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An LMI Criterion for the Global Stability Analysis of Nonlinear Polynomial Systems

R. Mtar *, M.M. Belhaouane, H. Belkhiria Ayadi and N. Benhadj Braiek

Laboratoire d'Etude et Commande Automatique de Processus – LECAP
Ecole Polytechnique de Tunis (EPT), BP.743, 2078 La Marsa, Tunis, Tunisie.

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Abstract: This paper presents an original practical criterion of global stability analysis of nonlinear polynomial systems. This criterion derived from the application of the Lyapunov direct method with a quadratic function generalizes the famous Lyapunov stability condition for linear systems. Useful mathematical transformations have allowed the formulation of the obtained conditions as an LMI (Linear Matrix Inequalities) problem according to the polynomial system parameters.

Keywords: *nonlinear polynomial systems; Lyapunov methods; global stability analysis; LMI approach.*

Mathematics Subject Classification (2000): 34D20, 93D20, 93D30.

1 Introduction

The problem of stability analysis of nonlinear systems has received considerable attention in the field of research in automatic control and different approaches have been proposed in the literature related with this subject [1]– [16]. The polynomial technique of studying stability of nonlinear systems is one of the most important developed approaches. It is based on the modeling of the considered nonlinear analytical systems by a polynomial system [17]– [27]. Notice that the class of polynomial systems is large enough to include the description of numerous physical processes such as electrical machines and robot manipulators [28]. Moreover, the description of polynomial systems can be simplified using the Kronecker product and power of vectors and matrices [17, 29, 30].

In previous works, sufficient algebraic conditions of global asymptotic stability of polynomial systems have been derived using the direct Lyapunov method with a quadratic

* Corresponding author: mtarriadh@yahoo.fr

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function [17, 18, 25, 29, 31, 32] or non quadratic function as polynomial or monomial Lyapunov functions [33, 34]. The advantage of the proposed criteria is that they are expressed according to the studied polynomial system parameters, generalizing the famous Lyapunov condition known for the linear systems. However, the implementation of the general form of the derived stability conditions of polynomial systems requires the resolution of nonlinear matrix inequalities [35, 36]. To overcome this difficulty, we propose in this paper a new development which leads to the formulation of a practical LMI stability condition for polynomial systems.

This paper is organized as follows: Section 2 is concerned with the description of the studied systems and some useful notations. Then, in the third section we present the derived global stability condition for polynomial systems. The fourth section shows how the obtained condition can be implemented as an LMI problem. Section 5 is devoted to a numerical example which illustrates the availability of the proposed approach.

2 System Description and Notations

2.1 System description

The considered nonlinear polynomial systems are described by the following state equation:

$$\dot{X} \quad = \quad f(X), \tag{2.1}$$

where $f(X)$ is a polynomial vector function of X.

$$f(X) = \sum_{i=1}^{r} A_i X^{[i]} = \sum_{i=1}^{r} \tilde{A}_i \tilde{X}^{[i]} \tag{2.2}$$

with $X = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $X^{[i]}$ is the Kronecker power of the vector X defined as:

$$\begin{cases} X^{[0]} = 1, \\ X^{[i]} = X^{[i-1]} \otimes X = X \otimes X^{[i-1]} \quad for \quad i \geq 1, \end{cases} \tag{2.3}$$

$\otimes$ designates the symbol of the Kronecker product [30], $\tilde{X}^{[i]}_{i=1,\ldots,r} \in \mathbb{R}^{n_i}$, $n_i = \begin{pmatrix} n+i-1 \\ i \end{pmatrix}$ is the nun-redundant Kronecker power of the state vector X defined as:

$$\begin{array}{l} \tilde{X}^{[1]} = X^{[1]} = X, \\ \forall\, i \geq 2,\, \tilde{X}^{[i]} = [x_1^i,\, x_1^{i-1}x_2,\, ...,\, x_1^{i-1}x_n,\, ...,\, x_1^{i-2}x_n^2,\, ...,\, x_1^{i-3}\, x_2^3,\, ...,\, x_n^i]^T, \end{array} \tag{2.4}$$

where the repeated components of the redundant (ith-power) $X^{[i]}$ are omitted, $A_{i,i=1,\ldots,r} \in \mathbb{R}^{n \times n^i}$ (resp. $\tilde{A}_i \in \mathbb{R}^{n \times n_i}$) are constant matrices. The polynomial order r is considered odd: $r = 2s - 1$, with $s \in \mathbb{N}^*$. Let's recall that this class of systems describes a large set of processes as electrical machines and robot manipulators and that any analytical system can be approached by a polynomial model.

2.2 Notations

In this section, we introduce some useful notations and needed rules and functions. Let the matrices and vectors be of the following dimensions: $A(p \times q)$, $B(r \times s)$, $C(q \times f)$, $E(n \times p)$, $X(n \times 1)$, $Y(m \times 1)$.

(i) We consider the following notations: I_n is an $(n \times n)$ identity matrix; $0_{n\times m}$ is an $(n \times m)$ zero matrix; 0 is a zero matrix of convenient dimension; A^T is a transpose of matrix A; $A > 0 (A \geq 0)$ is a symmetric positive definite (semi-definite) matrix; e_k^q is a q dimensional unit vector which has 1 in the kth element and zero elsewhere.

(ii) The relation between the redundant and the nun-redundant Kronecker power of the state vector X can be stated as follows:

$$\left\{ \begin{array}{llll} \forall i \in \mathbb{N} \ \exists & T_i & \in & \mathbb{R}^{n^i \times n_i} \\ & X^{[i]} & = & T_i \tilde{X}^{[i]} \end{array} \right\}, \tag{2.5}$$

where (n_i) stands for the binomial coefficient. A procedure of the determination of the matrix T_i is given in [37].

(iii) The permutation matrix denoted by $(U_{n\times m})$ is defined as:

$$U_{n\times m} = \sum_{i=1}^{n}\sum_{j=1}^{m} \left(e_i^n \cdot {e_j}^{mT} \right) \otimes \left(e_j^m \cdot {e_i}^{nT} \right). \tag{2.6}$$

This matrix is square $(nm \times nm)$ and has precisely a single 1 in each row and in each column. Among the main properties of this matrix presented in [30], we recall the following useful ones:

$$(B \otimes A) = U_{r\times p}(A \otimes B)U_{q\times s}, \tag{2.7}$$

$$(X \otimes Y) = U_{n\times m}(Y \otimes X), \tag{2.8}$$

$$\forall i \leq k \quad X^{[k]} = U_{n^i \times n^{k-i}} X^{[k]}. \tag{2.9}$$

(iv) An important vector valued function of matrix denoted by $vec(.)$ was defined in [30] as follows:

$$A = \left[\begin{array}{cccc} A_1 & A_2 & ... & A_q \end{array} \right] \in \mathbb{R}^{p\times q}, \quad A_i \in \mathbb{R}^p, \quad i \in \{1, ..., q\},$$

$$vec(A) = \left[\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_q \end{array} \right] \in \mathbb{R}^{pq}.$$

We recall the following useful rules [30] of *vec-function*:

$$vec(EAC) = (C^T \otimes E)vec(A), \tag{2.10}$$

$$vec(A^T) = U_{p\times q} vec(A). \tag{2.11}$$

(v) A special function $mat_{(n,m)}(.)$ can be defined as follows:
If V is a vector of dimension $p = nm$ then $M = mat_{(n,m)}(V)$ is the $(n \times m)$ matrix verifying: $V = vec(M)$.

(vi) For a polynomial vectorial function:

$$f(X)=\sum_{i=1}^{r} A_i X^{[i]}, \tag{2.12}$$

where $X \in \mathbb{R}^n$, $A_{i,\, i=1,\dots,r}$ are $(n\times n^i)$ constant matrices and $r=2s-1$, $s\in\mathbb{N}^*$, $\mathcal{M}(f)$ designates the set of matrices defined by:

$$\mathcal{M}(f)=\{\mathcal{M}_\lambda(f)\in\mathbb{R}^{\upsilon\times\upsilon} \quad ; \quad \lambda=[\lambda_{ij}]\in\mathbb{R}^{s\times s}\} \tag{2.13}$$

such that:

$$\mathcal{M}_\lambda(f)=\begin{bmatrix} \lambda_{11}M_{11} & \lambda_{12}M_{12} & \dots & \lambda_{1k}M_{1k} & \dots & \lambda_{1s}M_{1s} \\ \lambda_{21}M_{21} & \lambda_{22}M_{22} & \dots & \lambda_{2k}M_{2k} & \dots & \lambda_{2s}M_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{k1}M_{k1} & \lambda_{k2}M_{k2} & \vdots & \lambda_{kk}M_{kk} & \vdots & \lambda_{ks}M_{ks} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{s1}M_{s1} & \lambda_{s2}M_{s2} & \dots & \lambda_{sk}M_{sk} & \dots & \lambda_{ss}M_{ss} \end{bmatrix}, \tag{2.14}$$

$\upsilon=n+n^2+\dots+n^s$, and

- for $k=1,\dots,r=2s-1$,
- for $j=g_k,\dots,h_k$ where $g_k=\sup(1,k+1-s)$ and $h_k=\inf(s,k)$

we have:

$$M_{k+1-j,j}=\begin{bmatrix} mat_{(n^{k-j},n^j)}\left(A_k^{1T}\right) \\ mat_{(n^{k-j},n^j)}\left(A_k^{2T}\right) \\ \vdots \\ mat_{(n^{k-j},n^j)}\left(A_k^{nT}\right) \end{bmatrix}, \tag{2.15}$$

A_k^i is the i^{th} row of the matrix A_k:

$$A_k=\begin{bmatrix} A_k^1 \\ A_k^2 \\ \vdots \\ A_k^n \end{bmatrix}. \tag{2.16}$$

Notice that, for all integer numbers i and j such that $1\le i,j\le s$, there exist $k\in\mathbb{N}^*$ such that $1\le k\le 2s-1$, $i=k+1-j$ and $g_k\le j\le h_k$.
λ_{ij} are arbitrary reals verifying:

$$\sum_{j=g_k}^{h_k}\lambda_{k+1-j,j}=1. \tag{2.17}$$

(vii) We introduce the matrix $\mathcal{R}$ defined by:

$$\mathcal{R}=\tau_1^{+[2]}\cdot\mathcal{U}\cdot\mathcal{H}\cdot\tau_2, \tag{2.18}$$

where

$$\tau_1 = \begin{bmatrix} T_1 & & & & \\ & T_2 & & 0 & \\ & & T_3 & & \\ & 0 & & \ddots & \\ & & & & T_s \end{bmatrix}, \tag{2.19}$$

$$\tau_2 = \begin{bmatrix} T_2 & & & 0 \\ & T_3 & & \\ & & \ddots & \\ 0 & & & T_{2s} \end{bmatrix}, \tag{2.20}$$

$$\mathcal{U} = \begin{bmatrix} U_{n\times\eta_0} & & & 0 \\ & U_{n^2\times\eta_0} & & \\ & & \ddots & \\ 0 & & & U_{n^s\times\eta_0} \end{bmatrix}, \tag{2.21}$$

$$\mathcal{H} = \begin{bmatrix} I_{\eta_1} & & & & 0 \\ 0_{\eta_2\times\eta_1} & I_{\eta_2} & & & \\ 0_{\eta_3\times(\eta_1+\eta_2)} & & I_{\eta_3} & & \\ \vdots & & & \ddots & \\ 0_{\eta_s\times(\eta_1+\eta_2+\ldots+\eta_{s-1})} & & & & I_{\eta_s} \end{bmatrix}, \tag{2.22}$$

for $j = 1, \ldots.., s, : \eta_j = n^j \cdot \left(\sum\limits_{i=1}^{s} n^i\right)$,

τ_1^+ is the Moore-Penrose pseudo-inverse of τ_1.

We note Γ is the matrix defined by:

$$\Gamma = \left(I_{\eta^2} + U_{\eta\times\eta}\right)\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^T - I_{\eta^2}\right) \tag{2.23}$$

with $\eta = \sum\limits_{j=1}^{s} n_j = \sum\limits_{j=1}^{s} \begin{pmatrix} n+j-1 \\ j \end{pmatrix}$ and $\mathcal{R}^+$ is the Moore-Penrose pseudo-inverse of $\mathcal{R}$.

$\beta = rank(\Gamma)$ and $C_{i,\, i=1,\ldots,\beta}$ are β linearly independent columns of Γ.

(iix) For a $(n \times n)$ matrix P, we define the $(\upsilon \times \upsilon)$ matrix $\mathcal{D}_s(P)$ as:

$$\mathcal{D}_s(P) = \begin{bmatrix} P & & & 0 \\ & P\otimes I_n & & \\ & & \ddots & \\ & 0 & & P\otimes I_{n^{s-1}} \end{bmatrix}. \tag{2.24}$$

Notice that if P is a definite symmetric positive matrix then so is $\mathcal{D}_s(P)$.

3 Stability Criterion of Polynomial Systems

We consider the analytical nonlinear autonomous systems described by the following polynomial state-space equation:

$$\dot{X} = f(X) = \sum_{k=1}^{r} A_k X^{[k]}, \quad r = 2s-1. \tag{3.1}$$

The Lyapunov's direct method leads to a sufficient condition of the global asymptotic stability of the equilibrium ($X = 0$) of the polynomial system (3.1). This condition is stated in the following theorem.

Theorem 1 *Consider the nonlinear polynomial system defined by the equation (3.1) where the integer r is odd: $r = 2s-1$. If there exist:*

- *an $(n \times n)$-symmetric positive definite matrix P,*
- *an $(s \times s)$-matrix $\lambda = [\lambda_{ij}]$ verifying $\sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} = 1$,*
- *arbitrary parameters $\mu_{i,i=1,\dots,\beta} \in \mathbb{R}$*

such that the $(\eta \times \eta)$ symmetric matrix $\mathcal{Q}$ defined by:

$$\mathcal{Q} \quad = \quad -\tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1 + \sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) \tag{3.2}$$

is positive definite, then the equilibrium $X = 0$ of the considered system (3.1) is globally asymptotically stable.

Proof Consider the quadratic Lyapunov function:

$$V(X) = X^T P X. \tag{3.3}$$

Differentiating $V(X)$ along the trajectory of the system (3.1), one obtains:

$$\dot{V}(X) = \sum_{k=1}^{r} (X^T P A_k X^{[k]} + X^{[k]^T} A_k^T P X) = 2\sum_{k=1}^{r} X^T P A_k X^{[k]}. \tag{3.4}$$

Using the rule of the vec-function (2.10), the relation (3.4) can then be written as:

$$\dot{V}(X) = 2\sum_{k=1}^{r} V_k^T X^{[k+1]}, \tag{3.5}$$

where

$$V_k = vec(PA_k). \tag{3.6}$$

To ensure the global asymptotic stability of the equilibrium $(X = 0)$ of the system (3.1), it is sufficient to have $\dot{V}(X)$ negative definite for $\forall X \in \mathbb{R}^n$.

Let the following notations be used for $k = 1, \dots, 2s-1$ and $j = g_k, \dots, h_k$

$$N_{k+1-j,\,j} = mat_{(n^{k+1-j},\, n^j)}(V_k). \tag{3.7}$$

Then, using the relation (3.7), we can write:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} X^{[k+1-j]^T} N_{k+1-j} X^{[j]} \tag{3.8}$$

such that $\sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} = 1$. It can be shown [17] that one has:

$$N_{k+1-j,j} = mat_{(n^{k+1-j},n^j)}(Vec(PA_k)) = U_{n^{k-j}\times n}(P \otimes I_{n^{k-j}}).M_{k+1-j,j}, \tag{3.9}$$

where $M_{k+1-j,j}$ is defined in (2.15).

Using the result (3.9) and the relation (2.9), we can write:

$$\begin{aligned} X^{[k+1-j]^T} N_{k+1-j,j} X^{[j]} &= X^{[k+1-j]^T} U_{n^{k-j}\times n}(P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]} \\ &= X^{[k+1-j]^T} (P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]}. \end{aligned} \tag{3.10}$$

Consequently, we obtain:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} X^{[k+1-j]^T} (P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]} = \mathcal{X}^T \mathcal{D}_S(P) \mathcal{M}_k(\lambda) \mathcal{X}$$

with

$$\mathcal{X} = \begin{bmatrix} X^T & X^{[2]^T} & \cdots & X^{[s]^T} \end{bmatrix}^T \tag{3.11}$$

and

$$\mathcal{M}_k(\lambda) = \begin{bmatrix} 0 & & \lambda_{1k} M_{1k} \\ & \lambda_{k-1,2} M_{k-1,2} \;\; \iddots & \\ \lambda_{k1} M_{k1} & & 0 \end{bmatrix}. \tag{3.12}$$

Then $\dot{V}(X)$ can be written as:

$$\dot{V}(X) = 2 \sum_{k=1}^{2s-1} V_k^T X^{[k+1]} = \mathcal{X}^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^T \mathcal{D}_S(P)) \mathcal{X}, \tag{3.13}$$

where $\mathcal{M}_\lambda(f) = \sum_{k=1}^{r} \mathcal{M}_k(\lambda)$ is defined in (2.14).

Using the nun-redundant form, the vector $\mathcal{X}$ can be written as:

$$\mathcal{X} = \tau_1 \tilde{\mathcal{X}}, \tag{3.14}$$

where $\widetilde{\mathcal{X}} \in \mathbb{R}^\eta$, $\eta = \sum_{j=1}^{s} n_j$ and τ_1 is defined in (2.19).

Then $\dot{V}(X)$ can be written in the following form:

$$\dot{V}(X) = \widetilde{\mathcal{X}}^{\mathrm{T}} \tau_1^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}} \mathcal{D}_S(P)) \tau_1 \widetilde{\mathcal{X}}. \tag{3.15}$$

A sufficient condition of the global asymptotic stability of the equilibrium ($X = 0$) is that the quadratic form $\dot{V}(X)$ should be negative definite. This condition can be ensured if there exists a symmetric positive definite $\mathcal{Q} \in \mathbb{R}^{\eta\times\eta}$ such that:

$$\widetilde{\mathcal{X}}^{\mathrm{T}} \tau_1^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}} \mathcal{D}_S(P)) \tau_1 \widetilde{\mathcal{X}} = -\widetilde{\mathcal{X}}^{\mathrm{T}} \mathcal{Q} \widetilde{\mathcal{X}}. \tag{3.16}$$

Using the *vec*-function, the equality (3.16) can be expressed as:

$$vec^{\mathrm{T}}\left(\tau_1^{\mathrm{T}}(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1+\mathcal{Q})\right)\widetilde{\mathcal{X}}^{[2]}=0 \tag{3.17}$$

But, it can be easily checked that $\widetilde{\mathcal{X}}^{[2]}$ can be written as

$$\widetilde{\mathcal{X}}^{[2]}=\mathcal{R}\widetilde{\mathcal{X}}_2, \tag{3.18}$$

where

$$\widetilde{\mathcal{X}}_2=\begin{bmatrix}\widetilde{\mathcal{X}}^{[2]}\\ \vdots\\ \widetilde{\mathcal{X}}^{[s+1]}\\ \widetilde{\mathcal{X}}^{[s+2]}\\ \vdots\\ \widetilde{\mathcal{X}}^{[2s]}\end{bmatrix} \tag{3.19}$$

and $\mathcal{R}$ is the matrix defined in (2.18). Therefore the equality (3.17) yields the following equation:

$$\mathcal{R}^{\mathrm{T}}vec(S)=0 \tag{3.20}$$

with $S=\tau_1^T\left(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P)\right)\tau_1+\mathcal{Q}$. The η^2-vector $vec(S)$ which is a solution of (3.20) can be expressed as:

$$vec(S)=\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^{\mathrm{T}}-I_{\eta^2}\right)\mathcal{Y}, \tag{3.21}$$

where $\mathcal{Y}$ is an arbitrary vector of $\mathbb{R}^{\eta^2}$. On the other hand, the matrix S is symmetric since $\mathcal{Q}$ is symmetric, then we have

$$S=\tfrac{1}{2}(S+S^T) \tag{3.22}$$

and using the property (2.11) yields

$$vec(S)=\tfrac{1}{2}(I_{\eta^2}+U_{\eta\times\eta})vec(S)=\sum_{i=1}^{\beta}\mu_iC_i, \tag{3.23}$$

where $\beta=rank\left[\left(I_{\eta^2}+U_{\eta\times\eta}\right)\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^T-I_{\eta^2}\right)\right]$, $C_{i,i=1,\ldots,\beta}$ are β linearly independent columns of

$$\left(I_{\eta^2}+U_{\eta\times\eta}\right)\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^T-I_{\eta^2}\right), \tag{3.24}$$

$\mu_{i,i=1,\ldots,\beta}$ are arbitrary values. Consequently, the matrix $\mathcal{Q}$ verifying (3.20) is of the following form:

$$\mathcal{Q}=-\tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1+\sum_{i=1}^{\beta}\mu_i mat_{(\eta,\eta)}(C_i) \tag{3.25}$$

which ends the proof.

Remark. For $r=1$, the system (3.1) becomes linear ($\dot{X}=AX$) and by (3.25) we obtain the famous Lyapunov stability condition for linear system: *The asymptotic stability of the origin equilibrium of the system $\dot{X}=AX$ is ensured iff there exist symmetric positive definite matrices P and Q such that $A^TP+PA=-Q$.* Thus, the criterion stated in Theorem 1 generalizes this linear stability Lyapunov condition for polynomial systems.

4 LMI Formulation of the Global Stability Criterion of Polynomial Systems

In this section we show how the stated stability conditions of Theorem 1 can be formulated as LMI conditions. Les us notice that the proved stability condition can be presented as the following matrix inequality feasibility problem. Find:

- a $(n \times n)$ matrix P;
- $\lambda = [\lambda_{ij}] \in \mathbb{R}^{s \times s}$ verifying the relation (2.17);
- real parameters $\mu_{i,i=1,\dots,\beta}$;

such that:

$$\begin{cases} P > 0, \\ \\ \tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1 - \sum\limits_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) < 0. \end{cases} \tag{4.1}$$

However, these inequalities are nonlinear with respect of the unknown parameters P,λ_{ij} and μ_i, since the second inequality of (4.1) is bilinear on (P, λ_{ij}). To overcome this problem we make use of the separation lemma [38] and we exploit the generalized Schur's complement [35], in order to transform the BMI problem into an LMI one.

Let us remark that the coefficients λ_{ij} of the matrix λ verify the relations (2.17) which implies that

$$\lambda_{11} = 1, \quad \lambda_{ss} = 1, \tag{4.2}$$

and the matrix $\mathcal{M}_\lambda(f)$ can be written as

$$\mathcal{M}_\lambda(f) = \mathcal{N}(f) + \mathcal{N}_\lambda(f), \tag{4.3}$$

where:

$$\mathcal{N}(f) = \begin{bmatrix} M_{11} & & & \mathbf{0} \\ & M_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & M_{ss} \end{bmatrix} \tag{4.4}$$

and

$$\mathcal{N}_\lambda(f) = \begin{bmatrix} 0 & \lambda_{12}M_{12} & \cdots & \cdots & \lambda_{1s}M_{1s} \\ \lambda_{21}M_{21} & \alpha_2 M_{22} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{s-1}M_{s-1,s-1} & \lambda_{s-1s}M_{s-1,s} \\ \lambda_{s1}M_{s1} & \cdots & \cdots & \lambda_{ss-1}M_{s,s-1} & 0 \end{bmatrix} \tag{4.5}$$

for $k = 2, ..., s-1$,

$$\alpha_k = - \sum_{\substack{1 \le i,j \le s \\ i+j=2k \\ i \ne j}} \lambda_{ij}. \tag{4.6}$$

According to the relation (4.3), the second inequality of (4.1) becomes:

$$\begin{aligned} & -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ & + [\mathcal{D}_S(P)\tau_1]^T [\mathcal{N}_\lambda(f)\tau_1] + [\mathcal{N}_\lambda(f)\tau_1]^T [\mathcal{D}_S(P)\tau_1] < 0. \end{aligned} \tag{4.7}$$

Making use of the following separation lemma.

Lemma 1 *[38]: For any matrices A and B with appropriate dimensions and for any positive scalar $\epsilon > 0$, one has: $A^T B + B^T A \leq \epsilon A^T A + \epsilon^{-1} B^T B$.*

Then, the inequality (4.7) is satisfied if there exists a real $\epsilon > 0$ such that

$$\begin{array}{l} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ +\epsilon\left[\mathcal{D}_S(P)\tau_1\right]^T\left[\mathcal{D}_S(P)\tau_1\right] + \frac{1}{\epsilon}\left[\mathcal{N}_\lambda(f)\tau_1\right]^T\left[\mathcal{N}_\lambda(f)\tau_1\right] < 0. \end{array} \tag{4.8}$$

This inequality (4.8) can be put as

$$\begin{array}{l} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ -\left[\mathcal{D}_S(P)\tau_1\right]^T(-\epsilon I)\left[\mathcal{D}_S(P)\tau_1\right] - \left[\mathcal{N}_\lambda(f)\tau_1\right]^T\left(-\frac{1}{\epsilon}I\right)\left[\mathcal{N}_\lambda(f)\tau_1\right] < 0 \end{array} \tag{4.9}$$

Using Schur complement, inequality (4.9) holds if and only if

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & [\mathcal{D}_S(P)\tau_1]^T & [\mathcal{N}_\lambda(f)\tau_1]^T \\ \mathcal{D}_S(P)\tau_1 & -\frac{1}{\epsilon}I & 0 \\ \mathcal{N}_\lambda(f)\tau_1 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{4.10}$$

Multiplying $diag\left(I, I, \epsilon^{-1}I\right)$ for both sides of (4.10), we have

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & [\mathcal{D}_S(P)\tau_1]^T & \left[\widetilde{\mathcal{N}}_\lambda(f)\tau_1\right]^T \\ \mathcal{D}_S(P)\tau_1 & -\epsilon^{-1}I & 0 \\ \widetilde{\mathcal{N}}_\lambda(f)\tau_1 & 0 & -\epsilon^{-1}I \end{bmatrix} < 0, \tag{4.11}$$

where $\widetilde{\lambda}_{ij} = \epsilon^{-1}\lambda_{ij}$. This new inequality (4.11) is linear on the decision variables, and then we can state the following theorem.

Theorem 2 *The equilibrium ($X = 0$) of the system (3.1) is globally asymptotically stable if there exist:*

- *a $(s \times s)$-matrix $\widetilde{\lambda} = [\widetilde{\lambda}_{ij}]$ verifying $\sum_{j=g_k}^{h_k} \widetilde{\lambda}_{k+1-j,j} = 1$;*
- *a $(n \times n)$-symmetric positive definite matrix P ;*
- *arbitrary parameters $\mu_{i,i=1,\ldots,\beta} \in \mathbb{R}$;*
- *a real $\epsilon > 0$;*

such that:

$$P > 0 \tag{4.12}$$

and

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & [\mathcal{D}_S(P)\tau_1]^T & \left[\widetilde{\mathcal{N}}_\lambda(f)\tau_1\right]^T \\ \mathcal{D}_S(P)\tau_1 & -\epsilon^{-1}I & 0 \\ \widetilde{\mathcal{N}}_\lambda(f)\tau_1 & 0 & -\epsilon^{-1}I \end{bmatrix} < 0. \tag{4.13}$$

The stability analysis of polynomial systems using Theorem 2, can be carried out using Matlab software.

5 Illustrative Example

To illustrate the availability of the proposed method we consider the stability study of the origin equilibrium of the following second order polynomial systems:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & -x_1 - x_2 + x_1^2 + x_1x_2 - x_1^3 + x_1^2x_2 - x_1x_2^2 + 2x_2^3, \\ \dot{x}_2 & = & -x_1 - 1.5x_2 - 1.1x_1^2 + 0.3x_1x_2 - 1.8x_1^3 - 5.6x_1^2x_2 - 5.3x_1x_2^2 - 0.7x_2^3. \end{array}\right. \quad (5.1)$$

This system can be written in the following form:

$$\dot{X} = A_1X + A_2X^{[2]} + A_3X^{[3]} \quad (5.2)$$

with

$$A_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1.1 & 0.3 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ -1.8 & 0.9 & -5.2 & -1.8 & -1.3 & 4.3 & -8.3 & -0.7 \end{bmatrix}.$$

Solving the optimization problem formulated by Theorem 2, we obtain:

$$\left\{\begin{array}{l} \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 3.8529 \end{array}\right., \quad \left\{\begin{array}{l} \lambda_{11} = 1 \\ \lambda_{12} = 0.1419 \\ \lambda_{21} = 0.8581 \\ \lambda_{22} = 1 \end{array}\right., \quad \epsilon = 0.1864, \quad P = \begin{bmatrix} 1.9551 & -0.1723 \\ -0.1723 & 1.1529 \end{bmatrix},$$

which ensure the global asymptotic stability of the equilibrium $X = 0$.

6 Conclusion

In this paper, we have presented an original practical criterion for global stability analysis of nonlinear polynomial systems. This criterion is stated as sufficient conditions derived from a quadratic Lyapunov function. Furthermore, useful mathematical transformations have allowed the formulation of the obtained conditions as an LMI problem, which has facilitated the numerical implementation of the proposed criterion using Matlab LMI toolboxes.

Let's notice that the obtained results presented in this paper are developed with a quadratic Lyapunov function, but they can be easily extended for the case of polynomial Lyapunov functions. Also, we point out that a similar method can be elaborated for the stabilization and robust control of polynomial systems.

References

[1] Boyd, S. and Yang, Q. Structured and simultaneous Lyapunov functions for system stability problems. *Int. J. Control* **49**(6) (1989) 2215–2240.

[2] Banks, S. and Yi, Z. On the stability analysis of nonlinear systems. *Journal of Mathematical Control and Information* **8**(3) (1991) 275–285.

[3] Kiong, N.S. and Fu, M. Global quadratic stabilization of a class of nonlinear systems. *Int. J. Robust Nonlinear Control.* **8** (1998) 483–497.

[4] Rios-Patron, E. and Braatz, R. Global stability analysis for discrete-time nonlinear systems. *American Control Conference* **1** (1998) 338–342.

[5] Gleria, I., Figueiredo, A. and Filho, T.R. A numerical method for the stability analysis of quasi-polynomial vector fields. *Nonlinear Analysis: Theory, Methods and Applications* **52**(1) (2003) 329–342.

[6] Borne, P., Richard, J.P. and Radhy, N.E. Stability, stabilization, regulation using vector norm. *Nonlinear Systems, (Chapter 2)* **2** (1996) 45–90.

[7] Perruquetti, W., Richard, J.P., Grujic, Lj.T. and Borne, P. On practical stability with the settling time via vector norm. *International Journal of Control* **62**(1) (1995) 173–189.

[8] Borne, P. and Gentina, J.C. On the stability of large nonlinear systems structured and simultaneous lyapunov for system stability problems. *Joint. Aut. Cont. Conf, Austin, Texas*, 1974.

[9] Borne, P. and Benrejeb, M. On the stability of a class of interconnected systems. application to the forced working conditions. *Actes 4th IFAC Symposium MTS Frederiction, Canada*, 1977.

[10] Papachristodoulou, A. and Prajna, S. On the construction of Lyapunov function using sum of squares decomposition. *In Proceeding of IEEE Conference on Descion and Control, CDC'02*, 2002.

[11] Parrilo, P.A. Exploiting structure in sum of squares programs. *In Proceeding of the IEEE Conference on Descion and Control,CDC'03*, December 2003.

[12] Henrion, D. and Garulli, A. *Positive polynomials in Control*. Springler LNCIS, January 2005.

[13] Rotella, F. and Tanguy, G. Non linear systems: identification and optimal control. *Int. J. Control* **48**(2) (1988) 525–544.

[14] Martynyuk, A.A. and Slyn'ko, V.I. Stability Results for Large-Scale Difference Systems via Matrix-Valued Liapunov Functions. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 217–224.

[15] Liu, B. and Lui, F. Robusty Global Exponential Stability of Time-varying Linear Impulsive Systems with Uncertainty. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 187–196.

[16] Radu Balan. An Extension of Barbashin–Krasovskii–LaSalle Theorem to a class of Nonautonomous Systems. *Nonlinear Dynamics and Systems Theory* **8**(3) (2008) 255–268.

[17] Benhadj Braiek, N., Rotella, F. and Benrejeb, M. Algebraic criteria for global stability analysis of nonlinear systems. *Journal of Systems Analysis Modelling and Simulation, Gordon and Breach Science Publishers* **17** (1995) 221–227.

[18] Benhadj Braiek, N. On the global stability of nonlinear polynomial systems. *IEEE Conference On Decision and Control, CDC'96*, December 1996.

[19] Benhadj Braiek, N. and Rotella, F. Design of observers for nonlinear time variant systems. *IEEE Syst. Man and Cybernetics Conference* **4** (1993) 219–225.

[20] Benhadj Braiek, N. and Rotella, F. State observer design for analytical nonlinear systems. *IEEE Syst. Man and Cybernetics Conference* **3** (1994) 2045–2050.

[21] Belkhiria, H. and Benhadj Braiek, N. On the robust stability analysis of uncertain polynomial systems : an LMI Approach. *17th IMACS World Congress, Scientific Computation, Applied Mathematics and Simulation, Paris*, Juillet, 2005.

[22] Mtar, R., Belkhiria, H. and Benhadj Braiek, N. Robust stability of polynomial systems under nonlinear perturbations: an LMI Approach. *Fourth IEEE International Multi-Conference on Systems, Signals and Devices, SSD'2007*, Hammamet–Tunisie, 19–22 Mars, 2007.

[23] Mtar, R., Belhaouane, M., Belkhiria, H. and Benhadj Braiek, N. H_∞-Performance Analysis for Nonlinear Polynomial Systems: an LMI Approach. *Journées Tunisiennes d'Electrotechnique et d'Automatique, JTEA'2008*, Hammamet–Tunisie, 02–04 Mai, 2008.

[24] Benhadj Braiek, N, Bacha, A. and Jerbi, H. A Technique of a Stability Domain Determination for Nonlinear Discrete Polynomial System. *IFAC World Congress*,Seoul 6-11 Juillet 2008.

[25] Benhadj Braiek, N. Feedback stabilization and stability domain estimation of nonlinear systems. *Journal of the Franklin Institute* **332** (2) (1995) 183–193.

[26] Parrilo, P.A. Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. *PhD thesis, California Institue of Technologie, Pasadena*, CA, 2000.

[27] Zachary, W.J.W. Lyapunov based analysis and controllers synthesis for polynomial systems using sum of squares optimization. *PhD thesis, University of California, Berkeley*, 2001.

[28] Benhadj Braiek, N and Rotella, F. Robot model simplification by means of an identification method. In: *Robotics and Flexible Manufacturing Systems* (Eds. Gentina, J.C. and Tzafesta, S.G.). Elsevier Science Publishes B. V, North Holand (1992) 217–227.

[29] Benhadj Braiek, N and Rotella, F. Stabilization of nonlinear systems using a Kronecker product approach. *European Control Conference ECC'95*, September 1995, 2304–2309.

[30] Brewer, J. Kronecker product and matrix calculus in system theory. *IEEE Trans. Circ. Sys* **CAS-25** (1978) 722–781.

[31] Khalil, H.K. *Non-linear Systems*. 3rd ed. New York, Prentice–Hall, 2000.

[32] Bohner, M. and Martynyuk, A.A. Elements of Stability of A.M. Liapunov for Dynamic Equations on Time Scales. *Nonlinear Dynamics and Systems Theory* **7**(3) (2007) 225–252.

[33] Zelentsovsky, A.L. Non quadratic Lyapunov functions for robust stability analysis of linear uncertain systems. *IEEE Trans. Automat. Control* **39**(1) (1994).

[34] Bouzaouache, H. and Benhadj Braiek, N. On the stability analysis of nonlinear systems using polynomial Lyapunov functions. *Mathematics and Computers in Simulation* **76**(5–6) (2008) 316–329.

[35] Boyd, S., Ghaoui, L. and Balakrishnan, F. Linear Matrix Inequalities in System and Control Theory. *SIAM*, 1994.

[36] Karimi, H.R., Lohmann, B. and Buskens, C. An LMI Approach to H∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs. *Nonlinear Dynamics and Systems Theory* **7**(4) (2007) 351–368.

[37] Benhadj Braiek, N and Rotella, F. Logic: a nonlinear systems identification software. *Modelling and Simulation of Systems, Scientific Publishing Co.* (1990) 211–218.

[38] Zhou, K. and Khargonedkar, P. P. Robust stabilization of linear systems with normbounded time-varying uncertainty. *Sys. Contr. Letters* **10** (1988) 17–20.