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On Expansions in Eigenfunctions for Second Order Dynamic Equations on Time Scales

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Abstract: In this study, we explore an eigenvalue problem on a bounded time scales interval for self-adjoint second order dynamic equations with self-adjoint separated boundary conditions. Existence of the eigenvalues and eigenfunctions is shown. Next, mean square convergent and uniformly convergent expansions in eigenfunctions are established.

Keywords: *time scale; delta and nabla derivatives; Green's function; eigenvalue; eigenfunction.*

Mathematics Subject Classification (2000): 34L10.

1 Introduction

The first papers on eigenvalue problems for linear Δ-differential equations on time scales were fulfilled by Agarwal, Bohner, and Wong in [1] and Chyan, Davis, Henderson, and Yin in [6]. In [1], an oscillation theorem is offered for Sturm-Liouville eigenvalue problem on time scales with separated boundary conditions and Rayleigh's principle is established for the eigenvalues. In [6], the theory of positive operators with respect to a cone in a Banach space is applied to eigenvalue problems associated with the second order linear Δ-differential equations on time scales to prove existence of a smallest positive eigenvalue and then a theorem is established comparing the smallest positive eigenvalues for two problems of that type.

Recently, Guseinov [7] investigated eigenfunction expansions for the simple Sturm–Liouville eigenvalue problem

$$-y^{\Delta\nabla}(t) = \lambda y(t), \quad t \in (a,b), \tag{1}$$

$$y(a) = y(b) = 0, \tag{2}$$

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where a and b are some fixed points in a time scale $\mathbf{T}$ with $a < b$ and such that the time scale interval (a, b) is not empty. In that paper [7], existence of the eigenvalues and eigenfunctions for problem (1), (2) is proved and mean square convergent and uniformly convergent expansions in eigenfunctions are established.

In the present paper, we extend the results of [7] to the more general eigenvalue problem

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in (a, b], \tag{3}$$

$$y(a) - hy^{[\Delta]}(a) = 0, \quad y(b) + Hy^{[\Delta]}(b) = 0, \tag{4}$$

where $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$ is the so-called quasi Δ-derivative of $y(t)$.

We will assume that the following two conditions are satisfied.

(C1) $p(t)$ is continuous on $[a, b]$ and continuously ∇-differentiable on $(a, b]$, $q(t)$ is piecewise continuous on $[a, b]$, h and H are given real numbers.

(C2) $p(t) > 0$, $q(t) \geq 0$ for $t \in [a, b]$, and $h \geq 0$, $H \geq 0$.

The paper is organized as follows. In Section 2, the Hilbert-Schmidt theorem on self-adjoint completely continuous operators is applied to show that the eigenvalue problem (3), (4) has a system of eigenfunctions that forms an orthonormal basis for an appropriate Hilbert space. This yields mean square convergent (that is, convergent in an L^2-metric) expansions in eigenfunctions. In Section 3, uniformly convergent expansions in eigenfunctions are obtained when the expanded functions satisfy some smoothness conditions. In the last Section 4, two special cases are described.

Finally, for easy reference, we state here two integration by parts formulas on time scales which are employed in the subsequent sections.

Let $\mathbf{T}$ be a time scale and $a, b \in \mathbf{T}$ be fixed points with $a < b$ such that the time scale interval

$$(a, b) = \{t \in \mathbf{T} : a < t < b\}$$

is not empty. Throughout, all the intervals are time scale intervals. For standard notions and notations connected to time scales calculus we refer to [4, 5].

Theorem 1.1 (see [7, Theorem 2.4]). *Let f and g be continuous functions on $[a, b]$. Suppose that f is Δ-differentiable on $[a, b)$ with the continuous Δ-derivative f^{Δ} that is Δ-integrable over $[a, b)$ and g is ∇-differentiable on $(a, b]$ with the continuous ∇-derivative g^{∇} that is ∇-integrable over $(a, b]$. Then:*

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t) \mid_a^b - \int_a^b f(t)g^{\nabla}(t)\nabla t, \tag{5}$$

$$\int_a^b g^{\nabla}(t)f(t)\nabla t = g(t)f(t) \mid_a^b - \int_a^b g(t)f^{\Delta}(t)\Delta t. \tag{6}$$

2 $\mathbf{L^2}$-convergent Expansions

Denote by $\mathcal{H}$ the Hilbert space of all real ∇-measurable functions $y : (a, b] \to \mathbf{R}$ such that $y(b) = 0$ in the case b is left-scattered and $H = 0$, and that

$$\int_a^b y^2(t)\nabla t < \infty,$$

with the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\| = \sqrt{\langle y, y\rangle} = \left\{\int_a^b y^2\,(t)\,\nabla t\right\}^{1/2}.$$

Next denote by $\mathcal{D}$ the set of all functions $y \in \mathcal{H}$ satisfying the following three conditions:

(i) y is continuous on $[a, \sigma(b)]$, where σ denotes the forward jump operator.

(ii) $y^{\Delta}(t)$ is defined for $t \in [a, b]$ and

$$y(a) - hy^{[\Delta]}(a) = 0, \quad y(b) + Hy^{[\Delta]}(b) = 0, \tag{7}$$

where $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$.

(iii) $p(t)y^{\Delta}(t)$ is ∇-differentiable on $(a, b]$ and $[p(t)y^{\Delta}(t)]^{\nabla} \in \mathcal{H}$.

Obviously $\mathcal{D}$ is a linear subset dense in $\mathcal{H}$. Now we define the operator $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ as follows. The domain of definition of A is $\mathcal{D}$ and we put

$$(Ay)(t) = -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t), \quad t \in (a, b],$$

for $y \in \mathcal{D}$.

Definition 2.1 A complex number λ is called an eigenvalue of problem (3), (4) if there exists a nonidentically zero function $y \in \mathcal{D}$ such that

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in (a, b].$$

The function y is called an eigenfunction of problem (3), (4), corresponding to the eigenvalue λ.

We see that the eigenvalue problem (3), (4) is equivalent to the equation

$$Ay = \lambda y, \quad y \in \mathcal{D},\ y \neq 0. \tag{8}$$

Theorem 2.1 *Under the condition (C1) we have, for all* $y, z \in \mathcal{D}$,

$$\langle Ay, z\rangle = \langle y, Az\rangle\,, \tag{9}$$

$$\langle Ay, y\rangle = h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t. \tag{10}$$

Proof Using integration by parts formulas (5), (6), we have for all $y, z \in \mathcal{D}$,

$$\begin{aligned}
\langle Ay, z\rangle &= \int_a^b \{-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t)\}z(t)\nabla t \\
&= -p(t)y^{\Delta}(t)z(t)\mid_a^b + \int_a^b p(t)y^{\Delta}(t)z^{\Delta}(t)\Delta t + \int_a^b q(t)y(t)z(t)\nabla t \\
&= -p(t)y^{\Delta}(t)z(t)\mid_a^b + y(t)p(t)z^{\Delta}(t)\mid_a^b \\
&\quad - \int_a^b y(t)[p(t)z^{\Delta}(t)]^{\nabla}\nabla t + \int_a^b q(t)y(t)z(t)\nabla t \\
&= \int_a^b y(t)\{-[p(t)z^{\Delta}(t)]^{\nabla} + q(t)z(t)\}\nabla t = \langle y, Az\rangle\,,
\end{aligned}$$

where we have used the boundary conditions (7) for functions $y, z \in \mathcal{D}$.

Simultaneously we have also got

$$\begin{aligned}\langle Ay, y\rangle &= -p(t)y^{\Delta}(t)y(t)\mid_a^b + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t \\ &= h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t.\end{aligned}$$

The theorem is proved.

Relation (9) shows that the operator A is symmetric (self-adjoint), while (10) shows that, under the additional condition (C2), it is positive:

$$\langle Ay, y\rangle > 0 \quad \text{for all} \quad y \in \mathcal{D},\ y \neq 0.$$

Therefore all eigenvalues of the operator A are real and positive and any two eigenfunctions corresponding to the distinct eigenvalues are orthogonal. Besides, it can easily be seen that eigenvalues of problem (3), (4) are simple, that is, to each eigenvalue there corresponds a single eigenfunction up to a constant factor (equation (3) cannot have two linearly independent solutions satisfying the condition $y(a) - hy^{[\Delta]}(a) = 0$).

Now we are going to prove the existence of eigenvalues for problem (3), (4).

Note that

$$\ker A = \{y \in \mathcal{D} : Ay = 0\}$$

consists only of the zero element. Indeed, if $y \in \mathcal{D}$ and $Ay = 0$, then from (10) we have by the condition (C2) that $y^{\Delta}(t) = 0$ for $t \in [a, b)$ and hence $y(t) =$*constant* on $[a, b]$. Then using boundary conditions (7) we get that $y(t) \equiv 0$.

It follows that the inverse operator A^{-1} exists. To present its explicit form we introduce the Green function (see [2, 3])

$$G(t,s) = -\frac{1}{\omega}\begin{cases} u(t)v(s), & \text{if} \quad t \leq s, \\ u(s)v(t), & \text{if} \quad t \geq s, \end{cases} \tag{11}$$

where $u(t)$ and $v(t)$ are solutions of the equation

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = 0, \quad t \in (a, b],$$

satisfying the initial conditions

$$u(a) = h,\ u^{[\Delta]}(a) = 1; \quad v(b) = H,\ v^{[\Delta]}(b) = -1,$$

and

$$\omega = W_t(u, v) = u(t)v^{[\Delta]}(t) - u^{[\Delta]}(t)v(t),$$

the Wronskian of the solutions u, v, is constant so that

$$\omega = -v(a) + hv^{[\Delta]}(a) = -u(b) - Hu^{[\Delta]}(b).$$

Note that $\omega \neq 0$. Otherwise we would have $u \in \mathcal{D}$ and $Au = 0$ so that $u \in \ker A$. But this is a contradiction, since we showed above that $\ker A = \{0\}$, while u is not equal to the zero element (we have $u^{[\Delta]}(a) = 1$).

Then

$$(A^{-1}f)(t) = \int_a^b G(t,s)f(s)\nabla s \quad \text{for any} \quad f \in \mathcal{H}. \tag{12}$$

The equations (11) and (12) imply that A^{-1} is a completely continuous (or compact) self-adjoint linear operator in the Hilbert space $\mathcal{H}$.

The eigenvalue problem (8) is equivalent (note that $\lambda = 0$ is not an eigenvalue of A) to the eigenvalue problem

$$Bg = \mu g, \quad g \in \mathcal{H},\ g \neq 0,$$

where

$$B = A^{-1} \quad \text{and} \quad \mu = \frac{1}{\lambda}.$$

In other words, if λ is an eigenvalue and $y \in \mathcal{D}$ is a corresponding eigenfunction for A, then $\mu = \lambda^{-1}$ is an eigenvalue for B with the same corresponding eigenfunction y; conversely, if $\mu \neq 0$ is an eigenvalue and $g \in \mathcal{H}$ is a corresponding eigenfunction for B, then $g \in \mathcal{D}$ and $\lambda = \mu^{-1}$ is an eigenvalue for A with the same eigenfunction g.

Note that $\mu = 0$ cannot be an eigenvalue for B. In fact, if $Bg = 0$, then applying A to both sides we get that $g = 0$.

Next we use the following well-known Hilbert-Schmidt theorem (see, for example, [8, Section 24.3]): *For every completely continuous self-adjoint linear operator B in a Hilbert space $\mathcal{H}$ there exists an orthonormal system $\{\varphi_k\}$ of eigenvectors corresponding to eigenvalues $\{\mu_k\}$ $(\mu_k \neq 0)$ such that each element $f \in \mathcal{H}$ can be written uniquely in the form*

$$f = \sum_k c_k\varphi_k + \psi,$$

where $\psi \in \ker B$, that is, $B\psi = 0$. Moreover,

$$Bf = \sum_k \mu_k c_k \varphi_k$$

and if the system $\{\varphi_k\}$ is infinite, then $\lim \mu_k = 0$ $(k \to \infty)$.

As a corollary of the Hilbert-Schmidt theorem we have: *If B is a completely continuous self-adjoint linear operator in a Hilbert space $\mathcal{H}$ and if $\ker B = \{0\}$, then the eigenvectors of B form an orthogonal basis of $\mathcal{H}$.*

Applying the corollary of the Hilbert-Schmidt theorem to the operator $B = A^{-1}$ and using the above described connection between the eigenvalues and eigenfunctions of A and the eigenvalues and eigenfunctions of B we obtain the following result.

Theorem 2.2 *Under the conditions (C1) and (C2), for the eigenvalue problem (3), (4) there exists an orthonormal system $\{\varphi_k\}$ of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$. Each eigenvalue λ_k is positive and simple. The system $\{\varphi_k\}$ forms an orthonormal basis for the Hilbert space $\mathcal{H}$. Therefore the number of the eigenvalues is equal to $N = \dim \mathcal{H}$. Any function $f \in \mathcal{H}$ can be expanded in eigenfunctions φ_k in the form*

$$f(t) = \sum_{k=1}^{N} c_k\varphi_k(t), \tag{13}$$

where c_k are the Fourier coefficients of f defined by

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t. \tag{14}$$

In the case $N=\infty$ the sum in (13) becomes an infinite series and it converges to the function f in metric of the space $\mathcal{H}$, that is, in mean square metric:

$$\lim_{n\to\infty}\int_a^b\left[f(t)-\sum_{k=1}^n c_k\varphi_k(t)\right]^2\nabla t=0. \tag{15}$$

Note that since

$$\int_a^b\left[f(t)-\sum_{k=1}^n c_k\varphi_k(t)\right]^2\nabla t=\int_a^b f^2(t)\nabla t-\sum_{k=1}^n c_k^2,$$

we get from (15) the Parseval equality

$$\int_a^b f^2(t)\nabla t=\sum_{k=1}^N c_k^2. \tag{16}$$

Remark 2.1 Above in the definition of the Hilbert space $\mathcal{H}$ we required the condition $y(b)=0$ for functions $y:(a,b]\to\mathbf{R}$ in $\mathcal{H}$ in the case b is left-scattered and $H=0$. This is needed to ensure that $\mathcal{D}$ is dense in $\mathcal{H}$ which in turn is essential for the theory of operators.

Remark 2.2 It is easy to see that the dimension of the space $\mathcal{H}$ is finite if and only if the time scale interval $(a,b]$ consists of a finite number of points and in this case $\dim\mathcal{H}$ is equal to the number of points in the interval $(a,b]$ if $H\neq 0$, and to the number of points in the interval (a,b) if $H=0$.

Remark 2.3 If we denote by $\varphi(t,\lambda)$ the solution of equation (3) satisfying the initial conditions

$$\varphi(a,\lambda)=h,\quad \varphi^{[\Delta]}(a,\lambda)=1,$$

then the eigenvalues of problem (3), (4) will coincide with the zeros of the function $\chi(\lambda)=\varphi(b,\lambda)+H\varphi^{[\Delta]}(b,\lambda)$, the characteristic function of problem (3), (4). So we have proved existence of zeros of $\chi(\lambda)$ by proving existence of eigenvalues of problem (3), (4). It is possible (see [1]) to prove existence of zeros of $\chi(\lambda)$ directly and to get in this way existence of the eigenvalues.

3 Uniformly Convergent Expansions

In this section, assuming that the conditions (C1) and (C2) formulated in Section 1 are satisfied, we prove the following result (we assume that $\dim\mathcal{H}=\infty$, since in the case $\dim\mathcal{H}<\infty$ the series becomes a finite sum).

Theorem 3.1 *Let $f:[a,b]\to\mathbf{R}$ be a function such that it has a Δ-derivative $f^{\Delta}(t)$ everywhere on $[a,b]$, except at a finite number of points $t_1,t_2,\ldots,t_m$ belonging to (a,b), the Δ-derivative being continuous everywhere except at these points, at which f^{Δ} has finite limits from the left and right. Besides assume that f satisfies the boundary conditions*

$$f(a)-hf^{[\Delta]}(a)=0,\quad f(b)+Hf^{[\Delta]}(b)=0, \tag{17}$$

where $f^{[\Delta]}(t) = p(t)f^{\Delta}(t)$. *Then the series*

$$\sum_{k=1}^{\infty} c_k \varphi_k(t), \tag{18}$$

where

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t, \tag{19}$$

converges uniformly on $[a, b]$ *to the function* f.

Proof We employ a method applied in the case of the usual ($\mathbf{T} = \mathbf{R}$) Sturm-Liouville problem by V. A. Steklov (see [9, Section 182]). First for simplicity we assume that the function f is Δ-differentiable everywhere on $[a, b]$ and that f^{Δ} is continuous on $[a, b]$. Consider the functional

$$J(y) = h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t$$

so that we have $J(y) \geq 0$. Substituting in the functional $J(y)$

$$y = f(t) - \sum_{k=1}^{n} c_k \varphi_k(t),$$

where c_k are defined by (19), we obtain

$$\begin{aligned}
&J\left(f - \sum_{k=1}^{n} c_k\varphi_k\right) \\
&= h\left[f^{[\Delta]}(a) - \sum_{k=1}^{n} c_k\varphi_k^{[\Delta]}(a)\right]^2 + H\left[f^{[\Delta]}(b) - \sum_{k=1}^{n} c_k\varphi_k^{[\Delta]}(b)\right]^2 \\
&\quad + \int_a^b p\left(f^{\Delta} - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}\right)^2 \Delta t + \int_a^b q\left(f - \sum_{k=1}^{n} c_k\varphi_k\right)^2 \nabla t \\
&= h[f^{[\Delta]}(a)]^2 + H[f^{[\Delta]}(b)]^2 - 2\sum_{k=1}^{n} c_k[hf^{[\Delta]}(a)\varphi_k^{[\Delta]}(a) + Hf^{[\Delta]}(b)\varphi_k^{[\Delta]}(b)] \\
&\quad + \sum_{k,l=1}^{n} c_k c_l[h\varphi_k^{[\Delta]}(a)\varphi_l^{[\Delta]}(a) + H\varphi_k^{[\Delta]}(b)\varphi_l^{[\Delta]}(b)] \\
&\quad + \int_a^b pf^{\Delta 2}\Delta t + \int_a^b qf^2\nabla t - 2\sum_{k=1}^{n} c_k\left(\int_a^b pf^{\Delta}\varphi_k^{\Delta}\Delta t + \int_a^b qf\varphi_k\nabla t\right) \\
&\quad + \sum_{k,l=1}^{n} c_k c_l\left(\int_a^b p\varphi_k^{\Delta}\varphi_l^{\Delta}\Delta t + \int_a^b q\varphi_k\varphi_l\nabla t\right).
\end{aligned} \tag{20}$$

Next, applying integration by parts formula (5), we get

$$\begin{aligned}
&\int_a^b pf^\Delta\varphi_k^\Delta\Delta t+\int_a^b qf\varphi_k\nabla t\\
=\quad& p(t)f(t)\varphi_k^\Delta(t)\mid_a^b+\int_a^b f\left[-(p\varphi_k^\Delta)^\nabla+q\varphi_k\right]\nabla t\\
=\quad& f(b)\varphi_k^{[\Delta]}(b)-f(a)\varphi_k^{[\Delta]}(a)+\lambda_k\int_a^b f\varphi_k\nabla t\\
=\quad& -Hf^{[\Delta]}(b)\varphi_k^{[\Delta]}(b)]-hf^{[\Delta]}(a)\varphi_k^{[\Delta]}(a)+\lambda_k c_k,
\end{aligned}$$

$$\begin{aligned}
&\int_a^b p\varphi_k^\Delta\varphi_l^\Delta\Delta t+\int_a^b q\varphi_k\varphi_l\nabla t\\
=\quad& p(t)\varphi_k(t)\varphi_l^\Delta(t)\mid_a^b+\int_a^b \varphi_k[-(p\varphi_l^\Delta)^\nabla+q\varphi_l]\nabla t\\
=\quad& \varphi_k(b)\varphi_l^{[\Delta]}(b)-\varphi_k(a)\varphi_l^{[\Delta]}(a)+\lambda_l\int_a^b \varphi_k\varphi_l\nabla t\\
=\quad& -h\varphi_k^{[\Delta]}(a)\varphi_l^{[\Delta]}(a)-H\varphi_k^{[\Delta]}(b)\varphi_l^{[\Delta]}(b)+\lambda_l\delta_{kl},
\end{aligned}$$

where δ_{kl} is the Kronecker symbol and where we have used the boundary conditions (17),

$$\varphi_k(a)-h\varphi_k^{[\Delta]}(a)=0,\quad \varphi_k(b)+H\varphi_k^{[\Delta]}(b)=0, \tag{21}$$

and the equation

$$-[p(t)\varphi_k^\Delta(t)]^\nabla+q(t)\varphi_k(t)=\lambda_k\varphi_k(t).$$

Therefore we have from (20)

$$\begin{aligned}
J\left(f-\sum_{k=1}^n c_k\varphi_k\right)\quad=\quad& h[f^{[\Delta]}(a)]^2+H[f^{[\Delta]}(b)]^2\\
&+\int_a^b(pf^{\Delta 2}+qf^2)\Delta t-\sum_{k=1}^n\lambda_k c_k^2.
\end{aligned}$$

Since the left-hand side is nonnegative, we get the inequality

$$\sum_{k=1}^\infty\lambda_k c_k^2\le h[f^{[\Delta]}(a)]^2+H[f^{[\Delta]}(b)]^2+\int_a^b(pf^{\Delta 2}+qf^2)\Delta t \tag{22}$$

analogous to Bessel's inequality, and the convergence of the series on the left follows. All the terms of this series are nonnegative, since $\lambda_k>0$.

Note that the proof of (22) is entirely unchanged if we assume that the function f satisfies only the conditions stated in the theorem. Indeed, when integrating by parts, it is sufficient to integrate over the intervals on which f^Δ is continuous and then add all these integrals (the integrated terms vanish by (17), (21), and the fact that f, φ_k, and φ_k^Δ are continuous on $[a,b]$).

We now show that the series

$$\sum_{k=1}^\infty|c_k\varphi_k(t)| \tag{23}$$

is uniformly convergent on the interval $[a,b]$. Obviously from this the uniform convergence of series (18) will follow.

Using the integral equation

$$\varphi_k(t) = \lambda_k \int_a^b G(t,s)\varphi_k(s)\nabla s$$

which follows from $\varphi_k = \lambda_k A^{-1}\varphi_k$ by (12), we can rewrite (23) as

$$\sum_{k=1}^{\infty} \lambda_k \left| c_k g_k(t) \right|, \tag{24}$$

where

$$g_k(t) = \int_a^b G(t,s)\varphi_k(s)\nabla s$$

can be regarded as the Fourier coefficient of $G(t,s)$ as a function of s. By using inequality (22), we can write

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k g_k^2(t) \quad &\leq \quad h[p(a)G^{\Delta_s}(t,a)]^2 + H[p(b)G^{\Delta_s}(t,b)]^2 \\ &\quad + \int_a^b \left[p(s)G^{\Delta_s 2}(t,s) + q(s)G^2(t,s)\right] \Delta s, \end{aligned} \tag{25}$$

where $G^{\Delta_s}(t,s)$ is the delta derivative of $G(t,s)$ with respect to s. The function appearing under the integral sign is bounded (see (11)), and it follows from (25) that

$$\sum_{k=1}^{\infty} \lambda_k g_k^2(t) \leq M,$$

where M is a constant. Now replacing λ_k by $\sqrt{\lambda_k}\sqrt{\lambda_k}$, we apply the Cauchy-Schwarz inequality to the segment of series (24):

$$\begin{aligned} \sum_{k=m}^{m+p} \lambda_k \left| c_k g_k(t) \right| \quad &\leq \quad \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{\sum_{k=m}^{m+p} \lambda_k g_k^2(t)} \\ &\leq \quad \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{M}, \end{aligned}$$

and this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$ (see (22)), at once implies that series (24), and hence series (23) is uniformly convergent on the interval $[a,b]$.

Denote the sum of series (18) by $f_1(t)$:

$$f_1(t) = \sum_{k=1}^{\infty} c_k \varphi_k(t). \tag{26}$$

Since the series in (26) is convergent uniformly on $[a,b]$, we can multiply both sides of (26) by $\varphi_l(t)$ and then ∇ integrate it term-by-term to get

$$\int_a^b f_1(t)\varphi_l(t)\nabla t = c_l.$$

Therefore the Fourier coefficients of f_1 and f are the same. Then the Fourier coefficients of the difference $f_1 - f$ are zero and applying the Parseval equality (16) to the function $f_1 - f$ we get that $f_1 - f = 0$, so that the sum of series (18) is equal to $f(t)$.

4 Examples

1. In the case $\mathbf{T} = \mathbf{R}$ of reals, for functions $y : \mathbf{T} \to \mathbf{R}$ we have

$$y^{\Delta}(t) = y^{\nabla}(t) = y'(t), \quad t \in \mathbf{R},$$

and problem (3), (4) becomes

$$-[p(t)y'(t)]' + q(t)y(t) = \lambda y(t), \quad t \in (a,b], \tag{27}$$

$$y(a) - hy^{[1]}(a) = 0, \quad y(b) + Hy^{[1]}(b) = 0, \tag{28}$$

where $y^{[1]}(t) = p(t)y'(t)$, $p(t)$ is continuously differentiable and $q(t)$ is piecewise continuous on $[a,b]$, and

$$p(t) > 0,\ q(t) \geq 0 \quad \text{for } t \in [a,b], \quad \text{and} \quad h \geq 0,\ H \geq 0.$$

Theorem 2.2 and Theorem 3.1 give expansion results for the ordinary Sturm-Liouville problem (27), (28). Such results for problem (27), (28) in the case $h = H = 0$ were established earlier by V. A. Steklov (see [9, Section 182]).

2. In the case $\mathbf{T} = \mathbf{Z}$ of integers, for functions $y : \mathbf{T} \to \mathbf{Z}$ we have

$$y^{\Delta}(t) = y(t+1) - y(t), \quad y^{\nabla}(t) = y(t) - y(t-1), \quad t \in \mathbf{Z},$$

and problem (3), (4) can be written in the form

$$-p(t-1)y(t-1) + q_1(t)y(t) - p(t)y(t+1) = \lambda y(t), \quad t \in [a+1,b], \tag{29}$$

$$[1+hp(a)]y(a) - hp(a+1)y(a+1) = 0, \quad [1-Hp(b)]y(b) + Hp(b+1)y(b+1) = 0, \tag{30}$$

where $[a+1,b] = \{a+1, a+2, \ldots, b\}$ is a discrete interval, $\{y(t)\}_{t=a}^{b+1}$ is a desired solution, $q_1(t) = p(t-1) + p(t) + q(t)$,

$$p(t) > 0 \text{ for } t \in [a,b+1], \quad q(t) \geq 0 \text{ for } t \in [a+1,b], \quad \text{and} \quad h \geq 0,\ H \geq 0.$$

Consider two possible cases separately.

(i) Let $H \neq 0$. Then we have, from (30),

$$y(a) = \frac{hp(a+1)}{1+hp(a)}y(a+1), \quad y(b+1) = -\frac{1-Hp(b)}{Hp(b+1)}y(b). \tag{31}$$

Taking (31) into account in (29), we can rewrite problem (29), (30) in the form

$$Jy = \lambda y,$$

where J is the $(b-a)\times(b-a)$ matrix and y is the $(b-a)\times 1$ column vector of the form

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{b-a-2} & \beta_{b-a-2} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{b-a-2} & \alpha_{b-a-1} & \beta_{b-a-1} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{b-a-1} & \alpha_{b-a} \end{bmatrix}, \tag{32}$$

$$y = [y(a+1),\ y(a+2),\ \dots\ y(b-1),\ y(b)]^T,$$

where

$$\alpha_1 = q_1(a+1) - \frac{hp(a)p(a+1)}{1+hp(a)}, \quad \alpha_{b-a} = q_1(b) + \frac{p(b)[1-Hp(b)]}{Hp(b+1)}, \tag{33}$$

$$\alpha_i = q_1(a+i), \quad i \in \{2,3,\dots,b-a-1\}, \tag{34}$$

$$\beta_i = -p(a+i), \quad i \in \{1,2,\dots,b-a-1\}, \tag{35}$$

T denotes the transpose. Therefore Theorem 2.2 expresses simply an expansion in eigenvectors of the matrix J defined by (32).

(ii) If $H = 0$, then from (30) we have

$$y(b) = 0 \tag{36}$$

and equation (29) gives, for $t = b$,

$$-p(b-1)y(b-1) - p(b)y(b+1) = 0,$$

whence

$$y(b+1) = -\frac{p(b-1)}{p(b)}y(b-1). \tag{37}$$

Therefore, in the case $H = 0$, problem (29), (30) is equivalent to the eigenvalue problem

$$Jy = \lambda y, \tag{38}$$

where J is the $(b-a-1)\times(b-a-1)$ matrix and y is the $(b-a-1)\times 1$ column vector of the form

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{b-a-3} & \beta_{b-a-3} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{b-a-3} & \alpha_{b-a-2} & \beta_{b-a-2} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{b-a-2} & \alpha_{b-a-1} \end{bmatrix},$$

$$y = [y(a+1),\ y(a+2),\ \dots\ y(b-2),\ y(b-1)]^T, \tag{39}$$

where the numbers α_i, β_i are defined as in (33)–(35). In the case $H = 0$, the equivalence of problem (29), (30) to the problem (38) means that if $\{y(t)\}_{t=a}^{b+1}$ is a solution of problem (29), (30), then the column vector y of the form (39), formed by using that solution, is a solution of equation (38), and conversely, if a column vector y of the form (39) is a solution of (38) then $\{y(t)\}_{t=a}^{b+1}$ in which the values $y(a+1), y(a+2), \dots\ y(b-2), y(b-1)$ are taken from the vector (39) and the components $y(a)$, $y(b)$, and $y(b+1)$ are defined by (31), (36), and (37), respectively, is a solution of problem (29), (30).

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