque and their actual values. The DTC technique possesses advantages such as less parameter dependency, fast torque response and simple control scheme.

In this paper, we develop a DTC strategy based on a fuzzy logic ([6]) for double star synchronous machine to increase the system performances. A suitable transformation matrix is used to develop a simple dynamic model in view of control. A space vector decomposition control of VSIs fed DSSM is elaborated and DTC strategy is applied to get decoupled control of the flux and torque. In order to improve the static and dynamic control performance of the DSSM, the hysteresis controllers used in conventional DTC is replaced by a fuzzy controller. The main limitation of the DTC is the use the stator resistance for the estimation of stator flux. The variation of the stator resistance due to the temperature and frequency degrades the DTC controller performance especially at low speed. The DTC controller at low speed can be more reliable if the stator resistance is estimated on line and use it in the stator flux estimation algorithm. Several control schemes have been proposed to overcome this problem [8, 10, 14, 11]. To estimate the stator resistance we use a stator current error with PI estimator. The advantages of the proposed control system are shown by simulation involving 5kw DSSM.

2 Formulation Problem

The decoupled control scheme for double star synchronous machine supplied by two inverters is shown in Figure 2.1. The decoupled control bloc is based on DTC control

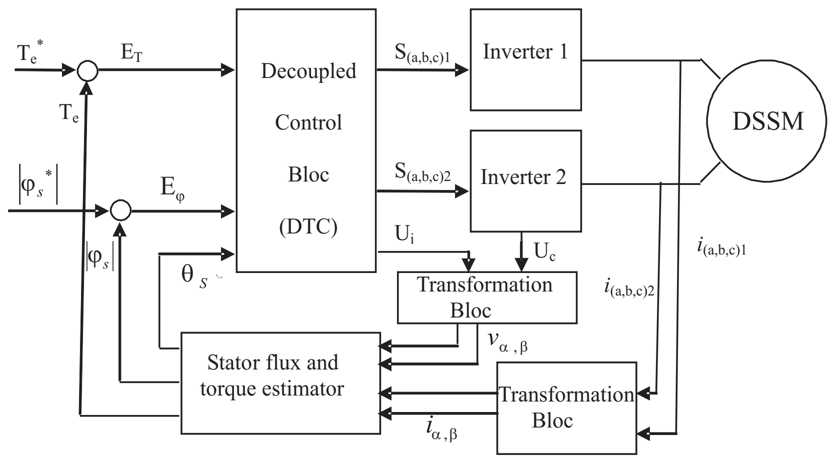

Figure 2.1: Decoupled control scheme for DSSM.

witch control the stator flux linkage and the torque directly, not via controlling the stator current.

2.1 Machine model

The studied system is a DSSM supplied by two VSIs (Figure 2.2). The DSSM is built with two symmetrical 3-phase armature winding systems, electrically shifted by 30° and its rotor is excited by constant current source (Figure 2.3).

In order to obtain a model of double star synchronous machine, we adopt the usual assumptions i.e.: the MMF in air-gap have a sinusoidal repartition and the saturation of the iron in machine is neglected [20, 1]. The stator voltage equation for six-phase can be written as:

$$[v_s] = [R_s]\,[i_s] + \frac{d}{dt}\left([L_{ss}]\,[i_s] + [M_{sr}]\,i_f\right) \tag{1}$$

with

$$[v_s] = [v_{a1}\ v_{a2}\ v_{b1}\ v_{b2}\ v_{c1}\ v_{c2}]^T, \quad [i_s] = [i_{a1}\ i_{a2}\ i_{b1}\ i_{b2}\ i_{c1}\ i_{c2}]^T.$$

The original six dimensional system of the machine can be decomposed into three orthogonal subspaces $(\alpha,\beta),(Z_1,Z_2)$ and (Z_3,Z_4) [1, 24]:

$$[F_\alpha\ F_\beta\ F_{Z1}\ F_{Z2}\ F_{Z3}\ F_{Z4}]^T = [T_s]\,[F_s], \tag{2}$$

where F_s can be voltage, courant or flux,

$$[T_s] = \frac{1}{\sqrt{3}}\begin{bmatrix} \cos(0) & \cos(\gamma) & \cos\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}+\gamma\right) & \cos\left(\frac{4\pi}{3}\right) & \cos\left(\frac{4\pi}{3}+\gamma\right) \\ \sin(0) & \sin(\gamma) & \sin\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}+\gamma\right) & \sin\left(\frac{4\pi}{3}\right) & \sin\left(\frac{4\pi}{3}+\gamma\right) \\ \cos(0) & \cos(\pi-\gamma) & \cos\left(\frac{4\pi}{3}\right) & \cos\left(\frac{\pi}{3}-\gamma\right) & \cos\left(\frac{2\pi}{3}\right) & \cos\left(\frac{5\pi}{3}-\gamma\right) \\ \sin(0) & \sin(\pi-\gamma) & \sin\left(\frac{4\pi}{3}\right) & \sin\left(\frac{\pi}{3}-\gamma\right) & \sin\left(\frac{2\pi}{3}\right) & \sin\left(\frac{5\pi}{3}-\gamma\right) \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{3}$$

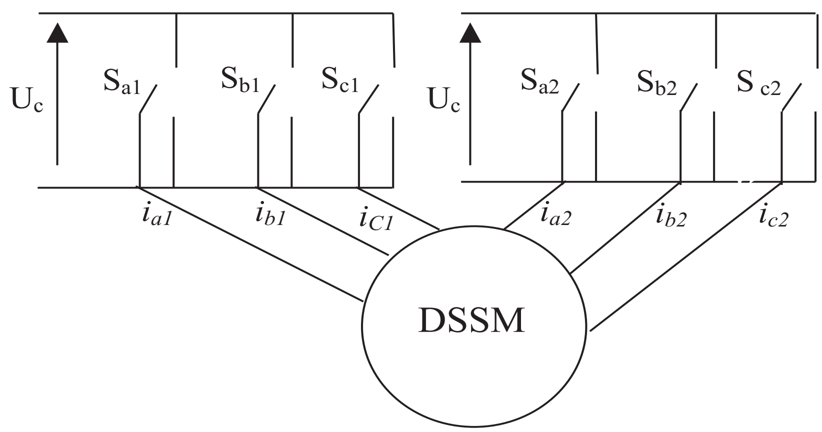

Figure 2.2: Electrical drive system.

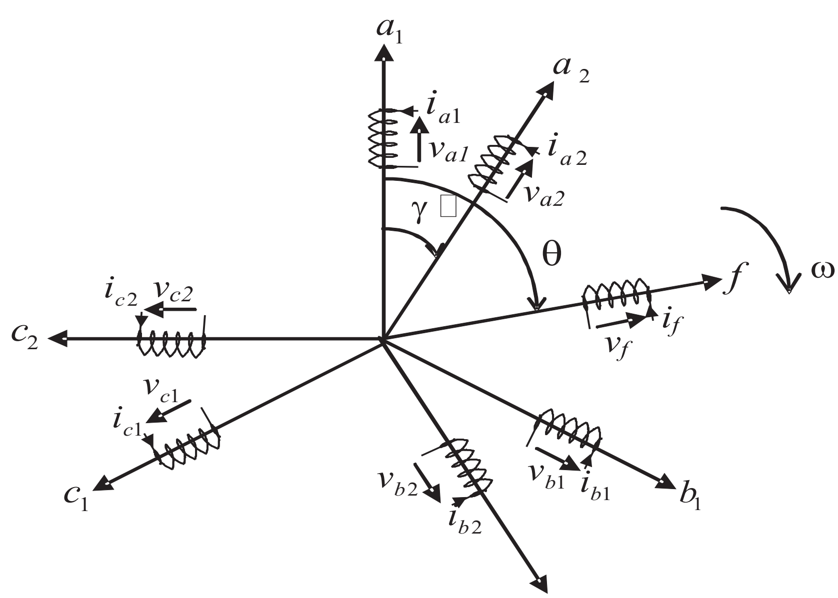

Figure 2.3: DSSM stator winding scheme.

From (1) and (2) the dynamic model describing the machine in $\alpha, \beta, Z_1, Z_2, Z_3, Z_4$ vector space can be given by

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_{Z_1} \\ v_{Z_2} \\ v_{Z_3} \\ v_{Z_4} \end{bmatrix} = R_s \begin{bmatrix} i_\alpha \\ i_\beta \\ i_{Z_1} \\ i_{Z_2} \\ i_{Z_3} \\ i_{Z_4} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} l_{fs}+3M_{ss} & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{fs}+3M_{ss} & 0 & 0 & 0 & 0 \\ 0 & 0 & l_{fs} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{fs} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{fs} & 0 \\ 0 & 0 & 0 & 0 & 0 & l_{fs} \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_{Z_1} \\ i_{Z_2} \\ i_{Z_3} \\ i_{Z_4} \end{bmatrix}$$

$$+M_{sfm}\frac{d}{dt} \begin{bmatrix} 3\cos(2\theta) & 3\sin(2\theta) & 0 & 0 & 0 & 0 \\ 3\sin(2\theta) & -3\cos(2\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_\alpha \\ i_\beta \\ i_{Z_1} \\ i_{Z_2} \\ i_{Z_3} \\ i_{Z_4} \end{bmatrix} + \sqrt{3}M_{sf}\frac{d}{dt} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} i_f.$$

It is observed from the above equations that all the electromechanical energy conversion related variable components are mapped into the α-β plane and the non electromechanical energy conversion related variable components are transformed to the Z_1, Z_2 and Z_3, Z_4 planes. Hence, the dynamic equations of the machine are totally decoupled. To express the stator and rotor equations in the same stationary reference frame, the following rotation transformation is appropriate

$$[P] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

With this transformation, the components of the α-β plane can be expressed in the d-q plan as

$$\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \begin{bmatrix} R_s + pL_d & -\omega L_q \\ \omega L_d & R_s + pL_q \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} + M_d\omega \begin{bmatrix} 0 \\ 1 \end{bmatrix} i_f \Bigg] .$$

The electromagnetic torque of DSSM is expressed as

$$T_e = P\left(\varphi_d i_q - \varphi_q i_d\right)$$

with $\varphi_d = L_d i_d + M_{fd}\, i_f$; $\varphi_q = L_q i_q$; $L_d = l_{sf} + 3M_{ss} + 3M_{sfm}$; $L_q = l_{sf} + 3M_{ss} - 3M_{sfm}$; $M_d = \sqrt{3}M_{sf}$.

By applying the following rotation transformation, witch transforms variable in the rotor flux reference frame $(d-q)$ to the stator flux reference frame $x-y$ (Figure 2.4):

$$\begin{bmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{bmatrix}.$$

The stator flux linkage and electromagnetic torque equations in $x-y$ reference frame are as follows [26]:

$$\begin{aligned} \begin{bmatrix} \varphi_x \\ \varphi_y \end{bmatrix} &= \begin{bmatrix} L_d\cos^2\delta + L_q\sin^2\delta & -L_d\cos\delta\sin\delta + L_q\sin\delta\cos\delta \\ -L_d\cos\delta\sin\delta + L_q\sin\delta\cos\delta & L_d\sin^2\delta + L_q\cos^2\delta \end{bmatrix} \begin{bmatrix} i_x \\ i_y \end{bmatrix} \\ &\quad + M_d \begin{bmatrix} \cos\delta \\ \sin\delta \end{bmatrix}, \end{aligned} \tag{4}$$

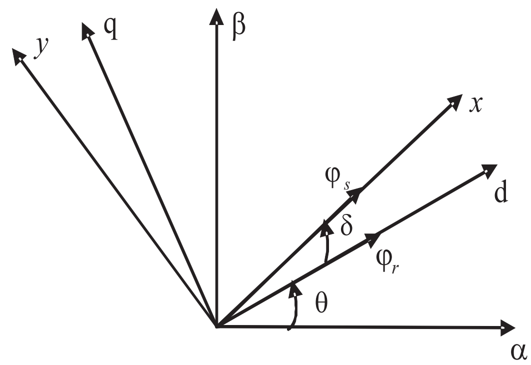

Figure 2.4: The stator and rotor flux linkages in different reference frames.

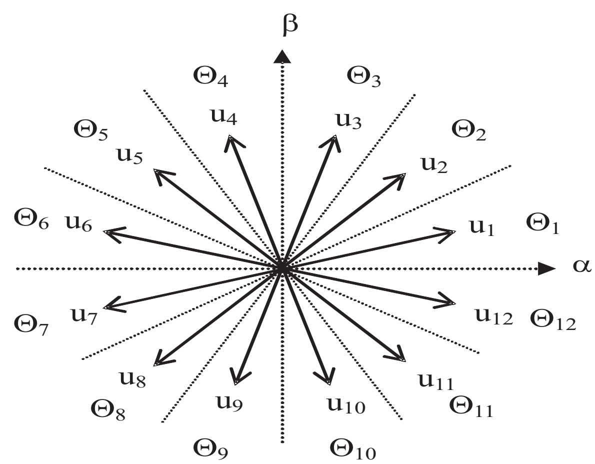

Figure 2.5: The chosen inverter voltage vectors projected on $\alpha - \beta$ plane.

$$T_e = P\,|\varphi_s|\,i_y.$$

The torque equation in terms of the stator flux linkage and load angle can be obtained by solving for i_y from the system equation (4) with $\varphi_y = 0$ and $\varphi_x = \varphi_s$, since the stator flux is along the x-axis [26]:

$$i_y = \frac{1}{2L_dL_q}\left[2M_d i_f\ L_q \sin\delta - |\varphi_s|\,(L_q - L_d)\sin 2\delta\right].$$

The torque equation is as follows:

$$T_e = \frac{P\,|\varphi_s|}{2L_dL_q}\left[2M_d i_f\ L_q \sin\delta - |\varphi_s|\,(L_q - L_d)\sin 2\delta\right]. \tag{5}$$

2.2 Modeling of the inverters

The DSSM is supplied by two VSIs. Each inverter can be controlled independently. However if we consider the two inverters as a six-phase voltage source inverter we obtain a total of 64 switching modes. By using the transformation matrix (3) the 64 voltage vectors corresponding to the switching modes are projected on three planes. From 64 vectors there are only 12 voltage vectors that offer a maximum voltage on the α-β plane and keep the harmonics on the Z_1, Z_2 plane at a minimum [24, 13].

The chosen switching modes are indicated in Table 2.1. The primary voltage $v_{a1}, v_{b1}, v_{c1}, v_{a2}, v_{b2}$ and v_{c2} are determined by the status of the six switches $S_{a1}, S_{b1}, S_{c1}, S_{a2}, S_{b2}, S_{c2}$. The non-zero voltage vectors are 30° a part from each other as in Figure 2.5.

$U[S_{a1}S_{b1}S_{c1}S_{a2}S_{b2}S_{c2}]$	$[v_{a1}v_{b1}v_{c1}v_{a2}v_{b2}v_{c2}].3/U_c$	$[v_\alpha\ v_\beta].1/Uc$
u_1[1 0 0 1 0 0]	[2 -1 -1 2 -1 -1]	[1.077 0.288]
u_2[1 1 0 1 0 0]	[1 1 -2 2 -1 -1]	[0.7887 0.7887]
u_3[1 1 0 1 1 0]	[1 1 -2 1 1 -2]	[0.2887 1.0774]
u_4[0 1 0 1 1 0]	[-1 2 -1 1 1 -2]	[-0.288 1.077]
u_5[0 1 0 0 1 0]	[-1 2 -1 -1 2 -1]	[-0.788 0.288]
u_6[0 1 1 0 1 0]	[-2 1 1 -1 2 -1]	[-1.077 0.2887]
u_7[0 1 1 0 1 1]	[-2 1 1 -2 1 1]	[-1.077 -0.288]
u_8[0 0 1 0 1 1]	[-1 -1 2 -2 1 1]	[-0.788 -0.788]
u_9[0 0 1 0 0 1]	[-1 -1 2 -1 -1 2]	[-0.288 -1.077]
u_{10}[1 0 1 0 0 1]	[1 -2 1 -1 -1 2]	[0.288 -1.077]
u_{11}[1 0 1 1 0 1]	[1 -2 1 1 -2 1]	[0.788 -0.788]
u_{12}[1 0 0 1 0 1]	[2 -1 -1 1 -2 1]	[1.077 -0.288]

Table 2.1: Chosen switching mode and primary voltage.

3 Direct Torque Control of DSSM

The main goal of DTC is to control the stator flux linkage and the torque directly, not via controlling the stator current. The change of torque can be controlled by keeping the amplitude of the stator flux linkage and by controlling the rotating speed of the stator flux linkage as fast as possible according to the equation (5).

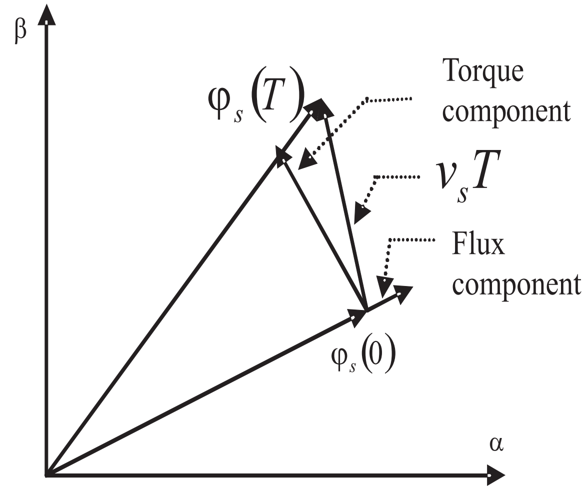

Figure 3.1: The control of stator flux linkage.

The stator flux linkage vector of DSSM in stationary reference frame is as follows:

$$\varphi_s(t) = \int_0^t (v_s - R_s i_s)\, dt + \varphi_s(0). \tag{6}$$

During the switching interval [0 T], v_s is constant and equation (6) became:

$$\varphi_s(T) = v_s T - R_s \int_0^T i_s dt + \varphi_s(0).$$

It can be seen from the formula that the end of stator flux linkage vector φ_s will move along the direction of voltage vector applied if the stator resistance is neglected as shown in Figure 3.1.

The basic principle of the DTC is to select proper voltage vectors using a pre-defined switching table. The selection is based on the hysteresis control of the stator flux linkage and the torque [25, 3]. For example, in region Θ_1, as shown in Figure 2.5, selection of vectors u_2, u_3 increases the amplitude of the stator flux linkage and increases torque. The selection of vectors u_4, u_5, u_6 decreases the amplitude of the stator flux linkage and increases torque. The selection of vectors u_8, u_9 decreases the amplitude of the stator flux linkage and decreases torque. The selection of vectors u_{10}, u_{11}, u_{12} increases the amplitude of the stator flux linkage and decreases torque. We have ten voltage vectors to control the amplitude of the stator flux linkage and torque, but with hysteresis controller we need only four voltage vectors to control the amplitude of the stator flux linkage and torque. The voltage vector plane is divided into twelve sectors so that each voltage vector divides each region into two equal parts as shown in Figure 2.5. In each sector, four of the twelve voltage vectors may be used. All possibilities can be tabulated into a switching table. The switching table used in this work is indicated in Table 3.1. The output of the flux hysteresis comparator is denoted as Φ, the output of the torque hysteresis comparator is denoted as τ. The flux hysteresis comparator is a two valued comparator. Φ=1 means that the actual value of the amplitude of the flux linkage is below the reference value and

Φ	$\tau \setminus \Theta$	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8	Θ_9	Θ_{10}	Θ_{11}	Θ_{12}
$\Phi=1$	$\tau=1$	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2
$\Phi=1$	$\tau=0$	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
$\Phi=0$	$\tau=1$	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4
$\Phi=0$	$\tau=0$	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8

Table 3.1: The switching states for inverters.

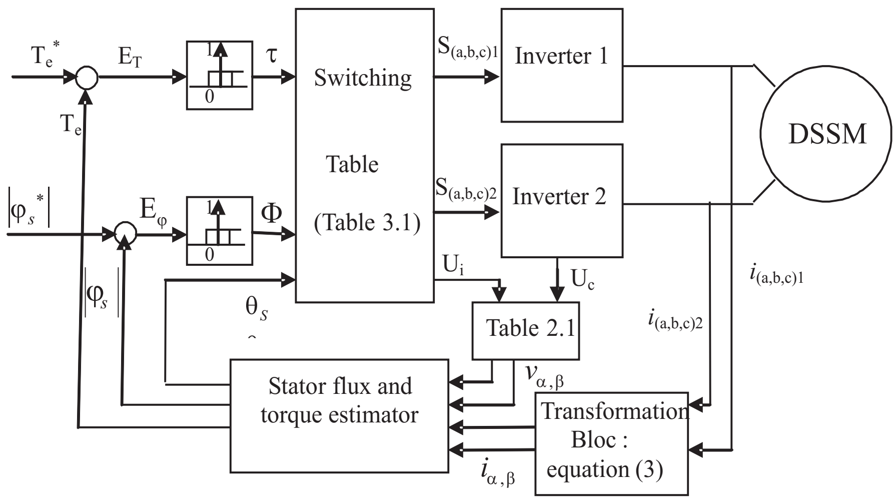

Figure 3.2: Direct torque control scheme for DSSM.

Φ=0 means that the actual value is above the reference value. The same is true for the torque. Θ_i denote the region numbers for the stator linkage positions.

The used control system is depicted in Figure 3.2, E_T and E_φ are the torque and the flux errors.

The stator flux linkage and the torque are estimated with

$$\varphi_\alpha(t) = \int_0^t (v_\alpha - R_s i_\alpha)\,dt + \varphi_\alpha(0), \quad \varphi_\beta(t) = \int_0^t (v_\beta - R_s i_\beta)\,dt + \varphi_\beta(0),$$

$$|\varphi_s| = \sqrt{\varphi_\alpha^2 + \varphi_\beta^2}, \quad tg\theta_s = \frac{\varphi_\beta(t)}{\varphi_\alpha(t)}, \quad T_e = P(\varphi_\alpha i_\beta - \varphi_\beta i_\alpha).$$

The simulation results in Figure 3.3 show that basic DTC regulates the torque and stator flux well. We can see that, this control approach ensure good decoupling between stator flux linkage and torque. However, in this approach we have used only four voltage vectors to control flux and torque. In order to improve the performance of DSSM, we propose a DTC based on fuzzy logic to control flux and torque. In the proposed approach we used ten voltage vectors to control flux and torque.

4 The Proposed DTC Based on Fuzzy Logic for DSSM

In DTC scheme proposed in Section 3 a hysteresis controller is used. The output of hysteresis controller has only two states, which naturally leads to tacking the same action

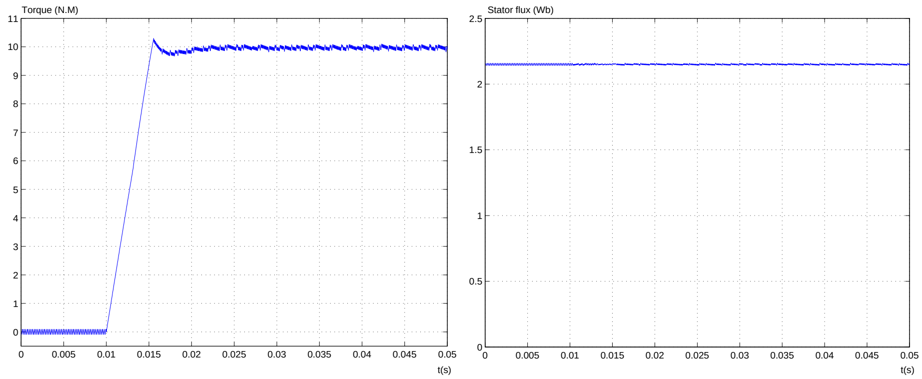

Figure 3.3: Performance of conventional DSSM DTC.

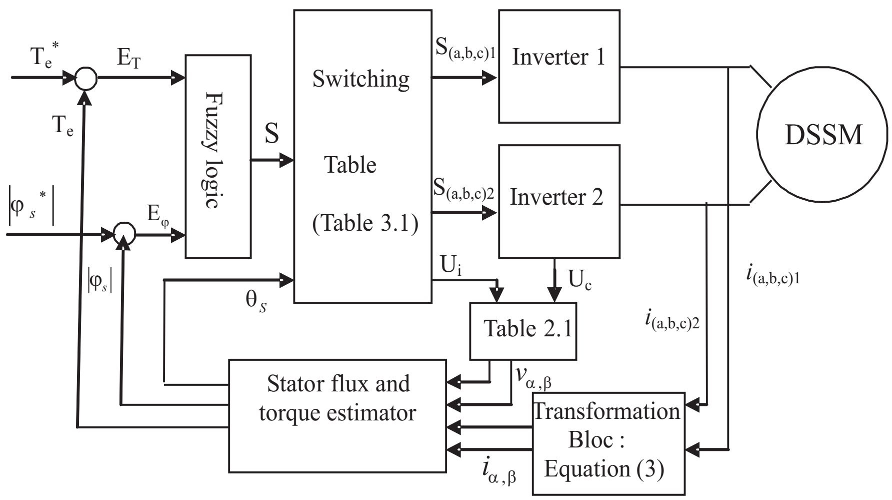

Figure 4.1: Direct torque control scheme based on fuzzy control for DSSM.

for the big torque error and small one. As consequence, a poor performances in response to step changes and large torque ripple. To improve the performances of DSSM, a DTC method based on a fuzzy control is used. The hysteresis controller is replaced by two input fuzzy controller and the vector output of the fuzzy controller is led to a switching table to decide which vector should be applied. This method has the advantage of fuzzy classification and has the advantage of simplicity and easy to implement [12]. The diagram of DTC incorporated with a fuzzy logic controller is shown in Figure 4.1. S denotes the vector output of the fuzzy controller.

4.1 Fuzzy controller

The fuzzy logic controller is comprised of fuzzification part, fuzzy inference part and defuzzification part.

A. Fuzzification.

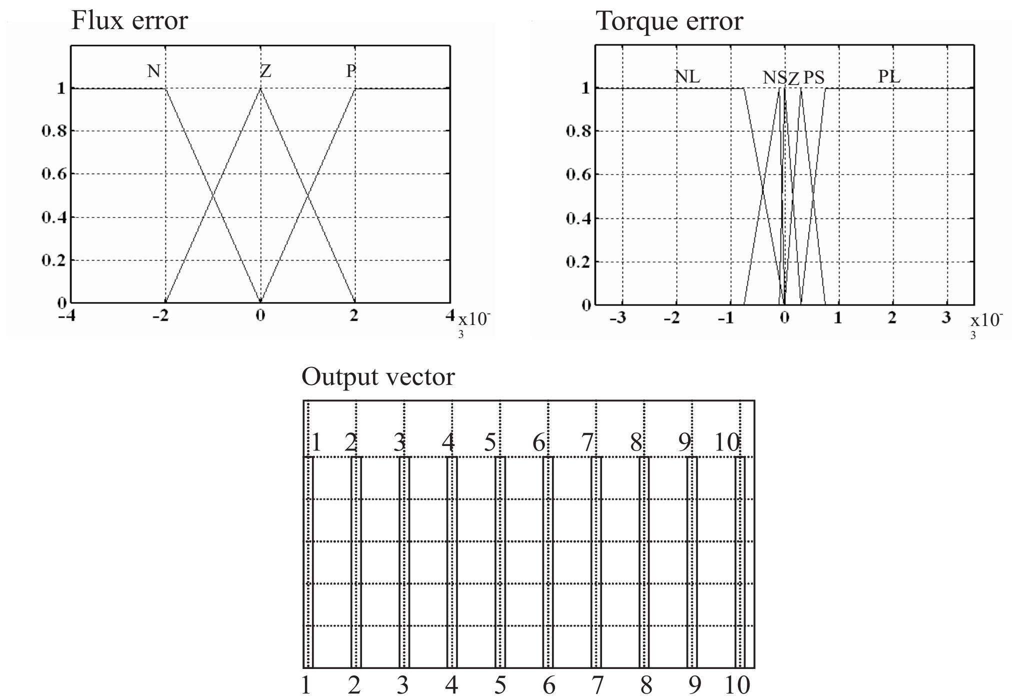

Figure 4.2: Membership function of fuzzy control.

The fuzzification is the process of a mapping from measured or estimated input to corresponding fuzzy set in the universe of discourse. As shown in Figure 4.1 there are two inputs E_φ and E_T. The member ship functions of the two fuzzy input variables are shown in Figure 4.2. The output variable can be classified into ten types, which are fuzzified as ten singleton fuzzy sets.

B. Fuzzy inference.

The fuzzy reasoning used is Mamdani's method. The fuzzy control rule-base is shown in Table 4.1.

$E_T \diagdown E_{\varphi s}$	P	Z	N
PB	1	2	3
PS	4	2	5
NZ	-	-	-
NS	6	7	8
NB	9	7	10

Table 4.1: Fuzzy rule-bases.

Where 1,2,...,10 denote the specified states of the vector output of the fuzzy controller. Note that the above strategy was used in [12] for synchronous machine with four states of the vector output of the fuzzy controller.

C. Defuzzification.

The maximum criterion method is used for defuzzification. By this method, the value of fuzzy output which has the maximum possibility distribution is used as control output.

4.2 Selection of voltage vectors

The voltage vector, for controlling both the amplitude and rotating direction of φ_s, are indicated in Table 4.2.

$S\diagdown\Theta$	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8	Θ_9	Θ_{10}	Θ_{11}	Θ_{12}
S = 1	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2
S = 2	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3
S = 3	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4
S = 4	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1
S = 5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5
S = 6	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
S = 7	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
S = 8	u_8	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
S = 9	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
S = 10	u_9	u_{10}	u_{11}	u_{12}	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8

Table 4.2: The switching states for inverters.

5 Comparative Study

In this section, we aim to compare the DTC based on fuzzy logic for DSSM to the conventional DTC for DSSM. We consider two situations:

Situation 1: Step change in torque. For the DTC based on fuzzy logic for DSSM we have simulated a step variation on the torque applied at t=0.2 ms. The obtained results are given in Figure 5.2, for the conventional DTC, see Figure 3.3. We can see that, both control approaches ensure good decoupling between stator flux linkage and torque. However the DTC based on fuzzy logic for DSSM decrease considerably the torque ripple and have faster torque response.

Situation 2: Stator resistance variation. For both DTC control schemes we have simulated variation on stator resistance as shown in Figure 5.5. The obtained results, shown in Figures 5.3 and 5.4, shows that the torque and flux are oscillating when stator resistance is increased. Thus incorrect resistance stator can causing instability. Several control scheme have been proposed to overcome this problem [8, 10, 14, 11]. The stator resistance estimator used in this paper is shown in Figure 5.1. The error in the stator current is used as an input to the PI estimator. The output of the PI estimator is continuously added to the previously estimated stator resistance.

$\hat{i}_s$ the estimated stator current Figures 5.5, 5.6 and 5.7 shows the actual and estimated stator resistance and their error. We can see that the estimation error is approximately 0.02 % . In Figures 5.8 and 5.9 we have inserted the estimated stator resistance in control scheme. The obtained results are very satisfactory.

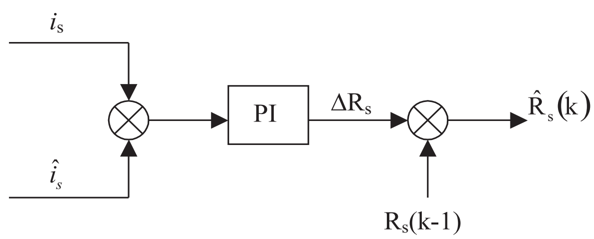

Figure 5.1: Block diagram of the stator resistance estimator.

	Torque ripple	Response time	Stator resistance variation
DTC based on fuzzy logic	0.4 %	4.5 ms	Unstable
Conventional DTC	2.4 %	10 ms	Unstable

Table 5.1: Comparative study between DTC based on fuzzy logic and conventional DTC for DSSM.

Table 5.1 summarizes the results of the comparative study. From the above table we can conclude that for the DSSM, the DTC based on fuzzy logic is more advantageous than the conventional DTC.

6 Conclusion

In this paper, we have developed a DTC for DSSM. First we have developed a conventional DTC for DSSM. In this approach we have used only four vectors voltage to control both torque and stator flux linkage. Secondly, in order to improve the performance of DSSM we have used ten vectors voltage to control torque and stator flux linkage. The proposed approach consist to replace the hysteresis controllers by two input fuzzy controller and the vector output of the fuzzy controller is led to a switching table to decide which vector should be applied. Thirdly, a comparative study demonstrates that the

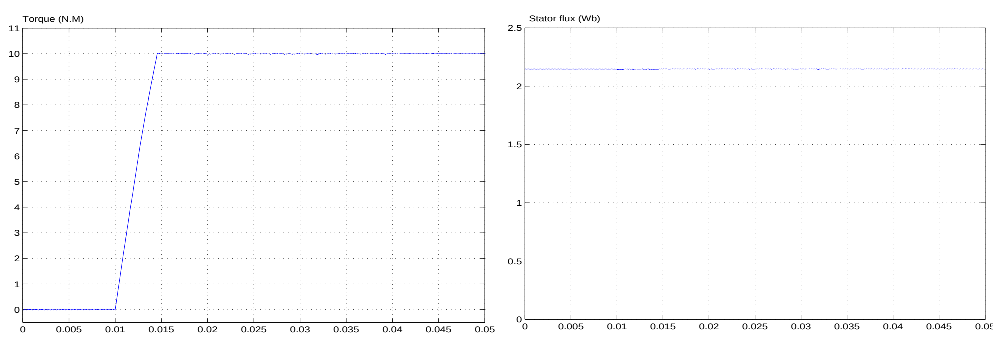

Figure 5.2: Performance of DSSM based on fuzzy control for situation 1.

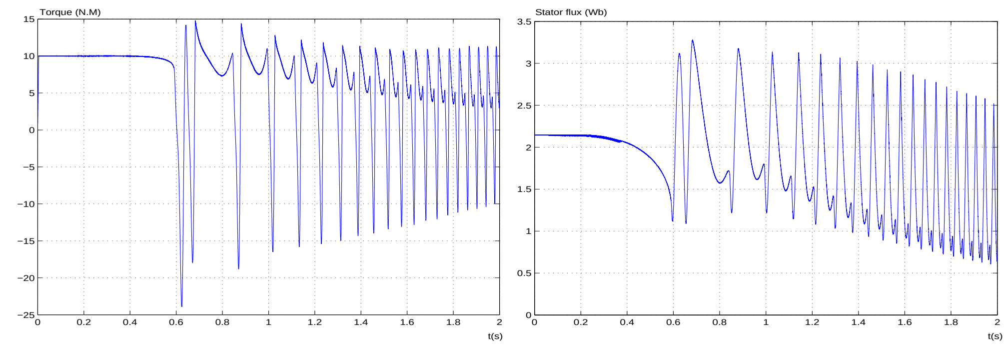

Figure 5.3: Performance of DSSM based on fuzzy control for situation 2.

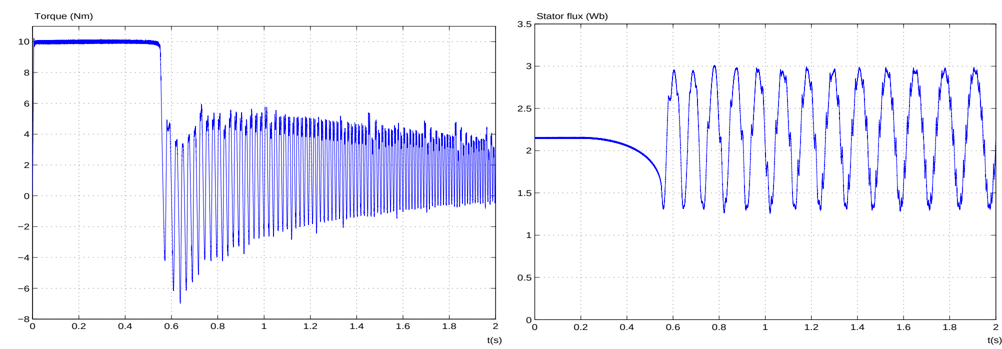

Figure 5.4: Performance of conventional DSSM DTC for situation 2.

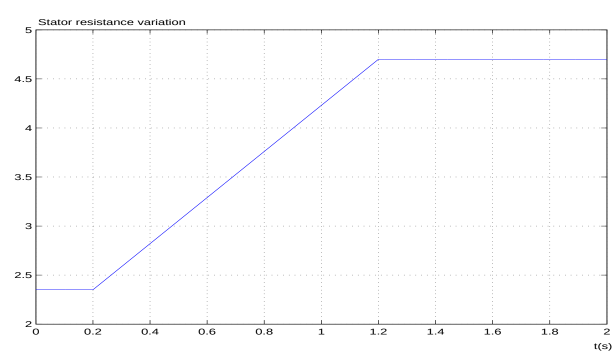

Figure 5.5: Actual stator resistance variation.

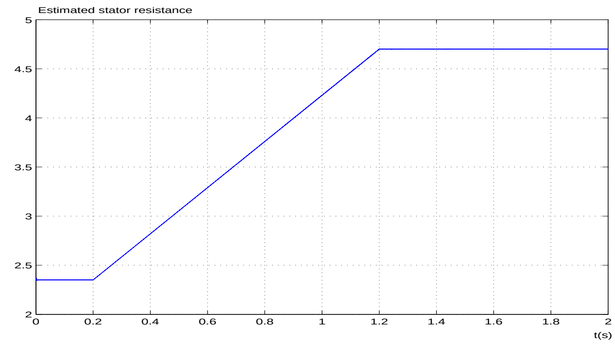

Figure 5.6: Estimated stator resistance.

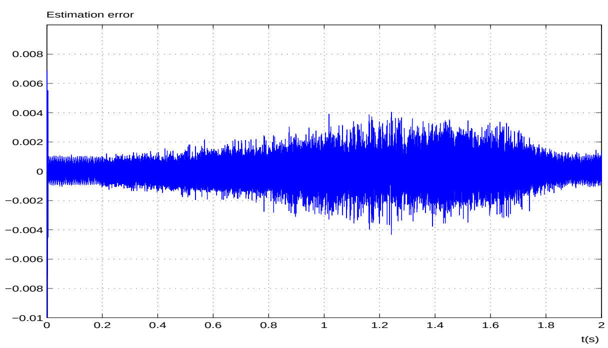

Figure 5.7: Estimation error.

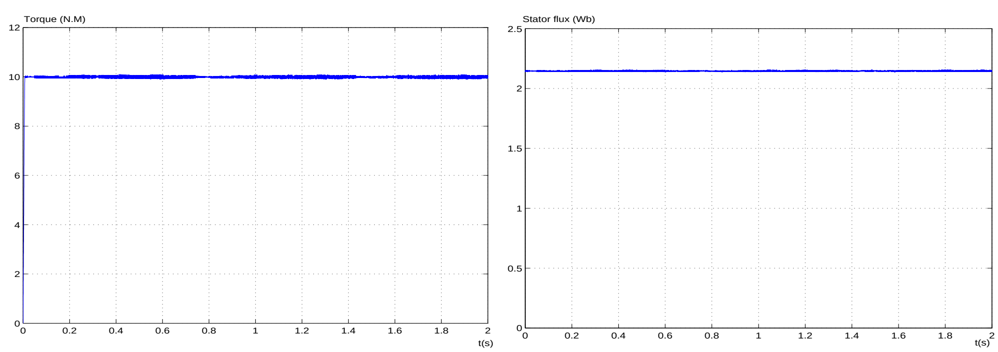

Figure 5.8: Performance of DSSM based on fuzzy control with estimated stator resistance.

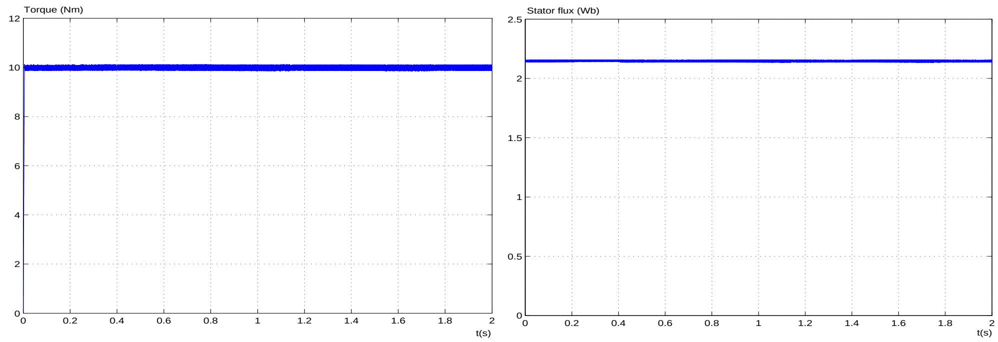

Figure 5.9: Performance of conventional DSSM DTC with estimated stator resistance.

DTC based on fuzzy logic for DSSM decrease the torque ripple and has a better dynamic and static performance. Nevertheless the variations of the stator resistance cause the DTC drive system to become unstable. So a PI stator resistance estimator is designed and applied to eliminate the effect of the stator resistance variation. It is shown that the stator flux and torque response is very satisfactory.

7 Appendix 1: List of Principal Symbols

v_{a1}, v_{b1}, v_{c1} : simple voltage of stator three phase first winding.
v_{a2}, v_{b2}, v_{c2} : simple voltage of stator three phase second winding.
i_{a1}, i_{b1}, i_{c1}: stator current a, b, c phase of first winding.
i_{a2}, i_{b2}, i_{c2} : stator current a, b, c phase of second winding.
$i_s, \hat{i}_s$: stator current vector, estimated stator current vector.
v_s : stator voltage vector.
v_d, v_q : stator voltages d-q axis.
v_α, v_β: stator voltages α-β axis.
v_x, v_y : stator voltages x-y axis.
$[L_{ss}]$: stator inductance matrix.
$[M_{sr}]$: stator-rotor mutual inductance matrix.
$[R_s]$: diag $(R_s\ R_s\ R_s\ R_s\ R_s\ R_s)$.
R_s: stator resistance.
L_d, L_q : d-q inductances.
R_f: rotor resistance.
T_e, T_e^* : electromagnetic torque, reference torque.
φ_s, φ_s^* : stator flux vector, reference flux vector.
φ_d, φ_q : stator flux d-q axis.
φ_α, φ_β: stator flux α-β axis.
φ_x, φ_y : stator flux x-y axis.
w : stator voltages synchronous pulsation.
Φ: output of the flux hysteresis comparator.
τ: output of the torque hysteresis comparator.
δ: angle between rotor and stator flux linkage.
θ_s: angle of stator flux linkage.

Θ_i : the region numbers for the stator linkage positions.
E_T : torque error.
E_φ : flux error.
J: friction coefficient.
f_r : moment of inertia.
P : pole pairs number.

7.1 Appendix 2: DSSM Parameters

$P_n = 5$ kW, $U_c = 232$ V, $i_f = 1$ A, $R_s = 2.35\ \Omega$, $R_f = 30.3$, $L_d = 0.3811$ H, $L_q = 0.211$ H, $L_f = 15$ H, $M_d = 2.146$ H, J = 0.05 Nms2/rd, $f_r = 0.001$Nms/rd, $P = 1$.

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Existence and Exponential Stability of Almost Periodic Solutions for a Class of Neural Networks with Variable Delays [1]

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Abstract: In this paper, some sufficient conditions for the existence and exponential stability of almost periodic solutions for Cohen–Grossberg neural networks with variable delays are obtained by applying Banach fixed point theory and differential inequality techniques. Some previous results are improved and extended. Moreover, an example is given to illustrate that our results are feasible.

Keywords: *Cohen–Grossberg neural networks; almost periodic solutions; exponential stability.*

Mathematics Subject Classification (2000): 03C50, 34C27, 34D23, 34K50, 92B20.

1 Introduction

Recently, the behavior of dynamical systems has been widely investigated [1, 2, 3, 4]. Cohen–Grossberg neural networks, which were first proposed by Cohen and Grossberg in [5] are typical dynamical systems and have received increasing interesting due to their promising potential applications in many fields such as optimization, associative memory, pattern recognition, signal and image processing. The stability of Cohen–Grossberg neural network with or without delays has been widely studied by many researchers [6, 7, 8, 9]. Moreover, many sufficient conditions on the stability of equilibrium point for Cohen–Grossberg neural networks with constant coefficients have been available [10, 11, 12].

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As is well known, the investigation on the neural dynamical systems not only involves a discussion of stability, but also involves many other dynamical behavior such as periodic oscillatory behavior, almost periodic oscillatory properties, chaos and so on. There exist some results on the existence of periodic solutions of Cohen–Grossberg neural networks with variable coefficients [13, 14, 15, 16]. In practice, almost periodic oscillatory is more accordant. Some authors have researched almost periodic solutions for neural networks, and obtained several interesting results [17, 18, 19, 20]. However, To the best of our knowledge, few authors discuss almost periodic solutions for Cohen–Grossberg neural networks with variable coefficients [21].

In this paper, our objective is to study further Cohen–Grossberg neural networks with variable delays. By applying Banach fixed point theory, differential inequality techniques, we get some sufficient conditions ensuring the existence and exponential stability of almost periodic solutions for Cohen–Grossberg neural networks with variable delays. These conditions obtained are easy to check and in practice. Moreover, in this paper, the assumptions of boundedness, monotonicity, and differentiability for the activation functions are not available.

The rest of the paper is organized as follows. In Section 2, some notations, definitions and model description are given. The existence and uniqueness of almost periodic solutions is established in Section 3. In Section 4, we derive some sufficient conditions on exponential stability of almost periodic solutions. Finally, an example is given to demonstrate the validity of our results in Section 5.

2 Model Description and Preliminaries

Consider the Cohen–Grossberg neural networks with variable delays as follows:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t-\tau_j(t))) + I_i(t)\right], \quad (1)$$

where $t \geq 0$, $i = 1, 2, \dots, n$; n is the number of neurons, $x_i(t)$ is the state of neuron i at the time t; $a_i(x_i(t))$ and $b_i(x_i(t))$ represent an amplification function and an appropriately behaved function at the time t, respectively; $f_j(x_j)$ is the activation function of the j-th unit; $c_{ij}(t)$ and $d_{ij}(t)$ denote the neural connection at the time t; $I_i(t)$ is the external inputs at the time t, $\tau_j(t) > 0$ is transmission delay.

The initial conditions of system (1) are of the form $x_i(t) = \varphi_i(t)$, $t \in [-\tau, 0]$, $\tau = \max_{1 \leq i \leq n} \tau_j(t)$, $\varphi_i \in C$ ($C \triangleq C[[-\tau, 0], R^n]$), and φ_i is assumed to be bounded and continuous on $[-\tau, 0]$.

Definition 2.1 [22, 23] Let $x(t)\colon\ R \to R^n$ be continuous in t. $x(t)$ is said to be *almost periodic* on R if, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$ such that, for any interval with length $l(\varepsilon)$, there is a number $\delta = \delta(\varepsilon)$ in this interval such that $|x(t+\delta) - x(t)| < \delta l$, for any $t \in R$.

Throughout this paper, we assume that $c_{ij}(t)$, $d_{ij}(t)$, $I_i(t)$, $\varphi_i(t)$ are continuous almost periodic functions. For an arbitrary continuous function $f(t)\colon\ R \to R$, we define

$$\overline{f} = \sup_{t \in R} |f(t)|, \quad \underline{f} = \inf_{t \in R} |f(t)|.$$

We list some assumptions which will be used in this paper as follows:

(H1) $a_i(t)$ is continuous and $0 < \underline{a}_i \le a_i(t) \le \overline{a}_i$ for all $t \in R$, $i = 1, 2, \dots, n$.

(H2) There are positive constants k_i such that $\dot{b}_i(\cdot) \ge k_i$, $\dot{b}_i(\cdot)$ denotes the derivative of $b_i(\cdot)$, and $b_i(0) = 0$, $i = 1, 2, \dots, n$.

(H3) There are constants $\alpha_j > 0$ such that $|f_j(x) - f_j(y)| \le \alpha_j |x - y|$ for any $x, y \in R$, and $f_j(0) = 0$, $j = 1, \dots, n$.

Definition 2.2 The almost periodic solutions $x^*(t)$ of system (1) is said to be *global exponentially stable*, if there exist constants $\varepsilon > 0$ and $M \ge 1$ such that

$$|x_i(t) - x^*| \le M\|\varphi - \varphi^*\|e^{-\varepsilon t}, \quad t > 0, \quad i = 1, 2, \dots, n,$$

where φ^* is the initial value of x^*, $\|\varphi - \varphi^*\| = \sup\limits_{-\infty \le s \le 0} \max\limits_{1 \le i \le n} |\varphi_i(s) - \varphi_i^*(s)|$.

Definition 2.3 [21] Let $y \in R^n$ and $P(t, y)$ be a $n \times n$ continuous matrix defined on $R \times R^n$. For any continuous function $v(t)\colon\ R \to R^n$, the following system

$$\dot{y}(t) = P(t, v(t))y(t)$$

is said to be an *exponential dichotomy* on R if there exist constants $k, l > 0$, projection S and the fundamental matrix $Y_v(t)$ satisfying

$$\begin{aligned} \|Y_v(t)SY_v^{-1}(s)\| &\le ke^{-l(t-s)} \quad \text{for} \quad t \ge s, \\ \|Y_v(t)(I - S)Y_v^{-1}(s)\| &\le ke^{-l(t-s)} \quad \text{for} \quad t \le s. \end{aligned}$$

Lemma 2.1 [21] *If the linear system $\dot{y}(t) = P(t, v(t))y(t)$ has an exponential dichotomy, then almost periodic system*

$$\dot{y}(t) = P(t, v(t))y(t) + g(t, v(t))$$

has a unique almost periodic solution $y(t)$ which can be expressed as follows:

$$y(t) = \int_{-\infty}^{t} Y_v(t)SY_v^{-1}(s)g(s, v(s))\, ds - \int_{t}^{\infty} Y_v(t)(I - S)Y_v^{-1}(s)g(s, v(s))\, ds.$$

Lemma 2.2 [22, 23] *Assume that $e_i(t)$ is an almost periodic function and*

$$\lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} e_i(s)\, ds > 0, \quad i = 1, 2, \dots, n.$$

Then the linear system $\dot{y}(t) = e(t)y(t)$ admits an exponential dichotomy, where $e(t) =$ $\operatorname{diag}\{e_i(t)\}$.

Definition 2.4 [24, 25] A real $n \times n$ matrix $W = (w_{ij})_{n \times n}$ is said to be an *M-matrix* if $w_{ij} \le 0$, $i, j = 1, 2, \dots, n$, $i \ne j$, and $W^{-1} \ge 0$, where W^{-1} denotes the inverse of W.

Lemma 2.3 [24, 25] *Let $W = (w_{ij})_{n \times n}$ with $w_{ij} \le 0$, $i, j = 1, 2, \dots, n$, $i \ne j$. Then the following statements are equivalent:*

(1) *W is an M-matrix;*

(2) *there exists a vector* $\eta = (\eta_1, \eta_2, \dots, \eta_n) > (0, 0, \dots, 0)$ *such that* $\eta W > 0$*;*

(3) *there exists a vector* $\xi = (\xi_1, \xi_2, \dots, \xi_n)^{\mathrm{T}} > (0, 0, \dots, 0)^{\mathrm{T}}$ *such that* $W\xi > 0$.

Lemma 2.4 [24, 25] *Let* $A \geq 0$ *be an* $n \times n$ *matrix and* $\rho(A) < 1$*, the* $(E_n - A)^{-1} \geq 0$*, where* $\rho(A)$ *denotes the spectral radius of* A*.*

From (H1), the antiderivative of $\frac{1}{a_i(x_i)}$ exists. We choose an antiderivative $g_i(x_i)$ of $\frac{1}{a_i(x_i)}$ that satisfies $g_i(0) = 0$. Obviously, $\dot{g}_i(x_i) = \frac{1}{a_i(x_i)}$. By $a_i(x_i) > 0$, we obtain that $g_i(x_i)$ is increasing with respect to x_i, and the inverse function $g_i^{-1}(x_i)$ of $g_i(x_i)$ is existential, continuous, and differentiable. So, $\dot{g_i}^{-1}(x_i) = a_i(x_i)$, where $\dot{g_i}^{-1}(x_i)$ is the derivative of $g_i^{-1}(x_i)$ with respect to x_i, and composition function $b_i(g_i^{-1}(z))$ is differentiable. Denote $u_i(t) = g_i(x_i(t))$. It is easy to see that $\dot{u}_i(t) = \dot{g}_i(x_i)\dot{x}_i(t) = \frac{\dot{x}_i(t)}{a_i(x_i)(t)}$ and $x_i(t) = g_i^{-1}(u_i)$. Substituting these equalities into system (1) gives that

$$
\begin{aligned}
\dot{u}_i(t) &= -b_i(g_i^{-1}(u_i(t))) + \sum_{j=1}^{n} c_{ij}(t) f_j(g_j^{-1}(u_j(t))) \\
&\qquad + \sum_{j=1}^{n} d_{ij}(t) f_j(g_j^{-1}(u_j(t - \tau_j(t)))) - I_i(t), \quad t \geq 0 \\
u_i(t) &= g_i(\varphi_i(t)) \triangleq \phi_i(t), \quad -\tau \leq t \leq 0.
\end{aligned} \tag{2}
$$

Considering $b_i(g_i^{-1}(u_i(t))) = \dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i} \cdot u_i(t)$, system (2) can be written as the following system:

$$
\begin{aligned}
\dot{u}_i(t) &= -e_i(u_i(t))u_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(g_j^{-1}(u_j(t))) \\
&\qquad + \sum_{j=1}^{n} d_{ij}(t) f_j(g_j^{-1}(u_j(t - \tau_j(t)))) - I_i(t), \quad t \geq 0, \\
u_i(t) &= \phi_i(t), \quad -\tau \leq t \leq 0,
\end{aligned} \tag{3}
$$

where $e_i(u_i)(t) \triangleq \dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i}$, $\dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i}$ denotes the derivative of $b_i(g_i^{-1}(z))$ at point $z = \varepsilon_i$, $z \in R$, ε_i is between 0 and $u_i(t)$.

Let $e_i(u_i)(t)$ be an almost periodic function, the system (1) has a unique almost periodic solution which is globally exponentially stable if and only if system (3) has a unique almost periodic solution which is globally exponentially stable.

It is easy to see that $|g_i^{-1}(u) - g_i^{-1}(v)| = |\dot{g}_i^{-1}(\mu)(u - v)| = |a_i(\mu)||u - v| \leq \overline{a}_i|u - v|$, where μ is between u and v.

For convenience, we introduce some notations. We will use $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in R^n$ to denote a column vector, in which the symbol $(^{\mathrm{T}})$ denotes the transpose of a vector. For matrix $A = (a_{ij})_{n \times n}$, A^{T} denotes the transpose of A, and E_n denotes the identity matrix of size n. A matrix or vector $A \geq 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A and B, $A \geq B$ (rep. $A > B$) means that $A - B \geq 0$ (rep. $A - B > 0$).

3 Existence and Uniqueness of Almost Periodic Solutions

In this section, we shall discuss the existence and uniqueness of the almost periodic solution of system (3).

Theorem 3.1 *Suppose that (H1)–(H3) are satisfied, and* $\rho(\underline{A}^{-1}(\overline{C}+\overline{D}))<1$*, where* $\overline{C}=(\overline{c}_{ij}\alpha_j\overline{a}_j)_{n\times n}$*,* $\overline{D}=(\overline{d}_{ij}\alpha_j\overline{a}_j)_{n\times n}$*,* $\underline{A}=\mathrm{diag}(k_1\underline{a}_1,k_2\underline{a}_2,\dots,k_n\underline{a}_n)$*. Then, there exists exactly one almost periodic solution of system (3).*

Proof Set the vector $\hat{u}(t)=(\hat{u}_1(t),\ \hat{u}_2(t),\dots,\ \hat{u}_n(t))^{\mathrm{T}}$, for $\forall\, x\in R^n$, we define the norm: $\|\hat{u}(t)\|=\max_{1\le i\le n}|\hat{u}_i(t)|$. Let $\Lambda=\{\hat{u}(t)=\mathrm{col}\{\hat{u}_i(t)\mid \hat{u}(t)\colon\ R\to R^n$, is continuous almost periodic function$\}$. For any $\hat{u}\in\Lambda$, we define its induced model as follows:

$$\|\hat{u}\|=\sup_{t\in R}\|\hat{u}(t)\|=\sup_{t\in R}\max_{1\le i\le n}|\hat{u}_i(t)|.$$

Obviously, $(\Lambda,\|\cdot\|)$ is a Banach space. For any $\{\hat{u}_i(t)\}\in\Lambda$, consider the following system:

$$\begin{aligned}\dot{u}_i(t)&=-e_i(\hat{u}_i(t))u_i(t)+\sum_{j=1}^{n}c_{ij}(t)f_j(g_j^{-1}(\hat{u}_j(t)))\\&\quad+\sum_{j=1}^{n}d_{ij}(t)f_j(g_j^{-1}(\hat{u}_j(t-\tau_j(t))))-I_i(t),\end{aligned}\tag{4}$$

where $i=1,2,\dots,n$. From H(1) and H(2), we get $e_i(u_i(t))\ge k_i\underline{a}_i>0$ and

$$\lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T}e_i(u_i(s))\,ds\ge\lim_{T\to+\infty}k_i\underline{a}_i>0.$$

Similar to the analysis of [21], we know that following system:

$$\dot{U}(t)=Q(\hat{u}(t))U(t)$$

has an exponential dichotomy on R, where

$$Q(\hat{u})(t))=\mathrm{diag}(e_1(\hat{u}_1(t)),\,e_2(\hat{u}_2(t)),\dots,\,e_n(\hat{u}_n(t))).$$

Thus by Lemma 2.1 and Lemma 2.2, system (4) has a unique almost periodic solution $u_{\hat{u}}(t)$ which can be expressed as follows:

$$\begin{aligned}u_{\hat{u}}(t)=\mathrm{col}\bigg\{\int_{-\infty}^{t}e^{-\int_s^t e_i(\hat{u}(\sigma))d\sigma}\bigg[&\sum_{j=1}^{n}c_{ij}(s)f_j(g_j^{-1}(\hat{u}_j(s)))\\&+\sum_{j=1}^{n}d_{ij}(s)f_j\big(g_j^{-1}(\hat{u}_j(s-\tau_{ij}(s)))\big)-I_i(s)\bigg]ds\bigg\}.\end{aligned}\tag{5}$$

Now define a mapping $T\colon\ \Lambda\to\Lambda$ by setting

$$T_{\hat{x}}(t)=x_{\hat{x}}(t),\quad\forall\,\hat{x}\in\Lambda.$$

Next, we prove that T is a contraction mapping. For any $\forall\,\hat{x},x^*\in\Lambda$, from (H3) we have

$$|T(\hat{u}(t))-T(u^*(t))|$$

$$
\begin{aligned}
&= \Bigg(\Bigg| \int_{-\infty}^{t} e^{-\int_s^t e_1(\hat{u}(\sigma))d\sigma} \Bigg[\sum_{j=1}^{n} c_{1j}(s)\big(f_j(g_j^{-1}(\hat{u}_j(s))) - f_j(g_j^{-1}(u_j^*(s)))\big) \\
&+ \sum_{j=1}^{n} d_{1j}(s)\big(f_j(g_j^{-1}(\hat{u}_j(s-\tau_{1j}(s)))) - f_j(g_j^{-1}(u_j^*(s-\tau_{1j}(s))))\big)\Bigg] ds \Bigg|, \ \dots, \\
&\Bigg| \int_{-\infty}^{t} e^{-\int_s^t e_n(\hat{u}(\sigma))d\sigma} \Bigg[\sum_{j=1}^{n} c_{nj}(s)\big(f_j(g_j^{-1}(\hat{u}_j(s)) - f_j g_j^{-1}(u_j^*(s))))\big) \\
&+ \sum_{j=1}^{n} d_{nj}(s)\big(f_j(g_j^{-1}(\hat{u}_j(s-\tau_{nj}(s)))) - f_j(g_j^{-1}(u_j^*(s-\tau_{nj}(s))))\big)\Bigg] ds \Bigg| \Bigg)^{\mathrm{T}} \\
&\le \Bigg(\int_{-\infty}^{t} e^{-k_1\underline{a}_1(t-s)} \Bigg[\sum_{j=1}^{n} \overline{c}_{1j}\alpha_1\overline{a}_1 |\hat{u}_j(s) - u_j^*(s)| \qquad (6) \\
&+ \sum_{j=1}^{n} \overline{d}_{1j}\alpha_1\overline{a}_1 |\hat{u}_j(s-\tau_{1j}(s)) - x_j^*(s-\tau_{1j}(s))| \Bigg] ds, \ \dots, \\
&\int_{-\infty}^{t} e^{-k_n\underline{a}_n(t-s)} \Bigg[\sum_{j=1}^{n} \overline{c}_{nj}\alpha_n\overline{a}_n |\hat{u}_j(s) - u_j^*(s)| \\
&+ \sum_{j=1}^{n} \overline{d}_{nj}\alpha_n\overline{a}_n |\hat{u}_j(s-\tau_{nj}(s)) - u_j^*(s-\tau_{nj}(s))| \Bigg] ds \Bigg)^{\mathrm{T}} \\
&\le \Bigg(\sum_{j=1}^{n} (k_1\underline{a}_1)^{-1}(\overline{c}_{1j} + \overline{d}_{1j})\alpha_1\overline{a}_1 \sup_{t\in R} |\hat{u}_j(t) - u_j^*(t)|, \ \dots, \\
&\sum_{j=1}^{n} (k_n\underline{a}_n)^{-1}(\overline{c}_{nj} + \overline{d}_{nj})\alpha_n\overline{a}_n \sup_{t\in R} |\hat{u}_j(t) - u_j^*(t)| \Bigg)^{\mathrm{T}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\Big(\sup_{t\in R} |(T(\hat{u}(t)) - T(u^*(t)))_1|, \ \dots, \ \sup_{t\in R} |(T(\hat{u}(t)) - T(u^*(t)))_n| \Big)^{\mathrm{T}} \\
&\qquad \le \Bigg(\sum_{j=1}^{n} (k_1\underline{a}_1)^{-1}(\overline{c}_{1j} + \overline{d}_{1j})\alpha_1\overline{a}_1 \sup_{t\in R} |\hat{u}_j(t) - u_j^*(t)|, \ \dots, \\
&\qquad \sum_{j=1}^{n} (k_n\underline{a}_n)^{-1}(\overline{c}_{nj} + \overline{d}_{nj})\alpha_n\overline{a}_n \sup_{t\in R} |\hat{u}_j(t) - u_j^*(t)| \Bigg)^{\mathrm{T}} \qquad (7) \\
&\qquad \le F \Big(\sup_{t\in R} |\hat{u}_1(t) - u_1^*(t)|, \ \dots, \ \sup_{t\in R} |\hat{u}_n(t) - u_n^*(t)| \Big)^{\mathrm{T}}
\end{aligned}
$$

where $F = \underline{A}^{-1}(\overline{C} + \overline{D})$. Let m be a positive integer. Then, from (7), we get

$$
\begin{aligned}
&\Big(\sup_{t\in R} |(T^m(\hat{u}(t)) - T^m(u^*(t)))_1|, \ \dots, \ \sup_{t\in R} |(T^m(\hat{u}(t)) - T^m(u^*(t)))_n| \Big)^{\mathrm{T}} \\
&= \Big(\sup_{t\in R} |(T(T^{m-1}(\hat{u}(t)) - T^m(u^*(t))))_1|, \dots, \sup_{t\in R} |(T(T^{m-1}(\hat{u}(t)) - T^m(u^*(t))))_n| \Big)^{\mathrm{T}}
\end{aligned}
$$

$$\leq F\Big(\sup_{t\in R}|(T^{m-1}(\hat{u}(t)) - T^{m-1}(u^*(t)))_1|, \dots, \sup_{t\in R}|(T^{m-1}(\hat{u}(t)) - T^{m-1}(u^*(t)))_n|\Big)^{\mathrm{T}}$$
$$\leq F^m\Big(\sup_{t\in R}|(T(\hat{u}(t)) - T(u^*(t)))_1|, \dots, \sup_{t\in R}|(T(\hat{u}(t)) - T(u^*(t)))_n|\Big)^{\mathrm{T}}$$
$$\leq F^m\Big(\sup_{t\in R}|\hat{u}_1(t) - u_1^*(t)|, \dots, \sup_{t\in R}|\hat{u}_n(t) - u_n^*(t)|\Big)^{\mathrm{T}}. \tag{8}$$

Since $\rho(F) < 1$, we obtain $\lim_{n\to+\infty} F^m = 0$, which implies that there exists a positive integer N and a positive integer $\beta < 1$ such that

$$F^N = (\underline{A}^{-1}(\overline{C} + \overline{D}))^N = (h_{ij})_{n\times n}, \quad \text{and} \quad \sum_{j=1}^{n} h_{ij} \leq \beta, \quad i = 1, 2, \dots, n. \tag{9}$$

In view of (8) and (9), we have

$$|(T^N(\hat{u}(t)) - T^N(u^*(t)))_i| \leq \sup_{t\in R}|(T^N(\hat{u}(t)) - T^N(u^*(t)))_i|$$
$$\leq \sum_{j=1}^{n} h_{ij} \sup_{t\in R}|\hat{u}_j(t) - u_j^*(t)|$$
$$\leq \Big(\sup_{t\in R}\max_{1\leq i\leq n}|\hat{u}_j(t) - u_j^*(t)|\Big)\sum_{j=1}^{n} h_{ij} \leq \beta\|\hat{u}(t) - u^*(t)\|,$$

for all $t \in R$, $i = 1, 2, \dots, n$. It follows that

$$\|T^N(\hat{u}(t)) - T^N(u^*(t))\| = \sup_{t\in R}\max_{1\leq i\leq n}|(T^N(\hat{u}(t)) - T^N(u^*(t)))_i| \leq \beta\|\hat{u}(t) - u^*(t)\|.$$

This implies that the mapping $T^N\colon\ \Lambda \to \Lambda$ is a contraction mapping.

By Banach fixed point theorem, there exists a unique fixed point $u^* \in \Lambda^*$ such that $Tu^* = u^*$. From (4) and (5), we know that u^* satisfies system (3), therefore, it is the unique almost periodic solution of system (3). We complete the proof. □

4 Exponential Stability of Almost Periodic Solutions

In this section, we shall discuss the global exponential stability of the almost periodic solution of system (3).

Theorem 4.1 *Suppose that (H1)–(H3) are satisfied, and the condition in Theorem 3.1 holds, then there exists exactly one almost periodic solution of system (3) which is exponentially stable, i.e. all other solutions of system (3) converge to this almost periodic solution exponentially.*

Proof By Theorem 3.1, we have known that system (3) has a unique almost periodic solution, then we shall prove the exponential stability of almost periodic solution.

Let $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^{\mathrm{T}}$ be an arbitrary solution and $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t))^{\mathrm{T}}$ be an almost periodic solution of system (3) with initial values $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^{\mathrm{T}}$ and $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))^{\mathrm{T}}$, respectively. Set

$$y_i(t) = u_i(t) - (u_i^*(t), \quad F_j(y_j(t)) = f_j(y_j(t) + (u_j^*(t)) - f_j(u_j^*(t)),$$

where $i, j = 1, 2, \ldots, n$. It is easy to see that system (3) can be reduced to the following system:

$$\dot{y_i}(t) = -e_i(u_i(t))y_i(t) + \sum_{j=1}^{n} c_{ij}(t)F_j(y_j(t)) + \sum_{j=1}^{n} d_{ij}(t)F_j(y_j(t-\tau_{ij}(t))). \tag{10}$$

Since $\rho(F) = \rho(\underline{A}^{-1}(\overline{C}+\overline{D})) < 1$, it follows from Lemma 2.4 that $E_n - \underline{A}^{-1}(\overline{C}+\overline{D})$ is an M-matrix. In view of Lemma 2.3, there exists a constant vector $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_n)^{\mathrm{T}} > (0, 0, \ldots, 0)^{\mathrm{T}}$ such that

$$(E_n - \underline{A}^{-1}(\overline{C}+\overline{D}))\overline{\xi} > (0, 0, \ldots, 0)^{\mathrm{T}}.$$

That is,

$$-k_i\underline{a}_i\overline{\xi}_i + \sum_{j=1}^{n} \overline{\xi}_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n.$$

Therefore, we can choose a constant $d > 1$ such that

$$\xi = d\overline{\xi} > \sup_{\tau \le t \le 0} |y_i(t)|, \quad i = 1, 2, \ldots, n,$$

and

$$-k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i = \Big[-k_i\underline{a}_i\overline{\xi}_i + \sum_{j=1}^{n} \overline{\xi}_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i \Big] d < 0,$$

where $i = 1, 2, \ldots, n$. Set

$$M_i(\varepsilon) = \varepsilon\xi_i - k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\varepsilon\tau})\alpha_i\overline{a}_i, \quad i = 1, 2, \ldots, n.$$

Clearly, $M_i(\varepsilon)$, $i = 1, 2, \ldots, n$, are continuous functions on $[0, \omega_0]$. Since

$$M_i(0) = -k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n,$$

we can choose a positive constant $\omega \in [0, \omega_0]$ such that

$$M_i(\omega) = (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n. \tag{11}$$

We consider the Lyapunov functional

$$V_i(t) = |y_i(t)|e^{\omega t}, \quad i = 1, 2, \ldots, n. \tag{12}$$

Obviously, for any $y_i(t) \neq 0$, $V_i(t) > 0$. Calculating the upper right derivative of $V_i(t)$ along the solution $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$ of system (10) with the initial value $\overline{\phi} = \phi - \phi^*$, we have

$$\begin{aligned} D^+(V_i(t)) &\le -k_i\underline{a}_i|y_i(t)|e^{\omega t} + \sum_{j=1}^{n} \overline{c}_{ij}|y_i(t)|e^{\omega t} + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t-\tau_{ij}(t))|e^{\omega t} + \omega|y_i(t)|e^{\omega t} \\ &= \Big[(\omega - k_i\underline{a}_i)|y_i(t)| + \sum_{j=1}^{n} \overline{c}_{ij}|y_i(t)|\alpha_i\overline{a}_i + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t-\tau_{ij}(t))|\alpha_i\overline{a}_i \Big] e^{\omega t} \end{aligned} \tag{13}$$

where $i = 1, 2, \ldots, n$. We claim that

$$V_i(t) = |y_i(t)|e^{\omega t} < \xi_i, \quad \text{for all} \quad t > 0, \quad i = 1, 2, \ldots, n. \tag{14}$$

Contrarily, there must exist $i \in \{i = 1, 2, \ldots, n\}$ and $t_i > 0$ such that

$$V_i(t_i) = \xi_i \quad \text{and} \quad V_j(t) < \xi_j, \quad \forall\, t \in (-\infty, t_i), \quad j = 1, 2, \ldots, n, \tag{15}$$

which implies that

$$V_i(t_i) - \xi_i = 0 \quad \text{and} \quad V_j(t) - \xi_j < 0, \quad \forall\, t \in (-\infty, t_i), \quad j = 1, 2, \ldots, n. \tag{16}$$

Together with (13) and (16), we obtain

$$\begin{aligned}
0 &\le D^+(V_i(t_i) - \xi_i) = D^+V_i(t_i) \\
&\le \left[(\omega - k_i\underline{a}_i)|y_i(t)| + \sum_{j=1}^{n} \overline{c}_{ij}|y_i(t)|\alpha_i\overline{a}_i + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t - \tau_{ij}(t))|\alpha_i\overline{a}_i\right]e^{\omega t} \\
&= (\omega - k_i\underline{a}_i)\xi_i + \alpha_i\overline{a}_i\left(\sum_{j=1}^{n} \overline{c}_{ij}|y_i(t_i)|e^{\omega t_i} + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t_i - \tau_{ij}(t_i))|e^{\omega(t_i - \tau_{ij}(t_i))}e^{\omega\tau_{ij}(t_i)}\right) \\
&\le (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i.
\end{aligned} \tag{17}$$

Thus

$$0 \le (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i$$

which contradicts (11). Hence, (14) holds. It follows that

$$|y_i(t)| < \max_{1 \le i \le n}\{\xi_i\}e^{-\omega t}. \tag{18}$$

Letting $\|\overline{\phi}\| = \|\phi - \phi^*\| > 0$, it follows from (18) that we can choose a constant $M > 1$ such that

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \le \max_{1 \le i \le n}\{\xi_i\}e^{-\omega t} \le M\|\phi - \phi^*\|e^{-\omega t}, \tag{19}$$

where $i = 1, 2, \ldots, n,\ t > 0$. Thus, the almost periodic solution of system (3) is globally exponentially stable.

We complete the proof. □

Corollary 4.1 Suppose that (H1)–(H3) are satisfied, and $E_n - \underline{A}^{-1}(\overline{C} + \overline{D})$ is an M-matrix, then there exists exactly an almost periodic solution of system (3) which is exponentially stable, i.e. all other solutions of system (3) converge to this almost periodic solution exponentially.

Proof Notice that $E_n - \underline{A}^{-1}(\overline{C} + \overline{D})$ is an M-matrix, it follows that there exists a vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}} > (0, 0, \ldots, 0)^{\mathrm{T}}$ such that

$$(E_n - \underline{A}^{-1}(\overline{C} + \overline{D}))\eta > (0, 0, \ldots, 0)^{\mathrm{T}}.$$

That is,

$$-k_i\underline{a}_i\eta + \sum_{j=1}^{n}(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i\eta < 0, \quad i = 1, 2, \ldots, n.$$

Therefore, Corollary 4.1 follows immediately from Theorem 4.1. □

Remark 4.1 In Theorem 4.1 and Corollary 4.1, we do not need the assumptions on boundedness, monotonicity, and differentiability for the activation functions. Clearly, the proposed results are different from those in [5, 6, 14] and the references cited therein. Therefore, our results are new and they complement previously known results.

5 An Example

In this section, we give an example to illustrate that our results are feasible.

Example 5.1 Consider the following system with continuously distributed delays:

$$\dot{x}_i(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{2} c_{ij}(t)f_j(x_j(t)) - \sum_{j=1}^{2} d_{ij}(t)f_j(x_j(t-\tau_j(t))) + I_i(t)\Big], \tag{20}$$

where $i = 1, 2$. Let $f_j(x) = \frac{1}{2}(|x+1| - |x-1|)$, we have $\alpha_j = 1$ $(j = 1, 2)$.

Taking

$$(a_1(x_1(t)), a_2(x_2(t)))^{\mathrm{T}} = \Big(2 - \frac{1}{10\pi}\arctan x_1(t), 2 + \frac{1}{10\pi}\arctan x_2(t)\Big)^{\mathrm{T}},$$

$$(b_1(x_1(t)), b_2(x_2(t)))^{\mathrm{T}} = (x_1, x_2)^{\mathrm{T}}, \quad I_1(t) = \frac{9}{5}\sin t, \quad I_2(t) = \frac{9}{5}\cos t,$$

thus we obtain $\underline{a}_1 = \underline{a}_2 = 1$, $\overline{a}_1 = \overline{a}_2 = 3$, $\underline{b}_1 = \underline{b}_2 = \overline{b}_1 = \overline{b}_2 = 1$, $\overline{I}_1 = \overline{I}_1 = \frac{9}{5}$, $k_1 = k_2 = 1$. Let

$$\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{13}\sin t & \frac{1}{13}\sin 2t \\ \frac{1}{13}\sin 3t & \frac{1}{13}\sin 4t \end{pmatrix},$$

$$\begin{pmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{13}\cos t & \frac{1}{13}\cos 2t \\ \frac{1}{13}\cos 3t & \frac{1}{13}\cos 4t \end{pmatrix}.$$

Noting that $\overline{c}_{11} = \overline{c}_{12} = \overline{c}_{21} = \overline{c}_{22} = \overline{d}_{11} = \overline{d}_{12} = \overline{d}_{21} = \overline{d}_{22} = \dfrac{1}{13}$, we get

$$\underline{A}^{-1}(\overline{C} + \overline{D}) = \begin{pmatrix} \frac{6}{13} & \frac{6}{13} \\ \frac{6}{13} & \frac{6}{13} \end{pmatrix}.$$

So, we have

$$\rho(\underline{A}^{-1}(\overline{C} + \overline{D})) = \frac{12}{13} < 1.$$

Thus, it follows from Theorem 3.1 and Theorem 4.1 that system (20) has exactly a unique almost periodic solution, which is globally exponentially stable.

Remark 5.1 System (20) is a simple form of Cohen-Grossberg neural networks with variable delays. In this system, $L_1^a = L_2^a = \frac{1}{30}$, $L_1^{ab} = L_2^{ab} = 1$. If we apply Corollary 4.1 in [15, 16], and choose $\eta = (\eta_1, \eta_2) = (1, 1)$, we obtain $\delta = \frac{26}{5}$, $\rho(K) = \frac{1800}{1781} > 1$, this doesn't satisfy the conditions in Corollary 4.1 in [15, 16]. So, the results in [15, 16] cannot be applicable to this system. This implies that our results are essentially new.

Remark 5.2 Since $f_1(x) = f_2(x) = \frac{1}{2}(|x+1| - |x-1|)$, we can easily verify that the assumptions of boundedness, monotonicity, and differentiability for the activation functions are not available. So, the proposed results in [5, 6, 14] and the references cited therein can not be applicable to system (20).

6 Conclusion

In this paper, the existence and exponential stability of almost periodic solutions for Cohen-Grossberg neural networks with variable delays are considered. Some new sufficient conditions are obtained by applying Banach fixed point theory and differential inequality techniques. Some previous results are improved and extended.

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Multimodel Approach using Neural Networks for Complex Systems Modeling and Identification

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Abstract: This article presents a new approach of systematic determination of models base for the multimodel approach. The application of this approach requires, first, to classify a numeric data by exploiting the self-adapting artificial Kohonen neural-networks. The obtained data relative to the clusters are then exploited for both structural and parametric estimation of base models. To resolve the estimation problem of the validity of the elementary models, used we proposed a new technique, based on the minimization of a quadratic criterion. This criterion exploits the centers of clusters obtained in the determination of the models base step. A comparative study with the residues approach showed the contribution in precision of the proposed validities computation technique. The satisfactory results obtained in numerical simulation, incited us to validate experimentally the contributions already mentioned. Indeed, an on-line experimental validation of the new proposed multimodel representation was carried out on an olive oil esterification reactor. The obtained results are very satisfactory in terms of precision and robustness.

Keywords: *Modeling; multimodel approach; models base; Kohonen card; validities; on line experimental validation*

Mathematics Subject Classification (2000): 92B20, 93B30, 93A30.

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1 Introduction

The development of mathematical models is a major problem for the application of advanced techniques for analysis, prediction, control, optimization, automatic fault detection and diagnostic in the industrial processes. Hence, there is a potential for improved quality and flexibility of final product if the cost of the model development can be reduced. Consequently, a strong demand for advanced modeling and identification methods arises. The multimodel approach is an efficient and a powerful way to resolve problem of modeling and control of complex, non-linear and/or ill-defined processes. This approach is based on a "divide and conquer" strategy [23]. A complex modeling problem is divided into a number of smaller sub-problems, which are solved independently by identifying simple models (generally linear). The obtained group of models forms the so-called models base. Afterwards, it is necessary to compute coefficients called validities of models. The simple models are, thereafter, combined, according to their estimated validities, together to obtain the global model. The past few years have shown an increase in the use of the multimodel representation [16]. This concept includes a number of approaches such as: Takagi and Sugeno Fuzzy Inference Systems [29], local model networks [16], gain-scheduled control, statistical mixture models, Smooth Threshold Auto-Regressive (STAR) models of Tong [30] and the state dependent models of Priestley [20]. For the majority of these approaches, the model parameters are obtained from prior knowledge, linearization of physical model or identified from measured data [21]. In many cases, the local models can be quite simple, such as linear or affine models. Besides, the multimodel concept coincides with engineering design in which the division of problems into manageable parts is the major design methodology [23]. The multimodel approaches were succeeded in different domains such as academic, biomedical, process industries, etc. However, they remain so confronted with several difficulties such as the determination of the models base. To resolve this problem, a modeling framework based on an operating decomposition of the system's operating range has interested Johansen in [10]. Indeed, he has proposed an algorithm that able to identify decomposition into operating regimes and local models on the base of empirical data. However, this algorithm requires that the regime must be d-dimensional boxes with orthogonal edges. Besides, the introduction of this last complex description of the regime limits will increase the number of parameters necessary to represent theses boundaries or local model validity functions. This leads, consequently, to a more complex identification problem [11]. Murray-Smith in [16] proposes to use learning systems able to model unknown nonlinear dynamic processes from their observed input-output behaviour. Local model networks use a number of simple and locally accurate models to represent a globally complex process. A major difficulty with local model nets is the optimization of the model structure. Heikki [8] has proposed an evolutionary self-organizing map capable of creating an organized model bank from a data set. However, the proposed algorithm is very complex and requires a very large knowledge such as genetic algorithm, self-organizing card, etc. Besides, the computing of one map is relatively very long. In 1995, Gawthrop considered the approximation of the continuous-time non-linear system in the vicinity of the equilibrium operating points by a continuous-time local model network [7]. One global inconvenience of most of these last strategies, is that the determination of these local models needs to a certain extend a priori knowledge of the system and its structure [3, 5, 1, 6, 2, 21, 19, 23]. Besides, we cannot found a systematic method for local models determination; which supposes several preliminary tests before its choice. Recently, it is proposed in [13] an approach of models base determination for the uncertain processes, which limit the number of base models to four or five models. This method is inspired from the algebric stability approach suggested by Kharitonov [13]. The models base is obtained by determining the four extreme models, and the average model, determined as an average of the boundary models. Mezghani in [17] proposed the extension of this last approach for discrete case using the d operator. These last approaches require the knowledge of the variation domains parameters of the uncertain process. But, this last information is always not still possible. Another inconvenience is that the last models base will contain models with the same structure. We propose, in this paper, a new systematic determination approach of a models base for the representation of uncertain discrete linear systems. This approach does not require the limits knowledge of the parameters. Besides, this method allows to generate automatically the number, the structures and the parameters of the elaborated models. Indeed, the proposed method requires three principle steps [25, 27, 28]. The first step consists in classifying numerical data by using a Self-adapting artificial Kohonen neural network. The second step is a structural and parametric estimations step in order to determine the base models. Also, to resolve the problem of validities computation we propose a new technique, based on the minimization of a quadratic criterion [26, 28]. This criterion exploits the centers of clusters obtained in the models base determination step. By comparison with the residues approaches, used by many researchers, we have demonstrated the efficiency and the precision of the suggested technique. In order to highlight the good performance in precision and the robustness under particularly severe conditions of the two suggested approaches, the theoretical study is, then, validated by numerical simulation and by experiments. This paper is organized as follows: in the section two, a principle of classification by using a Kohonen card is introduced. The new systematic approach determination of a models base is developed with details in the third section. The validities computation represents the subject of the fourth section. The principle of computation of multimodel'output is given in section five. A numerical example is presented in the section six. In section seven, an experimental validation, carried out on an olive oil esterification reactor, is considered. We finish the present work by a conclusion.

2 Classification of the Numerical Data by Using the Kohonen Card

The self-organizing Kohonen map is a well-known unsupervised algorithm used frequently for classification of data. The standard card can find the cluster centers and gives a visual interpretation of the distribution and clusters of the data. This classification strategy consists in applying the rule of Kohonen [18, 8, 22]. This rule is characterized by an unsupervised competitive learning. Where, a competition takes place before the modification of the network-weights. Only the neuron, which gained the competition, has the right to change their weight. The Kohonen rule has the property of self-adapting, which allows him to group together a set of data, presented to the corresponding network, around a certain number of representative centroides of these data clusters. The used neural network is formed by one input layer of p neurons and by one output layer of n neurons corresponding to the Kohonen card [18, 22]. The architecture of this network is given by (2.1) [25]. Each neuron of the Kohonen card receives p signals coming from the input layer. The weight w_{pn} is relative to the connection between the input neuron p and the card neuron n. The weight vector W_i associated to neuron i is then composed of p elements. The Kohonen rule works as follows [18, 22]:

1. The network receives a data set Y.

2. Each card neuron calculates the Euclidean distance between the weight vector W_i and the input vector Y.

3. The competition between card neurons starts. This competition is based on the winner-takes-all strategy. The neuron having the nearest weight vector Wi to the input Y wins the competition. The winner neuron output z_i is putting at 1 and the other ones are putting then at 0.

4. The different weights are modified according to the following relation:

$$W_i^{new} = W_i^{old} + \alpha\Big(Y - W_i^{old}\Big)z_i \qquad (1)$$

where α is a constant such that $0 < \alpha < 1$.

At the end of the training, the Kohonen network generates the representative vectors of different clusters and their centers.

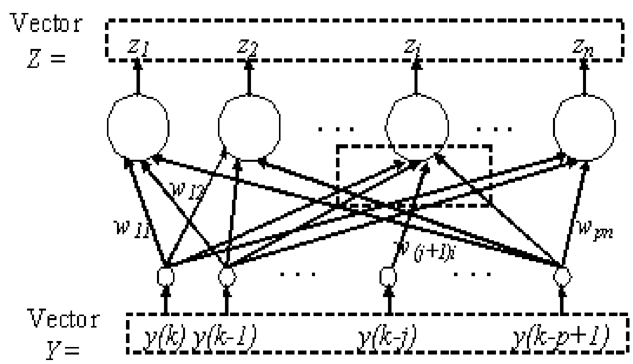

Figure 2.1: The retained architecture for the generation of different observations vectors for modeling.

3 A Systematic Determination Approach of a Models Base

The application of this approach requires firstly the determination of the clusters number. The classification of numerical data is the second stage. Then, there is a stage of structural and parametric estimation.

3.1 Determination of the clusters number

To classify the numerical data, it is necessary to pass throught the step of determination of the adequate clusters number and as consequence, the number of base models. To resolve the problem, we propose to consider a two-dimensional Kohonen card with a neurons number n in the output-layer which is relatively important. At the end of training, if the network gives badly repartition clusters, it will remove the cluster i having an elements number N_{Ci} verifying:

$$N_{Ci} < \frac{1}{2}\frac{N_H}{n}, \tag{2}$$

where N_H is the observations'number. Also, we increase the network structure and we restart the training.

3.2 Classification of the numerical data by exploiting the Kohonen card

After determining the suitable number of classes, consequently the base models number, it is the question of classifying the measurements. These last are related to the output of an uncertain or ill-defined discrete linear system using the proposed method described in Section 2. Therefore, we exploit a Kohonen network, which has neurons number in the output-layer equal to the clusters number, determined by the method described in the last section. This network is able of looking into the output of a set of representative vectors of different clusters with their respective centers. These vectors are, then, exploited for the structural and parametric identification of the elaborated base models.

3.3 Structural and parametric estimation

The order estimation method of the retained models is called instrumental determinants' ratio-test [4, 25, 15]. This method consists in building an information matrix Q_m, containing the input-output measurements couples given by:

$$Q_m = \frac{1}{N_H}\sum_{k=1}^{N_H}\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k-m+1) \\ u(k+m) \end{bmatrix}\begin{bmatrix} y(k+1) \\ u(k+1) \\ \vdots \\ y(k+m) \\ u(k+m) \end{bmatrix}^T. \tag{3}$$

The instrumental determinants'ratio $RDI(m)$ is given by the following relation:

$$RDI(m) = \left|\frac{det(Q_m)}{det(Q_{m+1}}\right|. \tag{4}$$

For every value of m, the order determination procedure computes Qm and Q_{m+1} matrices and estimates the ratio RDI, the retained order m is the value for which the ratio $RDI(m)$ quickly increases for the first time. Indeed, Q_{m+1} matrix becomes singular when m becomes identified with the exact order.

The retained parametric estimation method is the Recursive Least Squares' method RLS [4].

4 A New Approach for Validities' Computation

Several validities computation methods was proposed in the literature [5, 6, 12, 13, 16, 17, 19]. All these methods are based on the residues computation and they are based on measuring the distance between the current state of the process and the considered model M_i. The geometric distance can be calculated by several methods; the simplest one is the distance $r_i(k)$ between the process output $y(k)$ and the base models outputs $y_i(k)$:

$$r_i(k) = |y(k) - y_i(k)|. \tag{5}$$

Frequently, we choose the validities such as all the time their sum is equal to the unity. For example,

$$v_i(k) = \frac{|1 - r'_i(k)|}{C - 1}. \tag{6}$$

C represents the retained number of base models and is a normalized distance given by

$$r'_i(k) = \frac{r_i(k)}{\sum_{i=1}^{C} r_i(k)}. \tag{7}$$

The proposed method of validities computation is inspired from the fuzzy version of the "k-means" algorithm[18]. This method is based on the minimization of the following criterion:

$$J = \sum_{i=1}^{C} \sum_{k=1}^{N_H} v_i^2(k) \|y(k) - c_i\|^2 \tag{8}$$

with

$$\sum_{i=1}^{C} v_i(k) = 1, \tag{9}$$

where $v_i(k)$ represent the degree of validity of the model i at the instant k, c_i is the center of the class i.

It is a first order problem of optimization with equality constraint $g(v_i(k))$. The resolution of this type of problem requires the determination of the Lagrange's equation. In fact, so that $v_i(k)$ is a local extremum of the criterion J, it is necessary that there is a real λ such that the Lagrangian L of the problem can be written as follows:

$$L(v_i(k), \lambda) = J + \lambda g(v_i(k)) \tag{10}$$

is stationary with regard to $v_i(k)$ and λ. This leads to

$$\begin{cases} \frac{\partial(L(v_i(k),\lambda)}{\partial(v_i(k))} = 0, \\ \frac{\partial(L(v_i(k),\lambda))}{\partial(\lambda)} = 0, \end{cases} \tag{11}$$

where λ is the Lagrange's multiplier associated to the constraint. The relations (11) lead to the following system

$$\begin{cases} 2v_1(k) \|y(k) - c_1\|^2 + \lambda = 0, \\ 2v_2(k) \|y(k) - c_2\|^2 + \lambda = 0, \\ \vdots \\ 2v_i(k) \|y(k) - c_i\|^2 + \lambda = 0, \\ \vdots \\ 2v_C(k) \|y(k) - c_C\|^2 + \lambda = 0, \\ v_1(k) + v_2(k) + \ldots + v_i(k) + \cdots + v_C(k) = 1. \end{cases} \tag{12}$$

This problem becomes

$$\begin{cases} \left\{ v_i(k), \|y(k) - c_i\|^2 + \lambda = 0, i \in [1, C] \right\}, \\ \sum_{l=1}^{C} v_l(k) = 1. \end{cases} \tag{13}$$

The relations (13) give

$$v_i(k) = \frac{-\lambda}{2\,\|y(k) - c_i\|^2}. \tag{14}$$

This relation becomes

$$\sum_{l=1}^{C} \frac{-\lambda}{2\,\|y(k) - c_l\|^2} = 1. \tag{15}$$

Then λ is given from the relation (15) and replaced in the equation (14). Finally, we can conclude that the expression of validity degree for a model M_i can be written as follows:

$$v_i(k) = \frac{1}{\sum_{l=1}^{C} (A_i^2(k)/A_l^2(k))}, \tag{16}$$

where $A_i^2(k) = \|y(k) - c_i\|^2$ (see (4.1)).

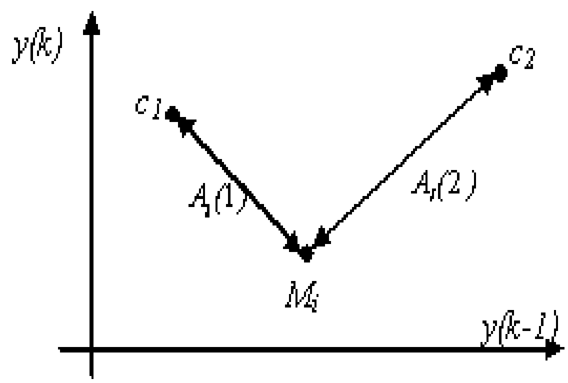

Figure 4.1: Euclidean distance illustrated by the new technique of validity computation.

5 Computation of Multimodel Output

The multimodel output is obtained by fusion of the local models pondered by their respective validities. The next relation (17) gives the expression of the final multimodel output:

$$y_{MM}(k) = \sum_{i=1}^{C} y_i(k) v_i(k). \tag{17}$$

6 Simulation Example in Stochastic Case

The object of this section is to demonstrate the interest and the robustness of both proposed methods: the multimodel representation and the validities computation technique. Let us consider a stochastic linear process with time varying parameters, described by the following equation [25, 15]:

$$y(k) = -a_1(k)y(k-1) - a_2(k)y(k-2) + b_1(k)u(k-1) + b_2(k)u(k-2) + e(k), \quad (18)$$

where $e(k)$ is a white noise $(0, \sigma^2)$ with covariance σ equal to 0.2. The variation laws of different parameters of the process are given by the Figure 6.1. The retained excitation signal $u(k)$ is a Pseudo Aleatory Binary Sequence.

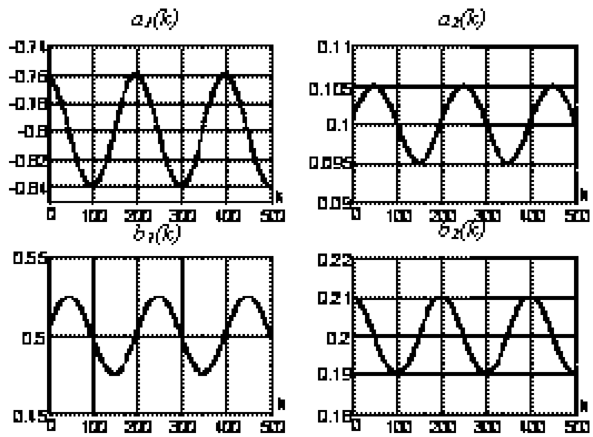

Figure 6.1: The variation laws of the considered process parameters.

6.1 Classification of the numerical data by exploiting the Kohonen card

The suggested approach for the systematic determination of the models base has been implemented. Indeed, the numerical noisy identification data obtained by exciting the system (18) by a Pseudo Aleatory Binary Sequence are presented to a Kohonen card formed by one input layer of two neurons and by one output layer of three neurons. The Figure 6.2 shows that three data sets relative to the various clusters are obtained at the end of learning of the neuronal network.

6.2 Structural and parametric estimation

From each of the data relative to the three clusters, we could determine the orders and the parameters of the transfer functions $H_1(q^{-1})$, $H_2(q^{-1})$ and $H_3(q^{-1})$ relative to the base models. Figure 6.3 shows the evolutions of the Instrumental Determinants' Ratio $RDI_i(m)(i = 1, 2 \text{ or } 3)$ for the three obtained clusters. We observe, clearly, that the orders of the three models are equal to 2.

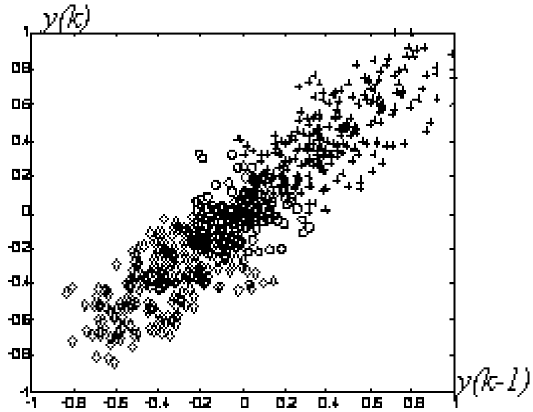

Figure 6.2: Three sets of numerical data relative to the different base models.

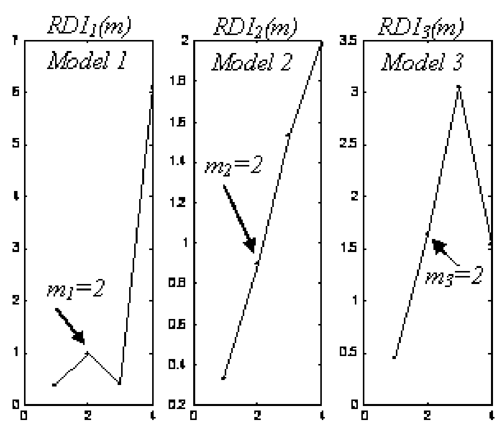

Figure 6.3: Evolutions of the RDI for the three obtained clusters.

After the parametric identification step, the obtained transfer functions $H_1(q^{-1}), H_2(q^{-1})$ and $H_3(q^{-1})$ can be written:

$$H_1(q^{-1}) = \frac{0.48765q^{-1} + 0.26243q^{-2}}{1 - 0.62912q^{-1} + 0.022475q^{-2}}, \tag{19}$$

$$H_2(q^{-1}) = \frac{0.49611q^{-1} + 0.22886q^{-2}}{1 - 0.70327q^{-1} + 0.019325q^{-2}}, \tag{20}$$

$$H_3(q^{-1}) = \frac{0.49987q^{-1} + 0.24861q^{-2}}{1 - 0.7443q^{-1} + 0.040774q^{-2}}. \tag{21}$$

6.3 Validation phase

The application of the following input sequence is the subject of validation step:

$$u(k) = 2 + \sin k/20. \tag{22}$$

The proposed approach for validities computation uses the clusters centers obtained in the stage of determination of a models base. The coordinates of the three obtained centers c_1, c_2, c_3 are: $c_1(-0.412; -0.4020); c_2(-0.0151; 0.0041); c_3(0.4738; 0.4687)$. The results of validation are given in the Figure 6.4. This figure shows that the multimodel output $y_{fn}(k)$ obtained by fusion of base models outputs pondered by the new technique validities, follows the real output $y_r(k)$ of the stochastic uncertain process with a relatively negligible error. In the case of modeling classical approach, we have exploited the same

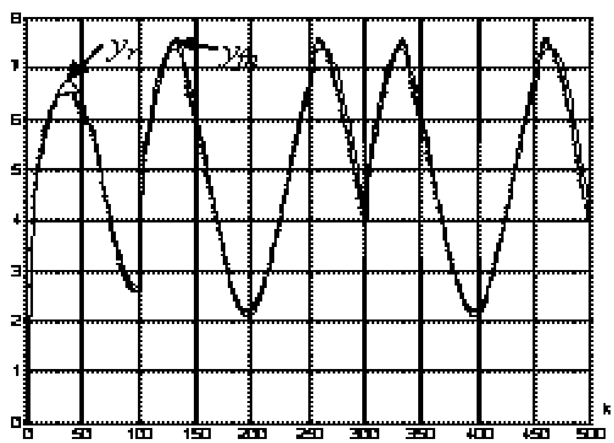

Figure 6.4: Evolutions of the real and multimodel outputs (New technique).

numerical noisy identification data used for the multimodel representation. By recourse to the instrumental determinants' ratio for the structural estimation and to the recursive least squares method for the parametric identification, the transfer function $H(q^{-1})$ of the global model "M" can be written as follows:

$$H(q^{-1}) = \frac{0.49457q^{-1} + 0.28186q^{-2}}{1 - 0.60115q^{-1} + 0.043232q^{-2}}. \tag{23}$$

The Figure 6.5 represents the evolutions of the relative errors between the real output and the global model "M" and the multimodel "MMn". This figure demonstrates that the multimodel representation offers a very satisfactory precision and robustness relatively to the case in which classical modeling, based on one global model "M", is considered. The evolutions of different validities of models are given by the Figure 6.6. This figure

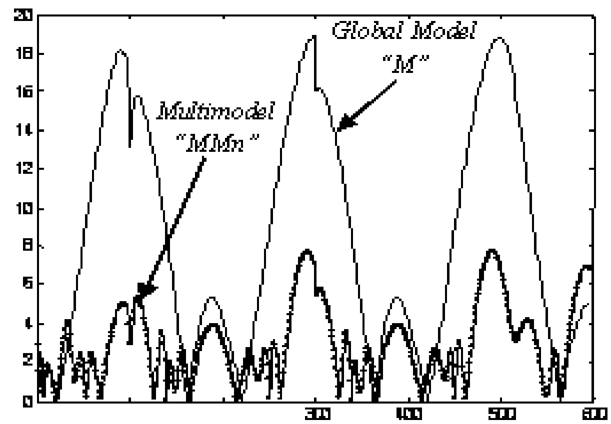

Figure 6.5: Evolutions of relative errors.

shows the complementarities of the different models in the different operation areas. It shows, also, that it is possible that one model can describe correctly the system (validity equal to the unity), the validities of the others models are equal to zero. This last result is not possible when the residues approach is applied. Indeed, in the Figure 6.7, we have presented the evolution of the three validities calculated by the residues approach in the same conditions. This figure shows that these validities cannot exceed $0,5$. This can be justified by the presence of term $'C-1'$ in the denominator of the validities expression (6). As consequent, the residues approach cannot evaluate correctly the contribution of every model of the base in the global behaviour of the system.

Figure 6.8 presents the evolutions of the prediction errors $er_1(k)$ and $er_2(k)$ of the two multimodel outputs respectively $y_{fc}(k)$ (residues approach) and $y_{fn}(k)$ (new technique) with regard to the real output. This figure shows the performance in precision and in robustness of the new technique of validities computation by comparison with the residues approach.

7 Experimental Validation: Olive Oil Esterification-Reactor

In order to show the contribution in precision and robustness of the suggested modeling strategy, we have implemented it practically in the case of modeling of an olive oil esterification-reactor. This discontinuous reactor carries out, by an alcohol, a chemical reaction of vegetable olive oil esterification. This type of reaction is given by the following scheme: $Acid + Alcohol \rightleftharpoons Ester + Water$. The obtained product is an ester with a very high benefit used mainly in the manufacture of cosmetic products. In previous work, the dynamic behaviour of this reactor has been modeled by a set of complex differential

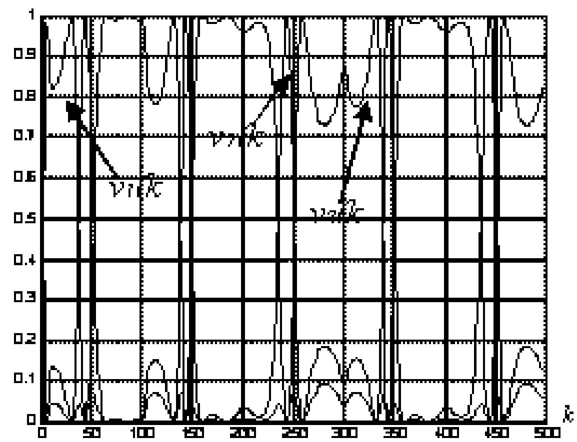

Figure 6.6: Evolutions of the validities (new technique).

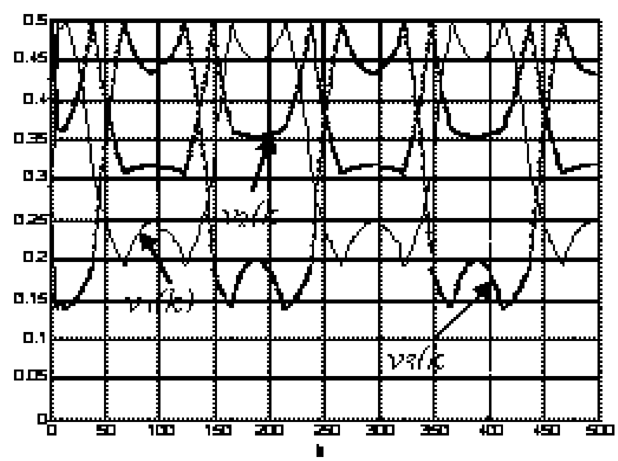

Figure 6.7: Evolutions of the validities (Residues approach).

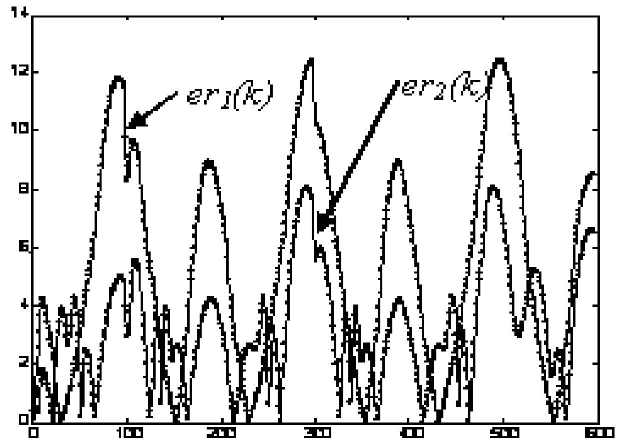

Figure 6.8: Evolution of the relative prediction errors.

equations. The static characteristic of the reactor is non-linear and, consequently, the classical modeling, based on one global model cannot lead to satisfactory results. To improve these results, we propose, in the next section, to use the suggested multimodel representation.

7.1 A modeling phase

In Figure 7.1, we have presented the input-output measurements picked out experimentally of the reactor for the identification step. By exploiting the last input-output

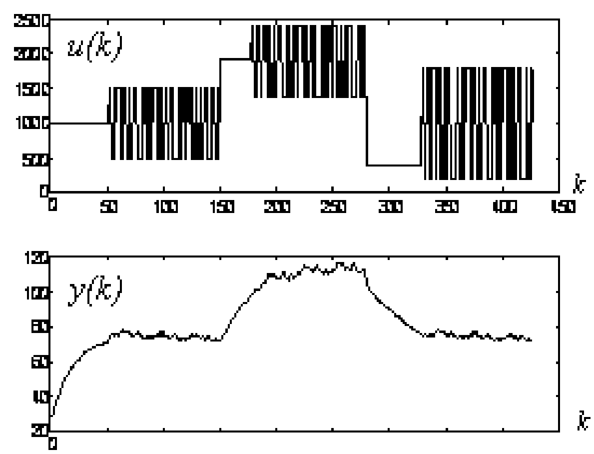

Figure 7.1: Evolutions of the input-output measurements u(k) and y(k).

measurements' file, the suggested approach for the determination of the models base has been implemented. Indeed, the experimental data are presented to a Kohonen network having two inputs and one-dimensional card with 3 neurons in the output layer. Figure 7.2 shows that three sets of data relative to the different clusters are obtained at the end of the neural network training. From each of the data relative to a cluster $c(c = 1, .., 3)$,

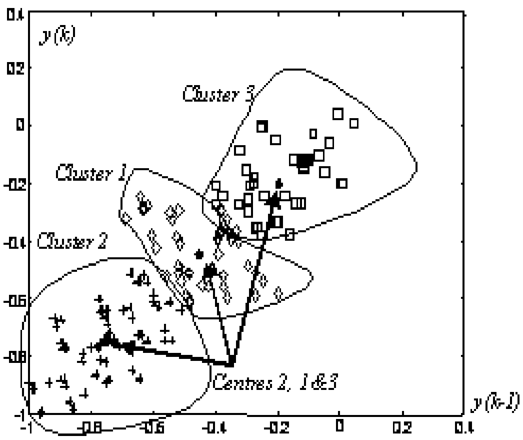

Figure 7.2: Three sets of the experimental data relative to the different base-models.

we could determine the transfer functions $(H_1(q^{-1}), H_2(q^{-1})$ and $H_3(q^{-1}))$ relative to the base-models. Figure 7.3 presents the evolutions of the Instrumental Determinants' Ratio $RDI_i(m)$ (i=1, 2 or 3) for the three obtained clusters. This figure shows that the adequate estimated orders of the three models are equal to 2. Finally, we have obtained

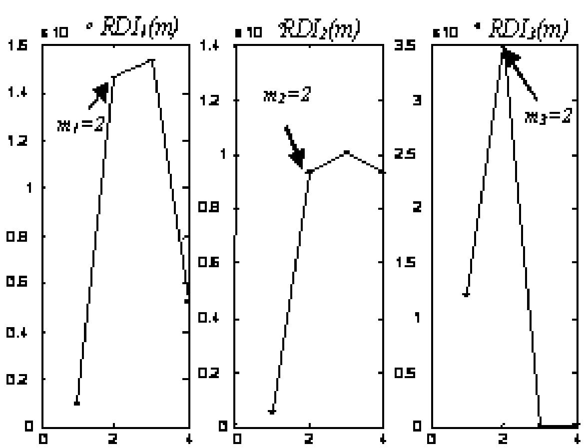

Figure 7.3: Evolutions of the RDI for the three obtained clusters.

the base formed by the models described by the following transfer functions:

$$H_1(q^{-1}) = \frac{0.0018269q^{-1} + 0.00043866q^{-2}}{1 - 1.3052q^{-1} + 0.32917q^{-2}}, \tag{24}$$

$$H_2(q^{-1}) = \frac{0.0018804q^{-1} + 4.2569.10^{-5}q^{-2}}{1 - 1.216q^{-1} + 0.24209q^{-2}}, \tag{25}$$

$$H_3(q^{-1}) = \frac{0.0011144q^{-1} + 0.00046594q^{-2}}{1 - 1.1185q^{-1} + 0.12743q^{-2}}. \tag{26}$$

In the case of modeling classical approach, the process is considered linear around an operation point. The non-linearity is consequently interpreted, under these conditions, as a parametric disturbance. By recourse to the instrumental determinants ratio test for the structural estimation, and to the recursive least squares method for the parametric identification, the transfer function $H(q^{-1})$of the global model "M", worked out by the exploitation of an input-output measurements'file experimentally picked out on the reactor, can be written as follows:

$$H(q^{-1}) = \frac{-0.00010162q^{-1} + 0.0012255q^{-2}}{1 - 1.0425q^{-1} + 0.058094q^{-2}}. \tag{27}$$

7.2 Evaluation of the modeling results

To validate the obtained models, we have considered a new input-output measurements'file picked out for the real system. The effective output $y_{MM}(k)$ of the multimodel "MM" is calculated by fusion of the three base outputs pondered by their respective validities. Figure 7.4 represents the evolutions of the real, the global model "M" and the multimodel "MM" outputs. This figure shows that the "MM" approach, using the elaborated base, offers a very satisfactory precision relatively to the case in which classical modeling, based on one global model "M", is considered. Indeed, the relative error between the real output and the model "M" and the multimodel "MM" outputs confirms this last conclusion (Figure 7.5). The evolutions of the different validities relative to the different models of the base are given on Figure 7.6. It gives information about the complementarities of the different models in the operation area of the reactor which can be divided into three zones of heating, reaction and cooling.

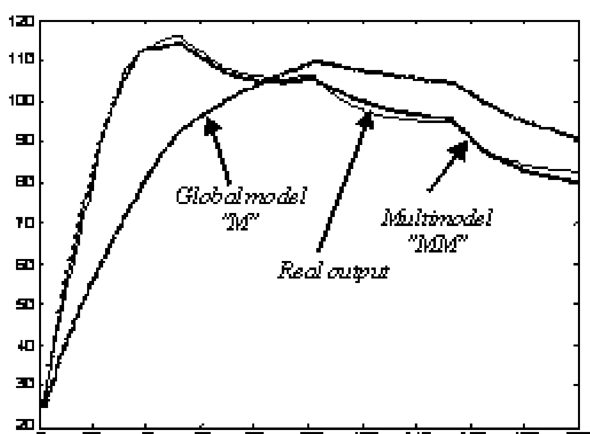

Figure 7.4: Experimental validation of the models (classical and multimodel approaches).

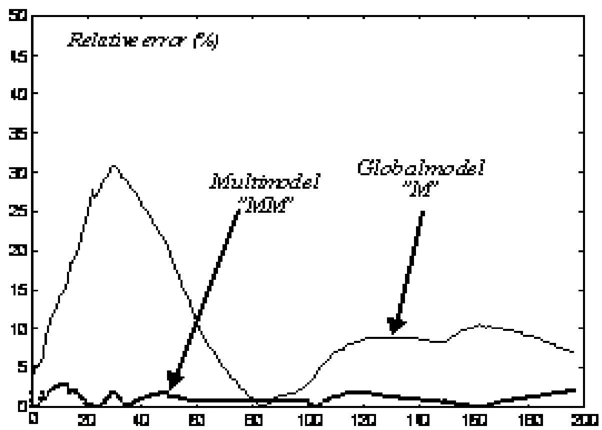

Figure 7.5: Evolutions of relative errors.

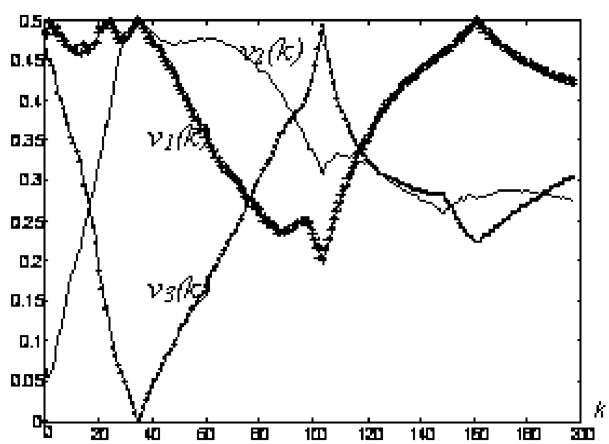

Figure 7.6: Evolutions of different models validities of the elaborate base.

8 Conclusion

In this paper, we have presented firstly a new systematic determination approach of models base for multimodel approach. This approach does not require a priori knowledge about the studied system and can generate automatically the number, the structures and the parameters of base models. Indeed, it can be applied on three steps. The primary step consists in determining the suitable number of base models. The second one consists in an off-line classification of identification data. The structural and parametric estimations of the base models from the obtained vectors in the classification step, form the third step. Secondly, a new technique of validities computation is developed. This last technique consists in minimizing a quadratic criterion exploiting the clusters centers obtained in the stage of determination of the models base. The application of these contributions is carried out, first, on a simulation example, then on a real process corresponding to a semi-batch chemical reactor. These applications showed the efficiency and the very good performances of the two proposed methods, with regard to the classical modeling method based on unique model and to the residues approach.

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