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A Comparison in the Theory of Calculus of Variations on Time Scales with an Application to the Ramsey Model

F.M. Atici * and C.S. McMahan

Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101, U.S.A.

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Abstract: The purpose of this paper is to provide a comparison between the existing results on the calculus of variations with the Δ and ∇ operators on time scales. We will also prove the theorems pertaining to free boundary conditions, for dynamic models missing one or both end points conditions. To illustrate our results we shall use a well known Ramsey model and an adjustment model in economics.

Keywords: *time scales; calculus of variations; nabla derivative; delta derivative; dynamic model.*

Mathematics Subject Classification (2000): 39J05, 49J15, 91B50, 91B62.

1 Introduction

The theory of calculus of variations on time scales has been developed in two directions, one with the Δ operator and one with the ∇ operator. It is possible to write one derivative in terms of the other derivative operator on time scales under certain continuity assumptions [4, Theorem 8.49]. On the other hand, it is not always possible to optimize the dynamic model on time scales by using the deterministic optimization method, namely calculus of variations, since the theory has been developed, to the authors' knowledge, for functionals of the form $\int_{[a,b]\cap\mathbb{T}} L(t, y(\sigma(t)), y^{\Delta}(t))\Delta t$ and $\int_{[a,b]\cap\mathbb{T}} N\left(t, y(\rho(t)), y^{\nabla}(t)\right)\nabla t$. As a result of this, here are some questions which need to be answered.

- Which is more advantageous using the Δ or ∇ derivative in dynamic modelling ?

* Corresponding author: ferhan.atici@wku.edu

• Does there exist a dynamic model which can be formed with the Δ and ∇ operators, solved and compared ?
The aim of this paper is to answer these questions by illustrating some familiar discrete or continuous models from economics. The reader may find the papers [1, 2, 5, 6] interesting to see the development of the theory and some nice applications. We will start with listing two main theorems, one is obtained by Bohner in [5] for the Δ operator and the other one is obtained by Atıcı, Biles and Lebedinsky in [1] for the ∇ operator.

Let $\mathbb{T}$ be a time scale which is nonempty closed subset of reals $\mathbb{R}$. We refer to the books by Bohner and Peterson for further reading on time scales [3, 4].

Assume that $L(t,u,v)$ for each $t \in [\sigma(a), \sigma^2(b)] \subseteq \mathbb{T}$ is a class C^2_Δ function of (u,v). Let $y \in C^1_\Delta[a, \sigma^2(b)]$ with $y(\sigma(a)) = A$, $y(\sigma^2(b)) = B$, where

$$C^1_\Delta[a,\sigma^2(b)] = \{y : [a,\sigma^2(b)] \to \mathbb{R} \ \mid \ y^\Delta \text{ is continuous on } [a,\sigma^2(b)]^\kappa\}.$$

Theorem 1.1 *If a function $y(t)$ provides a local extremum to the functional*

$$J[y] = \int_{\sigma(a)}^{\sigma^2(b)} L(t, y(\sigma(t)), y^\Delta(t))\Delta t$$

where $y \in C^2_\Delta[a,\sigma^2(b)]$ and $y(\sigma(a)) = A$ and $y(\sigma^2(b)) = B$, then y must satisfy the Euler-Lagrange equation

$$L_{y^\sigma}(t, y^\sigma, y^\Delta) - L^\Delta_{y^\Delta}(t, y^\sigma, y^\Delta) = 0 \tag{1}$$

for $t \in [a, \sigma(b)]^\kappa_\kappa$.

Assume that $N(t,u,v)$ is a class C^2_∇ function of (u,v) for each $t \in [\rho^2(a), \rho(b)] \subseteq \mathbb{T}$. Let $y \in C^1_\nabla[\rho^2(a), \rho(b)]$ with $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, where

$$C^1_\nabla[\rho^2(a),\rho(b)] = \{y : [\rho^2(a),\rho(b)] \to \mathbb{R} \ \mid \ y^\nabla \text{ is continuous on } [\rho^2(a),\rho(b)]_\kappa\}.$$

Theorem 1.2 *If a function $y(t)$ provides a local extremum to the functional*

$$J[y] = \int_{\rho^2(a)}^{\rho(b)} N\left(t, y(\rho(t)), y^\nabla(t)\right)\nabla t$$

where $y \in C^2_\nabla\left[\rho^2(a),\rho(b)\right]$ and $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, then y must satisfy the Euler-Lagrange equation

$$N_{y^\rho}\left(t, y^\rho, y^\nabla\right) - N^\nabla_{y^\nabla}\left(t, y^\rho, y^\nabla\right) = 0 \tag{2}$$

for $t \in [\rho(a), b]^\kappa_\kappa$.

The plan of this paper is as follows. In Section 2, we will introduce the Ramsey model [7] on time scales and write the model with the ∇ and Δ derivatives, respectively. We will then solve each model using the Euler-Lagrange equations (1) and (2). We shall then compare the solutions of each model on $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$. In Section 3, we shall state the free boundary conditions for the dynamic model which is missing one or two end-point conditions. We shall illustrate our results with an adjustment model [8].

2 The Ramsey Model

In this section, we study the Ramsey model which determines the behavior of saving/consumption as the result of optimal inter temporal choices by individual households. Before writing the model on time scales we will present its discrete and continuous versions so that one can see how the time scale model unifies its discrete and continuous counterparts.

Discrete Model:

We maximize the Ramsey model which is

$$\sum_{t=0}^{T-1}(1+p)^{-t}U[C_t]$$

subject to initial wealth W_0 that can always be invested for an exogeneously-given certain rate of yield r; or subject to the constraint

$$C_t = W_t - \frac{W_{t+1}}{1+r}, \tag{3}$$

or

$$\max_{[W_t]} \sum_{t=0}^{T-1}(1+p)^{-t}U\left[W_t - \frac{W_{t+1}}{1+r}\right],$$

where the quantities are defined as

C_t — consumption,
p — discount rate,
U_t — instantaneous utility function,
W_t — production function.

The Euler-Lagrange equation for the discrete model is as follows

$$\frac{r-p}{1+r}U'[C(t)] + \Delta[U'[C(t)]] = 0.$$

Continuous Model:

We maximize the Ramsey model

$$\int_0^T e^{-pt}U[C(t)]dt$$

subject to

$$C(t) = rW(t) - W'(t) \tag{4}$$

or

$$\max_{[W(t)]} \int_0^T e^{-pt}U[rW(t) - W'(t)]dt.$$

The Euler-Lagrange equation is as follows

$$(r-p)U'[C(t)] + [U'[C(t)]]' = 0.$$

Let's now develop two formulations of the time scale Ramsey model, in order to employ both the nabla and delta calculus of variations.

The Ramsey Model with the Nabla Derivative:

Consider the constraint (4) for the continuous case which can be rewritten as

$$C(t) = -e^{rt}[e^{-rt}W(t)]'. \tag{5}$$

Now consider the discrete constraint (3) which can be rewritten as

$$C_{t-1} = -(1+r)^{t-1}\nabla\left[\frac{W(t)}{(1+r)^t}\right]. \tag{6}$$

Using the new formulations (5) and (6), of the continuous and discrete constraints, a generalization can be made in order to develop the time scale constraint

$$C(\rho(t)) = -\left[\hat{e}_{-r}(\rho(t),0)\right]^{-1}\left[\hat{e}_{-r}(t,0)W(t)\right]^{\nabla}.$$

Then by taking the nabla derivative of $[\hat{e}_{-r}(t,0)W(t)]$ the following is obtained

$$\begin{aligned} C(\rho(t)) &= -\left[\hat{e}_{-r}(\rho(t),0)\right]^{-1}\left[\hat{e}_{-r}^{\nabla}(t,0)W(\rho(t)) + \hat{e}_{-r}(t,0)W^{\nabla}(t)\right] \\ &= -\left[(1+\nu(t)r)\hat{e}_{-r}(t,0)\right]^{-1}\left[-r\hat{e}_{-r}(t,0)W(\rho(t)) + \hat{e}_{-r}(t,0)W^{\nabla}(t)\right]. \end{aligned}$$

Then by distributing through by $-\left[(1+\nu(t)r)\hat{e}_{-r}(t,0)\right]^{-1}$ the constraint for the nabla version of the Ramsey model is obtained

$$C(\rho(t)) = \frac{rW(\rho(t))}{1+\nu(t)r} - \frac{W^{\nabla}(t)}{1+\nu(t)r}.$$

The Ramsey model with the nabla derivative is

$$\max_{[W(t)]}\int_{\rho^2(0)}^{\rho^2(T)} \hat{e}_{-p}(\rho(t),0)U\left[\frac{rW(\rho(t))}{1+\nu(t)r} - \frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\nabla t. \tag{7}$$

Note that this model includes the discrete case and the continuous case as special cases. First we derive the Euler-Lagrange equation using Theorem 1.2. In this model,

$$N(t, W^{\rho}, W^{\nabla}) = \hat{e}_{-p}(\rho(t),0)U\left[\frac{rW^{\rho}}{1+\nu(t)r} - \frac{W^{\nabla}}{1+\nu(t)r}\right]$$

so we obtain the following dynamic equation

$$\begin{aligned} &\hat{e}_{-p}(\rho(t),0)U'\left[\frac{rW(\rho(t))}{1+\nu(t)r} - \frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\left(\frac{r}{1+\nu(t)r}\right) \\ &+ \left[\hat{e}_{-p}(\rho(t),0)U'\left[\frac{rW(\rho(t))}{1+\nu(t)r} - \frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla} = 0. \end{aligned}$$

Then by substituting $C(\rho(t))$ in for $\frac{r}{1+\nu(t)r}W(\rho(t)) - \frac{1}{1+\nu(t)r}W^{\nabla}(t)$ the following is obtained

$$\hat{e}_{-p}(\rho(t),0)U'\left(C(\rho(t))\right)\left(\frac{r}{1+\nu(t)r}\right)+\left[\hat{e}_{-p}(\rho(t),0)U'\left(C(\rho(t))\right)\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla}=0.$$

Then by using the product rule and by taking the nabla derivative of the nabla exponential function we have

$$\left[U'\left(C(\rho(t))\right)\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla}=\frac{p(1-\nu^{\nabla}(t))-r}{(1+\nu(t)r)(1+\nu(\rho(t))p)}U'\left(C(\rho(t))\right),$$

where we assume that ν is a nabla differentiable function, note that ν is not necessarily nabla differentiable in general.
We let $\alpha(t):=\dfrac{1}{1+\nu(t)r}$. Then again using the product rule the following is obtained

$$\alpha(\rho(t))\left[U'\left(C(\rho(t))\right)\right]^{\nabla}+\alpha^{\nabla}(t)\left[U'\left(C(\rho(t))\right)\right]=\frac{p(1-\nu^{\nabla}(t))-r}{(1+\nu(t)r)(1+\nu(\rho(t))p)}U'\left(C(\rho(t))\right)$$

which is the same as

$$\left[U'\left(C(\rho(t))\right)\right]^{\nabla}=\left(\frac{p(1-\nu^{\nabla}(t))-r-\alpha^{\nabla}(t)(1+\nu(t)r)(1+\nu(\rho(t))p)}{(1+\nu(t)r)(1+\nu(\rho(t))p)\alpha(\rho(t))}\right)U'\left(C(\rho(t))\right),$$

then by substituting $\dfrac{1}{1+\nu(\rho(t))r}$ in for $\alpha(\rho(t))$ and rearranging the following is obtained

$$\frac{\left[U'\left(C(\rho(t))\right)\right]^{\nabla}}{U'\left(C(\rho(t))\right)}=\left(\frac{\left(p(1-\nu^{\nabla}(t))-r\right)(1+\nu(\rho(t))r)+\nu^{\nabla}(t)r(1+\nu(\rho(t))p)}{(1+\nu(t)r)(1+\nu(\rho(t))p)}\right) \tag{8}$$

for $t\in[\rho^2(0),\rho^2(T)]^{\kappa}_{\kappa}$.

The Ramsey Model with the Delta Derivative:

Consider the constraint for the continuous case (4) which can be rewritten as

$$C(t)=-e^{rt}[e^{-rt}W(t)]'. \tag{9}$$

Now consider the discrete constraint (3) which can be rewritten as

$$C_t=-(1+r)^{t-1}\Delta\left[\frac{W(t)}{(1+r)^{t-1}}\right]. \tag{10}$$

Using the new formulations (9) and (10), of the continuous and discrete constraints, a generalization can be made in order to develop the time scale constraint

$$C(t)=-[\hat{e}_{-r}(\rho(t),0)]^{-1}[\hat{e}_{-r}(\rho(t),0)W(t)]^{\Delta}.$$

Then by taking the delta derivative of $[\hat{e}_{-r}(\rho(t),0)W(t)]$ the following is obtained,

$$C(t)=-[\hat{e}_{-r}(\rho(t),0)]^{-1}\left[\frac{-r(1+\nu(t)r)+r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\hat{e}_{-r}(\rho(t),0)W(\sigma(t))+\hat{e}_{-r}(\rho(t),0)W^{\Delta}(t)\right],$$

where ν is assumed to be a delta differentiable function. Then by distributing through by $-[\hat{e}_{-r}(\rho(t),0)]^{-1}$ the constraint for the delta version of the Ramsey model is obtained which is

$$C(t) = \left[\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W(\sigma(t)) - W^{\Delta}(t)\right].$$

The Ramsey model with the delta derivative is

$$\max_{[W(t)]} \int_0^T \hat{e}_{-p}(t,0) U\left[\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W(\sigma(t)) - W^{\Delta}(t)\right] \Delta t. \tag{11}$$

Note that this model includes the discrete and continuous model as special cases. First we derive the Euler-Lagrange equation using Theorem 1.1. In this model,

$$L(t, W^{\sigma}, W^{\Delta}) = \hat{e}_{-p}(t,0) U\left[\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W^{\sigma} - W^{\Delta}\right]$$

so we obtain the following dynamic equation

$$\hat{e}_{-p}(t,0) U'\left[\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W(\sigma(t)) - W^{\Delta}(t)\right]\left(\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\right)$$

$$+\left[\hat{e}_{-p}(t,0) U'\left(\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W(\sigma(t)) - W^{\Delta}(t)\right)\right]^{\Delta} = 0.$$

Then by substituting $C(t)$ in for $\frac{r(1+\nu(t)r) - r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)} W(\sigma(t)) - W^{\Delta}(t)$, using the product rule and taking the delta derivative of the nabla exponential we have

$$\frac{[U'(C(t))]^{\Delta}}{U'(C(t))} = \frac{(r\nu^{\Delta}(t) - r(1+\nu(t)r))(1+\mu(t)p) + p(1+\mu(t)r)(1+\nu(t)r)}{(1+\mu(t)r)(1+\nu(t)r)} \tag{12}$$

for $t \in [0,T]^{\kappa}_{\kappa}$.

We will end this section by comparing the solutions of the two models (7) and (11). The first comparison of the two solutions, (8) and (12), will be made where $\mathbb{T} = \mathbb{R}$. The solution (8) obtained from the Ramsey model with the nabla derivative becomes

$$\frac{[U'(C(t))]'}{U'(C(t))} = p - r$$

for $t \in [0,T]$. The solution (12) obtained from the Ramsey model with the delta derivative becomes

$$\frac{[U'(C(t))]'}{U'(C(t))} = p - r.$$

for $t \in [0,T]$. So when $\mathbb{T} = \mathbb{R}$ the two solutions are the same.

The next comparison will be made where $\mathbb{T} = h\mathbb{Z}$. The solution to the Ramsey model with the nabla derivative and $\mathbb{T} = h\mathbb{Z}$ is as follows

$$\frac{\nabla[U'(C(\rho(t)))]}{U'(C(\rho(t)))} = \frac{p-r}{1+hp}.$$

Then by taking the indicated backward difference we have

$$U'(C(\rho(t))) = \frac{1+hp}{1+hr} U'(C(\rho(\rho(t))))$$

for $t \in [-\frac{1}{h}, T - \frac{3}{h}]$. The solution to the Ramsey model with the delta derivative and $\mathbb{T} = h\mathbb{Z}$ is as follows

$$\frac{\Delta[U'(C(t))]}{U'(C(t))} = \frac{p-r}{1+hr}.$$

Then by taking the indicated forward difference we have

$$U'(C(\sigma(t))) = \frac{1+hp}{1+hr} U'(C(t))$$

for $t \in [\frac{1}{h}, T - \frac{1}{h}]$.

3 Free Boundary Conditions

In this section, we will form an adjustment model with Δ derivative on time scales. The Euler-Lagrange equation turns out to be a second order dynamic equation which currently has no closed solution. So to circumvent this issue we will consider a time scale $\mathbb{T} = \{[0,6) \cap h_1\mathbb{Z}\} \cup \{[6,14) \cap h_2\mathbb{Z}\} \cup \{[14,30] \cap h_3\mathbb{Z}\}$ where $h_1 = 1$, $h_2 = 0.5$, and $h_3 = 0.001$ in order to solve and compare the obtained solution and the desired target solution.

Discrete Model:

We want to minimize the dynamic model of adjustment

$$J[y] = \sum_{t=1}^{T} r^t [\alpha(y(t) - \overline{y}(t))^2 + (y(t) - y(t-1))^2],$$

where $y(t)$ is the output state variable, $r > 1$ is the exogenous rate of discount, $\overline{y}$ is the desired target level (which for the purposes of this paper we will consider two cases which are that $\overline{y}$ is either linear or exponential), and T is the horizon. The first component of the loss function above is the disequilibrium cost due to deviations from the desired target and the second component characterizes the agent's aversion to output fluctuations. The Euler-Lagrange equation for the discrete model is as follows

$$ry(t+1) - (r + \alpha + 1)y(t) + y(t-1) + \alpha\overline{y}(t) = 0.$$

Continuous Model:

We want to minimize the dynamic model of adjustment

$$J[y] = \int_0^T e^{(r-1)t}[\alpha(y(t) - \overline{y}(t))^2 + (y'(t))^2]dt.$$

The Euler-Lagrange equation becomes

$$y''(t) + (r-1)y'(t) - \alpha y(t) + \alpha\overline{y}(t) = 0.$$

Time Scales Model:

The time scale model which we wish to minimize is

$$J[y] = \int_{\sigma(0)}^{\rho(T)} e_{r-1}(\sigma(t),0)[\alpha(y(\sigma(t)) - \overline{y}(\sigma(t)))^2 + (y^{\Delta}(t))^2]\Delta t.$$

Note that this model includes the discrete case and the continuous case as special cases. First we derive the Euler-Lagrange equation using Theorem 1.1. In this model,

$$L(t, y(\sigma(t)), y^{\Delta}(t)) = e_{r-1}(\sigma(t),0)[\alpha(y(\sigma(t)) - \overline{y}(\sigma(t)))^2 + (y^{\Delta}(t))^2],$$

so we obtain the following dynamic equation

$$e_{r-1}(\sigma(t),0)[2\alpha(y(\sigma(t)) - \overline{y}(\sigma(t))] - 2[e_{r-1}(\sigma(t),0)]^{\Delta} y^{\Delta}(\sigma(t)) - 2e_{r-1}(\sigma(t),0)y^{\Delta\Delta}(t) = 0.$$

Then using the identity $e^{\Delta}_{r-1}(\sigma(t),0) = (r-1)(\mu^{\Delta}(t)+1)e_{r-1}(\sigma(t),0)$, where μ is assumed to be a delta differentiable function, we have

$$e_{r-1}(\sigma(t),0)[2\alpha(y(\sigma(t)) - \overline{y}(\sigma(t))] - 2(r-1)(\mu^{\Delta}(t)+1)[e_{r-1}(\sigma(t),0)y^{\Delta}(\sigma(t)) \\ -2e_{r-1}(\sigma(t),0)y^{\Delta\Delta}(t) = 0,$$

then dividing through by $-2e_{r-1}(\sigma(t),0)$ the equation simplifies to

$$y^{\Delta\Delta}(t) + (r-1)(\mu^{\Delta}(t)+1)y^{\Delta}(\sigma(t))) - \alpha y(\sigma(t)) + \alpha\overline{y}(\sigma(t)) = 0.$$

This model differs from others that have been studied in the literature since there is no constraint or boundary condition. Next we derive the free boundary conditions and then apply the results to this adjustment model.

Theorem 3.1 *If*

$$J[y] = \int_{\sigma(a)}^{\sigma^2(b)} L(t, y(\sigma(t)), y^{\triangle}(t))\triangle t,$$

where $y \in C^2[a, \sigma^2(b)]$ and $y(\sigma(a)) = A$, has a local extremum at $y(t)$, then $y(t)$ satisfies the Euler-Lagrange equation for $t \in [a, \sigma(b)]^{\kappa}_{\kappa}$, $y(\sigma(a)) = A$ and $y(t)$ satisfies the condition

$$(\sigma^2(b) - \sigma(b))L_{y^{\sigma}}(\sigma(b), y(\sigma^2(b)), y^{\triangle}(\sigma(b))) + L_{y^{\triangle}}(\sigma(b), y(\sigma^2(b)), y^{\triangle}(\sigma(b))) = 0 \quad (13)$$

Proof As in the proof of Theorem 1.1, $J_1[h] = 0$ for all $h \in C^1[\sigma(a), \sigma^2(b)]$ with $h(\sigma(a)) = 0$. While getting Euler-Lagrange equation, if we use $h(\sigma(a)) = 0$, we get

$$\int_{\sigma(a)}^{\sigma^2(b)} \{L_{y^{\sigma}}(t, y^{\sigma}, y^{\triangle}) - L^{\triangle}_{y^{\triangle}}(t, y^{\sigma}, y^{\triangle})\}h^{\sigma}(t)\triangle t \\ +\{(\sigma^2(b) - \sigma(b))L_{y^{\sigma}}(\sigma(b), y(\sigma^2(b)), y^{\triangle}(\sigma(b))) \\ +L_{y^{\triangle}}(\sigma(b), y(\sigma^2(b)), y^{\triangle}(\sigma(b)))\}h(\sigma^2(b)) = 0$$

for all $h \in C^1[\sigma(a), \sigma^2(b)]$. The conclusion of the theorem follows. □

Theorem 3.2 *If*

$$J[y] = \int_{\sigma(a)}^{\sigma^2(b)} L(t, y(\sigma(t)), y^{\triangle}(t))\triangle t,$$

where $y \in C^2[a, \sigma^2(b)]$ *and* $y(\sigma^2(b)) = B$*, has a local extremum at* $y(t)$*, then* $y(t)$ *satisfies the Euler-Lagrange equation for* $t \in [a, \sigma(b)]_\kappa^\kappa$*,* $y(\sigma^2(b)) = B$ *and* $y(t)$ *satisfies the condition*

$$L_{y^\triangle}(\sigma(a), y(\sigma^2(a)), y^\triangle(\sigma(a))) = 0. \tag{14}$$

In a similar way, we have the following theorem.

Theorem 3.3 *If* $y(t)$ *is a local extremum for* $J[y]$ *where* $y \in C^2[a, \sigma^2(b)]$*, then* $y(t)$ *satisfies the Euler-Lagrange equation for* $t \in [a, \sigma(b)]_\kappa^\kappa$ *and the conditions (*13*) and (*14*).*

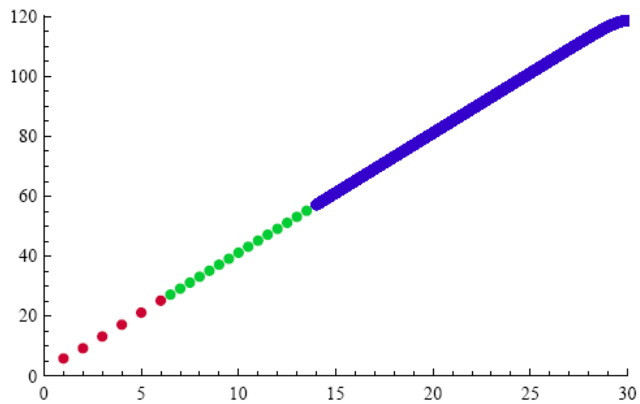

Figure 3.1: Linear case.

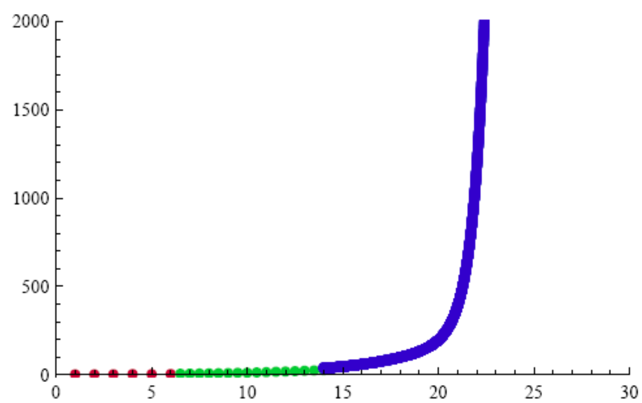

Figure 3.2: Exponential case.

So we have that the free boundary conditions for this model are $y^\Delta(\sigma(a)) = 0$ and $y^\Delta(\sigma(b)) = h\alpha[\overline{y}(\sigma^2(b)) - y(\sigma^2(b))]$. We will now illustrate the optimized solution of this problem for a time scale where $\mathbb{T} = \{[0, 6) \cap h_1\mathbb{Z}\} \cup \{[6, 14) \cap h_2\mathbb{Z}\} \cup \{[14, 30] \cap h_3\mathbb{Z}\}$ where

$h_1 = 1$, $h_2 = 0.5$, and $h_3 = 0.001$. This is accomplished by considering the optimized solution to be the following piecewise defined function

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [1, 6), \\ y_2(t) & \text{if } t \in [6, 14), \\ y_3(t) & \text{if } t \in [14, 30 - 0.002], \end{cases}$$

being optimized on the three separate intervals. Figures 3.1 and 3.2 are the graphs of the linear and exponential case with $r = 1.9$, $\alpha = 4$, $b = 0.25$, and $v = 4$ whose target functions are $\overline{y}(t) = vt + b$ and $\overline{y}(t) = e_b(t, 0)$.

4 Concluding Remarks

The techniques of modelling with dynamic equations on time scales are not widely used in economics. This may be due to the view that ordinary differential equations and difference equations are sufficient for modelling most interesting events in economy. However, economists encounter situations in which discrete and continuous models do not capture all the essential features of the events. In this sense, modelling with dynamic equations on time scales provides a more "complete" model for events at all level of time domains.

In Section 2, a well-known Ramsey model of economics has been used to illustrate that it is possible to write a model in economics with Δ operator as well as with ∇ operator. The existing theory, theory of calculus of variations on time scales, allows us to solve both models and compare the obtained solutions on time scales $\mathbb{R}$ and $h\mathbb{Z}$. Our calculations show that the solutions are exactly the same on certain time scales. In Section 3, we studied a model of economics where we cannot use both derivative operators. This is due to the fact that the theory of calculus of variations on time scales is very much a work in progress. At this time, the adjustment model can be solved if it is modelled with Δ operator only.

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