Nonlinear Dynamics and Systems Theory, **9** (1) (2009) 99–108

Limit-Point Criteria for a Second Order Dynamic Equation on Time Scales

J. Weiss *

Department of Mathematics and Statistics, University of Nebraska-Kearney
Kearney, NE 68849-1296

Received: November 7, 2007; Revised: January 14, 2009

Abstract: In this paper, we establish some criteria under which the second order formally self-adjoint dynamic equation

$$(p(t)x^{\Delta})^{\nabla} + q(t)x = 0$$

is of limit-point type on a time scale $\mathbb{T}$. As a special case when $\mathbb{T} = \mathbb{R}$, our results include those of Wong and Zettl [11] and Coddington and Levinson [5]. Our results are new in a general time scale setting and can be applied to difference and q-difference equations.

Keywords: *time scales; limit-point; limit-circle; second-order equation.*

Mathematics Subject Classification (2000): 34A99.

1 Introduction

In this paper, assume that $\inf \mathbb{T} = t_0$, and $\sup \mathbb{T} = \infty$. We will sometimes refer to $\mathbb{T}$ as $[t_0, \infty)$ which we mean to be the real interval $[t_0, \infty)$ intersected with $\mathbb{T}$. Assume that $p(t) \neq 0$ and $q(t) \neq 0$ for $t \in \mathbb{T}$ are continuous functions on $\mathbb{T}$. We will consider the formally self-adjoint equations

$$Lx = \left(p(t)x^{\Delta}\right)^{\nabla} + q(t)x = 0 \tag{1.1}$$

and

$$\widetilde{L}y = \left(\frac{1}{q(t)}y^{\nabla}\right)^{\Delta} + \frac{1}{p(t)}y = 0. \tag{1.2}$$

* Corresponding author: weissjj@unk.edu

Let $\mathbb{D}$ be the set of functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T} \to \mathbb{R}$ is continuous, and $(px^{\Delta})^{\nabla} : \mathbb{T}_{\kappa} \to \mathbb{R}$ is continuous. Let $\widetilde{\mathbb{D}}$ be the set of functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\nabla} : \mathbb{T}_{\kappa} \to \mathbb{R}$ is continuous, and $\left(\frac{1}{q}x^{\nabla}\right)^{\Delta} : \mathbb{T}_{\kappa} \to \mathbb{R}$ is continuous. We say (1.1) and (1.2) are *reciprocal equations* of each other. See [7] for more on reciprocal equations and [8], [9] and [10] for other results dealing with second-order equations, and [6] and [4] for more on general theories used in this paper.

These equations are said to be formally self-adjoint because they satisfy the following Lagrange identity.

Theorem 1.1 (Lagrange identity)

(i) Let $u, v \in \mathbb{D}$. Then

$$u(t)Lv(t) - v(t)Lu(t) = \{u; v\}^{\nabla}(t)$$

for $t \in \mathbb{T}_{\kappa}$, where the Lagrange bracket $\{u; v\}$ is defined by

$$\{u; v\}(t) := p(t)W(u, v)(t),$$

where

$$W(u, v)(t) := \begin{vmatrix} u(t) & v(t) \\ u^{\Delta}(t) & v^{\Delta}(t) \end{vmatrix}.$$

(ii) Let $\widetilde{u}, \widetilde{v} \in \widetilde{\mathbb{D}}$. Then

$$\widetilde{u}(t)\widetilde{L}\widetilde{v}(t) - \widetilde{v}(t)\widetilde{L}\widetilde{u}(t) = \left(\frac{1}{q(t)}\widetilde{W}(\widetilde{u}, \widetilde{v})(t)\right)^{\Delta}$$

for $t \in \mathbb{T}_{\kappa}$, where

$$\widetilde{W}(\widetilde{u}, \widetilde{v})(t) := \begin{vmatrix} \widetilde{u}(t) & \widetilde{v}(t) \\ \widetilde{u}^{\nabla}(t) & \widetilde{v}^{\nabla}(t) \end{vmatrix}.$$

For a proof of Theorem 1.1 (i), see Theorem 4.33 in [3].

Corollary 1.1 (Abel's formula)
(i) If x and y both solve (1.1) *then*

$$p(t)W(x, y)(t) = a \quad t \in \mathbb{T},$$

where a is a constant.
(ii) If x and y both solve (1.2) *then*

$$\frac{1}{q(t)}\widetilde{W}(x, y)(t) = a \quad t \in \mathbb{T}_{\kappa},$$

where a is a constant.

For a proof of Corollary 1.1 for the case of (1.1), see Corollary 4.34 in [3].

Definition 1.1 The set $L^2[t_0, \infty)$ is defined to be the set of all functions $f(t)$ such that the Lebesgue integral

$$\int_{t_0}^{\infty} f^2(t)\Delta t < \infty.$$

We define the L^2-norm of a function $f \in L^2[t_0, \infty)$ by

$$\|f\|_{L^2} = \|f\| := \left(\int_{t_0}^{\infty} f^2(t) \Delta t \right)^{1/2}.$$

Definition 1.2 We say that the operator L is (Δ-)*limit-circle type* if for every solution x of $Lx = 0$, we have the Lebesgue integral

$$\int_{t_0}^{\infty} x^2(t) \Delta t < \infty.$$

If not, we say that the operator L is (Δ-)*limit-point type*.

Refer to Wong and Zettl, [11], and Coddington and Levinson, [5], for an analysis of the differential equations case.

2 Preliminary Lemmas

Lemma 2.1 *If there exists a function $\beta(t)$ with $\frac{1}{\beta} \notin L^2[t_0, \infty)$ such that $px^{\Delta}(t) = O(\beta(t))$ as $t \to \infty$ for every solution x of* (1.1)*, then L is limit-point type.*

Proof Suppose (1.1) is limit-circle type, and let x_1, x_2 be linearly independent solutions of (1.1), so we have by Corollary 1.1 part (i)

$$p(t)(x_1(t)x_2^{\Delta}(t) - x_2(t)x_1^{\Delta}(t)) \equiv a \quad t \in \mathbb{T}.$$

Then there exist constants $c, d \geq 0$ such that

$$\begin{aligned} a &\leq |x_1(t)||p(t)x_2^{\Delta}(t)| + |x_2(t)||p(t)x_1^{\Delta}(t)| \\ &\leq c\beta(t)|x_1(t)| + d\beta(t)|x_2(t)| \quad \text{ for large } t \in \mathbb{T}. \end{aligned}$$

Thus, for large $t \in \mathbb{T}$,

$$\frac{a}{\beta(t)} \leq c|x_1(t)| + d|x_2(t)|.$$

It follows that for T large,

$$\begin{aligned} a \int_T^t \frac{1}{\beta^2(s)} \Delta s &\leq \int_T^t [c^2 x_1^2(s) + 2cd x_1(s) x_2(s) + d^2 x_2^2(s)] \Delta s \\ &\leq c^2 \|x_1\|^2 + 2cd \|x_1\| \|x_2\| + d^2 \|x_2\|^2 \\ &< \infty \end{aligned}$$

by the Cauchy-Schwarz inequality (Theorem 6.15, [2]). This contradicts the fact that $\frac{1}{\beta} \notin L^2[t_0, \infty)$, so L is limit-point type.

Lemma 2.2 *Suppose $q \in C^1[t_0, \infty)$. If there exists a positive function β with $\frac{1}{\beta} \notin L^2[t_0, \infty)$ such that $y(t) = O(\beta(t))$ as $t \to \infty$ for every solution y of* (1.2)*, then L is limit-point type.*

Proof Let x be a solution of (1.1), and put $y = px^{\Delta}$. Then $y^{\nabla} = -qx$ and

$$\left(\frac{1}{q}y^{\nabla}\right)^{\Delta} = -x^{\Delta} = -\frac{y}{p}.$$

Hence, y solves (1.2). Thus,

$$y(t) = (px^{\Delta})(t) = O(\beta(t)) \text{ as } t \to \infty.$$

Thus, by Lemma 2.1, L is limit-point type.

A useful corollary to these lemmas is obtained by letting $\beta(t) \equiv 1$.

Corollary 2.1 *If $(px^{\Delta})(t)$ is bounded for every solution x of* (1.1)*, or if every solution y of* (1.2) *is bounded, then L is limit-point type.*

3 Riccati Substitution

Suppose y is a solution of (1.2) with $q(t)y(t)y^{\sigma}(t) > 0$ for $t \geq t_0$. We can then make the Riccati substitution

$$z(t) = \frac{y^{\nabla}(t)}{q(t)y(t)} \quad \text{for} \quad t \in [t_0, \infty).$$

Then, we have

$$\begin{aligned} z^{\Delta}(t) &= \left(\left(\frac{y^{\nabla}(t)}{q(t)}\right)\left(\frac{1}{y(t)}\right)\right)^{\Delta} \\ &= \left(\frac{y^{\nabla}(t)}{q(t)}\right)^{\Delta}\left(\frac{1}{y(t)}\right) + \left(\frac{y^{\nabla}(t)}{q(t)}\right)^{\sigma}\left(\frac{1}{y(t)}\right)^{\Delta} \\ &= -\frac{1}{p(t)} + \left(\frac{y^{\nabla}(t)}{q(t)}\right)^{\sigma}\left(\frac{-y^{\Delta}(t)}{y(t)y^{\sigma}(t)}\right) \\ &= -\frac{1}{p(t)} - \frac{z^{\sigma}(t)y^{\Delta}(t)}{y(t)}. \end{aligned}$$

We now use the following lemma, due to Atici and Guseinov [1]:

Lemma 3.1 *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^{\kappa}$ and if f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is ∇-differentiable on $\mathbb{T}_{\kappa}$ and*

$$f^{\nabla}(t) = f^{\Delta\rho}(t) \quad t \in \mathbb{T}_{\kappa}.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_{\kappa}$ and if g^{∇} is continuous on $\mathbb{T}_{\kappa}$, then g is Δ-differentiable on $\mathbb{T}^{\kappa}$ and

$$g^{\Delta}(t) = g^{\nabla\sigma}(t) \quad t \in \mathbb{T}^{\kappa}.$$

See also Corollary 4.11 and Theorem 4.8 and Corollary 4.10 in [3] for a generalization of this result.

Thus, we get

$$\frac{z^\sigma(t)y^\Delta(t)}{y(t)} = \frac{z^\sigma(t)y^\Delta(t)}{y^\sigma(t)-\mu(t)y^\Delta(t)} = \frac{z^\sigma(t)\frac{y^{\nabla\sigma}(t)}{y^\sigma(t)}}{1-\mu(t)\frac{y^{\nabla\sigma}(t)}{y^\sigma(t)}}$$
$$= \frac{q^\sigma(t)(z^\sigma(t))^2}{1-\mu(t)q^\sigma(t)z^\sigma(t)} = \frac{(z^\sigma(t))^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma(t)}.$$

Hence, we get that $z(t)$ solves the so-called Riccati equation associated with (1.2)

$$z^\Delta + \frac{1}{p(t)} + \frac{(z^\sigma)^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma} = 0. \tag{3.1}$$

Notice that $\dfrac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) > 0$ for all $t \ge t_0$:

$$\frac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) = \frac{1}{q^\sigma(t)} - \mu(t)\frac{y^\Delta(t)}{q^\sigma(t)y^\sigma(t)}$$
$$= \frac{1}{q^\sigma(t)y^\sigma(t)}[y^\sigma(t) - \mu(t)y^\Delta(t)]$$
$$= \frac{y(t)}{q^\sigma(t)y^\sigma(t)} > 0.$$

Hence, we have proven the following lemma:

Lemma 3.2 *If $y(t)$ is a solution of (1.2) with $q(t)y(t)y^\sigma(t) > 0$ then $z(t) := \dfrac{y^\nabla(t)}{q(t)y(t)}$ is a solution of (3.1) that satisfies $\dfrac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) > 0$ for all $t \in \mathbb{T}$.*

4 Main Results

Theorem 4.1 *Suppose that $p(t) > 0$ and $q(t) > 0$ on $[t_0, \infty)$, and $\displaystyle\int_{t_0}^\infty \frac{1}{p(t)}\Delta t = \infty$.*
(a) If (1.2) is nonoscillatory, then L is limit-point.
(b) If (1.1) is nonoscillatory, then L is limit-point.

Proof Suppose (1.2) is nonoscillatory. Let y be a positive solution of (1.2) on $[t_0, \infty)$, and make the Riccati substitution $z(t) = \dfrac{y^\nabla(t)}{q(t)y(t)}$. Then z solves

$$z^\Delta = -\frac{1}{p(t)} - \frac{(z^\sigma)^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma}.$$

Integrate both sides from t_0 to t:

$$z(t) - z(t_0) = -\int_{t_0}^t \frac{1}{p(s)}\Delta s - \int_{t_0}^t \frac{(z^\sigma(s))^2}{\frac{1}{q^\sigma(s)}-\mu(s)z^\sigma(s)}\Delta s. \tag{4.1}$$

Since

$$\frac{(z^\sigma(t))^2}{\frac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t)} \geq 0$$

for all $t \geq t_0$, we get that the right hand side of (4.1) goes to $-\infty$ as t goes to ∞. Thus, $z(t) \to -\infty$ as $t \to \infty$, so z, and hence y^∇, is eventually negative. Thus, eventually $y(t) > 0$ and $y^\nabla(t) < 0$, hence y is bounded. Thus, by Corollary 2.1 we get that L is limit-point.

Now suppose (1.1) is nonoscillatory. Let x be a positive solution of (1.1) on $[t_0, \infty)$. Since $q(t) > 0$, we have $\left(p(t)x^\Delta(t)\right)^\nabla = -q(t)x(t) < 0$ on $[t_0, \infty)$.
Claim: $p(t)x^\Delta(t) \geq 0$ on $[t_0, \infty)$.
To see this, suppose not. Then there exists $t_1 \geq t_0$ with $p(t_1)x^\Delta(t_1) < 0$. Since $p(t)x^\Delta(t)$ is decreasing, $p(t)x^\Delta(t) \leq p(t_1)x^\Delta(t_1) < 0$ on $[t_1, \infty)$. Then, dividing by $p(t)$ and integrating, we get

$$x(t) - x(t_1) \leq p(t_1)x^\Delta(t_1) \int_{t_1}^t \frac{1}{p(s)} \Delta s.$$

Thus, $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. Hence the claim holds and we see then that $p(t)x^\Delta(t)$ is bounded, so by Corollary 2.1, L is limit-point.

Definition 4.1 The set $L^2_\nabla[t_0, \infty)$ is defined to be the set of all functions $f(t)$ such that the Lebesgue integral

$$\int_{t_0}^\infty f^2(t)\nabla t < \infty.$$

We define the L^2_∇-norm of a function $f \in L^2_\nabla[t_0, \infty)$ by

$$\|f\|_{L^2_\nabla} := \left(\int_{t_0}^\infty f^2(t)\nabla t\right)^{1/2}.$$

Definition 4.2 The operator L is said to be ∇-*limit-circle* if all solutions of $Lx = 0$ satisfy $x, x^\rho \in L^2_\nabla[t_0, \infty)$. We say L is ∇-*limit-point* if there is a solution $x(t)$ of $Lx = 0$ such that $x \notin L^2_\nabla[t_0, \infty)$ or $x^\rho \notin L^2_\nabla[t_0, \infty)$.

Theorem 4.2 *Let M be a positive ∇-differentiable function and $k_1, k_2 > 0$ such that there is a $T \in \mathbb{T}$, sufficiently large such that*
(i) $q(t) \leq k_1 M(t)$ for $t \in [T, \infty)$,
(ii) $\int_T^\infty (p^\rho M^\rho)^{-1/2} \nabla s = \infty$,
(iii) $\left| \left(\frac{p^\rho(t)}{M^\rho(t)}\right)^{1/2} \frac{M^\nabla(t)}{M(t)} \right| \leq k_2$ for $t \in [T, \infty)$.
Then L is ∇-limit-point.

Proof Suppose x is a solution of $Lx = 0$ and $x, x^\rho \in L^2_\nabla[t_0, \infty)$. Since $(px^\Delta)^\nabla = -qx$, we get that for some $c > 0$,

$$\int_c^t \frac{(px^\Delta)^\nabla x}{M} \nabla s = -\int_c^t \frac{q}{M} x^2 \nabla s \geq -k_1 \int_c^t x^2 \nabla s. \tag{4.2}$$

Using the integration by parts formula ([2], Theorem 8.47 (vi))

$$\int_a^b f(s)g^{\nabla}(s)\nabla s = f(s)g(s)\big|_a^b - \int_a^b f^{\nabla}(s)g^{\rho}(s)\nabla s,$$

we get from (4.2)

$$\begin{aligned}\frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t (px^{\Delta})^{\rho}\left(\frac{x}{M}\right)^{\nabla}\nabla s &= \frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t p^{\rho}x^{\Delta\rho}\left(\frac{x^{\nabla}M - xM^{\nabla}}{MM^{\rho}}\right)\nabla s \\ &= \frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s + \int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s \\ &\geq -k_1\int_c^t x^2\nabla s.\end{aligned}$$

Thus, multiplying by -1, we get

$$-\frac{x}{M}px^{\Delta}\Big|_c^t + \int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s - \int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s \leq k_1\|x\|^2 < k_3$$

for some $k_3 > 0$.

Let $H(t) = \displaystyle\int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s\right|^2 &= \left|\int_c^t \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} M^{-1}M^{\nabla}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} xx^{\nabla}\nabla s\right|^2 \\ &\leq k_2^2\left(\int_c^t \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} xx^{\nabla}\nabla s\right)^2 \qquad \text{by } (iii) \\ &\leq k_2^2 H(t)\int_c^t x^2\nabla s.\end{aligned}$$

Thus, there exists a constant $k_4 > 0$ such that

$$-\frac{px^{\Delta}x}{M} + H - k_4H^{1/2} < k_3.$$

If $H(t) \to \infty$ as $t \to \infty$, then for all large t, $\dfrac{px^{\Delta}x}{M} > \dfrac{H}{2}$. Then x and x^{Δ} have the same sign for all large t, which contradicts $x \in L^2_{\nabla}[t_0, \infty)$. Thus,

$$H(\infty) = \int_{t_0}^{\infty} \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s < \infty.$$

Now suppose L is ∇-limit-circle. Let ϕ, ψ be two linearly independent solutions of $Lx = 0$ with $p(t)\left(\phi(t)\psi^{\Delta}(t) - \psi(t)\phi^{\Delta}(t)\right) = 1$ and ϕ, ϕ^{ρ}, ψ, $\psi^{\rho} \in L^2_{\nabla}[t_0, \infty)$. Then

$$\begin{aligned}1 &= p^{\rho}(t)\left(\phi^{\rho}(t)\psi^{\Delta\rho}(t) - \psi^{\rho}(t)\phi^{\Delta\rho}(t)\right) \\ &= p^{\rho}(t)\left(\phi^{\rho}(t)\psi^{\nabla}(t) - \psi^{\rho}(t)\phi^{\nabla}(t)\right).\end{aligned}$$

So, if we divide both sides by $(p^{\rho}M^{\rho})^{1/2}$, we get

$$\frac{1}{(p^{\rho}M^{\rho})^{1/2}} = \phi^{\rho}(t)\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\psi^{\nabla}(t) - \psi^{\rho}(t)\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\phi^{\nabla}(t). \tag{4.3}$$

If we integrate both sides of (4.3) from t_0 to ∞, we get

$$\int_{t_0}^{\infty}\frac{1}{(p^{\rho}M^{\rho})^{1/2}}\nabla s = \int_{t_0}^{\infty}\phi^{\rho}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\psi^{\nabla}\nabla s - \int_{t_0}^{\infty}\psi^{\rho}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\phi^{\nabla}\nabla s. \tag{4.4}$$

By assumption, the left-hand side of (4.4) is infinite. But, by the Cauchy-Schwarz inequality, the right-hand side becomes

$$\begin{aligned}&\left|\int_{t_0}^{\infty}\phi^{\rho}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\psi^{\nabla}\nabla s - \int_{t_0}^{\infty}\psi^{\rho}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2}\phi^{\nabla}\nabla s\right|\\ &\leq \|\phi^{\rho}\|_{L^2_{\nabla}}\left(\int_{t_0}^{\infty}\frac{p^{\rho}}{M^{\rho}}(\psi^{\nabla})^2\nabla s\right)^{1/2} + \|\psi^{\rho}\|_{L^2_{\nabla}}\left(\int_{t_0}^{\infty}\frac{p^{\rho}}{M^{\rho}}(\phi^{\nabla})^2\nabla s\right)^{1/2}\\ &< \infty.\end{aligned}$$

This is a contradiction to the assumption that L is ∇-limit-circle. Thus, we have that L is ∇-limit-point

5 Example

Fix $q > 1$. Let $\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}$. Consider the dynamic equation

$$x^{\Delta\nabla} + (t\ln t)^2 x = 0.$$

Here, we have $p(t) \equiv 1$, and $q(t) = (t\ln t)^2$. We need to show that the three assumptions in Theorem 4.2 hold. Fix $N > 0$ sufficiently large and let $T = q^N$. Also, let $M(t) = (t\ln t)^2$. For (i), if we take $k_1 = 1$, we get that $q(t) = M(t) = (t\ln t)^2$ for all $t \in \mathbb{T}$, so certainly $q(t) \leq M(t)$ for $t \geq T$.

For (ii), consider

$$\begin{aligned}\int_T^{\infty}(p^{\rho}(s)M^{\rho}(s))^{-1/2}\nabla s &= \int_T^{\infty}\frac{1}{(M^{\rho}(s))^{1/2}}\nabla s = \int_T^{\infty}\frac{1}{((\rho(s)\ln\rho(s))^2)^{1/2}}\nabla s\\ &= \int_T^{\infty}\frac{1}{\rho(s)\ln\rho(s)}\nabla s = \sum_{k=N+1}^{\infty}\frac{1}{q^{k-1}\ln q^{k-1}}\nu(q^k)\\ &= \sum_{k=N+1}^{\infty}\frac{1}{q^{k-1}\ln q^{k-1}}(q^k - q^{k-1}) = \sum_{k=N+1}^{\infty}\frac{q^{k-1}(q-1)}{q^{k-1}\ln q^{k-1}}\\ &= \frac{q-1}{\ln q}\sum_{k=N+1}^{\infty}\frac{1}{k-1} = \frac{q-1}{\ln q}\sum_{k=N}^{\infty}\frac{1}{k} = \infty.\end{aligned}$$

Notice,

$$\begin{aligned} M^{\nabla}(t) &= \frac{(q^k \ln q^k)^2 - (q^{k-1}\ln q^{k-1})^2}{q^k - q^{k-1}} \\ &= \frac{(q^k \ln q^k - q^{k-1}\ln q^{k-1})(q^k \ln q^k + q^{k-1}\ln q^{k-1})}{q^k - q^{k-1}} \\ &= \frac{(q^{k-1})^2(\ln q)^2(qk-(k-1))(qk+(k-1))}{q^{k-1}(q-1)} \\ &= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{q-1}. \end{aligned}$$

Thus, for part (iii), we have for $k \geq N$

$$\begin{aligned} \left| \left(\frac{p^{\rho}(t)}{M^{\rho}(t)}\right)^{1/2} \frac{M^{\nabla}(t)}{M(t)} \right| &= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{(q-1)q^{k-1}\ln(q^{k-1})\, q^{2k}(\ln q^k)^2} \\ &= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{(q-1)q^{k-1}q^{2k}(k-1)k^2\ln q(\ln q)^2} \\ &= \frac{q^2k^2-(k-1)^2}{(q-1)k^2(k-1)q^{2k}\ln q} \\ &\leq \frac{q^2k^2}{(q-1)k^2(k-1)q^{2k}\ln q} \\ &\leq \frac{q^2}{(q-1)(k-1)q^{2k}\ln q} \\ &\leq \frac{1}{(q-1)(k-1)q^{2k-2}\ln q} \\ &\leq \frac{1}{(q-1)(N-1)q^{2N-2}\ln q} := k_2 \end{aligned}$$

Thus, the assumptions of Theorem 4.2 hold, so we get that

$$x^{\Delta\nabla} + (t\ln t)^2 x = 0$$

is ∇-limit-point.

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