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Some Linear and Nonlinear Integral Inequalities on Time Scales in Two Independent Variables

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Abstract: We establish some linear and nonlinear integral inequalities of Gronwall–Bellman–Bihari type for functions with two independent variables on general time scales. The results are illustrated with examples, obtained by fixing the time scales to concrete ones. An estimation result for the solution of a partial delta dynamic equation is given as an application.

Keywords: *integral inequalities; Gronwall–Bellman–Bihari inequalities; time scales; two independent variables.*

Mathematics Subject Classification (2000): 26D15, 45K05.

1 Introduction

Inequalities have always been of great importance for the development of several branches of mathematics. For instance, in approximation theory and numerical analysis, linear and nonlinear inequalities, in one and more than one variable, play an important role in the estimation of approximation errors [12].

Time scales, which are defined as nonempty closed subsets of the real numbers, are the basic but fundamental ingredient that permits to define a rich calculus that encompasses both differential and difference tools [8, 9]. At the same time one gains more (cf., e.g., Corollary 3.1). For an introduction to the calculus on time scales we refer the reader to [6] and [4, 5], respectively for functions of one and more than one independent variables.

Integral inequalities of Gronwall–Bellman–Bihari type for functions of a single variable on a time scale can be found in [2, 3, 7, 11, 14]. To the best of the authors knowledge, no such results exist in the literature of time scales when functions of two independent variables are considered. It is our aim to obtain here a first insight on this type of inequalities.

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2 Linear Inequalities

Throughout the text we assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales with at least two points and consider the time scales intervals $\tilde{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\tilde{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$, and $a_2 \in \mathbb{T}_2$. We also use the notations $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $e_p(t,s)$ denotes the usual exponential function on time scales with $p \in \mathcal{R}$, i.e., p is a regressive function [6].

Theorem 2.1 *Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$ with $a(t_1,t_2)$ nondecreasing in each of its variables. If*

$$u(t_1,t_2) \leq a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 \tag{1}$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2) \leq a(t_1,t_2) e_{\int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2}(t_1,a_1)\,, \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{2}$$

Proof Since $a(t_1,t_2)$ is nondecreasing on $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, inequality (1) implies, for an arbitrary $\varepsilon > 0$, that

$$r(t_1,t_2) \leq 1 + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)r(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2,$$

where $r(t_1,t_2) = \frac{u(t_1,t_2)}{a(t_1,t_2)+\varepsilon}$. Define $v(t_1,t_2)$ by the right hand side of the last inequality. Then,

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) = f(t_1,t_2)r(t_1,t_2) \leq f(t_1,t_2)v(t_1,t_2),\ (t_1,t_2) \in \tilde{\mathbb{T}}_1^k \times \tilde{\mathbb{T}}_2^k. \tag{3}$$

From (3), and taking into account that $v(t_1,t_2)$ is positive and nondecreasing, we obtain

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \leq f(t_1,t_2),$$

from which it follows that

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \leq f(t_1,t_2) + \frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\frac{\partial v(t_1,t_2)}{\Delta_2 t_2}}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)}\right) \leq f(t_1,t_2).$$

Delta integrating with respect to the second variable from a_2 to t_2 (we observe that t_2 can be the maximal element of $\tilde{\mathbb{T}}_2$, if it exists), and noting that $\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\mid_{(t_1,a_2)} = 0$, we have

$$\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)} \leq \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \le \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 v(t_1,t_2).$$

Fixing $t_2 \in \tilde{\mathbb{T}}_2$ arbitrarily, we have that $p(t_1) := \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 \in \mathcal{R}^+$. Because $v(a_1,t_2)=1$, by [2, Theorem 5.4] $v(t_1,t_2) \le e_p(t_1,a_1)$. Inequality (2) follows from

$$u(t_1,t_2) \le [a(t_1,t_2)+\varepsilon]v(t_1,t_2)$$

and the arbitrariness of ε. □

Corollary 2.1 (cf. Lemma 2.1 of [10]) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and assume that the functions $u(x,y), a(x,y), f(x,y) \in C([x_0,\infty)\times[y_0,\infty),\mathbb{R}_0^+)$ with $a(x,y)$ nondecreasing in its variables. If*

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(t,s)u(t,s)dtds$$

for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$, then

$$u(x,y) \le a(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} f(t,s)dtds\right)$$

for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$.

Corollary 2.2 (cf. Theorem 2.1 of [13]) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and assume that the functions $u(m,n), a(m,n), f(m,n)$ are nonnegative and that $a(m,n)$ is nondecreasing for $m \in [m_0,\infty)\cap\mathbb{Z}$ and $n \in [n_0,\infty)\cap\mathbb{Z}$, $m_0, n_0 \in \mathbb{Z}$. If*

$$u(m,n) \le a(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)u(s,t)$$

for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$, then

$$u(m,n) \le a(m,n)\prod_{s=m_0}^{m-1}\left[1+\sum_{t=n_0}^{n-1} f(s,t)\right]$$

for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$.

Remark 2.1 We note that, following the same steps of the proof of Theorem 2.1, one can obtained other bound on the function u, namely

$$u(t_1,t_2) \le a(t_1,t_2)e_{\int_{a_1}^{t_1} f(s_1,t_2)\Delta_1 s_1}(t_2,a_2). \tag{4}$$

When $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then the bounds in (2) and (4) coincide (see Corollary 2.1). If, for example, we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the bounds obtained can be different. Moreover, at different points one bound can be sharper than the other and vice-versa (see Example 2.1).

Example 2.1 *Let $f(t_1, t_2)$ be a function defined by $f(0,0) = 1/4$, $f(1,0) = 1/5$, $f(2,0) = 1$, $f(0,1) = 1/2$, $f(1,1) = 0$, and $f(2,1) = 5$. Set $a_1 = a_2 = 0$. Then, from (2) we get*

$$u(2,1) \leq a(2,1)\frac{3}{2}, \quad u(3,2) \leq a(3,2)\frac{147}{10},$$

while from (4) we get

$$u(2,1) \leq a(2,1)\frac{29}{20}, \quad u(3,2) \leq a(3,2)\frac{637}{40}.$$

Other interesting corollaries can be obtained from Theorem 2.1.

Corollary 2.3 *Let $\mathbb{T}_1 = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, for some $q > 1$, and $\mathbb{T}_2 = \mathbb{R}$. Assume that the functions $u(t,x)$, $a(t,x)$ and $f(t,x)$ satisfy the hypothesis of Theorem 2.1 for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$ with $a_1 = 1$ and $a_2 = 0$. If*

$$u(t,x) \leq a(t,x) + \sum_{s=1}^{t/q}(q-1)s\int_0^x f(s,\tau)u(s,\tau)d\tau$$

for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t,x) \leq a(t,x)\prod_{s=1}^{t/q}\left[1 + (q-1)s\int_0^x f(s,\tau)d\tau\right]$$

for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$.

We now generalize Theorem 2.1. If in Theorem 2.2 we let $f \equiv 1$ and g not depending on the first two variables, then we obtain Theorem 2.1.

Theorem 2.2 *Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with a and f nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S, \mathbb{R}_0^+)$ be nondecreasing in t_1 and t_2, where $S = \{(t_1,t_2,s_1,s_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$. If*

$$u(t_1,t_2) \leq a(t_1,t_2) + f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2) \leq a(t_1,t_2)e_{\int_{a_2}^{t_2} f(t_1,t_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1)\,, \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{5}$$

Proof We start by fixing arbitrary numbers $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$, and considering the following function defined on $[a_1,t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2,t_2^*] \cap \tilde{\mathbb{T}}_2$ for an arbitrary $\varepsilon > 0$:

$$v(t_1,t_2) = a(t_1^*,t_2^*) + \varepsilon + f(t_1^*,t_2^*)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1^*,t_2^*,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2\,.$$

From our hypothesis we see that

$$u(t_1,t_2) \leq v(t_1,t_2), \text{ for all } (t_1,t_2) \in [a_1,t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2,t_2^*] \cap \tilde{\mathbb{T}}_2.$$

Moreover, delta differentiating with respect to the first variable and then with respect to the second, we obtain

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) = f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)u(t_1,t_2)$$
$$\leq f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)v(t_1,t_2),$$

for all $(t_1,t_2) \in [a_1,t_1^*]^k \cap \tilde{\mathbb{T}}_1 \times [a_2,t_2^*]^k \cap \tilde{\mathbb{T}}_2$. From this last inequality, we can write

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \leq f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)\,.$$

Hence,

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))} \leq f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2) + \frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\frac{\partial v(t_1,t_2)}{\Delta_2 t_2}}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)}\right) \leq f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2).$$

Delta integrating with respect to the second variable from a_2 to t_2 and noting that $\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\,|_{(t_1,a_2)}= 0$, we have

$$\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)} \leq \int_{a_2}^{t_2} f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \leq \int_{a_2}^{t_2} f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2 v(t_1,t_2).$$

Fix $t_2 = t_2^*$ and put $p(t_1) := \int_{a_2}^{t_2^*} f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2 \in \mathcal{R}^+$. By [2, Theorem 5.4]

$$v(t_1,t_2^*) \leq (a(t_1^*,t_2^*)+\varepsilon)e_p(t_1,a_1).$$

Letting $t_1 = t_1^*$ in the above inequality, and remembering that t_1^*, t_2^* and ε are arbitrary, it follows (5). □

3 Nonlinear Inequalities

Theorem 3.1 *Let $u(t_1,t_2)$ and $f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$. Moreover, let $a(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}^+)$ be a nondecreasing function in each of the variables. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t_1,t_2) \leq a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2 \tag{6}$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2) \leq a^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}},\ (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{7}$$

Proof Since $a(t_1,t_2)$ is positive and nondecreasing on $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, inequality (6) implies that

$$u^p(t_1,t_2)\le a(t_1,t_2)\left(1+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2\right).$$

Define $v(t_1,t_2)$ on $\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$ by

$$v(t_1,t_2)=1+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2.$$

Then

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)=f(t_1,t_2)\frac{u^q(t_1,t_2)}{a(t_1,t_2)}\le f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v^{\frac{q}{p}}(t_1,t_2)\,,$$

and noting that $v^{\frac{q}{p}}(t_1,t_2)\le v(t_1,t_2)$ we conclude that

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)\le f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v(t_1,t_2).$$

We can now follow the same procedure as in the proof of Theorem 2.1 to obtain

$$v(t_1,t_2)\le e_p(t_1,a_1),$$

where $p(t_1)=\int_{a_2}^{t_2} f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2$. Noting that

$$u(t_1,t_2)\le a^{\frac{1}{p}}(t_1,t_2)v^{\frac{1}{p}}(t_1,t_2),$$

we obtain the desired inequality (7). □

Theorem 3.2 *Let $u(t_1,t_2),a(t_1,t_2),f(t_1,t_2)\in C(\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2,\mathbb{R}_0^+)$, with a and f nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2)\in C(S,\mathbb{R}_0^+)$ be nondecreasing in t_1 and t_2, where $S=\{(t_1,t_2,s_1,s_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2\times\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2 : a_1\le s_1\le t_1, a_2\le s_2\le t_2\}$. If p and q are two positive real numbers such that $p\ge q$ and if*

$$u^p(t_1,t_2)\le a(t_1,t_2)+f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 \tag{8}$$

for all $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2)\le a^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,s_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}}$$

for all $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$.

Proof Since $a(t_1,t_2)$ is positive and nondecreasing on $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, inequality (8) implies that

$$u^p(t_1,t_2)\le a(t_1,t_2)\left(1+f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2\right).$$

Fix $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$ arbitrarily and define a function $v(t_1, t_2)$ on $[a_1, t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*] \cap \tilde{\mathbb{T}}_2$ by

$$v(t_1, t_2) = 1 + f(t_1^*, t_2^*) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1^*, t_2^*, s_1, s_2) \frac{u^q(s_1, s_2)}{a(s_1, s_2)} \Delta_1 s_1 \Delta_2 s_2.$$

Then

$$\begin{aligned} \frac{\partial}{\Delta_2 t_2} \left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right) &= f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) \frac{u^q(t_1, t_2)}{a(t_1, t_2)} \\ &\leq f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1, t_2) v^{\frac{q}{p}}(t_1, t_2). \end{aligned}$$

Since $v^{\frac{q}{p}}(t_1, t_2) \leq v(t_1, t_2)$, we have that

$$\frac{\partial}{\Delta_2 t_2} \left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right) \leq f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1, t_2) v(t_1, t_2).$$

We can follow the same steps as done before to reach the inequality

$$\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \leq \int_{a_2}^{t_2} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1, s_2) \Delta_2 s_2 v(t_1, t_2).$$

Fix $t_2 = t_2^*$ and put $p(t_1) := \int_{a_2}^{t_2^*} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1, s_2) \Delta_2 s_2 \in \mathcal{R}^+$. Again, an application of [2, Theorem 5.4] gives

$$v(t_1, t_2^*) \leq e_p(t_1, a_1),$$

and putting $t_1 = t_1^*$ we obtain the desired inequality. $\Box$

We end this section by considering a particular time scale. Let $\{\alpha_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers and let

$$t_0^\alpha \in \mathbb{R}, \quad t_k^\alpha = t_0^\alpha + \sum_{n=1}^{k} \alpha_n, \ k \in \mathbb{N},$$

where we assume that $\lim_{k \to \infty} t_k^\alpha = \infty$. Then, we define the following time scale: $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$. For $p \in \mathcal{R}$ we have (cf. [1, Example 4.6]):

$$e_p(t_k^\alpha, t_0^\alpha) = \prod_{n=1}^{k} (1 + \alpha_n p(t_{n-1})), \ \text{for all } k \in \mathbb{N}_0. \tag{9}$$

Given two sequences $\{\alpha_k, \beta_k\}_{k \in \mathbb{N}}$ and two numbers $t_0^\alpha, t_0^\beta \in \mathbb{R}$ as above, we define the two time scales $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$ and $\mathbb{T}^\beta = \{t_k^\beta : k \in \mathbb{N}_0\}$. We state now our last corollary.

Corollary 3.1 *Let $u(t, s)$, $a(t, s)$, and $f(t, s)$, defined on $\mathbb{T}^\alpha \times \mathbb{T}^\beta$, be nonnegative with a and f nondecreasing. Further, let $g(t, s, \tau, \xi)$, where $(t, s, \tau, \xi) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta \times \mathbb{T}^\alpha \times \mathbb{T}^\beta$ with $\tau \leq t$ and $\xi \leq s$, be nonnegative and nondecreasing in the first two variables t and s. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t, s) \leq a(t, s) + f(t, s) \sum_{\tau \in [t_0^\alpha, t)} \sum_{\xi \in [t_0^\beta, s)} \mu^\alpha(\tau) \mu^\beta(\xi) g(t, s, \tau, \xi) u^q(\tau, \xi) \tag{10}$$

for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$*, where* μ^α *and* μ^β *are the graininess functions of* $\mathbb{T}^\alpha$ *and* $\mathbb{T}^\beta$*, respectively, then*

$$u(t,s) \leq a^{\frac{1}{p}}(t,s)\left[e_{\int_{t_0^\beta}^s f(t,s)a^{\frac{q}{p}-1}(t,\xi)g(t,s,t,\xi)\Delta^\beta\xi}(t,t_0^\alpha)\right]^{\frac{1}{p}}$$

for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$*, where* e *is given by (9).*

Remark 3.1 In (10) we are slightly abusing on notation by considering $[t_0^\alpha, t) = [t_0^\alpha, t) \cap \mathbb{T}^\alpha$ and $[t_0^\beta, t) = [t_0^\beta, t) \cap \mathbb{T}^\beta$.

4 An Application

Let us consider the partial delta dynamic equation

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial u^2(t_1,t_2)}{\Delta_1 t_1}\right) = F(t_1,t_2,u(t_1,t_2)) \tag{11}$$

under given initial boundary conditions

$$u^2(t_1,0) = g(t_1),\ u^2(0,t_2) = h(t_2),\ g(0) = 0,\ h(0) = 0, \tag{12}$$

where we are assuming $a_1 = a_2 = 0$, $F \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \mathbb{R}_0^+, \mathbb{R}_0^+)$, $g \in C(\tilde{\mathbb{T}}_1, \mathbb{R}_0^+)$, $h \in C(\tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with g and h nondecreasing functions and positive on their domains except at zero.

Theorem 4.1 *Assume that on its domain,* F *satisfies*

$$F(t_1,t_2,u) \leq t_2 u.$$

If $u(t_1,t_2)$ *is a solution of the IBVP (11)-(12) for* $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$*, then*

$$u(t_1,t_2) \leq \sqrt{(g(t_1)+h(t_2))}\left[e_{\int_0^{t_2} s_2(g(t_1)+h(s_2))^{-\frac{1}{2}}\Delta_2 s_2}(t_1,0)\right]^{\frac{1}{2}} \tag{13}$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$*, except at the point* $(0,0)$*.*

Proof Let $u(t_1,t_2)$ be a solution of the IBVP (11)-(12). Then, it satisfies the following delta integral equation:

$$u^2(t_1,t_2) = g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} F(s_1,s_2,u(s_1,s_2))\Delta_1 s_1 \Delta_2 s_2.$$

The hypothesis on F imply that

$$u^2(t_1,t_2) \leq g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} s_2 u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2.$$

An application of Theorem 3.1 with $a(t_1,t_2) = g(t_1)+h(t_2)$ and $f(t_1,t_2) = t_2$ gives (13). □

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