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Frequent Oscillatory Solutions of a Nonlinear Partial Difference Equation

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Abstract: This paper is concerned with a class of nonlinear delay partial difference equations with variable coefficients, which may change sign. By making use of frequency measures, some new oscillatory criteria are established.

Keywords: *partial difference equations; frequency oscillatory; frequency measures; nonlinear.*

Mathematics Subject Classification (2000): 39A11.

1 Introduction

Let Z be the set of integers, $Z[k,l] = \{i \in Z | i = k, k+1, ..., l\}$ and $Z[k,\infty) = \{i \in Z | i = k, k+1, ...\}$.

In [1], authors considered oscillations of the partial difference equation with several nonlinear terms of the form

$$u_{m+1,n} + u_{m,n+1} - u_{m,n} + \sum_{i=1}^{h} p_i(m,n) \left|u_{m-k_i,n-l_i}\right|^{\alpha_i} \operatorname{sgn} u_{m-k_i,n-l_i} = 0.$$

In this paper, we investigate the equation of the following form

$$u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} - u_{m,n} + \sum_{i=1}^{h} p_i(m,n) \left|u_{m-k_i,n-l_i}\right|^{\alpha_i} \operatorname{sgn} u_{m-k_i,n-l_i} = 0, \quad (1)$$

where $m, n \in Z[0,\infty)$, $P_i\,(m,n) \geq 0$ $(i = 1, 2, \cdots, h)$ and

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(H$_1$) $\alpha_h > \alpha_{h-1} > \cdots > \alpha_k > 1 > \alpha_{k-1} > \cdots > \alpha_1 > 0$;

(H$_2$) k_i, l_i $(i = 1, 2, \cdots, h)$ are nonnegative integers.

Such an equation arises in several mathematical models (see e.g.[3]) including interconnected neuron units placed on an arbitrary large board, heat transfer in lattice of molecules, population migration among cities, and discrete simulation of the heat equation *et al.*

The usual concepts of oscillation or stability of steady state solutions do not catch all their fine details, and it is necessary to use the concept of frequency measures introduced in [2] to provide better descriptions. In this paper, by employing frequency measures, some new oscillatory criteria of (1) are established.

In addition to (H$_1$) and (H$_2$), we also assume

(H$_3$) $p_i = \{p_i(m,n)\}_{m,n \in Z[0,\infty)}$ $(i = 1, 2, \cdots, h)$ are real double sequences;

(H$_4$) Suppose there exists $a_i > 0$ $(i = 1, 2, \cdots, h)$ such that $\sum_{i=1}^{h} a_i = 1$ and $\sum_{i=1}^{h} a_i \alpha_i = 1$;

(H$_5$) If $p_i = \{p_i(m,n)\}$ has negative components, then a_i is chosen such that a_i is a quotient of odd positive integers.

Let

$$\overline{k} = \max_{1 \le i \le h} \{k_i\} > 0,\ \overline{l} = \max_{1 \le i \le h} \{l_i\} > 0,\ \underline{k} = \min_{1 \le i \le h} \{k_i\},\ \underline{l} = \min_{1 \le i \le h} \{l_i\}$$

and

$$\gamma = \min \left\{ \frac{1}{a_1}, \cdots, \frac{1}{a_h} \right\}.$$

Since $0 < a_i < 1$, we see that $\gamma > 1$.

Our plan is as follows. In the next section, we recall some of the terminologies and basic results related to the frequency measures. Then we derive several criteria for all solutions of (1) to be frequently oscillatory or unsaturated. In the final section, we give some examples to illustrate our results.

For the sake of convenience, $Z[-\overline{k}, \infty) \times Z[-\overline{l}, \infty)$ will be denoted by Ω in the sequel. Given a double sequence $\{u_{m,n}\}$, the partial differences $u_{m+1,n} - u_m$ and $u_{m,n+1} - u_{m,n}$ will be denoted by $\Delta_1 u_{m,n}$ and $\Delta_2 u_{m,n}$ respectively.

2 Preliminaries

The union, intersection and difference of two sets A and B will be denoted by $A + B$, $A \cdot B$ and $A \backslash B$ respectively. The number of elements of a set S will be denoted by $|S|$. Let Φ be a subset of Ω. Then

$$X^m \Phi = \{(i + m, j) \in \Omega |\, (i, j) \in \Phi\}, \quad Y^m \Phi = \{(i, j + m) \in \Omega |\, (i, j) \in \Phi\}$$

are the translations of Φ. Let α, β, λ and δ be integers satisfying $\alpha \le \beta$ and $\lambda \le \delta$. The union $\sum_{i=\alpha}^{\beta} \sum_{j=\lambda}^{\delta} X^i Y^j \Phi$ will be denoted by $X_\alpha^\beta Y_\lambda^\delta \Phi$. Clearly,

$$(i, j) \in \Omega \backslash X_\alpha^\beta Y_\lambda^\delta \Phi \Leftrightarrow (i - s, j - t) \in \Omega \backslash \Phi$$

for $\alpha \le s \le \beta$ and $\lambda \le t \le \delta$.

For any $m, n \in Z[0, \infty)$, we set $\Phi^{(m,n)} = \left\{(i, j) \in \Phi |\, -\overline{k} \le i \le m, -\overline{l} \le j \le n\right\}$. If

$$\limsup_{m,n \to \infty} \frac{\left|\Phi^{(m,n)}\right|}{mn}$$

exists, then the superior limit, denoted by $\mu^*(\Phi)$, will be called the upper frequency measure of Φ. Similarly, if

$$\liminf_{m,n\to\infty} \frac{\left|\Phi^{(m,n)}\right|}{mn}$$

exists, then the inferior limit, denoted by $\mu_*(\Phi)$, will be called the lower frequency measure of Φ. If $\mu_*(\Phi) = \mu^*(\Phi)$, then the common limit is denoted by $\mu(\Phi)$ and is called the frequency measure of Φ.

Clearly, $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and $0 \le \mu_*(\Phi) \le \mu^*(\Phi) \le 1$ for any subset Φ of Ω, furthermore if Φ is finite, then $\mu(\Phi) = 0$.

The following results are concerned with the frequency measures and their proofs are similar to those in [3].

Lemma 2.1 *Let Φ and Γ be subsets of Ω. Then $\mu^*(\Phi + \Gamma) \le \mu^*(\Phi) + \mu^*(\Gamma)$. Furthermore, if Φ and Γ are disjoint, then*

$$\mu_*(\Phi) + \mu_*(\Gamma) \le \mu_*(\Phi + \Gamma) \le \mu_*(\Phi) + \mu^*(\Gamma) \le \mu^*(\Phi + \Gamma) \le \mu^*(\Phi) + \mu^*(\Gamma),$$

so that

$$\mu_*(\Phi) + \mu^*(\Omega \backslash \Phi) = 1.$$

Lemma 2.2 *Let Φ be a subset of Ω and α, β, λ and δ be integers such that $\alpha \le \beta$ and $\lambda \le \delta$. Then*

$$\mu^*\left(X_\alpha^\beta Y_\lambda^\delta \Phi\right) \le (\beta - \alpha + 1)(\delta - \lambda + 1)\mu^*(\Phi)$$

and

$$\mu_*\left(X_\alpha^\beta Y_\lambda^\delta \Phi\right) \le (\beta - \alpha + 1)(\delta - \lambda + 1)\mu_*(\Phi).$$

Lemma 2.3 *Let $\Phi_1, ..., \Phi_n$ be subsets of Ω. Then*

$$\mu^*\left(\sum_{i=1}^n \Phi_i\right) \le \sum_{i=1}^n \mu^*(\Phi_i) - (n-1)\mu^*\left(\prod_{i=1}^n \Phi_i\right)$$

and

$$\mu_*\left(\sum_{i=1}^n \Phi_i\right) \le \mu_*(\Phi_1) + \mu^*\left(\sum_{i=2}^n \Phi_i\right) - (n-1)\mu^*\left(\prod_{i=1}^n \Phi_i\right).$$

Lemma 2.4 *Let Φ and Γ be subsets of Ω. If $\mu_*(\Phi) + \mu^*(\Gamma) > 1$, then the intersection $\Phi \cdot \Gamma$ is infinite.*

For any real double sequence $\{v_{i,j}\}$ defined on a subset of Ω, the level set $\{(i,j) \in \Omega |\, v_{i,j} > c\}$ is denoted by $(v > c)$. The notations $(v \ge c)$, $(v < c)$, $(v \le c)$ are similarly defined. Let $u = \{u_{i,j}\}_{(i,j)\in\Omega}$ be a real double sequence. If $\mu^*(u \le 0) = 0$, then u is said to be frequently positive, and if $\mu^*(u \ge 0) = 0$, then u is said to be frequently negative.

u is said to be frequently oscillatory if it is neither frequently positive nor frequently negative.. If $\mu^*(u > 0) = \omega \in (0,1)$, then u is said to have unsaturated upper positive part, and if $\mu_*(u > 0) = \omega \in (0,1)$, then u is said to have unsaturated lower positive part. u is said to have unsaturated positive part if $\mu^*(u > 0) = \mu_*(u > 0) = \omega \in (0,1)$.

The concepts of frequently oscillatory and unsaturated double sequences were introduced in [2-6]. It was also observed that if a double sequence $u = \{u_{i,j}\}_{(i,j)\in\Omega}$ is frequently oscillatory or has unsaturated positive part, then it is oscillatory, that is, u is not positive for all large m and n, nor negative for all large m and n. Thus if we can show that every solution of (1) is frequently oscillatory or has unsaturated positive part, then every solution of (1) is oscillatory.

3 Frequently Oscillatory Solutions

An inequality, which can be found in [7], will be used in deriving the following results:

$$\sum_{i=1}^{h} \sigma_i x_i \geq \prod_{i=1}^{h} x_i^{\sigma_i}, \tag{2}$$

where $\sigma_i > 0, \sum_{i=1}^{h} \sigma_i = 1, x_i \geq 0, i = 1, 2, \cdots, h.$

Lemma 3.1 *Suppose there exist* $m_0 \geq 2\overline{k}$ *and* $n_0 \geq 2\overline{l}$ *such that*

$$p_i(m,n) \geq 0 \ \text{ for } (m,n) \in Z[m_0 - 2\overline{k}\ , m_0+1] \times Z[n_0 - 2\overline{l}, n_0+1],\ i = 1, 2, \cdots, h.$$

Let $\{u_{m,n}\}$ *be a solution of (1). If* $u_{m,n} \geq 0$ *for* $(m,n) \in Z[m_0 - 2\overline{k}, m_0+1] \times Z[n_0 - 2\overline{l}, n_0+1]$, *then*

$$\Delta_1 u_{m,n} \leq 0, \Delta_2 u_{m,n} \leq 0 \ \text{ for } (m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0],$$

and if $u_{m,n} \leq 0$ *for* $(m,n) \in Z[m_0 - 2\overline{k}, m_0+1] \times Z[n_0 - 2\overline{l}, n_0+1]$, *then*

$$\Delta_1 u_{m,n} \geq 0, \Delta_2 u_{m,n} \geq 0 \ \text{ for } (m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0].$$

Proof If $u_{m,n} \geq 0$ for $(m,n) \in Z[m_0 - 2\overline{k}, m_0+1] \times Z[n_0 - 2\overline{l}, n_0+1]$, it follows from (1) that

$$\begin{aligned} u_{m,n} &= u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} + \sum_{i=1}^{h} p_i(m,n) u^{\alpha_i}_{m-k_i,n-l_i} \\ &\geq u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} \\ &\geq u_{m+1,n} + u_{m,n+1}. \end{aligned}$$

Hence $\Delta_1 u_{m,n} \leq 0$, $\Delta_1 u_{m,n} \leq 0$ for $(m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0]$.

Similarly, we also have $\Delta_1 u_{m,n} \geq 0, \Delta_2 u_{m,n} \geq 0$ for $(m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0]$. Let

$$\prod_{i=1}^{h} p_i^{a_i} = \left\{ \prod_{i=1}^{h} p_i^{a_i}(m,n) \right\}_{m,n \in Z[0,\infty)}.$$

Under the assumption (H_5), $\prod_{i=1}^{h} p_i^{a_i}$ is well defined. We remark that if $p_i(m,n) \geq 0$, the assumption (H_5) is not needed.

Theorem 3.1 *Suppose there exist constants* ω_i $(i = 1, 2, \cdots, h)$ *and* ω *such that*

$$\mu^*(p_i < 0) = \omega_i \ (i = 1, 2, \cdots, h),\ \mu_* \left(\prod_{i=1}^{h} (p_i < 0) \right) = \omega,$$

$$\mu_* \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1 \right) > 4(\overline{k}+1)(\overline{l}+1) \left(\sum_{i=1}^{h} \omega_i - (h-1)\, \omega \right).$$

Then every nontrivial solution of (1) is frequently oscillatory.

Proof Suppose to the contrary that $u = \{u_{m,n}\}$ is a frequently positive solution of (1). Then $\mu^*(u \le 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}
1 &= \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\} \\
&\quad +\mu_*\left\{X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\} \\
&\le \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\} \\
&\quad +4(\overline{k}+1)(\overline{l}+1)\left\{\mu_*\left(\sum_{i=1}^{h}(p_i<0)\right)+\mu^*(u\le 0)\right\} \\
&\le \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}+4(\overline{k}+1)(\overline{l}+1)\left(\sum_{i=1}^{h}\omega_i-(h-1)\,\omega\right) \\
&< \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}+\mu_*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right).
\end{aligned}$$

Therefore by Lemma 4, the intersection

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}\cdot\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right)$$

is infinite. This implies that there exist $m_0 \ge 2\overline{k}$ and $n_0 \ge 2\overline{l}$ such that

$$\gamma\prod_{i=1}^{h}p_i^{a_i}(m_0,n_0)>1 \tag{3}$$

and

$$p_i(m,n)\ge 0\ (i=1,2,\cdots,h), u_{m,n}>0. \tag{4}$$

for $(m,n) \in Z[m_0-2\overline{k}, m_0+1]\times Z[n_0-2\overline{l}, n_0+1]$. In view of (4) and Lemma 3.1, we may then see that $\Delta_1 u_{m,n} \le 0$ and $\Delta_2 u_{m,n} \le 0$ for $(m,n) \in Z[m_0-\overline{k}, m_0]\times Z[n_0-\overline{l}, n_0]$, and hence $u_{m_0-k_i,n_0-l_i} \ge u_{m_0-\underline{k},n_0-\underline{l}} \ge u_{m_0,l_0}$ $(i = 1,2,\cdots,h)$, so that by (2) and (4),

$$\begin{aligned}
0 &\ge u_{m_0+1,n_0+1}+u_{m_0+1,n_0}+u_{m_0,n_0+1}-u_{m_0,n_0}+\sum_{i=1}^{h}p_i(m_0,n_0)u_{m_0-\underline{k},n_0-\underline{l}}^{\alpha_i} \\
&\ge u_{m_0+1,n_0+1}+u_{m_0+1,n_0}+u_{m_0,n_0+1}-u_{m_0,n_0}+\gamma\prod_{i=1}^{h}p_i^{a_i}(m_0,n_0)u_{m_0,n_0} \\
&\ge \left(\gamma\prod_{i=1}^{h}p_i^{a_i}(m_0,n_0)-1\right)u_{m_0,n_0}>0,
\end{aligned}$$

which is a contradiction.

In a similar manner, if $u = \{u_{m,n}\}$ is a frequently negative solution of (1) such that $\mu^*(u \geq 0) = 0$, then we may show that

$$\left\{\Omega \backslash X_{-1}^{2k_1} Y_{-1}^{2l_1} \left[\sum_{i=1}^{h} (p_i < 0) + (u \geq 0)\right]\right\} \cdot \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right)$$

is infinite. Again we may arrive at a contradiction as above. The proof is complete.

Theorem 3.2 *Suppose there exist constants ω_i $(i = 1, 2, \cdots, h)$ and ω such that*

$$\mu^*(p_i < 0) = \omega_i \ (i = 1, 2, \cdots, h), \ \mu^* \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) = \omega,$$

$$\mu_* \left(\prod_{i=1}^{h} (p_i < 0) \cdot \left(\gamma \prod_{j=1}^{h} p_j^{a_j} \leq 1\right)\right) > \frac{\sum\limits_{i=1}^{h} \omega_i + \omega}{h} - \frac{1}{4h(\overline{k}+1)(\overline{l}+1)}.$$

Then every nontrivial solution of (1) is frequently oscillatory.

Proof Suppose to the contrary that $u = \{u_{m,n}\}$ be an eventually positive solution of (1). Then $\mu^*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we get

$$\begin{aligned}
& \mu^* \left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h} (p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\} \\
= \ & 1 - \mu_* \left\{X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h} (p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\} \\
\geq \ & 1 - 4(\overline{k}+1)(\overline{l}+1) \left\{\mu_* \left[\sum_{i=1}^{h} (p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right)\right] + \mu^* (u \leq 0)\right\} \\
\geq \ & 1 - 4(\overline{k}+1)(\overline{l}+1) \left[\sum_{i=1}^{h} \mu^* (p_i < 0) + \mu^* \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right)\right. \\
& \left. - h\mu_* \left(\prod_{i=1}^{h} (p_i < 0) \cdot \left(\gamma \prod_{j=1}^{h} p_j^{a_j} \leq 1\right)\right)\right] > 0.
\end{aligned}$$

Thus, by Lemma 2.4, the intersection

$$\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h} (p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\}$$

is infinite. This implies that there exist $m_0 \geq 2\overline{k}$ and $n_0 \geq 2\overline{l}$ such that (3) and

$$p_i(m, n) \geq 0 \ (i = 1, 2, \cdots, h), \ u_{m,n} > 0$$

hold for $(m, n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$. By similar discussions as in the proof of Theorem 3.1, we may arrive at a contradiction against (3).

In case $u = \{u_{m,n}\}$ is eventually negative, then $\mu^*(u \geq 0) = 0$. In an analogous manner, we may see that

$$\left\{\Omega \backslash X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \geq 0)\right]\right\}$$

is infinite. This can lead to a contradiction again. The proof is complete.

4 Unsaturated Solutions

The methods used in the above proofs can be modified to obtain the following results for unsaturated solutions.

Theorem 4.1 *Suppose there exist constants ω_i $(i = 1, 2, \cdots, h), \omega$ and $\omega_0 \in (0, 1)$ such that*

$$\mu^*(p_i < 0) = \omega_i \ (i = 1, 2, \cdots, h), \ \mu_* \left(\prod_{i=1}^{h} (p_i < 0)\right) = \omega,$$

$$\mu_* \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right) > 4(\bar{k} + 1)(\bar{l} + 1) \left(\sum_{i=1}^{h} \omega_i + \omega_0 - (h-1)\,\omega\right).$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Proof Let $u = \{u_{m,n}\}$ be a nontrivial solution of (1). We assert that $\mu^*(u > 0) \in (\omega_0, 1)$. Suppose not, then $\mu^*(u > 0) \leq \omega_0$ or $\mu^*(u > 0) = 1$. In the former case, applying arguments similar to the proof of Theorem 3.1, we may then arrive at the fact that

$$\left\{\Omega \backslash X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + (u > 0)\right]\right\} \cdot \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right)$$

is infinite and a subsequent contradiction. In the latter case, we have $\mu_*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}
1 \;&=\; \mu^* \left\{\Omega \backslash X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + (u \leq 0)\right]\right\} \\
&\quad + \mu_* \left\{X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + (u \leq 0)\right]\right\} \\
&\leq\; \mu^* \left\{\Omega \backslash X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + (u \leq 0)\right]\right\} \\
&+\; 4\left(\bar{k} + 1\right)(\bar{l} + 1)\mu_* \left[\sum_{i=1}^{h} (p_i < 0) + (u \leq 0)\right] \\
&\leq\; \mu^* \left\{\Omega \backslash X_{-1}^{2\bar{k}} Y_{-1}^{2\bar{l}} \left[\sum_{i=1}^{h} (p_i < 0) + (u \leq 0)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
&+\quad 4\left(\overline{k}+1\right)\left(\overline{l}+1\right)\left\{\mu^*\left[\sum_{i=1}^{h}(p_i<0)\right]+\mu_*\left(u\le 0\right)\right\}\\
&\le\quad \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}\\
&+\quad 4\left(\overline{k}+1\right)\left(\overline{l}+1\right)\left(\sum_{i=1}^{h}\omega_i+\omega_0-(h-1)\,\omega\right)\\
&<\quad \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}+\mu_*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right).
\end{aligned}$$

Therefore by Lemma 2.4, we know that the set

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}\cdot\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right)$$

is infinite. Then by discussions similar to those in the proof of Theorem 3.1 again, we may arrive at a contradiction. This completes the proof. Combining Theorem 3.2 and 4.1, we have the following theorem the proof of which is omitted.

Theorem 4.2 *Suppose there exist constants* ω_i $(i=1,2,\cdots,h),\omega$ *and* $\omega_0\in(0,1)$ *such that*

$$\mu^*(p_i<0)=\omega_i\ (i=1,2,\cdots,h),\ \mu^*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)=\omega,$$

$$\mu_*\left(\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{j=1}^{h}p_j^{a_j}\le 1\right)\right)>\frac{\sum\limits_{i=1}^{h}\omega_i+\omega+\omega_0}{h}-\frac{1}{4h(\overline{k}+1)(\overline{l}+1)}.$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Theorem 4.3 *Suppose there exist constants* ω_i $(i=1,2,\cdots,h),\omega',\omega''$ *and* $\omega_0\in(0,1)$ *such that*

$$\mu^*(p_i<0)=\omega_i\ (i=1,2,\cdots,h),\ \mu^*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)=\omega',$$

$$\mu_*\left(\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{j=1}^{h}p_j^{a_j}\le 1\right)\right)=\omega'',\ 4(\overline{k}+1)(\overline{l}+1)\left(\sum_{i=1}^{h}\omega_i+\omega'+\omega_0-h\omega''\right)<1.$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Proof We claim that $\mu^*(u>0)\in(\omega_0,1)$. First, we prove that $\mu^*(u>0)>\omega_0$.

Otherwise, if $\mu^*(u>0)\le\omega_0$, by Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}
&\mu_*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}+\mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}\\
=\;& 2-\mu_*\left\{X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}-\mu_*\left\{X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}\\
\ge\;& 2-4(\overline{k}+1)(\overline{k}+1)\left\{\sum_{i=1}^{h}\mu^*(p_i<0)+\mu_*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)+\mu_*(u>0)\right.\\
&\left.-h\mu_*\left[\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}>1.
\end{aligned}$$

Hence, by Lemma 2.4, we see that

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}\cdot\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}$$

is infinite. Then there exist $m_0\ge 2\overline{k}$ and $n_0\ge 2\overline{l}$ such that (3) and

$$p_i(m,n)\ge 0\ (1,2,\cdots,h),\ u_{m,n}\le 0$$

hold for $(m,n)\in Z[m_0-2\overline{k},m_0+1]\times Z[n_0-2\overline{l},n_0+1]$. Applying similar discussions as in the proof of Theorem 3.1, we can get a contradiction. Next, we prove that $\mu^*(u>0)<1$. Otherwise, $\mu_*(u\le 0)=0$. Analogously, we see that

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}\cdot\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u\le 0)\right]\right\}$$

is infinite. Then, we can also come to a contradiction. The proof is complete. We remark that very nontrivial solution of (1) has unsaturated lower positive part under the same conditions as in Theorem 4.1, Theorem 4.2 or Theorem 4.3.

5 Examples

We give two examples to illustrate our previous results.

Example 5.1 Consider the partial difference equation

$$\begin{aligned}
&u_{m+1,n+1}+u_{m+1,n}+u_{m,n+1}-u_{m,n}+p_1(m,n)|u_{m-4,n-3}|^{\frac14}\mathrm{sgn}u_{m-4,n-3}\\
&+p_2(m,n)|u_{m-3,n-2}|^{\frac12}\mathrm{sgn}u_{m-3,n-2}+p_3(m,n)|u_{m-1,n-1}|^{\frac32}\mathrm{sgn}u_{m-1,n-1}=0, \qquad (5)
\end{aligned}$$

where $p_1(m,n)=2^{\frac14(n-1)}+2^{\frac14(5n-3)}+2^{\frac14(3n+7)}$, $p_2(m,n)=p_3(m,n)=1$. Obviously, $\alpha_1=1/4,\alpha_2=1/2,\alpha_3=3/2$. Let $a_1=1/5,a_2=1/4,a_3=11/20$. It is easy to see that $\sum_{i=1}^{3}a_i\alpha_i=1,\ \gamma=20/11$. It is clear that

$$\mu_*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}>1\right)=1,\quad \mu_*\left(\prod_{i=1}^{3}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\le 1\right)\right)=0,$$

$$\mu^*(p_1<0)=\mu^*(p_2<0)=\mu^*(p_3<0)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\right)=\mu^*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\le 1\right)=0.$$

Therefore, by Theorem 3.1 or 3.2, every nontrivial solution of (5) is frequently oscillatory. Furthermore, let $\omega_0\in(0,1/80)$, we see that all conditions in Theorem 4.1, 4.2 or 4.3 are satisfied. Thus, every nontrivial solution of (5) has unsaturated upper positive part. Indeed, $u=\{(-1)^m 2^n\}$ is such a solution with $\mu^*(u>0)=1/2$.

Example 5.2 Consider the partial difference equation

$$\begin{aligned}&u_{m+1,n+1}+u_{m+1,n}+u_{m,n+1}-u_{m,n}+p_1(m,n)|u_{m-3,n-3}|^{\frac{1}{3}}\mathrm{sgn}u_{m-3,n-3}\\&+p_2(m,n)|u_{m-3,n-2}|^{\frac{1}{2}}\mathrm{sgn}u_{m-3,n-2}+p_3(m,n)|u_{m-1,n-1}|^{2}\mathrm{sgn}u_{m-1,n-1}=0,\end{aligned}\tag{6}$$

where

$$p_1(m,n)=p_3(m,n)=1,\quad p_2(m,n)=\begin{cases}-1, & m=10s \text{ and } n=13t,\ s,t\in Z[0,\infty),\\ 1, & \text{otherwise.}\end{cases}$$

Choose $a_1=3/10$, $a_2=1/3$, $a_3=11/30$. It is easy to see that $\sum_{i=1}^{3}a_i=1$, $\sum_{i=1}^{3}a_i\alpha_i=1$ and $\gamma=30/11$. Clearly,

$$\mu^*(p_1<0)=\mu^*(p_3<0)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\right)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\le 1\right)\right)=0,$$

$$\mu^*(p_2<0)=\ \mu^*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\le 1\right)=\frac{1}{130},\quad \mu_*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}>1\right)=\frac{129}{130}.$$

Then by Theorem 3.1 or 3.2, every nontrivial solution of (6) is frequently oscillatory. Furthermore, when given $\omega_0=1/4161$, applying Theorem 4.1, 4.2 and 4.3, we may see that every nontrivial solution of (6) has unsaturated upper positive part.

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References

[1] Zhang, B.G. and Xing, Q.J. Oscillation of certain partial difference equations. *J. Math. Anal. Appl.* **329** (1) (2007) 567–580.

[2] C. J. Tian, S. L. Xie and S. S. Cheng. Measures for oscillatory sequences. *Comput. Math. Appl.* **36** (1998) 149–161.

[3] Cheng, S.S. *Partial Difference Equations.* London, New York, Taylor and Francis, 2003.

[4] Zhu, Z.Q. and Cheng, S.S. Frequent oscillation in a neutral difference equation. *Southeast Asian Bull. Math.* **29** (3) (2005) 627–634.

[5] Zhu, Z.Q. and Cheng, S.S. Frequently oscillatory solutions for multi-level partial difference equations. *International Math. Forum* **1** (31) (2006) 1497–1509.

[6] Zhu, Z.Q. and Cheng, S.S. Unsaturated solutions for partial difference equations with forcing terms. *Central European J. Math.,* to appear.

[7] Beckenbach, Edwin F. and Bellman, R. *Inequalities.* Berlin, Springer–Verlag, 1961.