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PERSONAGE IN SCIENCE

Academician N.N. Bogoliubov

(*to the 100th Birthday Anniversary*)

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This paper is dedicated to the memory of N.N. Bogoliubov in recognition of the significance of his efforts in the development of nonlinear mechanics and theoretical physics, his remarkable and versatile genius, as well as the novelty and depth of his contribution to the world science.

N.N. BOGOLIUBOV was born on August 21, 1909 in the city of Nizhny Novgorod. He grew up as a prodigy. At the age of only 13, Bogoliubov participated in a seminar led by Academician N.M. Krylov, a widely recognized scientist and teacher. In 1924, at the age of 15, Bogoliubov wrote his first scholarly work, "On the behavior of solutions to linear differential equations at infinity". Between 1925 and 1951, Bogoliubov was an employee in the mathematical physics division of the Ukrainian Academy of Sciences. During a period of his collaboration with Academician N.M. Krylov, Bogoliubov conducted fundamental research in the area of boundary value problems, approximation theory of differential equations, dynamical systems, and direct methods of variational calculus.

During those years, Bogoliubov created a new direction in the theory of uniform, almost periodic functions. Thereby he established a close link to the general behavior of linear combinations of arbitrary bounded functions. In 1930, one of Bogoliubov's first works was awarded the A. Merlani Prize by the Bologna's Academy of Sciences. In the same year, at the age of 21, with no formal thesis defense, Bogoliubov earned an honorable Habilitation degree in Mathematics from the Presidium of Ukrainian Academy of Sciences.

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The progress in science and technology created new avenues in telecommunication and electrical engineering, as well as mechanics of complex oscillating systems and aerospace. So, from 1932, Bogoliubov, together with his supervisor and mentor, N.M. Krylov, began developing an entirely new branch of mathematical physics — the theory of nonlinear oscillations, which they called "nonlinear mechanics". Their first work in this new direction was dealt with the theory of oscillations in power machinery and mechanical systems. The research in nonlinear mechanics was carried out in two directions: developing new asymptotic integration techniques of nonlinear equations of motion in oscillatory systems and laying a foundation for these methods based on measure theory.

Having overcome copious difficulties, Krylov and Bogoliubov extended the tools of perturbation theory to more general nonconservative systems and they created new and well established asymptotic methods of nonlinear mechanics. Unlike the popular Van der Pol's method, the corresponding solutions could be obtained not only in the first but in higher approximations as well. These methods became very useful in the studies of both periodic and quasiperiodic oscillatory processes. Moreover, they met practical needs in terms of simplicity and transparency of associated numerical algorithms.

Krylov and Bogoliubov quickly applied their asymptotic methods to many open and crucial problems. Among them, they obtained second approximation formulas for the frequency of stationary vibrations in electrical generators, which could estimate the overtone effect on stability of frequency. Furthermore, the results had an impact on the studies of resonances of frequency scaling and internal resonances in the systems with many degrees of freedom. A primal attention in resonance theory was paid to applications of nonlinear elements for controlling resonance in mechanical engineering. The asymptotic methods were employed to solve problems on aircraft longitudinal stability, vibrations and stability of rods, diverse frame structures and other engineering constructions.

The general measure theory in nonlinear mechanics developed by Bogoliubov and Krylov was a driving force in further development of the theory of dynamical systems. It also explained such properties of stationary motions as recurrence, i.e. strong stability in the sense of Poisson. Applying Lyapunov–Poincare and Poincare–Denjoy theory of trajectories on torus they studied the nature of the exact stationary solution near a proximate solution for a small parameter value and established existence and stability theorems for quasiperiodic solutions.

No wonder that the results of this research have become classical in modern theory of dynamical systems. Furthermore, the development of efficient methods of asymptotic integration for a wide class of nonlinear equations was due to Bogoliubov and Krylov's fundamental results. Bogoliubov also created new mathematical tools to study the behavior of general nonconservative systems with small parameter, which explained the nature of the stationary solution near a proximate solution.

Krylov's and Bogoliubov's studies on resonances in nonlinear oscillations are of especial importance, as so are their studies of the related phenomena of synchronization, demultiplication and diminishing of oscillation amplitudes in resonance under the presence of nonlinear elements in oscillatory system. Another sound result was their prediction of a possibility to observe a new phenomenon, called by them an anomalous perturbation, later on confirmed in practice. This phenomenon states that an equilibrium point, stable in the traditional sense, looses its stability under the effect of small sinusoidal perturbing forces.

The averaging method formulated and developed by Bogoliubov in the context of standard form equations contained in its essence a solution to the following two problems.

Firstly, it is the establishment of conditions under which the norm of the difference between the solutions of an exact and the associated averaged systems, for sufficiently small parameter values, remains arbitrarily large on a finite interval. Secondly, it is the establishment of a relationship between various properties of the solutions to exact and averaged equations on a finite interval.

As early as in 1945, Bogoliubov proved a fundamental existence theorem on main properties of a single-parametric integral manifold of a system of nonlinear differential equations in the standard form. He investigated periodic and quasi-periodic solutions on a one-dimensional manifold. This laid a foundation for a new method of nonlinear mechanics — the "method of integral manifolds".

In 1950, Bogoliubov developed a perturbation method in nonlinear mechanics, which he applied to a pendulum problem with a vibrating pivot point. In this problem, Bogoliubov was the first to prove that any unstable upper position of the pendulum can be made stable by means of a high vibration frequency of the pivot point. This breakthrough laid foundation to the theory of stability raise of elastic systems by vibrations.

Bogoliubov also obtained key results for systems of differential equations with a rapidly switching phase. Here the construction of solutions to averaged equations was rendered by separation of slow and rapid motions.

In 1963, Bogoliubov presented a new idea on application of accelerated convergence techniques to nonlinear mechanics. As early as in 1934, Bogoliubov, jointly with Krylov, developed a mixture of various changes of variables. Its application to the method of integral manifolds solved an existence problem of multi-frequent conditionally periodic solutions of nonlinear oscillatory systems, not only in the asymptotic, but also in a strict sense. When establishing this theory, Bogoliubov combined the method of integral manifolds with the iteration method. The latter has been proposed and used by A.N. Kolmogorov and V.I. Arnold for Hamiltonian systems by that time. This combined method gave rise to yet another method of accelerated convergence in nonlinear mechanics, which allowed Bogoliubov efficiently exclude the effect of small denominators that occur when using the change of variables.

Bogoliubov's ideas and methods expressed by him during his lectures in the international workshop in nonlinear mechanics (that took place in 1963 in Kanev) gave rise to their further development and applications to many vital problems in nonlinear mechanics. Among them, the problems of reducibility of a nonlinear system to a linear one with constant coefficients, reducibility of linear differential equations with quasi-periodic coefficients, as well as the problem of the behavior of integral curves in a vicinity of analytic and smooth manifolds.

The creative ideas and fundamental results of Bogoliubov in nonlinear mechanics laid the foundation to the global research in such areas as general mechanics, continuum mechanics, celestial mechanics, mechanics of rigid bodies and gyroscopic systems, motion stability, control theory, regulation and stabilization, mechanics of space flights, oscillations of mechanical systems, aero- and rocket construction, mathematical ecology, as well as other branches of science and engineering.

The word "nonlinear mechanics" has entered the scientific lexicon and is being widely used by mechanical and electrical engineers who are involved in the construction of systems with small perturbations. It is also being used by mathematicians who deal with differential equations containing small additive perturbation terms. The use of "nonlinear mechanics" has been further extended to include other mathematical disciplines such as nonlinear analysis and nonlinear dynamics.

The methods developed by Bogoliubov for the investigation of dynamical systems opened new avenues to problems of classical statistical physics. As early as in 1945, when studying the impact of a random force on a harmonic oscillator, Bogoliubov developed and applied for the first time an idea about the time hierarchy in the statistical theory of irreversible processes.

Chaining of recursive equations method proposed by Bogoliubov in 1946 for the distribution function of complexes of one, two, and more particles proved to be most efficient in modern statistical mechanics of processes in their equilibrium or transient state.

Discarding Ludwig Boltzmann's hypothesis on molecular chaos and having suggested a new idea of using boundary conditions that reduce correlation, Bogoliubov arrived at a method which allowed to include higher order terms of expansion in density powers. The same methods Bogoliubov utilized in statistical quantum mechanics in 1946–1948. Here he proposed a version of a secondary quantization with numerous forthcoming applications. He also developed a generalized method of self-conformed field presently referred to as the Hartry–Fock–Bogoliubov's method.

Bogoliubov's name is inseparable with the appearance of modern theory of nonideal quantum macrosystems. His scientific dissertation on such significant physical phenomena as superfluidity (1946) and superconductivity (1957) was a core element of this theory.

Bogoliubov constructed an appropriate mathematical apparatus based on a special canonic transformation of birth-death operators now widely known as Bogoliubov's (u, v)-transformation. This transformation is extensively applied in theoretical physics, in particular, in recent works on quantum theory of gravitation and the theory of nonideal Bose-condensate in magneto-optical traps.

Further development of superconductivity theory as superfluidity of the Fermi systems led Bogoliubov to the discovery of a new principal phenomenon — the superfluidity of nuclear substance. The notion of superfluidity of nuclear substance formed the basis for the modern theory of nucleus.

When studying a stabilization problem of condensate in nonideal systems, Bogoliubov developed the method of quasi-mean (1961). The latter turned out to be a universal tool for the investigation of systems whose main state is unstable under small perturbations.

In the fiftieth, Bogoliubov formulated quantum field theory with new causality condition. This condition is widely known today as the Bogoliubov's microcausality condition. The axiomatic theory of perturbations and in quantum field theory created by Bogoliubov was based on the disperse matrix and it reshaped the development of this theory to the present day. Bogoliubov proved a theorem stating that the disperse matrix in all orders of perturbation theory is well defined by the conditions of relativistic invariance, spectrality, unitarity and causality up to quasi-local operators. It yielded a source of ultra-violet divergences of the disperse matrix and gave rise to the method of their successive elimination. This method was called the R-operation (1955).

Bogoliubov was among the first scientists who had dealt with axiomatic quantum field theory (neither making any assumption on the weakness of interactions nor using perturbation theory). Furthermore, he only partially modified the existing system of axioms in perturbation theory by augmenting it with the stability condition on vacuum and single-particle state and reformulating the causality condition (1956). Bogoliubov used this to establish expressions for pion-nucleolus dispersion. The latter in turn required the development of mathematical tool of analytical continuation for generalized functions of many complex variables. His famous "edge of the wedge" theorem was formulated and proved in 1956 among his purely mathematical results and today this theorem is named after him. His work in quantum field theory gave rise to the new direction — the theory of strong interactions.

The scope of Bogoliubov's scientific accomplishments is not limited by the areas cited herein. It also includes a fundamental research on the theory of plasma and on kinetic equations which are of great practical significance.

Among his most profound creations was the idea of spontaneous violation of symmetry, initially proposed in the framework of statistical mechanics when developing superfluidity theory in 1948. It was a core part of the theory of weak electrical interaction, various versions of the theory of Great Unification. It also laid foundation to the most significant research on elementary particles, the theory of nucleus, and theory of phase transitions in early Universe models.

For many years Bogoliubov has carried out an enormous work on training young scientists. Being the department head at Kiev and then Moscow University he systematically gave lectures which were received with great interest. He also presented talks in England, Belgium, Bulgaria, Hungary, Italy, India, Poland, the USA, Finland, Germany, Yugoslavia, Japan and many other countries. Each one of them was a major scientific event.

Another Bogoliubov's distinct achievement was his creation of several scientific schools. During his employment in the Ukraine he established a school of mathematical physics and nonlinear mechanics in Kiev and then — the schools of elementary particles (theoretical physics) in Moscow and Dubna. Many well-known scientists proudly and respectfully regard Bogoliubov as their teacher.

Academician Bogoliubov was a dedicated organizer of science in the former Soviet Union. He was a member of the Presidium and the Academician-Secretary of mathematics division of the USSR Academy of Sciences exerting a beneficial influence in promoting the development of research in mathematics and physics in the Ukraine. Over 25 years did he head the largest International Scientific Center — the Joint Institute of Nuclear Research in Dubna, and was the founder and the first director of the Institute of Theoretical Physics of the Ukrainian Academy of Sciences. He was the founding chief editor of the two world renowned journals: "Theoretical and Mathematical Physics" and "Physics of Elementary Particles and Atom Nucleus".

Despite his deep involvement in teaching and an immense scholarly work, Bogoliubov was a dedicated volunteer in public work and political life. He was a representative of the Supreme Soviet Parliament and a member of the Pagoush piece movement of scientists. The Government of the former USSR rightfully recognized Bogoliubov's scientific and public achievements and awarded him with two Gold Star Medals and two titles of the prestigious Social Labor Hero and five Lenin's Medals of Honor. He was also highly decorated with many other prestigious awards and medals.

Bogoliubov was an Honorary Member of several foreign Academies and various scientific societies. He earned an Honorary Doctorate of numerous foreign Universities and he was a Laureate of prestigious awards and medals.

Bogoliubov is regarded a triune personality in science. He was a great mathematician, physicist and a pioneer in mechanics. He mastered fine problems of mathematical modeling and became a ground breaker of new horizons in modern mathematical physics. Finally, he was well-acquainted with both theoretical and practical needs of mechanics and technology.

Monographs and Books by N.N. Bogoliubov in the area of nonlinear mechanics

- Investigation of longitudinal stability of an aircraft.— Moskow–Leningrad: Gosaviaavtotraktizdat, 1932.— 60 p. (with N.M. Krylov).
- On the oscillations of synchronous machines. 2. On stability of parallel work of n-synchronous machines.— Kharkiv–Kyiv: Energovydav, 1932.— 98 p. (with N.M. Krylov).
- Fundamental problems of nonlinear mechanics.— Moskow–Leningrad: GTTI, 1932.— 23 p. (Wth N.M. Krylov).
- Méthodes nouvelles pour la solution de quelques problèmes mathématiques se rencontrant dans la science des constructions.— Kiev, 1932.— 96 p. (with N.M. Krylov).
- New methods for solution of some mathematical problems found in engineering.— Kharkov–Kiev: Budvydav, 1933.— 96 p.
- New methods of nonlinear mechanics and their application to the investigation of the work of electronic generators. Part 1.— Moskow–Leningrad: GTTI, 1934.— 243 p. (with N.M. Krylov).
- Application of the methods of nonlinear mechanics to the theory of stationary oscillations.— Kiev: Izd-vo VUAN, 1934.— 112 p. / In-t stroit. mekh. VUAN. Dept. Mat. Phys., No 8 (with N.M. Krylov).
- On some formal expansions of nonlinear mechanics.— Kyiv: Vyd-vo VYAN, 1934.— 89 p. (with N.M. Krylov).
- L'application des méthodes de la mèchanique nonlinéarire à la théorie des perturbations des systémes canoniques. — Kiev: Acad. Sci. d' Ukraine, 1934.— 57 p.— / Acad. Sci. d' Ukraine, Inst. de mécanique des constructions. Chaire de phys. mat. No 4 (with N.M. Krylov).
- Introduction to nonlinear mechanics.— Kiev: Izd-vo AN USSR, 1937.— 365 p. (with N.M. Krylov).
- General theory of measure in nonlinear mechanics.— In: Collection of papers on nonlinear mechanics.— Kyiv: Vyd-vo AN URSR, 1937.— pp. 55–112.
- Introduction to Non-Linear Mechanics by N. Kryloff and N. Bogoluboff. A free Translation by Solomon Lefschets of Excerpts from two Russian Monographs.— London: Princeton Univ. Press, 1943.— 105 p. (with N.M. Krylov).
- On some statistical mehods in mathematical physics.— Kiev: Izd-vo AN USSR, 1945.— 139 p.
- Introduction to Non-Linear Mechanics.— Repr. Princeton: Princeton Univ. press, 1947.— 106 p. (with N.M. Krylov).
- Asymptotic methods in the theory of nonlinear oscillations.— Moskow: Gostekhizdat, 1955.— 448 p. (with Yu.A. Mitropolsky).
- Asymptotic methods in the theory of nonlinear oscillations.— 2nd ed., corr. and compl.— Moskow: Fizmatgiz, 1958.— 408 p. (with Yu.A. Mitropolsky).
- Method of integral manifolds in nonlinear mechanics.— Kiev, 1961.— 126 p. (with Yu.A. Mitropolsky).
- Asymptotic Methods in the Theory of Nonlinear Oscillations.— Dehli: Hindustan Publ. Corp., 1961. (with Yu.A. Mitropolsky).
- Les méthodes asymptotiques en théorie des oscillations non linéaires.— Paris: Gauthier–Villars, 1962.— VIII, 518 p. (Co-auteur Yu.A. Mitropolski).

- On Certain Statistical Methods in Mathematical Physics / Studies in Statistical Mechanics (Eds. I. de Boer, G.R. Uhlenbeck).— Amsterdam.— 1962, vol. 3.
- Asymptotic Methods in the Theory of Nonlinear Oscillations — 3rd ed., corr. and compl.— Moskow: Fizmatgiz, 1963.— 410 p. (with Yu.A. Mitropolsky).
- Method of integral manifolds in nonlinear mechanics.— In: Proceedings of the International Symposiun on Nonlinear Oscillations.— Kiev, 1961, Vol. 1.— Kiev: Izd-vo AN USSR, 1963.— pp. 93–154. (with Yu.A. Mitropolsky).
- On quasiperiodic solutions in problems of nonlinear mechanics.— In: The first summer mathematicaln school.— Part 1.— Kiev: Naukova Dumka, 1964.— pp. 11–101.
- Asymptotische Methoden in der Theorie der Nichtlinearen Schwingungen.— Berlin: Academie–Verlag, 1965.— XII, 453 s. (with Yu.A. Mitropolsky).
- The method of accelerated convergence in nonlinear mechanics.— Kiev: Naukova Dumka, 1969.— 247 p. (with Yu.A. Mitropolsky and A.M. Samoylenko).
- Selected papers: In 3 vol. (Ed. Yu.A. Mitropolsky). Vol. 1.— Kiev: Naukova Dumka, 1969.— 647 p.
- Methods of Accelerated Convergence in Nonlinear Mechanics (Ed. I. N. Sneddon). Transl. from Russian by V. Kumar.— Berlin etc.: Springer–Verlag, 1976.— VIII.— 291 p. (with Yu.A. Mitropolsky and A.M. Samoylenko).
- Selected works. Part I. Dynamical theory (Eds. N.N. Bogoliubov (Jr.), A.M. Kurbatov). Transl. from the Russian by A. Ermilov. Classics of Soviet Mathematics, 2. New York: Gordon and Breach Science Publishers, 1990.— x+386 pp.
- Selected works: In 4 parts, transl. from russian (Eds. N.N. Bogoliubov (Jr.), A.M. Kurbatov). New York: Gordon and Breach Science Publl, 1990–1995.
- Nonlinear mechanics and pure mathematics (Ed. V.S. Vladimirov). New York: Gordon and Breach Science Publ., 1995.— VIII, 551 p.
- Introduction to nonlinear mechanics: Approximate and asymptotic methods of nonlinear mechanics.— Moskva–Izhevsk: RHD, 2004.— 352 p. (with N.M. Krylov).
- Nonlinear mechanics, 1932–1940 (Eds. Yu.A. Mitropolsky and A.D. Sukhanov).— Moskow: Nauka, 2005.— 828 p. (with N.M. Krylov).
- Asymptotic methods in the theory of nonlinear oscillations (Eds. Yu.A. Mitropolsky and A.D. Sukhanov).— Moskow: Nauka, 2005.— 605 p. (with Yu.A. Mitropolsky).
- Nonlinear mechanics, 1945–1974 (Eds. Yu.A. Mitropolsky and A.D. Sukhanov).— Moskow: Nauka, 2006.— 432 p.

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Novel Qualitative Methods of Nonlinear Mechanics and their Application to the Analysis of Multifrequency Oscillations, Stability, and Control Problems $^\diamond$

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Abstract: The method of oriented manifolds is developed to study geometric properties of the sets of trajectories of nonlinear differential systems with control. This method is conceptually connected with the classical methods of Lyapunov, Poincaré, and Levi–Civita and is a natural extension and development of results of the Donetsk school of mechanics. In terms of the method of oriented manifolds, sufficient conditions for stabilizability of nonlinear control systems are established.

A new method for stability investigation of nonlinear differential systems of perturbed motions is created on the basis of the concept of matrix-valued Lyapunov functions. This method is generalized for the systems with impulse action and aftereffect, differential equations with explosive right-hand sides and hybrid systems.

New conditions of practical stability of motion for nonlinear systems with impulse action are established on the basis of two auxiliary Lyapunov functions and the condition of exponential stability for linear impulse systems in a Hilbert space.

$^\diamond$ Series of works honoured with the State Prize of Ukraine in the Field of Science and Technology in 2008.

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General theory of the Fredholm boundary-value problems is constructed for systems of functional-differential equations, a classification of resonance boundary-value problems is elaborated, efficient coefficient criteria of existence of solutions are obtained and bifurcation and branching conditions for solutions to such problems are established.

New matrix methods are developed for the analysis of stability, localization of spectrum and representation of solutions of arbitrary order linear differential and difference systems. The methods of comparison and robust stability analysis are worked out for nonlinear dynamic systems in partially ordered space.

The averaging technique and the method of integral manifolds are developed for nonlinear resonance oscillating systems with slowly varying frequencies. New exact error estimations are established for the averaging technique in the initial and boundary-value problems for multifrequency systems and systems with impulse action.

New statements on stability and instability of linear approach to solutions of evolutionary equations in a Banach space are made. Absolute stability conditions are established for systems with aftereffect. In particular, a process of aircraft undercarriage galloping is studied at landing on the ground airfield with constant velocity. Also, stability conditions are established for the metal cutting process at turning behind a track with constant angular velocity of spindle rotation.

Keywords: *stability; robust stability; practical stability of motion; initial and boundary-value problems; differential and difference systems; systems with impulse action and aftereffect; comparison principle; Lyapunov functions; matrix equation; generalized Lyapunov equation; cone inequality.*

Mathematics Subject Classification (2000): 15A04, 15A18, 15A22, 15A24,15A42, 15A48, 34K06, 34K11,34K20, 37N15, 93A15, 93B05, 93B07, 93B18, 93B25, 93B55, 93C10, 93C35, 93D09, 93D15, 93D20, 93D30.

1 Introduction

This paper presents a survey of main results of a series of investigations competing for the State Prize of Ukraine in the Field of Science and Technology in 2008.

First, it should be noted that the fruitful ideas by Lyapunov have enabled his successors to develop constructive approaches for the analysis of dynamical behaviour of nonlinear systems.

Remarkable results of N.M. Krylov and N.N. Bogoliubov, which became a groundwork for a new direction in the field of mathematical physics, called "nonlinear mechanics", have become a source of many investigations of systems with small parameter, both of theoretical and practical importance.

The discovery of the principle of maximum in the mathematical theory of optimal control made by L.S. Pontryagin proved to be a profound synthesis of the theory of differential equations and the variational calculus whose development is associated with the name of outstanding mathematician of the 18-th century L. Euler.

A range of problems whose solutions are proposed in the monographs [1–12] and papers [13–43] was formed according to the needs of new fields of science and technology such as exploration of the near-Earth and outer space, automatic control of production processes by computers, mathematical biology, etc. A key role in the solution of these problems is played by the ideas and methods set out in the remarkable works by Lyapunov–Bogoliubov–Pontryagin.

Several hundred references in the publications [1–43] give an idea about the directions of the investigations mentioned in the title of this series of works and bring the reader to the boundary beyond which new areas for further investigations are opened up in these challenging scientific directions which constitute the basis for the technological advance in the beginning of the third millenium.

2 Qualitative Theory of Nonlinear Control Systems

In the papers [1, 2, 13, 14, 16], [19]–[24], qualitative properties of the family of trajectories of nonlinear systems of differential equations of the type

$$\dot{x} = f(x, u), \quad x \in D \subset \mathbb{R}^n, \ u \in U \subset \mathbb{R}^m, \tag{2.1}$$

are studied, where x is the state vector and u is the control. The function $f(x, u)$ is assumed to be continuously differentiable sufficient number of times in $D \times \overline{U}$. In papers by A.M. Kovalev [13, 14], the notion of a set oriented with respect to control system was introduced and the method of oriented manifolds was proposed. This method is conceptually connected with the method of Lyapunov functions and the Poincaré–Levi–Civita method of invariant relations.

Definition 2.1 A manifold $K \subset D$ is called oriented with respect to system (2.1) in the domain D if it coincides with its positive ($K = Or^+K$) or negative ($K = Or^-K$) orbit. Positive orbit Or^+K of the set K is a set of points attainable from the set K along the trajectories of system (2.1) and negative orbit Or^-K is a set of points from which the set K can be attained.

By means of the method of oriented manifolds, a general controllability criterion for nonlinear systems is proved.

Theorem 2.1 [13] *System (2.1) is controllable iff there are no manifolds K with smooth boundary oriented with respect to this system such that $K \neq \emptyset, D$.*

As compared with known results in the control theory, Theorem 2.1 does not assume infinite differentiability (or analiticity) of the vector fields of a control system. The equations of oriented manifolds obtained in [13] are of independent interest. Their relationship with the Levi–Civita equations of invariant manifolds and Lyapunov equations for functions ensuring motion instability is established. This relationship was used in the investigation of the problem on sufficient conditions for stabilizability of nonlinear controlled systems and the synthesis of a feedback law with respect to all and a part of variables [22]. To formulate the main result of the paper, we designate the ε-neighborhood of the point $x = 0$ by $B(0, \varepsilon)$.

Theorem 2.2 [22] *Let $0 \in \operatorname{int} D$, $0 \in U$, $f(0, 0) = 0$, U be a compact and, for some $\varepsilon > 0$, each point of the set $B(0, \varepsilon) \setminus \{0\}$ is a point of local controllability of system (2.1). Then there exists a feedback control $u : B(0, \varepsilon) \to U, u(0) = 0$ (generally speaking, discontinuous) which ensures non-asymptotic stability of the solution $x = 0$ of the closed-loop system*

$$\dot{x} = f(x, u(x)). \tag{2.2}$$

Besides, the solutions of system (2.2) are defined in the sense of A.F. Filippov.

Examples are constructed which demonstrate that this result can not be refined (i.e. it is final). For a control affine system, it is proved that the set of discontinuity points of the feedback is contained in some set whose dimensions are smaller than the dimension of the state space.

In order to generalize controllability conditions for the case of manifolds with smooth boundary, properties of attainability domains of linear systems in the presence of joint restrictions on the control and the initial state were studied in [2, Ch. 1]. A formula for the gage function of attainability set was obtained which simplifies the further analysis and allows one to construct the external and internal estimates of the attainability set. In monographs [1, 2], problems on motion control for a rigid body and systems of bodies were considered with the application of estimates of attainability sets. New estimates of attainability sets of a system of differential equations modelling the rotational motion of a rigid body under the action of a control torque were proposed. A problem in restricted statement and a case of translational and rotational motion were studied. In particular, in [13] equations of rigid body motion with respect to a center of masses under the action of jet force were considered without taking into account mass changes

$$\begin{aligned} A_1\dot{\omega}_1 &= (A_2 - A_3)\omega_2\omega_3 + e_1 u, \\ A_2\dot{\omega}_2 &= (A_3 - A_1)\omega_1\omega_3 + e_2 u, \\ A_3\dot{\omega}_3 &= (A_1 - A_2)\omega_1\omega_2 + e_3 u, \end{aligned} \tag{2.3}$$

where A_1, A_2, A_3 are the principal central moments of inertia of the body; ω_1, ω_2, ω_3 are the projections of the angular velocity vector ω on the main central axes; $e = (e_1, e_2, e_3)$ is a unit vector of direction of the jet force moment; u is a control characterizing the magnitude of the jet moment. It is established that system (2.3) is uncontrollable under any of the conditions

$$A_1(A_2 - A_3)e_3^2 = A_3(A_1 - A_2)e_1^2, \tag{2.4}$$

$$A_2(A_3 - A_1)e_1^2 = A_1(A_2 - A_3)e_2^2, \tag{2.5}$$

$$A_3(A_1 - A_2)e_2^2 = A_2(A_3 - A_1)e_3^2. \tag{2.6}$$

In paper [13], it is shown that if the parameters of system (2.3) do not satisfy conditions (2.4)–(2.6) then system (2.3) is controllable according to Theorem 2.1. As compared with the previous papers, the application of the method of oriented manifolds enabled a unified description of controllability conditions for system (2.3) to be obtained in all cases of dynamically symmetric and asymmetric rigid body.

The evolution of geometric methods of nonlinear control theory led to the necessity of constructive description of the class of flat-systems, i.e. the systems which admit exact linearization by means of an endogenous feedback. The theory of flat-systems, appeared in the works by M. Fliess, J. Lévine, P. Rouchon, Ph. Martin, is being developed in the papers [2, 14, 16]. In these works, the method of invariant relations is applied for solving inverse control problems, observation, identification, convertibility, and functional controllability problems. General theorem on identifiability of nonlinear systems was proved. It states the identifiability of any system with respect to the measurements of its phase vector under a condition of its nonrepresentability by means of a smaller number of parameters. Conditions of observability and identifiability of nonlinear systems with respect to a part of variables are established [2, Ch. 5]. For general type systems, a functional controllability criterion is proposed, a property of invertibility is studied, a notion of inverse system is introduced, and an algorithm of its construction is presented

[14]. A generalized flat-algorithm proposed in [16] allows a considerable extension of the class of nonlinear control systems which admit explicit solution of motion planning problems. The concept of a generalized flat-system on the trajectory set is applied to study the problems of observation and identification of phase coordinates and parameters of motion equation of a rigid body in the force field. Observability conditions are used to substantiate the choice of output functions which are measured at probe navigation. In this direction, a class of problems on the determination of the mass center motion and rigid body orientation is solved [2, Ch. 6]. The results obtained in the field of identification of nonlinear systems are used to investigate problems of determining the moments of inertia and aerodynamical characteristics of a rigid body by the available information about motion.

A method of transforming the dynamical system with impulse control to the system with jumps realized on some surfaces in a phase space is proposed in [19], and new notions of impulses of high degrees and orders are introduced which are necessary for the investigation of systems nonlinear with respect to control. By employing impulse effects, a series of control and stabilization problems are solved and numerical methods are justified which can be used for an approximate construction of solutions to impulse systems. The results are applied for the problems on controlled stabilization of mechanical systems. In particular, a solution for the problem on stabilization of the Brockett integrator is obtained. An algorithm is proposed for constructing control system for nonholonomic models with independent quasivelocities as a control.

The notion of a control Lyapunov function with respect to a part of variables is introduced in [20]. These functions are employed in the proof of the theorem on partial stabilizability of nonlinear nonautonomous system. For control affine systems, an efficient method of constructing a stabilizing feedback is proposed. This result extends a theorem of Z. Artstein for the case of partial stabilization. The apparatus of control Lyapunov functions allowed one to solve a series of model problems on partial stabilization of a rigid body orientation. In particular, a model problem is considered for the motion of a satellite as an absolutely rigid body around its center of mass in the restricted statement under the action of jet control moments. Also, the case is studied when the control moments are implemented by means of a pair of flywheels [21]. In [23], control and stabilization algorithms are developed for motion of a satellite with elastic antennas and rods. The proposed control technique incorporates the mathematical model of a hybrid mechanical system in the form of differential Euler–Lagrange equations with infinite number of degrees of freedom. For a preassigned arbitrary number of elastic modes, an approximated finite-dimensional nonlinear system is constructed for which a stabilizing control with feedback is found. The above-mentioned control ensures asymptotic stability of the equilibrium state with respect to the combinations of elastic coordinates and body orientation. Besides, stability in the sense of Lyapunov is reached with respect to all phase coordinates. Observability of a model of hybrid system is proved with respect to the measurements of sensors of elastic element deformations. This allows one to substantiate the possibility of technical implementation of the proposed control laws.

A new approach is proposed in [24] for the investigation of stabilizability conditions for nonlinear controlled system by means of critical Hamiltonians. New stabilizability conditions are obtained for nonlinear affine control system defined by two homogeneous vector fields.

3 Rigid Body Dynamics and Motion Stability of Mechanical Systems

In [18] the author stated and solved a problem on the inclusion of given invariant manifold into the family of integral manifolds for a system of ordinary differential equations of the type

$$\dot{x} = \varphi(x), \quad x \in D \subset \mathbb{R}^n. \tag{3.1}$$

Assume that for every $x_0 \in D$ system (3.1) has a unique solution $x(t; x_0)$, $t \geq 0$ satisfying the initial condition $x(0; x_0) = x_0$. We shall introduce necessary definition.

Definition 3.1 A manifold $M \subset D$ is called invariant for system (3.1) if $x(t; x_0) \in M$ for all $t \geq 0$ and $x_0 \in M$. If $F_i(x)$, $i = 1, 2, ..., k$ are independent integrals of system (3.1), the set $N = \{x : F_i(x) = c_i,\ i = 1, 2, ..., k\}$ is called the integral manifold of system (3.1).

Now we shall formulate the main result on the inclusion of an invariant manifold into the family of integral manifolds.

Theorem 3.1 [18] *Any integral manifold M of dimension $n - k$ in a neighborhood of a nonsingular point of system (3.1) is contained in some k-parametric set of integral manifolds.*

It is proved that such an inclusion is locally possible only if the invariant manifold under consideration is not a $(n - 1)$-dimensional manifold consisting of singular points.

By means of the Levi–Civita equations of integral manifolds assertions describing the structure of the including family are proved. The results obtained are applied in the investigation of motion equations of the Hess gyroscope in special coordinate axes [18]

$$\begin{aligned}
&\dot{x} = -b_1 z x, \\
&\dot{y} = (a - a_*)zx + b_1 yz - \nu_3 \Gamma, \\
&\dot{z} = -(a - a_*)yx + b_1(x^2 + y^2) + \nu_2 \Gamma, \\
&\dot{\nu}_1 = a_* z \nu_2 - (a_* y + b_1 x)\nu_3, \\
&\dot{\nu}_2 = (ax + b_1 y)\nu_3 - a_* z \nu_1, \\
&\dot{\nu}_3 = (a_* y + b_1 x)\nu_1 - (ax + b_1 y)\nu_2,
\end{aligned} \tag{3.2}$$

where x, y, z are components of the kinetic moment vector in special coordinate axes; ν_1, ν_2, ν_3 are coordinates of the unit vector colinear to the direction of force field; a, a_*, b_1 are components of gyration tensor; the constant Γ characterizes intensity of the force field (action of gravity force). The following three integrals of the Euler-Poisson system of differential equations (3.2) are known

$$ax^2 + a_*(y^2 + z^2) + 2b_1 yx - 2\nu_1 \Gamma = 2h;$$

$$x\nu_1 + y\nu_2 + z\nu_3 = k;$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Besides, system (3.2) possesses the invariant Hess manifold $x = 0$.

Theorem 3.2 [18] *System of differential equations (3.2) possesses an additional integral of the form $I = xV$, where V is a solution of the differential equation $L_\varphi V = b_1 z V$. Partial cases of this integral are the Euler and Lagrange integrals and the Hess and Dokshevich solutions.*

In the above theorem L_φ means the operator of function differentiation along the trajectories of system (3.2). In the paper cited first approximation of the integral I is also obtained in the neighborhood of the uniform rotations curve belonging to the invariant Hess manifold.

In [17] the author developed the results by V.V. Kozlov and V.N. Koshlyakov on the application of the Rodrigues–Hamilton parameters in the motion investigation of a rigid body possessing a fixed point. By introducing in a special way a fixed system of coordinates a new form was obtained for motion equations of a rigid body which have a symmetric form and are quadratic in main variables. By means of these equations linear and nonlinear vibrations of a rigid body are studied in the Rodrigues–Hamilton parameters. To study stability of stationary motions of the Hamiltonian systems reducible to the two-dimensional ones a theorem generalizing the known Arnold–Moser result on stability of the equilibrium state of two-dimensional Hamiltonian system was proved. Application of this theorem to stability investigation of uniform rotations of a heavy rigid body with a fixed point allowed closing with this classical problem which has attracted the attention of investigators since the beginning of the 20-th century [15].

4 Stability, Control, and Stabilization of Infinite-Dimensional Systems

To study the motion of distributed parameter mechanical systems, the property of asymptotic stability with respect to a continuous functional is analyzed in [27] for generalized dynamical systems on a metric space. In particular, dynamical systems whose evolution is described by differential equations in some Banach space E are considered. Let X be a closed subset of E containing a sphere $B_R = \{x \in E \mid \|x\| \le R\}$, $R > 0$, and let $F : D(F) \to E$ be a nonlinear closed operator with dense in X domain of definition $D(F)$. For initial conditions $x_0 \in X$, we consider the abstract Cauchy problem

$$\frac{dx(t)}{dt} = Fx(t), \quad t \in \mathbb{R}_+ = [0, +\infty),\ x(0) = x_0. \tag{4.1}$$

We assume that the operator F is the infinitesimal generator of a continuous semigroup of nonlinear operators $\{S(t)\}_{t\ge 0}$ in X, therefore the Cauchy problem (4.1) is well-posed and its mild solutions are written in the form $x(t) = S(t)x_0$.

Definition 4.1 Let $y : X \to \mathbb{R}_+$ be a continuous functional, $F(0) = 0$. The singular point $x = 0$ of differential equation (4.1) is called asymptotically stable with respect to y if

(i) for arbitrary given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $y(S(t)x_0) < \varepsilon$ for all $t \in \mathbb{R}_+$;

(ii) there exists $\Delta > 0$ such that $\|x_0\| < \Delta$ implies

$$\lim_{t\to\infty} y(S(t)x_0) = 0. \tag{4.2}$$

The above definition of partial stability is associated with the development of abstract approach to the definition of stability in two metrics. The absence of the condition of positive definiteness of the functional y enables one to consider Definition 4.1 as a generalization of the notion of asymptotic stability with respect to a part of variables in the sense of Lyapunov and Rumyantzev for the case of infinite-dimensional systems.

Let $V : E \to \mathbb{R}$ be a Fréchet differentiable functional. Then the time derivative of V along the trajectories of (4.1) can be written as

$$\dot{V}(x(t)) = (Fx(t), \nabla_{x(t)} V), \tag{4.3}$$

where $(\cdot, \cdot) : E \times E^* \to \mathbb{R}$ denotes the duality pairing of E and E^*, i.e. $(\xi, \nabla_x V)$ is the value of linear functional $\nabla_x V \in E^*$ at the point $\xi \in E$.

In order to formulate partial stability conditions, we use the class of Hahn functions $\mathcal{K}$ consisting of all continuous strictly increasing functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ possessing the property $\alpha(0) = 0$.

Theorem 4.1 [27] *Let F be the infinitesimal generator of a continuous semigroup $\{S(t)\}$ of nonlinear operators in X, $F(0) = 0$, and let $y : X \to \mathbb{R}_+$ be a continuous functional. We assume that there exists a Frechet differentiable functional $V : E \to \mathbb{R}$ satisfying the following conditions:*

1) For some functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}$, the inequality

$$\alpha_1(y(x)) \le V(x) \le \alpha_2(\|x\|), \forall x \in X.$$

is satisfied.

2) $\dot{V}(x) \le 0$ for all $x \in D(F)$.

3)There exists a $\Delta > 0$ such that, for any $\|x_0\| < \Delta$, the corresponding set

$$\bigcup_{t \ge 0} \{S(t)x_0\}$$

is precompact in X.

4) The set $\operatorname{Ker} y = \{x \in X \,|\, y(x) = 0\}$ is invariant for (4.1), i.e. if $y(S(\tau)x_0) = 0$, $\tau \ge 0$ then $y(S(t)x_0) = 0$ for all $t \in \mathbb{R}_+$.

5) The set

$$M = \overline{\{x \in D(F) \,|\, \dot{V}(x) = 0\}} \setminus \operatorname{Ker} y$$

does not contain any semitrajectory of system (4.1) defined for $t \in \mathbb{R}_+$.

Then the singular point $x = 0$ of differential equation (4.1) is asymptotically stable with respect to y.

This theorem generalizes results by C. Risito and V.V. Rumyantzev for the case of partial stability of infinite-dimensional system. Theorem 4.1 is used for the synthesis of control functionals for mathematical models of hybrid mechanical systems. Such mechanical systems consisting of rigid and elastic bodies are widely applied in space industry and robot technology. In[26, 28], the author considered models of rotational motion of a satellite with an arbitrary number of elastic elements, i.e. antennas in the form of the Euler–Bernoulli beams. If all the beams have the same mechanical parameters, the system under investigation is not asymptotically stable and, under these conditions, the stabilization problem with respect to the norm of some projection operator onto an infinite-dimensional subspace of the state space was solved in [26]. In the case of beams with nonresonant parameters, the approximate controllability was proved and a control functional was proposed which ensures strong asymptotic stability of the equilibrium state [28]. From the mechanical point of view, such a control implements the stabilization of the body-carrier orientation with simultaneous damping of beams vibrations. In

[25], equations of the spatial motion of an elastic robot-manipulator were studied with allowance for the telescopic displacement of its links under the effect of control forces. The Euler-Bernoulli and Timoshenko beams with mixed boundary conditions were considered as models of link deformations. A scheme of stabilization with the help of an observer in the feedback chain was proposed for the model equilibrium state. It is proved that this approach ensures asymptotic stability of the unperturbed solution of the system for an arbitrary number of generalized coordinates corresponding to the elastic beam vibrations.

5 The Method of Matrix-Valued Lyapunov Functions and the Analysis of Dynamic Properties of Nonlinear Systems

Stability analysis of zero solution of nonlinear system in the normal form

$$\frac{dx}{dt} = X(t, x), \quad x(t_0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^n$, $X \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $X(t, 0) = 0$ for all $t \geq t_0$, is a challenging task if the dimension of the vector x is large enough. One of the approaches to solution of this problem is the decomposition of system (5.1) to the form

$$\frac{dx_i}{dt} = f_i(t, x_i) + g_i(t, x_1, ..., x_m), \quad i = 1, 2, ..., m, \tag{5.2}$$

where $x_i \in \mathbb{R}^{n_i}$, $f_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m} \to \mathbb{R}^{n_i}$, $\sum\limits_{i=1}^{m} n_i = n$.

The monographs [3, 4] and Chapter 5 of the monograph [5] presented the results of development of the direct Lyapunov method in terms of auxiliary matrix-valued function

$$V(t, x) = [v_{ij}(t, x)], \quad i, j = 1, 2, ..., m, \tag{5.3}$$

which is considered to be a suitable medium for construction of both scalar and vector Lyapunov functions solving the problem on stability of the state $x_i = 0$ of system (5.2).

It is proposed to take the elements $v_{ij}(t, x)$ of matrix-valued function (5.3) such that to satisfy the estimates

$$\underline{\gamma}_{ij}\psi_{ij}(\|x_i\|)\psi_{ji}(\|x_j\|) \leq v_{ij}(t, x) \leq \overline{\gamma}_{ij}\psi_{ij}(\|x_i\|)\psi_{ji}(\|x_j\|),$$

where $\underline{\gamma}_{ii}$, $\underline{\gamma}_{ij} > 0$, $\overline{\gamma}_{ij}$, $\overline{\gamma}_{ij}$ are constants for $i \neq j$, $(\psi_{ij}, \psi_{ji}) \in K(KR)$-Hahn class for all $i, j = 1, 2, ..., m$. If conditions (5.4) are satisfied, then for the function

$$V(t, x, y) = y^T U(t, x) y, \quad y \in \mathbb{R}^m_+, \tag{5.5}$$

the bilateral estimate

$$\psi_1^T(\|x\|) Y^T \underline{G} Y \psi_1(\|x\|) \leq V(t, x, y) \leq \psi_2^T(\|x\|) Y^T \overline{G} Y \psi_2(\|x\|), \tag{5.6}$$

is valid, where

$\psi_1(\|x\|) = (\psi_{11}(\|x_1\|), \dots, \psi_{1m}(\|x_m\|))^T$, $\psi_2(\|x\|) = (\psi_{21}(\|x_1\|), \dots, \psi_{2m}(\|x_m\|))^T$,
$Y = \mathrm{diag}(y_1, \dots, y_m)$, $\underline{G} = [\underline{\gamma}_{ij}]$, $\overline{G} = [\gamma_{ij}]$, $i, j = 1, 2, \dots, m$.

For function (5.5) the total derivative

$$D^+V(t,x,y) = y^T D^+U(t,x)y, \tag{5.7}$$

is considered, where $D^+U(t,x) = [D^+v_{ij}(t,x)]$, $i,j = 1,2,...,m$, and $D^+v_{ij}(t,x) = \limsup\{[v_{ij}(t+\theta, x+\theta(f_i(t,x_i)+g_i(t,x_1,...,x_m))]\theta^{-1} : \theta \to 0^+\}$.

For certain restrictions on function (5.5) and its total derivative (5.7) by virtue of system (5.2) sufficient conditions are established for various types of stability of zero solution to system (5.2)((5.1) respectively).

Theorem 5.1 *Let the vector-function X in system (5.1) be continuous on $\mathbb{R} \times N$ ($N \subset \mathbb{R}^n$) and admit decomposition of system (5.1) to the form (5.2).*

If for function (5.5) estimates (5.6) are valid and

$$D^+V(t,x,y) \le \psi_3^T(\|x\|)A_3(y)\psi(\|x\|), \tag{5.8}$$

for all $(t,x) \in \mathbb{R}_+ \times N$, where $A_3(y)$ is an $m \times n$-constant matrix then:

(1) the state $x = 0$ of system (5.1) is stable if the matrices $A_1 = Y^T \underline{G} Y$ u $A_2 = Y^T \overline{G} Y$ are positive definite and the matrix $A_3(y)$ is negative definite;

(2) the state $x = 0$ of system (5.1) is uniformly stable if the matrices A_1, A_2 are positive and the matrix A_3 is negative semidefinite.

Similarly to Theorem 5.1 the results on asymptotic stability, exponential stability and instability of the state $x = 0$ of system (5.1) are formulated and proved.

For polystability analysis of the state $x = 0$ of system (5.2) it is proposed to apply the vector function

$$L(t,x,b) = AU(t,x)b, \tag{5.9}$$

where A is a constant $m \times m$-matrix, $b \in \mathbb{R}^m_+$, $U \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^{m\times m})$, and its total derivative

$$D^+L(t,x,b) = AD^+U(t,x)b \tag{5.10}$$

by virtue of system (5.2). A detailed polystability analysis for system (5.2) was carried out in the cases $m = 2,3,4$, and sufficient conditions were established for various types of polystability of the state $x = 0$ of system (5.2).

Solution of the problem of constructing a suitable matrix-valued function (5.3) is considered in the following cases:

Case 1. The elements $v_i(t,x_i)$, $i = 1,2,...,m$ are put in correspondence with the independent subsystems

$$\frac{dx_i}{dt} = f_i(t,x_i), \quad i = 1,2,...,m, \tag{5.11}$$

of system (5.2) and the elements $v_{ij}(t,x_i,x_j)$, $i \ne j$, $i,j = 1,2,...,m$ are put in correspondence with the (i,j)-pairs of the independent subsystems

$$\frac{dx_i}{dt} = q_i(t,x_i,x_j),$$
$$\frac{dx_j}{dt} = q_j(t,x_i,x_j), \quad (i \ne j) \in [1,m],$$

where $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, $q_i \in (\mathbb{R}_+ \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_j}, \mathbb{R}^{n_j})$.

Case 2. Subsystems (5.11) are decomposed into M_i second level subsystems

$$\frac{dx_{ij}}{dt} = f_{ij}(t, x_{ij}) + h_{ij}(t, x_i), \quad j = 1, 2, ..., m_i \tag{5.12}$$

where $x_{ij} \in \mathbb{R}^{n_{ij}}$, $f_{ij} \in C(\mathbb{R} \times \mathbb{R}^{n_{ij}}, \mathbb{R}^{n_{ij}})$, $h_{ij} \in C(\mathbb{R} \times \mathbb{R}^{n_i}, \mathbb{R}^{n_{ij}})$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n_i$. The elements $v_{ii}(t, x_i)$ are put in correspondence with the free subsystems of the second level of decomposition

$$\frac{dx_{ij}}{dt} = f_{ij}(t, x_{ij}), \quad j = 1, 2, ..., m, \tag{5.13}$$

and the elements $v_{ij}(t, x)$, $(i \neq j) \in [1, m]$, are constructed with allowance for the interconnection functions $h_{ij}(t, x_i)$ in system (5.12).

Case 3. For the class of systems of (5.2) type [30]

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, ..., x_m),$$

$f_i \in \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $f_i(0) = 0$, $i = 1, 2, ..., m$, the elements $v_{ii}(x_i)$ are put in correspondence with the independent subsystems

$$\frac{dx_i}{dt} = f_i(x_i), \quad i = 1, 2, ..., m, \tag{5.14}$$

and the elements $v_{ij}(t, x_i, x_j)$ are found by the equations

$$\begin{aligned} &D_t v_{ij}(t, x_i, x_j) + (D_{x_i} v_{ij}(t, x_i, x_j))^T f_i(x_i) + (D_{x_j} v_{ij}(t, x_i, x_j))^T f_j(x_j) + \\ &+ \frac{y_i}{2y_j}(D_{x_i} v_{ii}(x_i))^T g_{ij}(t, x_i, x_j) + \frac{y_j}{2y_i}(D_{x_j} v_{jj}(x_j))^T g_{ji}(t, x_i, x_j) = 0, \quad (i \neq j) \in [1, m], \end{aligned}$$

where $g_{ij}(t, x_i, x_j) = g_i(t, 0, ..., x_i, ..., x_j, ..., 0)$, $i \neq j$, $i, j = 1, 2, ..., m$.

In all the above cases new conditions are established for various types of stability of the state $x = 0$ of system (5.1), without assuming on exponential stability of the state $x = 0$ of subsystems (5.11), (5.13) or (5.14). As is known this condition is necessary for the application of the vector Lyapunov function and appropriate comparison system.

Also, the method of matrix-valued Lyapunov functions was developed for:

— time discrete systems in terms of semidefinite positive functions (5.3), whose elements are linear forms, and hierarchical matrix Lyapunov functions;

— large-scale impulse systems of the form

$$\begin{aligned} \frac{dx_j}{dt} &= f_j(t, x_j) + f_j^*(t, x), \quad t \neq \tau_k(x), \quad j = 1, 2, ..., m, \\ \Delta x_j &= I_{kj}(x_j) + I_{kj}^*(x), \quad t = \tau_k(x), \quad k = 1, 2, ... \end{aligned}$$

in terms of auxiliary functions satisfying conditions (5.4), and also in terms of hierarchical matrix Lyapunov functions whose method of construction is indicated;

— systems with random parameters in the Ito form and Katz–Krasovsky form in terms of stochastic matrix-valued function;

— singularly perturbed systems of the form

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, y, \mu), \\ \mu\frac{dy}{dt} &= g(t, x, y, \mu), \end{aligned} \tag{5.15}$$

where $(x^T, y^T)^T$ is a state vector of system (5.15), $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times M, \mathbb{R}^n)$, $g \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \times M, \mathbb{R}^m)$, $\mu \in [0,1] = M$, in terms of the matrix-valued function

$$V(t,x,y,\mu) = \begin{pmatrix} v_{11}(t,x) & v_{12}(t,x,y,\mu) \\ v_{21}(t,x,y,\mu) & v_{22}(t,y,\mu) \end{pmatrix}.$$

Stability conditions of the state $x = y = 0$ of system (5.15), and large-scale system of Lurie–Postnikov type are obtained in terms of sign-definiteness of special matrices. Moreover, the upper bound μ^* of the values of parameter μ is calculated for which an appropriate type of stability of slow variables and boundary layer takes place.

The developed technique is illustrated by numerous examples and applications to the problems of mechanics, electric power industry, population biology, etc.

6 Generalization of the Direct Lyapunov Method and Comparison Method for Non-classical Stability Theories

The classical stability theory developed by A.M. Lyapunov is based on three fundamental concepts:

(1) deviations of perturbed motion from the nominal one should be infinitely small;
(2) in the course of motion perturbing forces are absent;
(3) motion is considered on unbounded interval.

We refer all other stability theories which are based on other concepts to the non-classical ones. One of such theories is the theory of practical stability based on the following concepts:

(1) initial and further deviations of perturbing motion from the nominal one are final;
(2) system motion is performed under persistent perturbations;
(3) interval of system functioning is unbounded.

In the monograph [6] general theory of practical stability of motion is presented with the applications in mechanics. The system of perturbed motion equations

$$\frac{dx}{dt} = X(t,x) + R(t,x), \tag{6.1}$$

is considered, where $x \in \mathbb{R}^n$; $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; $R : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and it is not assumed that $R(t,0) \neq 0$, i.e. $x = 0$ is not a solution of system (6.1), but it is a solution of the system

$$\frac{dx}{dt} = X(t,x). \tag{6.2}$$

For given estimates of the domains ($S_0(t)$, $S(t)$, $\Pi(t)$, $\mathbb{R}_+$) unperturbed motion of system (6.2) is practically stable under persistent perturbations if for $t_0 \in \mathbb{R}_+$ and any

$$x(t_0) \in S_0(t_0), \quad R(t,x) \in \Pi(t),$$

the solution $x(t, t_0, x_0)$ of system (6.1) remains inside the domain $S(t)$, i.e. $x(t) \in \mathrm{int}S(t)$ for all $t \geq t_0$.

Practical stability of unperturbed motion of system (6.2) is determined as a motion property opposite to practical stability.

To solve the problem on practical stability of systems of (6.1), (6,2) type or their partial form three approaches were developed in the monograph [6]:

Approach 1 is based on the representation of general solution to system (6.2) as series of special form.

Approach 2 is based on the application of the direct Lyapunov method and locally large auxiliary function.

Approach 3 is based on the reduction of system (6.1) or (6.2) to the other one called a comparison system with further analysis of its solutions. Here both scalar and vector Lyapunov functions are applied as a nonlinear transformation of the initial system.

In the framework of Approach 1 practical stability conditions are established for system (6.2) with uniformly bounded and uniformly analytic right-hand side and for a system with integrable approximation of the form

$$\frac{dx}{dt} = A(t)x + g(t,x), \tag{6.3}$$

where $x \in \mathbb{R}^n$; $A(t)$ is an $n \times n$-continuous and bounded matrix, $g(t,x)$ satisfies the estimate

$$\|g(t,x)\| \le b(t)\|x\|^{\alpha}, \quad \alpha > 0, \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times S(t).$$

These conditions are based on representation of solution to system (6.2) by series of the form

$$x(t) = x_0 + \sum_{m=1}^{\infty} C_m(x_0)\psi^m,$$

where $\psi = \{\exp[\lambda(t-t_0)] - 1\}\{\exp[\lambda(t-t_0)] + 1\}^{-1}$, λ is a positive number, with further application of the Schur theorem on convergence of series (6.4). Practical stability conditions for the state $x = 0$ of system (6.3) are based on the estimates associated with nonlinear integral inequality.

The results obtained are employed for the analysis of dynamics of large scale systems with integrable approximation.

In the framework of Approach 2 the direct Lyapunov method is applied with necessary modifications. For locally large function $V(t,x)$ the quantitative estimates

$$V_M^{\widehat{S}}(t) = \sup(V(t,x) \quad \text{for} \quad x \in \partial S(t)),$$
$$V_m^{\widehat{S}_0}(t) = \inf(V(t,x) \quad \text{for} \quad x \in \partial S_0(t)),$$

are introduced, where $S_0(t) \subset S(t)$ and $\partial S_1 \cap \partial S_0 = \emptyset$ for all $t \in \mathbb{R}_+$.

Theorem 6.1 *Assume that*

(1) $V(t,x) \in C(\mathbb{R}_+ \times S(t), \mathbb{R}_+)$, $V(t,x)$ is locally large and locally Lipschitz in x;

(2) $D^+V(t,x) < D^+\eta(t)$ for $(t,x) \in \mathbb{R}_+ \times S(t)$, where $\eta \in C(\mathbb{R}_+, (0,\infty))$ and $\eta(t)$ is nondecreasing in $t \in \mathbb{R}_+$;

(3) for some $t_0 \in \mathbb{R}_+$ the estimate $\eta(t_0) \le V_M^{S_0}(t_0)$ is valid and $\eta(t) \le V_m^{\partial S}(t)$ for all $t \ge t_0$.

Then the unperturbed motion of system (6.2) is practically stable with respect to the domains $(S_0(t), S(t))$.

Similar theorems are proved for various types of practical stability and instability of solutions for systems (6.1) and (6.2) with respect to different domains $S_0(t)$, $S(t)$.

In the realization of Approach 3 scalar (vector) comparison equations are incorporated which satisfy quasimonotonicity condition. Practical stability conditions are expressed in the form of quantitative restrictions on variation of solutions to comparison equation.

Moreover, alongside systems (6.1) and (6.2) the systems with first integrals are considered. General concept of practical stability is formulated in terms of extended system (6.1) and stability with respect to a part of variables.

With regard to practical stability the problems of stabilization of controlled systems are solved for some classes of linear and nonlinear systems on the basis of the principle of comparison with mixed monotonicity of comparison system. For the axiomatically determined system of processes conditions of practical stability are established with respect to two vector measures whose components may take negative values.

In [31, 32] practical stability of some classes of hybrid systems consisting of time-continuous and discrete components is studied. In [32] a nonlinear system of differential equations of perturbed motion with impulsive effect

$$\begin{aligned} &\frac{dx}{dt} = f(t,x), \quad t \neq \tau_k, \\ &\Delta x = I_k(x), \quad t = \tau_k, \end{aligned} \tag{6.4}$$

is considered, where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$ for all $t \in \mathbb{R}_+$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_k(0) = 0$, $k = 1, 2, \dots$, $0 < \tau_1 < \tau_2 < \dots < \tau_k < \dots \to +\infty$ for $k \to \infty$. It is assumed that the solution $x(t) = x(t; t_0, x_0)$ of the Cauchy problem (1) exists and is unique and the length of the maximal interval $[t_0, t_0 + J(t_0, x_0))$ of existence of solution to the Cauchy problem for system of equations (6.4) when the impulse effect is absent satisfies the inequality $J(t_0, x_0) > \theta_2$ for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = [0, +\infty)$.

In the space $\mathbb{R}^n$ let the sets $S_0 = \{x| \quad x \in \mathbb{R}^n, \quad \|x\| < \lambda\}$; $S = \{x| \quad x \in \mathbb{R}^n, \quad \|x\| < A\}$ be defined for given constants A, $\lambda > 0$, $\lambda < A$.

Let $G \subset \mathbb{R}_+ \times \mathbb{R}^n$ and for any $t \in \mathbb{R}_+$ we define the set $G(t) = \{x \in \mathbb{R}^n \quad | (t,x) \in G\}$ and the set $\mathcal{G} = \bigcup\limits_{i=1}^{\infty} G(\tau_i)$.

Practical stability is studied by means of the Lyapunov function for which the following assumptions are made:

a) function $v(t,x)$ is continuous and differentiable in $(t,x) \in [t_0, \infty) \times S$;

b) function $v(t,x)$ is locally large in the domain of values $(t,x) \in [t_0, \infty) \times S$, i.e. there exists a positive constant N such that for any c, $0 < c < N$, $t_0 \in \mathbb{R}_+$ there exists a positive number $\delta(t_0, c)$ such that outside the sphere $K_\delta = \{x : \|x\| \leq \delta\}$ the inequality $v(t,x) > c$ is satisfied for all $t \in [t_0, \infty)$;

c) total derivative $\frac{dv}{dt}\Big|_{(6.4)}$ of function $v(t,x)$ along solutions of system (6.4)

$$\frac{dv}{dt}\Big|_{(6.4)} = \frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)^T f(t,x)$$

vanishes together with the function $v(t,x)$ for $x = 0$;

d) function $v(t,x)$ is positive definite in the domain $\mathbb{R}_+ \times S$ in the sense of Lyapunov;

e) $a(\|x\|) \leq v(t,x) \leq b(t, \|x\|)$, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, где $a(.)$ is a function of Khan class $\mathcal{K}$, $b(t,.)$ is a function continuous and nondecreasing in the second argument.

Theorem 6.2 *Let system of equations (6.4) be such that:*

1) there exists a function $v(t,x)$ for which conditions (a)–(e) are satisfied;

2) there exist an invariant set G^+ and functions $\varphi_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\psi_1(.)$ is a nondecreasing function such that the *estimates below are satisfied*

$$\frac{dv}{dt}\Big|_{(6.4)} \le p_1(t)\varphi_1(v) \quad \textit{for all} \quad (t,x) \in \overline{G}^+,$$

$$v(\tau_k, x + I_k(x)) \le \psi_1(v(t,x)) \quad \textit{for all} \quad x \in \overline{\mathcal{G}}^+;$$

3) there exist functions $\varphi_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$*,*$\psi_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$*,* $p_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *such that the estimates below are satisfied*

$$\frac{dv}{dt}\Big|_{(6.4)} \le -p_2(t)\varphi_2(v) \quad \textit{for all} \quad (t,x) \in \overline{G}^-,$$

$$v(\tau_k, x + I(x)) \le \psi_2(v(t,x)) \quad \textit{for all} \quad x \in \overline{\mathcal{G}}^-,$$

where $G^- = \operatorname{ext} G^+$*;*

4) constants $\lambda, A > 0, \lambda < A < A^0$ *satisfy the estimates:*

a) for all $\eta \in [0, b(t_0, \lambda))$*,* $k = 0, 1, 2, ..., \tau_0 = t_0$*,*

$$\int\limits_{\eta}^{\psi_2(\eta)} \frac{ds}{\varphi_2(s)} \le \int\limits_{\tau_k}^{\tau_{k+1}} p_1(t)\,dt,$$

b) $\int\limits_{b(t_0,\lambda)}^{a(A)} \frac{ds}{\varphi_1(s)} \ge \int\limits_{\tau_k}^{\tau_{k+1}} p_2(t)\,dt$*;*

c) $\psi_1(a(A)) < b(t_0, \lambda)$ *.*

Then system (6.4) is $(S_0, S, [t_0, \infty))$*-stable.*

Theorem 6.2 generalizes the results of the paper [30] where conditions of Lyapunov stability were established in terms of two auxiliary functions. Conditions of Lyapunov stability for linear differential perturbed motion equations with impulse effect obtained in [34] and motion stability conditions for nonlinear system of perturbed motion equations of (6.4) type obtained in [30] enable one to investigate stability of the system in the case when continuous and discrete components of the system are not stable.

In [31] a hybrid system of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t,x) + B_k u(k), \quad t \in [\tau_k, \tau_{k+1}), \\ u(k+1) &= C_k u(k) + D_k x(\tau_k), \end{aligned} \tag{6.5}$$

is considered, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in C([0,\infty), \mathbb{R}^{n\times n})$, $B_k \in \mathbb{R}^{n\times m}, C_k \in \mathbb{R}^{m\times m}, D_k \in \mathbb{R}^{m\times n}$, $g \in C([0,\infty) \times \mathbb{R}^n; \mathbb{R}^n)$. Here $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of switching moments possessing a unique limiting point at infinity.

By means of the methods of the theory of integral inequalities practical stability conditions with respect to a part of variables and with respect to all variables of system (6.5) are established in terms of estimates of the Cauchy matrix of linear approximation of system (6.5).

In the monograph [5] stability conditions are obtained for systems with small parameters of the following types: systems standard by Bogoliubov, systems with slow and quick variables, systems with small perturbing forces. These conditions are based on the ideas of the direct Lyapunov method, the averaging technique and the method of comparison for auxiliary scalar functions.

7 Boundary-Value Problems in the Nonlinear Oscillations Theory

This part of the paper involves the theory of nonlinear oscillations which is one of the most important branches of nonlinear mechanics. For the first time in the field the most complete theory of the (Fredholm) boundary-value problems was constructed for the systems of differential equations with impulse effect in which the number of boundary conditions does not coincide with the number of unknowns. Most complicated and not well-studied resonance boundary-value problems, both underdetermined and overdetermined ones, are considered:

$$\begin{gathered}\dot{z} = A(t)z + f(t) + \varepsilon Z(z,t,\varepsilon), \quad t \neq \tau_i, \quad t, \tau_i \in [a,b], \\ \Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i + \varepsilon J_i(z(\tau_i - 0, \varepsilon), \varepsilon), \ i = 1, ..., p, \\ lz = \alpha + \varepsilon J(z(\cdot, \varepsilon), \varepsilon).\end{gathered} \tag{7.1}$$

Here $A(t)$ и $f(t) \in C([a,b] \setminus \{\tau_i\}_I)$ are $n \times n$-dimensional matrix functions and n-dimensional vector functions respectively; $Z(z,t,\varepsilon)$ is a nonlinear n-dimensional vector function continuously differentiable with respect to the first argument in the neighbourhood of solutions to generating boundary-value problem

$$\begin{gathered}\dot{z} = A(t)z + f(t) \quad t \neq \tau_i, \quad t, \tau_i \in [a,b], \\ \Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i, \ i = 1, ..., p, \quad lz = \alpha,\end{gathered} \tag{7.2}$$

$Z(z,t,\varepsilon)$ is continuous or piece-wise continuous in the second argument with first kind discontinuities for $t = \tau_i$ and continuous in $\varepsilon \in [0, \varepsilon_0]$; $\Delta z\Big|_{t=\tau_i} = z(\tau_i + 0) - z(\tau_i - 0)$, $S_i are (n \times n)$- constant matrices: $\det(E + S_i) \neq 0$, $a_i \in \mathbb{R}^n$; l is a linear continuous m-dimensional vector functional; $J(z(\cdot,\varepsilon),\varepsilon)$, $J_i(z(\tau_i - 0, \varepsilon), \varepsilon)$ are m-dimensional nonlinear vector functionals continuously differentiable (by Frechet) in z in the neighbourhood of solution of generating boundary-value problem (7.2)continuous in $\varepsilon \in [0, \varepsilon_0]$.

For the first time a problem was solved on establishing the existence (branching) conditions for solutions $z = z(t,\varepsilon) : z(\cdot,\varepsilon) \in C^1([a,b] \setminus \{\tau_i\}_I)$, $z(t,\cdot) \in C[0,\varepsilon_0]$ of the problems which, for $\varepsilon = 0$, become one of the solutions $z_0(t, c_r) : z(t, 0) = z_0(t, c_r)$, $c_r \in \mathbb{R}^r$ of generating boundary-value problem (7.2) and algorithms for their obtaining are proposed.

Theorem 7.1 (on branching of solutions) *Let boundary-value problem (7.1) be such that the critical (resonance) case (* $\operatorname{rank}[Q := lX(\cdot)] < m$*), takes place and generating problem (7.2) has r-parametric family of linearly independent solutions* $z_0(t, c_r)$, $(r = n - \operatorname{rank} Q)$. *Then for every value of the vector* $c_r = c_r^0 \in \mathbb{R}^r$, *which is a simple real root of the equation*

$$P_{Q^*}\Big\{J(z_0(\cdot,c_r^0),0) - l\int_a^b K(\cdot,\tau)Z(z_0(\tau,c_r^0),\tau,0)d\tau - l\sum_{i=1}^p \bar{K}(\cdot,\tau_i)J_i(z_0(\tau_i - 0, c_r^0),0)\Big\} = 0, \tag{7.3}$$

boundary-value problem (7.1) has at least one solution $x(t,\varepsilon) : x(\cdot,\varepsilon) \in C^1([a,b] \setminus \{\tau_i\}_I)$, $x(t,\cdot) \in C[0,\varepsilon]$ *which becomes generating with the vector constant* $c_r^0 : x(t,0) = z_0(t, c_r^0)$. *This solution can be found with the help of the iteration process convergent on* $[0, \varepsilon_*]$.

Here $X(t)$ is a normal fundamental matrix of homogeneous differential system (7.2), $K(t,\tau)$ is a Cauchy matrix, P_{Q^*} is an orthoprojector on the co-kernel of matrix Q.

In the case of periodic boundary-value problem (7.1) without impulses [7, 35] Theorem 7.1 yields the classical result of A. Lyapunov and I. Malkin. If equation (7.3) has a physical meaning then the constants c_r^0 are the amplitudes of generating solutions and, therefore, in the periodical case this equation is called the equation for generating amplitudes. In the case when generating boundary-value problem (7.2) has no solutions bifurcation conditions were established for solutions to linearly perturbed (Fredholm) boundary-value problem

$$\dot{z} = A(t)z + \varepsilon A_1(t)z + f(t), \quad t \neq \tau_i, \tag{7.4}$$

$$\Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i + \varepsilon A_{1i} z(\tau_i - 0), \quad lz = \alpha + \varepsilon l_1 z.$$

Theorem 7.2 (on bifurcation of solutions) *Let boundary-value problem (7.2) generating for (7.4) have no solutions for arbitrary functions $f(t) \in C([a,b]\setminus\{\tau_i\}_I)$, $a_i \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$. Then under the condition*

$$\operatorname{rank}\Big[B_0 := P_{Q^*}\Big[l_1 X_r(\cdot) - l\int_a^b K(\cdot,\tau)A_1(\tau)X_r(\tau)d\tau -$$

$$-l\sum_{i=1}^{p}\bar{K}(\cdot,\tau_i)A_{1i}X_r(\tau_i - 0)\Big] = m - \operatorname{rank} Q, \tag{7.5}$$

for arbitrary nonhomogeneities $f(t) \in C([a,b]\setminus\{\tau_i\}_I)$, $a_i \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$ boundary-value problem (7.4) has a parametric family $\rho = m - n$ of linear independent solutions in the form of a part of the Laurent series

$$z(t,\varepsilon) = \sum_{i=k}^{\infty} \varepsilon^i z_i(t) + P_\rho c_\rho, \ \forall c_\rho \in \mathbb{R}^\rho, \ k = -1, \tag{7.6}$$

which converges for fixed sufficiently small $\varepsilon \in (0, \varepsilon_*]$.

Similar results were obtained in the investigation of boundary-value problems for systems of ordinary differential equations with delaying argument [8, pp. 170–194], [38] and for difference systems [8, pp. 93–96], [36], as well as for systems with boundary conditions at infinity [8, pp. 257–304], [37] when the appropriate homogeneous differential system is exponentially dichotomous on semi-axes. These results complete and generalize essentially the known results of R.J. Sacker and K.J. Palmer.

8 Methods of Matrix Equations and Cone Comparisons in the Stability Theory

8.1 Analogues of Matrix Lyapunov Equation and Their Application ([9], [39])

The method of Lyapunov functions for linear differential and difference systems is formulated in terms of positive definite solutions to the matrix equations

$$-AX - XA^* = Y, \quad X - AXA^* = Y.$$

The known Lyapunov theorem provides criteria for placing a spectrum of such systems inside the left-hand half-plane and a unit disk. Matrix algebraic and differential Lyapunov equations are widely applied in the theory of qualitative systems and control theory.

The monograph [9] deals with the methods of constructing, investigating and applying in motion stability theory the analogues of the matrix Lyapunov equations and their generalizations of the form

$$\sum_{i,j} \gamma_{ij} A_i X A_j^* = Y, \tag{8.1}$$

where A_i is a set of matrices, in particular, $A_i = f_i(A)$ are the analytic functions of the matrix A.

The monograph presents criteria of localization and distribution of the matrix spectrum with respect to the sets

$$\Lambda_f^+ = \left\{\lambda : f(\lambda,\bar\lambda) > 0\right\}, \quad \Lambda_f^- = \left\{\lambda : f(\lambda,\bar\lambda) < 0\right\}, \quad \Lambda_f^0 = \left\{\lambda : f(\lambda,\bar\lambda) = 0\right\}.$$

The Lyapunov theorem and the inertia theorem of Ostrovsky–Schneider are generalized for the maximal possible classes of analytical Hermitian functions $f \in \mathcal{H}_0^m$ and $f \in \mathcal{H}_2^m$ determined by the corresponding conditions

$$\|1/f(\mu_i,\overline{\mu}_j)\|_{i,j=1}^m \geq 0, \ \forall\, \mu_1,\dots,\mu_m \in \Lambda_f^+; \quad i_\pm\left(\|f(\mu_i,\bar\mu_j)\|_1^m\right) \leq 1, \ \forall\, \mu_1,\dots,\mu_m \notin \Lambda_f^0;$$

where $i_\pm(\cdot)$ are the inertia indices of the Hermitian matrix that equal to the number of its positive and negative eigenvalues. If $f(\lambda,\bar\lambda) = \sum_{ij}\gamma_{ij} f_i(\lambda)\overline{f_j(\lambda)}$ then $f \in \mathcal{H}_0^m$ and $f \in \mathcal{H}_2^m$ under the corresponding restrictions $i_+(\Gamma) = 1$ and $i_\pm(\Gamma) \leq 1$.

Theorem 8.1 [9] *Let the matrix $A \in C^{n\times n}$, the function $f \in \mathcal{H}_0^m$ and the arbitrary positive definite matrix $Y = Y^* > 0$ be given. Then the spectrum $\sigma(A)$ is located in the domain Λ_f^+ if and only if there exists a unique positive definite solution $X = X^* > 0$ of the matrix equation*

$$L_f X \triangleq -\frac{1}{4\pi^2}\oint_{\omega_1}\oint_{\omega_2} f(\lambda,\bar\mu)(A-\lambda I)^{-1} X (A-\mu I)^{-1^*} d\lambda\, d\bar\mu = Y, \tag{8.2}$$

where ω_1 (ω_2) is a closed contour embracing and not intersecting $\sigma(A)$ ($\sigma(A)$).

All known results in the direction are the partial cases of Theorems 1–3 set out in the monograph. Also, correlations of the type of linear system controllability conditions are constructed which extend essentially the possibilities of the method of generalized Lyapunov equation in spectrum localization problems. Equation (8.2) which can be represented in the form of (8.1) was used for the first time in the problems of linear system optimization with respect to output [39].

$$\dot x = Ax+Bu,\ y = Cx,\ u = -Ky,\ J(u) = \int_\Delta \rho(x_0)\int_0^\infty (x^*Qx+u^*Ru)\,dt\,dx_0 \to \min_u. \tag{8.3}$$

In terms of generalized Lyapunov theorem and matrix Atans–Levine system a relationship of the quadratic quality functional and the domain of desirable location of closed loop system spectrum is established. Optimization algorithms are constructed controlling the system spectrum location in complex domain.

A general technique of constructing the analogues of Lyapunov equation is developed for polynomial and analytic matrix functions. Operators of such equations are presented in the form of the Cauchy type integrals of logarithmic derivative and also by means of special algebraic systems of spectrum splitting and the so-called right-hand and left-hand pairs of matrix functions. We shall formulate an analogue of Lyapunov theorem with the application of the left-hand eigen pairs of ($U \in C^{m\times m}$, $T \in C^{m\times n}$) of the matrix function $F(\lambda)$ of controllability index r determined by the conditions

$$TF(\lambda) \equiv (\lambda I - U)\Phi(\lambda), \quad \operatorname{rank} E = r, \quad E = [T, UT, ..., U^{m-1}T].$$

In this case $\sigma(U) = \sigma_r(F) \subseteq \sigma(F)$ and, besides, the spectra $\sigma(U)$ and $\sigma(F)$ coincide if rank $\begin{bmatrix} F(\lambda) \\ \Phi(\lambda) \end{bmatrix} = n$, $\forall\lambda \in \sigma(F)$. We introduce a set of matrices $\mathcal{K} = \{X : EXE^* \geq 0\}$.

Theorem 8.2 [9] *If the matrices $X \in \mathcal{K}$ and $X \in \mathcal{K}$ satisfy the correlations*

$$\sum_{i,j} \gamma_{ij} f_i(U) EXE^* f_j^*(U) = EYE^*, \tag{8.4}$$

$$S_\lambda = EYE^* + (\lambda I - U)EE^*(\lambda I - U)^* \geq 0, \quad \operatorname{rang} S_\lambda \equiv m,$$

then $\sigma_r(F) \subset \Lambda_f^+$, where $f(\lambda, \overline{\lambda}) = \sum_{i,j} \gamma_{ij} f_i(\lambda)\overline{f_j(\lambda)}$. Conversely, if $\sigma_r(F) \subset \Lambda_f^+$ and $f \in \mathcal{H}_0^m$, then for any matrix $Y \in \mathcal{K}$ equation (8.4) has the solution $X \in \mathcal{K}$.

The eigen pairs (U, T) of the matrix functions $F(\lambda)$ are also employed in the construction and investigation of solutions to dynamical systems of the type of $F(D)x = g$, where D is an operator of differentiation or displacement in time t.

For the linear descriptor systems $B\dot{x} = Ax$, $Bx_{k+1} = Ax_k$ and second order differential systems

$$Ax + B\dot{x} + C\ddot{x} = g, \tag{8.5}$$

modelling the dynamics of many objects of mechanics and physics new methods are developed for stability analysis, Lyapunov function construction and estimation of spectrum location with respect to algebraic curves. In particular, for the rotative system of the Lavale rotor type described as (8.5) with the matrix coefficients

$$A = K + iS, \quad B = D + iG, \quad C = M,$$

necessary and sufficient stability conditions are constructed in analytical form in terms of the corresponding mechanical parameters. Here $M = M^T > 0$ is a mass matrix, $D = D^T = D_0 + D_1 \geq 0$, $G = G^T = \omega G_0 \geq 0$ is a gyroscopic matrix, $K = K^T > 0$ is a rigidity matrix, $S = S^T \geq 0$ is a circulation matrix, D_0 and D_1 are constituents of the internal and external dampings, ω is the angular velocity of rotor rotation. The proposed technique refines the known estimate of the critical frequency of rotor rotation at which stability is lost. Also a regulator of the type of $g = Ru$, $u = K_0x + K_1\dot{x}$, is constructed which stabilizes closed loop system.

For the linear differential-difference systems

$$\dot{x} = Ax + \sum_i A_i x(t - \tau_i)$$

an analogue of the Lyapunov equation is constructed and in terms of its solutions absolute stability conditions are formulated (see [9], Chapters 2 and 3).

The theory of linear equations and operators in the matrix space is developed ([9], Chapter 4). Systems of matrix equation transformations are constructed allowing the description of their solvability conditions and inertia properties of the Hermitian solutions. A class of equations with special families of matrix coefficients is indicated and the Hill and Schneider theorems on inertia of their Hermitian solutions are generalized. A class of linear equations in the space with cone is studied [9], Appendix 2). The method of successive approximations is used to estimate solutions and their characteristics of the type of Hermitian matrix inertia. Structure of positive and positive invertible operators in the matrix space is studied ([9], Appendix 2).

8.2 Cone Inequalities in the Stability Theory([[40], [41], [42], [43])

For the modelling of physical objects differential and difference systems of equations are employed the phase space of which contains invariant sets, in particular, cones. The peculiarities of the systems such as positiveness and monotonicity should be taken into account in stability and control analysis problems. Examples of the positive systems with respect to a cone of symmetric negative definite matrices are the differential Lyapunov and Riccati equations and second moments equation for stochastic systems of Ito type. Positive and monotone systems appear, also due to the application of the comparison technique as a generalization of the Lyapunov functions methods in stability theory.

The main results of the paper [40] are positiveness conditions and algebraic criteria of asymptotic stability of linear systems in the Banach space $\mathcal{E}$ with normal generating cone $\mathcal{K}$

$$\dot{X} + M(t)X = 0, \quad t \geq t_0 \geq 0, \quad \mathcal{K} \subset \mathcal{E}. \tag{8.6}$$

These conditions are formulated in terms of positive and positive invertible operators.

Theorem 8.3 [40] *Positive stationary system (8.6) is exponentially stable iff the operator M is positive invertible. If the operator $M + \gamma E$ is positive invertible for any $\gamma \geq 0$ then system (8.6) is positive and exponentially stable.*

Stability investigation of linear positive reducible systems and nonstationary systems with functional commutative operators is reduced to solution of algebraic equations and cone comparison of their solutions: $MX = Y$, $X \overset{\mathcal{K}}{\geq} 0$, $Y \overset{\mathcal{K}}{>} 0$. A method of robust stability analysis is proposed as well as analogues of the known comparison systems in the space with cone.

Generalizations of the class of nonlinear monotone systems in partially ordered space are introduced:

$$\dot{X} = F(X, t), \quad t \geq t_0 \geq 0, \tag{8.7}$$

their characterization by means of linear positive functionals is presented and analogues of the Lyapunov theorem on stability of equilibrium state of such systems in first approximation are formulated. Comparison methods are developed for the solutions of differential systems with the use of constant and variable cones. As a corollary robust stability conditions are formulated for the families of systems of (8.7) type described by the cone inequalities [41, 42]

$$\underline{F}(X,t) \overset{\mathcal{K}}{\leq} F(X,t) \overset{\mathcal{K}}{\leq} \overline{F}(X,t), \quad \underline{F} \in \underline{\mathcal{F}}_1, \; \overline{F} \in \overline{\mathcal{F}}_1, \quad t \geq 0,$$

where $\underline{\mathcal{F}}_1, \overline{\mathcal{F}}_1$ are generalized classes of upper and lower systems of comparison with respect to the cone $\mathcal{K}$.

In [43] the methods for positiveness and stability investigation are developed for linear dynamic systems in partially ordered space. For stability analysis of positive systems special methods are worked out which are based on spectral properties of positive and positive invertible operators. Invariance conditions are found for the cones of circular type and their generalizations which allow, in particular, solution of the problem on positive stabilization of systems with respect to given cones by means of dynamical compensators. Invariance conditions for ellipsoidal cones and exponential stability conditions for linear differential and difference systems are formulated in terms of matrix inequalities. The notion of maximal eigen pairs of a matrix polynomial is used to establish algebraic conditions of exponential stability of linear arbitrary order differential systems.

9 Multifrequency Oscillations of Nonlinear Systems

Consider a multifrequency nonlinear system of ordinary differential equations with slow and quick variables of the form

$$\frac{dx}{d\tau} = a(x, \varphi, \tau, \varepsilon), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + b(x, \varphi, \tau, \varepsilon), \tag{9.1}$$

where x and φ are n- and m-dimensional vectors respectively, ε is a small positive parameter, $\tau = \varepsilon t$ is a "slow" time, real functions a, b, ω belong to some classes of smooth and almost periodic in φ functions. Systems of the type appear in the investigation of oscillatory processes in many problems of mechanics, electrical engineering, biology, etc.

We write an averaged in φ system

$$\frac{d\bar{x}}{d\tau} = \bar{a}(\bar{x}, \tau, \varepsilon), \quad \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \bar{b}(\bar{x}, \tau, \varepsilon), \tag{9.2}$$

where

$$\bar{a}(x, \tau, \varepsilon) = \lim_{k\to\infty} k^{-m} \int_0^k \dots \int_0^k a(x, \varphi, \tau, \varepsilon) d\varphi_1 \dots d\varphi_m,$$

and designate by $W_p(\tau)$ and $W_p^T(\tau)$ the $p \times m$-matrix

$$\left(\frac{d^{j-1}}{d\tau^{j-1}} \omega_\nu(\tau) \right)_{j,\nu=1}^{p,m}$$

and the transpose matrix respectively. Here $\omega = (\omega_1, ..., \omega_m)$.

Under the assumption that $\det \left(W_p^T(\tau) W_p(\tau) \right) > 0, \ \tau \in [0, L]$, we obtain an exact estimate with respect to the order in ε [10]

$$\|x(\tau, \varepsilon) - \bar{x}(\tau, \varepsilon)\| + \|\varphi(\tau, \varepsilon) - \bar{\varphi}(\tau, \varepsilon)\| \le c\varepsilon^{\frac{1}{p}}, \ \ \tau \in [0, L], \ \varepsilon > 0, \tag{9.3}$$

where (x, φ) and $(\bar{x}, \bar{\varphi})$ are solutions of systems (9.1) and (9.2), coinciding for $\tau = 0$. For the proof of inequality (9.3) uniform estimates of oscillation integrals are essentially used [10].

The averaging technique was applied for solution of boundary-value problems for system (9.1) with multipoint and integral boundary conditions. Moreover, in the case of integral boundary conditions the averaged problem is constructed via averaging of not only differential equations but boundary conditions as well [10].

If system (9.1) is given for $\tau \in R$ and

$$\left\| \left(W_p^T(\tau) W_p(\tau)\right)^{-1} W_p^T(\tau) \right\| \le c_1 = \text{const},\ \tau \in R,$$

then existence of the integral manifold $x = X(\varphi, \tau, \varepsilon)$ of system (9.1) is proved on which the equations of quick variables become

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + b(X(\varphi, \tau, \varepsilon), \varphi, \tau, \varepsilon).$$

Under the assumption that the functions a, b, ω have continuous bounded partial derivatives in all variables up to the order of $l \ge 2$ it is proved that the function X is $l - 1$ times differentiable and [10]

$$\left\| D_\varphi^s \frac{\partial^q}{\partial \tau^q} \frac{\partial^r}{\partial \varepsilon^r} X(\varphi, \tau, \varepsilon) \right\| \le c_2 \varepsilon^{\frac{1}{p} - q - 2r},\ 1 \le s + q + r \le l - 1,$$

and the derivatives of $(l-1)$-th order satisfy Lipschitz condition in $\varphi, \tau, \varepsilon$. Also, conditional asymptotic stability of integral manifold is studied and decomposition of slow and quick variables is accomplished in the neighbourhood of asymptotically stable integral manifold [10].

The averaging method for initial and boundary-value problems and the method of integral manifolds are justified as well in the case of systems of (9.1) type with impulse effect at fixed moments of time $\tau_j = \varepsilon t_j$, $t_{j+1} - t_j = \theta = \text{const} > 0$ and moreover,

$$\Delta x|_{\tau = \tau_j} = \varepsilon p(x, \varphi, \tau_j),\ \ \Delta \varphi|_{\tau = \tau_j} = \varepsilon q(x, \varphi, \tau_j).$$

It should be noted that in this case the average system is smooth and not subject to the impulse effect [10]

$$\frac{d\bar{x}}{d\tau} = \bar{a}(\bar{x}, \tau) + \frac{1}{\theta}\bar{p}(\bar{x}, \tau),\ \ \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \bar{b}(\bar{x}, \tau) + \frac{1}{\theta}\bar{q}(\bar{x}, \tau).$$

10 Absolute Stability, Stability and Instability by Linear Approximation and Essential Instability of Motion for Nonlinear Infinite Dimensional Systems

10.1 Absolute stability of systems with aftereffect.

In practice one have sometimes to study stability of dynamical systems at arbitrary parameter values. If the systems are stable at arbitrary values of the corresponding parameters these systems are called absolutely stable (with respect to these parameters). Mathematical models of a wide class of dynamical systems are differential delay equations and delays are the corresponding parameters.

In [11] spectral criteria of absolute stability (with respect to constant deviations of argument) are obtained for solutions of linear autonomous differential difference equations of delay and neutral type

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k),$$

$$B_0 \frac{dx(t)}{dt} + \sum_{k=1}^{m} B_k \frac{dx(t-\delta_k)}{dt} = C_0 x(t) + \sum_{k=1}^{m} C_k x(t-\tau_k)$$

in a Banach space the separate case of which is the known theorem of Yu.M. Repin. Here A_k, B_k, C_k, $k = \overline{0,m}$, are linear continuous operators, δ_k, τ_k, $k = \overline{0,m}$, are arbitrary positive or nonnegative constants. Also, classes of systems with arbitrary slowly changing operator coefficients and argument deviations are constructed whose solutions are strong absolutely asymptotically stable. Algebraic criteria of absolute asymptotic stability and instability are obtained for solutions to the scalar equation

$$\frac{d^n x(t)}{dt^n} + \sum_{k=0}^{n-1} a_k \frac{d^k x(t)}{dt^k} + \sum_{k=0}^{n} \sum_{j=1}^{m} b_{kj} \frac{d^k x(t-\tau_j)}{dt^k} = 0,$$

which strengthen the known result of L.A. Zhivotovskii. It is shown that absolute exponential stability of solutions to the equations under consideration is preserved as well for small nonlinear perturbations of equations. The results of investigation are applied in stability investigation of the equilibrium states of mechanical systems. In particular, undercarriage galloping at aircraft uniform motion on an even ground air strip is studied and stability conditions are established for the equilibrium state at steady cutting at trace turning for arbitrary constant angular velocity of spindle rotation. Note that in these two examples the oscillation processes under some restrictions are described by differential difference equation of the type

$$\frac{d^2 x(t)}{dt^2} + a\frac{dx(t)}{dt} + bx(t) + cx(t-\tau) = f\left(x(t), \frac{dx(t)}{dt}, x(t-\tau), \frac{dx(t-\tau)}{dt}\right),$$

where $a, b, c \in R$ and $f(x_1, x_2, x_3, x_4) = o(|x_1| + |x_2| + |x_3| + |x_4|)$ for $x_k \to 0$, $k = \overline{1,4}$.

The investigations are based on the analogue of the maximum principle for the spectrum of operator holomorphic function (see [11]).

10.2 Stability and instability in linear approximation and essential instability of evolutionary systems.

New conditions of stability and instability in linear approximation are established for solutions to differential and difference equations of the type

$$\frac{dx(t)}{dt} = Ax(t) + F(t, x(t)),\ t \geq 0,$$

$$x_{n+1} = Ax_n + G_n(x_n),\ n \geq 0,$$

and similar functional differential equations in a Banach space which generalize and strengthen the results of A.M. Lyapunov, M.G. Krein and Yu.L. Daletskii. In these equations A is a continuous linear operator and $F(t,\cdot)$ and G_n are nonlinear operators for which

$$\lim_{x\to 0} \frac{\sup\limits_{t\geq 0} \|F(t,x)\|}{\|x\|} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\sup\limits_{n\geq 0} \|G_n(x)\|}{\|x\|} = 0.$$

Examples of autonomous nonlinear systems with asymptotically stable solutions are set out for the linear approximations of which these solutions are unstable and are spectrum points of operators generated by linear approximations with positive real parts (differential case) or absolute values larger than one (difference case) [12]. Theorems on stability in linear approximation are applicable to the investigation of oscillation processes of a series of nonlinear mechanical systems, in particular, vibroimpact ones whose constituents are components with distributed parameters, systems with impulse loadings, etc. A mathematical apparatus is so far created for solution of a wide class of problems of the theory of nonlinear oscillations of complex mechanical systems.

A notion of essentially unstable solution to evolution equation in infinite dimensional case is introduced which is associated with the essentially approximate spectrum of operator. Such equations have the property that arbitrary absolutely continuous and some other perturbations can not influence essentially unstable solution so that it becomes stable. The notion of essentially unstable solution allowed new results on instability of solutions which have no analogues in the finite dimensional case [12].

The Belitskii–Lyubich hypothesis on smooth mapping of a convex compact subset of finite dimensional space was disproved. The hypothesis claimed that in the case when spectral radius of the Frechet derivative of the mapping at all points of the subset is smaller than a unit, the sequences generated by this iteration mapping converge to the unique point of the subset. It is shown that in general case the iteration sequences can diverge and the mapping can have an arbitrary number of cycles. Mappings of the type occur in practice in the computer investigation of oscillation processes in nonlinear mechanical systems. Additional conditions are found under which the hypothesis is true. Also, global asymptotic stability conditions are established for solutions to nonlinear differential and difference equations in a Banach space [12].

11 Concluding Remarks

The paper provides review of results obtained by the authors in the field of nonlinear mechanics. The development of the Lyapunov's methods and the averaging theory allowed solutions to a wide range of problems of the mathematical stability theory, motion control theory, dynamics of a rigid body and systems of bodies, theory of boundary-value problems and multifrequency oscillation theory to be described from a unique methodological point of view. New approaches set out in the paper are applied not only to the investigation of systems of ordinary differential equations, but also to a huge class of hybrid dynamical systems including the systems with impulse effect, delay equations and differential difference equations in a Banach space. It seems reasonable to develop further the presented methods for description of dynamical properties of complex systems in abstract spaces and to apply the obtained results to motion stability and control problems for mechanical objects with distributed parameters.

The worked out method of oriented manifolds reduced the controllability problem to the investigation of solvability of differential equations with respect to auxiliary functions under general assumptions on regularity of vector fields of controlled system. For this method to be constructively used it is of interest to develop approaches for constructing basic systems for arbitrary nonlinear control processes. The results obtained in the paper demonstrate efficiency of applying the method of trajectory set for solution of inverse problems of control theory. Generalization of theorems of the direct Lyapunov method yielded a complete description of conditions of strong and partial stabilizability of the class of plane mechanical systems with elastic beams. Meanwhile, the problem on compactness of limiting trajectory sets of nonlinear differential equations with non-monotone and unbounded right-hand sides in a Banach space should be further investigated.

Note that the method of matrix-valued Lyapunov functions allows to extend maximally the assumptions on dynamical properties of subsystems in large scale system and assumptions on interconnection functions between the subsystems. As compared with the other approaches developed in stability theory of large scale systems this method has the following advantages: it does not require the application of quasimonotone comparison systems which is a necessary condition when the vector Lyapunov functions are applied; it allows extension of the class of auxiliary functions by means of which an appropriate Lyapunov function can be constructed for the problem under consideration; it provides a possibility of taking into account the effect of interconnection functions between subsystems on the whole system dynamics; the method also allows to take into account the effect of pairs of subsystems appearing as result of first level decomposition on the whole system dynamics.

It is known that the a priori determination of the domains of initial and subsequent deviations of solutions from zero equilibrium states (or given nominal solution) and the domain of persistent perturbations is characteristic for nonclassical stability theories such as technical and practical ones. Moreover, the interval of system functioning is also fixed. An efficient application of the direct Lyapunov method in the practical stability problems by A.A. Martynyuk yielded significant extensions of this method, which are follows: an extension of the class of auxiliary functions suitable for the studying practical stability of motion; elimination of the property of having a fixed sign of the total derivative of an auxiliary function along with solutions of the system under investigation; establishing a relationship between the quantitative values of the auxiliary function in given (finite) domains of the phase space and decrement (increment) of this function along with solutions of the system under investigation.

The importance of practical application of the theory of boundary-value problems in various fields (nonlinear oscillation theory, motion stability theory, control theory, a series of economical and biological problems) attracts a great interest to the investigations in the theory of boundary-value problems for a wide class of systems of functional differential equations.

General theory of under- and over-determined resonance boundary-value problems is constructed, natural classification of the problems is worked out, efficient coefficient criteria of existence of solutions to both linear and nonlinear problems are obtained and algorithms for their construction are developed [7, 8]. Perturbation theory for such problems is constructed and bifurcation and branching conditions are established for solutions of boundary-value problems (including the problems with conditions at infinity) with the Fredholm operator in linear part. The application of the apparatus of generalized inverse operators based on classical results of A.M. Lyapunov and I.J. Malkin on nonlinear periodic oscillation theory provided the development of the qualitative theory of boundary-value problems for the systems of ordinary differential [7, 8] and difference [36] equations, systems of differential equations with delaying argument [38] and differential systems with impulse effect [35]. Further original application of this theory was to the known problem on bounded on the whole real axis solutions to differential and difference equations of appropriate homogeneous system under the dichotomy condition on semiaxes [8, pp. 257–304].

Originality and importance of the main results of the papers [9], [39]–[43] are as follows. The author generalizes the Lyapunov and Ostrovsky–Schneider theorems on localization of matrix spectrum for the classes of analytic domains including the previously known ones and being maximally admissible in the framework of the method of matrix equations. Generalized Lyapunov equation is used in the problem on quadratic optimization of linear controlled systems. The analogues of generalized Lyapunov equation constructed for analytic matrix-functions enable formulation of new algebraic methods for stability and localization analysis of spectrum of different classes of differential, difference and differential-difference systems. The elaborated transformation systems and generalized inertia theory provide new techniques for classification of linear matrix equations with respect to their solvability conditions and properties of solutions employed in the applied investigations. Stability criteria are obtained for linear dynamic systems in partially ordered space in terms of positive and positive invertible operators. New methods for stability analysis and generalized principle of comparison of nonlinear differential systems with the use of cone inequalities are formulated. The results obtained allow one to describe algebraically the classes of stable systems in the parameter space, to compare their dynamics and to construct stabilizing controls.

Scientific novelty of the results presented in the monograph [10] is as follows. New uniform estimates are obtained for oscillation integrals and parameter dependent sums. These estimates are used to substantiate the method of averaging with respect to all quick parameters on a segment and semiaxis for nonlinear oscillation systems with slowly varying frequencies in the resonance case. A new construction technique is developed for integral manifolds of resonant multifrequency systems and their smoothness and stability are studied. Solvability conditions are established for boundary-value problems of multifrequency systems with multipoint and integral boundary conditions and new error estimates are proved for the averaging method for such problems. The averaging method and the method of integral manifolds are justified for oscillation systems with slowly varying frequencies and impulse effect.

In the monographs [11, 12] functional analytical methods are developed for investigation of absolute stability of dynamic systems with aftereffect, stability, instability and essential instability of trajectories of dynamic systems in infinite dimensional phase space. These methods allow, first of all, obtaining general results on asymptotic behaviour of trajectories of nonlinear systems under investigation, constructing a mathematical apparatus for investigation of dynamic processes in complex nonlinear systems and finding out general regularities of the evolutionary processes going on in many real systems where motion occurs. Besides, they open up new possibilities for investigating oscillation of trajectories of nonlinear dynamic systems and studying invertibility of nonlinear functional operators.

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Monographs:

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[8] A.A. Boichuk, A.M. Samoilenko. Generalized inverse operators and Fredholm boundary-value problems.— VSP, Utrecht–Boston, The Netherlands.— 2004.— 317 pp.

[9] A.G. Mazko. Localization of Spectrum and Stability of Dynamical Systems.— Kiev: Institute of Mathematics of NAS of Ukraine, 1999.— 216 p. [Russian]

[10] A.M. Samoilenko, R.I. Petryshyn. Mathematical Aspects of Nonlinear Oscillation Theory.— Kiev: Naukova Dumka, 2004.– 474 p. [Ukrainian]

[11] V.Yu. Slyusarchuk. Absolute Stabilty of Dynamical Systems with Aftereffect.— Rivne: National University of Water Management and Nature Resources Use, 2003.— 288 p. [Ukrainian]

[12] V.Yu. Slyusarchuk. Instability of Solutions of Evolution Equations.— Rivne: National University of Water Management and Nature Resources Use, 2004.— 416 p. [Ukrainian]

Papers:

[13] A.M. Kovalev. Controllability criteria and sufficient conditions for dynamical systems to be stabilizable // J. Appl. Maths Mechs.— 1995, Vol. 59, No. 3.— P. 379–386.

[14] A.M. Kovalev. Criteria for the functional controllability and invertibility of non-linear systems // Journal of Applied Mathematics and Mechanics.— 1998.— Vol. 62, No. 1.— P. 103–113.

[15] A.M. Kovalev. Stability of stationary motions of mechanical systems with rigid body as basic element // Nonlinear Dynamics and Systems Theory.— 2001.— Vol. 1.— P. 81–97.

[16] A.M. Kovalev. Solution of a two-point control problem using a measurable function on several trajectories // Mekh. Tverd. Tela.— 2001, No. 31.— P. 119–125. [Russian]

[17] A.M. Kovalev. Normal oscillations of a rigid body in the Rodrigues–Hamilton parameters. // Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki. — 2002, No. 3. — P. 61–65. [Russian]

[18] A.M. Kovalev. Invariant and Integral Manifolds of Dynamical Systems and the Problem of Integration of the Euler–Poisson Equations // Reg. & Chaot. Dyn.— 2004.— Vol. 9, No 1.— P. 59–72.

[19] A.M. Kovalev, V.N. Nespirnyy. Impulsive Discontinuous Stabilization of the Brockett Integrator // Journal of Computer and Systems Sciences International.— 2005.— Vol. 44, No 5.— P. 671–681.

[20] A.L. Zuyev. Stabilization of Non-Autonomous Systems with Respect to a Part of Variables by Means of Controlled Lyapunov Functions // Journal of Automation and Information Sciences.— 2000.— Vol. 32, No. 10.— P. 18–25.

[21] A.L. Zuyev. On partial stabilization of satellite orientation by means of two control torques // Space Science and Technology.— 2001.— Vol. 7, No. 1.— P. 76–81. [Russian]

[22] A.M. Kovalev, A.L. Zuyev. On Nonasymptotic Stabilizability of Controllable Systems. In: *Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems MTNS–2000.*— Perpignan (France).— 2000.— CD-ROM file B80.pdf.

[23] A.M. Kovalev, A.L. Zuyev, V.F. Shcherbak. The Synthesis of Stabilizing Control of a Rigid Body with Attached Elastic Elements // Journal of Automation and Information Sciences.— 2002.— Vol. 34, No. 11.— P. 1–10.

[24] B. Jakubczyk, A. Zuyev. Stabilizability conditions in terms of critical Hamiltonians and symbols // Systems & Control Letters.— 2005.— Vol. 54.— P. 597–606.

[25] A.L. Zuyev. Control of a flexible manipulator within the framework of the Timoshenko beam model // International Applied Mechanics.— 2005.— Vol. 41, No. 12.— P. 1418–1425.

[26] A.L. Zuyev. Partial asymptotic stabilization of nonlinear distributed parameter systems // Automatica.— 2005.— Vol. 41, No.1.— P. 1–10.

[27] A.L. Zuyev. Partial asymptotic stability of abstract dynamical processes // Ukrainian Mathematical Journal.— 2006.— Vol. 58, No. 5.— P. 709–717.

[28] A.L. Zuyev. Control of a system with elastic components in the non-resonance case // Ukrainian Mathematical Bulletin.— 2006.— Vol. 3, No. 1.— P. 131–144.

[29] A.A. Martynyuk, V.I. Slynko. On Motion Stability of a Large-Scale System // Diffrential Equations.— 2003.— Vol. 39, No 6.— C. 754–758. (Russian)

[30] A.A. Martynyuk, V.I. Slynko. On Motion Stability of a Nonlinear Impulse System // Prikladnaya Mekhanika.— 2004.— Vol. 40, No 2.— C. 134–144. (Russian)

[31] V.I. Slynko. On Conditions of Practical Stability for a Quasilinear Hybrid System // Prikladnaya Mekhanika.— 2005.— Vol. 41, No 2.— C. 131–143. (Russian)

[32] V.I. Slynko. On Sufficient Conditions of Practical Stability for Nonlinear Impulse Systems // Prikladnaya Mekhanika.— 2004.— Vol. 40, No 10.— C. 131–134. (Russian)

[33] V.I. Slynko. On Motion Stability Conditions for Nonlinear Impulse Systems with Delay // Prikladnaya Mekhanika.— 2005.— Vol. 41, No 6. — C. 130–138. (Russian)

[34] V.I. Slynko. On Exponential Stability of a Linear Impulse System in the Hilbert Space // Dopovidi NAN Ukrainy.— 2002.— No 12.— C. 44–47. (Russian)

[35] A.A. Boichuk, N.A. Perestyuk, A.M. Samoilenko. Periodic Solutions of Impulse Differential Systems in Critical Cases // Differential Equations.— 1991.— Vol. 27, No 9.— C. 1516–1521. (Russian)

[36] A.A. Boichuk. Boundary-Value Problems for Systems of Difference Equations // Ukrainian Mathematical Journal.— 1997.— Vol. 49, No 6.— P. 832–835. (Russian)

[37] A.A. Boichuk. Existence Condition for a Unique Function of the Green-Samoilenko Problem on the Invariant Torus // Ukrainian Mathematical Journal.— 2001.— Vol. 53, No 4.— C. 556–559. (Russian)

[38] A.A. Boichuk. Perturbed Fredholm BVP's for Delay Differential Systems. In: *Equadiff–2003, International Conference on Differential Equations*. Hasselt–Belgium, July 22–26, 2003. World Scientific. New Jersey–London–Singapore.— 2005.— P. 1033–1035.

[39] A.G. Mazko. Matrix algorithm for design of optimal linear systems with specified properties // Avtomatika i Telemechanika.— 1981.— No 5.— P. 33–41. [Russian]

[40] A.G. Mazko. Stability and comparison of systems in partially ordered space // Problems of Nonlinear Analysis in Engineering Systems.— 2002.— Vol. 8, No 1(15).— C. 24–36.

[41] A.G. Mazko. Stability of positive and monotone systems in partially ordered space // Ukrainian Mathematical Journal.— 2004.— Vol. 56, No 4.— P. 462–475. [Russian]

[42] A.G. Mazko. Stability and comparison of states of dynamic systems with respect to time-varying cone // Ukrainian Mathematical Journal.— 2005.— Vol. 57, No 2. — P. 198–213. [Russian]

[43] A.M. Aliluyko, A.G. Mazko. Invariant cones and stability of linear dynamical systems // Ukrainian Mathematical Journal.— 2006.— Vol. 58, No 11.— P. 1446–1461. [Ukrainian]

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State Dependent Generalized Inversion-Based Liapunov Equation for Spacecraft Attitude Control

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Abstract: Parametrization of nonunique linear equations solution via generalized inversion is utilized in nonlinear spacecraft control system design. A stable linear time-invariant ordinary differential equation in an attitude deviation norm measure is formed and is evaluated along the trajectories defined by the spacecraft mathematical model, yielding a linear relation in the control variables. Generalized inversion of the relation results in a control law that consists of auxiliary and particular parts. The *null-control vector* in the auxiliary part is designed by solving a state dependent Liapunov equation involving a *perturbed nullprojector* and by utilizing a *damped controls coefficient generalized inverse*, yielding globally uniformly ultimately bounded attitude trajectory tracking errors.

Keywords: *spacecraft attitude control; Moore–Penrose controls coefficient generalized inverse; null-control vector; damped controls coefficient generalized inverse; state dependent Liapunov equation; perturbed controls coefficient nullprojection.*

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1 Introduction

Throughout the second half of the twentieth century, numerous control methodologies have been employed for spacecraft control, benefiting from the rapid development in nonlinear system theory. Among the methodologies applied to the attitude control problem of rigid spacecraft with known inertia parameters were those based on geometrical concepts, energy principles, optimal control, and feedback linearizing transformations.

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The present article introduces an algebraic control methodology that aims to utilize the simplicity of linear system theory by casting the nonlinear spacecraft control problem in a pointwise-linear form and utilizing a simple linear algebra relation to tackle the control problem. The primary tool used is the Moore–Penrose generalized matrix inverse (MPGI).

The procedure begins by defining a norm measure function of the spacecraft's attitude variables deviations from their desired values, and prespecifying a stable second-order linear differential equation in the measure function, resembling the desired attitude deviation dynamics. The differential equation is then transformed to a relation that is linear in the control vector by differentiating the norm measure function along the trajectories defined by the solution of the spacecraft's state space mathematical model. The MPGI is utilized thereafter to invert this relation for the control law required to realize the desired stable linear attitude deviation norm measure dynamics.

In addition to its algebraic simplicity, the derived control law has a special geometrical structure. It consists of auxiliary and particular parts, residing in the nullspace of the *controls coefficient* row vector and the range space of its generalized inverse, respectively. The auxiliary part contains a free nullvector, named the *null-control vector*, and is being projected onto the controls coefficient nullspace by means of a nullprojection matrix. Therefore, the choice of the null-control vector does not affect the dynamics of the attitude deviation norm measure function, and it parameterizes *all* control laws that are capable of realizing that dynamics.

The control problem is a problem of nonuniqueness; that is, if a dynamical system is controllable then there exists no unique strategy to control it. The MPGI was reintroduced in [1] to parameterize this *redundancy in control authority* in the context of program, or servo-constraints. The procedure is generalized in this work to the gas jet-actuated spacecraft control problem by considering nulling the deviation from desired spacecraft kinematics to be the servo-constraint that is to be realized.

Generalized inversion of the controls coefficient implies outer kinematics tracking exponential stability. However, not all choices from the infinite set of null-control vectors guarantee stability of the spacecraft internal dynamics. An observation is made in [1] that the null-control vector choice substantially affects the inner system states. Therefore, the primary objective in utilizing the null-control vector design freedom is to subdue internal instability of the closed loop control system.

To fulfill the internal stability objective, and inspired by the control law's affinity in the null-control vector, the later is chosen in this work to be proportional to the spacecraft angular velocity vector. The state dependent proportionality matrix is constructed by solving a state dependent Liapunov equation that is produced by a quadratic Liapunov function in the spacecraft angular velocity vector.

A fundamental property of the resulting Liapunov equation is its dependency on the controls coefficient generalized inverse (CCGI) and the corresponding nullprojector. This dependency is a source of two difficulties in the way of solving the equation. The first difficulty is due to rank deficiency of the controls coefficient nullprojector, and it is overcomed by perturbing the nullprojector to disencumber its rank deficiency.

The second difficulty is due to an inherent characteristic of the MPGI. Although well-defined for any matrix, regardless of its size or rank, the MPGI mapping of a matrix that is continuous in its elements suffers from a discontinuity, whenever the matrix changes rank. This appears as a divergence of the generalized inverse matrix elements to infinite values as the mapped matrix changes rank. Robustness against this generalized inversion instability is achieved by modifying the structure of the controls coefficient MPGI by means of a damping factor that limits its growth as steady state response is approached. Depending on the amount of modification, this *damped* CCGI results in a tradeoff between trajectory tracking accuracy and generalized inversion stability.

Modifying the definition of the controls coefficient MPGI results in an approximate realization of the desired spacecraft attitude deviation norm measure dynamics. It is shown that the closed loop attitude trajectories tracking errors resulting from applying the proposed generalized inversion-based control law are globally uniformly ultimately bounded, and that the ultimate bound is inversely proportional to the damping factor by which the generalized inverse is modified.

The present article introduces a nonlinear spacecraft attitude tracking control law, derived in the generalized inversion framework via a novel state dependent Liapunov equation, in a continuous development of Liapunov thoughts and results that remain after a century and a half from his birth anniversary to be the most famous criteria for nonlinear motion stability [2].

2 Spacecraft Mathematical Model

The spacecraft mathematical model is given by the following system of kinematical and dynamical differential equations

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0, \tag{1}$$

$$\dot{\omega} = J^{-1}\omega^{\times} J\omega + \tau, \quad \omega(0) = \omega_0, \tag{2}$$

where $\rho \in \mathbb{R}^{3\times1}$ is the spacecraft vector of modified Rodrigues attitude parameters (MRPs) [3], $\omega \in \mathbb{R}^{3\times1}$ is the vector of spacecraft angular velocity components in its body reference frame, $J \in \mathbb{R}^{3\times3}$ is a diagonal matrix containing spacecraft's body principal moments of inertia, and $\tau := J^{-1}u \in \mathbb{R}^{3\times1}$ is the vector of scaled control torques, where $u \in \mathbb{R}^{3\times1}$ contains the applied gas jet actuator torque components about the spacecraft's principal axes. The cross product matrix $x^{\times}$ which corresponds to a vector $x \in \mathbb{R}^{3\times1}$ is skew symmetric of the form

$$x^{\times} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

and the matrix valued function $G(\rho) : \mathbb{R}^{3\times1} \to \mathbb{R}^{3\times3}$ is given by

$$G(\rho) = \frac{1}{2}\left(\frac{1-\rho^T\rho}{2}\, I_{3\times3} - \rho^{\times} + \rho\rho^T\right).$$

The MRPs are used as the attitude state variables, because of their validity in describing any angular displacement about the spacecraft's body axes up to 2π rad, such that $G(\rho)$ remains finite and invertible for any value of ρ that corresponds to such spacecraft angular displacement.

3 Attitude Deviation Norm Measure Dynamics

Let $\rho_d(t) \in \mathbb{R}^{3\times1}$ be a prescribed desired spacecraft attitude vector such that $\rho_d(t)$ is at least twice continuously differentiable in t. The spacecraft attitude deviation vector from

$\rho_d(t)$ is defined as

$$z(\rho, t) := \rho - \rho_d(t). \tag{3}$$

Define the scalar attitude deviation norm measure function $\phi : \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}$ to be half the squared Euclidean norm of $z(\rho, t)$

$$\phi = \frac{1}{2} \parallel z(\rho, t) \parallel^2 = \frac{1}{2} \parallel \rho - \rho_d(t) \parallel^2 . \tag{4}$$

The first two time derivatives of ϕ along the spacecraft trajectories given by the solution of Eqs. (1) and (2) are

$$\dot{\phi} = \frac{\partial \phi}{\partial \rho} G(\rho)\omega + \frac{\partial \phi}{\partial t} = z^T(\rho, t) \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \tag{5}$$

and

$$\begin{aligned} \ddot{\phi} = {} & \left[G(\rho)\omega - \dot{\rho}_d(t)\right]^T \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \\ & + z^T(\rho, t) \left[\dot{G}(\rho, \omega)\omega + G(\rho)\left[J^{-1}\omega^{\times} J\omega + \tau\right] - \ddot{\rho}_d(t)\right], \end{aligned} \tag{6}$$

where $\dot{G}(\rho, \omega)$ is the time derivative of $G(\rho)$ obtained by differentiating the individual elements of $G(\rho)$ along the kinematical subsystem given by Eqs. (1). The procedure is to prespecify a desired stable linear second-order dynamics of ϕ in the form

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \quad c_1, c_2 > 0. \tag{7}$$

With ϕ, $\dot{\phi}$, and $\ddot{\phi}$ given by Eqs. (4), (5), and (6), it is possible to write Eq. (7) in the quasi-linear form

$$\mathcal{A}(\rho, t)\tau = \mathcal{B}(\rho, \omega, t), \tag{8}$$

where the vector valued function $\mathcal{A}(\rho, t) : \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}^{1\times 3}$ is given by

$$\mathcal{A}(\rho, t) = z^T(\rho, t)G(\rho) \tag{9}$$

and the scalar valued function $\mathcal{B}(\rho, \omega, t) : \mathbb{R}^{3\times 1} \times \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}$ is

$$\begin{aligned} \mathcal{B}(\rho, \omega, t) = {} & - \left[G(\rho)\omega - \dot{\rho}_d(t)\right]^T \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \\ & - z^T(\rho, t) \left[\dot{G}(\rho, \omega)\omega + G(\rho)J^{-1}\omega^{\times} J\omega - \ddot{\rho}_d(t)\right] \\ & - c_1 z^T(\rho, t) \left[G(\rho)\omega - \dot{\rho}_d(t)\right] - \frac{c_2}{2} \parallel z(\rho, t) \parallel^2 . \end{aligned}$$

The row vector function $\mathcal{A}(\rho, t)$ is named the *controls coefficient* of the attitude deviation norm measure dynamics given by Eq. (7) along the spacecraft trajectories, and the scalar function $\mathcal{B}(\rho, \omega, t)$ is the corresponding *controls load*.

4 Linearly Parameterized Attitude Control Laws

The quasi-linear form given by Eq. (8) makes it feasible to assess realizability of the linear attitude deviation norm measure dynamics given by Eq. (7) in a pointwise manner.

Definition 4.1 For a given desired spacecraft attitude vector $\rho_d(t)$, the linear attitude deviation norm measure dynamics given by Eq. (7) is said to be realizable by the spacecraft equations of motion (1) and (2) at specific values of ρ and t if there exists a control vector τ that solves Eq. (8) for these values of ρ and t. If this is true for all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, then the linear attitude deviation norm measure dynamics is said to be globally realizable by the spacecraft equations of motion.

Proposition 4.1 *For any desired spacecraft attitude vector $\rho_d(t)$, the linear attitude deviation norm measure dynamics given by Eq. (7) is globally realizable by the spacecraft equations of motion (1) and (2). Furthermore, the infinite set of all control laws realizing that dynamics by the spacecraft equations of motion is parameterized by an arbitrarily chosen nullvector $y \in \mathbb{R}^{3\times 1}$ as*

$$\tau = \mathcal{A}^+(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y, \tag{10}$$

where $\mathcal{A}^+$ stands for the MPGI *of the controls coefficient given by*

$$\mathcal{A}^+(\rho, t) = \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\mathcal{A}(\rho,t)\mathcal{A}^T(\rho,t)}, & \mathcal{A}(\rho, t) \neq \mathbf{0}_{1\times 3}, \\ \mathbf{0}_{3\times 1}, & \mathcal{A}(\rho, t) = \mathbf{0}_{1\times 3}, \end{cases} \tag{11}$$

and $\mathcal{P}(\rho, t) \in \mathbb{R}^{3\times 3}$ is the corresponding nullprojector given by

$$\mathcal{P}(\rho, t) = I_{3\times 3} - \mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t). \tag{12}$$

Proof A necessary and sufficient condition for the existence of a control vector τ that solves Eq. (8) at specific values of ρ and t is consistency of the equation at these values, i.e., $\mathcal{B}(\rho, \omega, t)$ is in the range space of $\mathcal{A}(\rho, t)$. This is guaranteed for all values of $\omega \in \mathbb{R}^{3\times 1}$, provided that $\mathcal{A}(\rho, t)$ does not vanish at the specified values of ρ and t, at which the linear attitude deviation norm measure dynamics given by Eq. (7) is realizable by the spacecraft equations of motion (1) and (2) according to definition 4.1. Since the matrix $G(\rho)$ is invertible for all values of ρ, it has a trivial nullspace, which implies from Eq. (9) that $\mathcal{A}(\rho, t)$ vanishes if and only if $z(\rho, t)$ does. Therefore, Eq. (8) is consistent at all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, and the linear attitude deviation norm measure dynamics is globally realizable by the spacecraft equations of motion according to definition 4.1. Consequently, the infinite set of *all* control laws that realize the linear attitude deviation norm measure dynamics by the spacecraft equations of motion at all ρ and t such that $\mathcal{A}(\rho, t) \neq \mathbf{0}_{1\times 3}$ is given by Eq. (10) [4].

Since any choice of the nullvector y in the control law expression given by Eq. (10) yields a solution to Eq. (8), the choice of y does not affect realizability of the linear attitude deviation norm measure dynamics given by Eq. (7). Nevertheless, the choice of y substantially affects the spacecraft transient state response [1]. In particular, an inadequate choice of y can destabilize the spacecraft internal dynamics given by Eq. (2) or causes unsatisfactory closed loop performance. Due to the importance of the nullvector y in the present control system design development as a control vector by itself, we name it the *null-control vector*.

Corollary 4.1 *The infinite set of spacecraft closed loop systems equations realizing the linear attitude deviation norm measure dynamics given by Eq. (7) is parameterized by the null-control vector y as*

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0, \tag{13}$$

$$\dot{\omega} = J^{-1}\omega^\times J\omega + \mathcal{A}^+(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y, \quad \omega(0) = \omega_0. \tag{14}$$

Proof Equations (13) and (14) are obtained by substituting the control laws expressions given by Eqs. (10) in the spacecraft's mathematical model given by Eqs. (1) and (2).

5 Perturbed Controls Coefficient Nullprojector

The concept of perturbed controls coefficient nullprojector (PCCN) is crucial in the present development of the generalized inversion-based spacecraft control law.

Definition 5.1 The PCCN $\widetilde{\mathcal{P}}(\rho, \delta, t)$ is defined as

$$\widetilde{\mathcal{P}}(\rho, \delta, t) := I_{3\times 3} - h(\delta)\mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t), \tag{15}$$

where $h(\delta) : \mathbb{R}^{1\times 1} \to \mathbb{R}^{1\times 1}$ is any continuous function such that

$$h(\delta) = 1 \quad \text{if and only if} \quad \delta = 0.$$

Proposition 5.1 *The* PCCN $\widetilde{\mathcal{P}}(\rho, \delta, t)$ *is of full rank for all* $\delta \neq 0$.

Proof The singular value decomposition of $\mathcal{A}(\rho, t)$ is given by

$$\mathcal{A}(\rho, t) = \mathbf{\Sigma}(\rho, t)\mathcal{V}^T(\rho, t),$$

where

$$\mathbf{\Sigma}(\rho, t) = \begin{bmatrix} \parallel \mathcal{A}(\rho, t) \parallel & 0 & 0 \end{bmatrix}$$

and $\mathcal{V}(\rho, t) \in \mathbb{R}^{3\times 3}$ is orthonormal, i.e.,

$$\mathcal{V}^{-1}(\rho, t) = \mathcal{V}^T(\rho, t), \quad \text{and} \quad \det \mathcal{V}(\rho, t) = 1.$$

By inspecting the four conditions defining the MPGI [4], it can be easily verified that it is given for $\mathcal{A}(\rho, t)$ by

$$\mathcal{A}^+(\rho, t) = \mathcal{V}(\rho, t)\mathbf{\Sigma}^+(\rho, t),$$

where $\mathbf{\Sigma}^+(\rho, t)$ is the MPGI of $\mathbf{\Sigma}(\rho, t)$ given by

$$\mathbf{\Sigma}^+(\rho, t) = \begin{bmatrix} \dfrac{1}{\parallel \mathcal{A}(\rho, t) \parallel} & 0 & 0 \end{bmatrix}^T.$$

Therefore,

$$\mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t) = \mathcal{V}(\rho, t)\mathbf{\Sigma}^+(\rho, t)\mathbf{\Sigma}(\rho, t)\mathcal{V}^T(\rho, t). \tag{16}$$

The right hand side of Eq. (16) is a singular value decomposition of $\mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t)$, where the diagonal matrix $\mathbf{\Sigma}^+(\rho, t)\mathbf{\Sigma}(\rho, t)$ contains the singular values of $\mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t)$ as its diagonal elements

$$\mathbf{\Sigma}^+(\rho, t)\mathbf{\Sigma}(\rho, t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, the PCCN $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is

$$\begin{aligned}\widetilde{\mathcal{P}}(\rho,\delta,t) &= I_{3\times3} - h(\delta)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\\ &= I_{3\times3} - h(\delta)\mathcal{V}(\rho,t)\mathbf{\Sigma}^{+}(\rho,t)\mathbf{\Sigma}(\rho,t)\mathcal{V}^{T}(\rho,t)\\ &= \mathcal{V}(\rho,t)[I_{3\times3} - h(\delta)\mathbf{\Sigma}^{+}(\rho,t)\mathbf{\Sigma}(\rho,t)]\mathcal{V}^{T}(\rho,t)\\ &= \mathcal{V}(\rho,t)\begin{bmatrix}1-h(\delta) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\mathcal{V}^{T}(\rho,t),\end{aligned}$$

which is of full rank for all $\delta \neq 0$.

Lemma 5.1 *The controls coefficient nullprojector $\mathcal{P}(\rho,t)$ commutes with its perturbation $\widetilde{\mathcal{P}}(\rho,\delta,t)$ for all $\delta\in\mathbb{R}$. Furthermore, their matrix multiplication is the controls coefficient nullprojector itself, i.e.,*

$$\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t) = \widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t) = \mathcal{P}(\rho,t). \tag{17}$$

Proof Equations (17) are verified by direct evaluation of the $\mathcal{P}(\rho,t)$ and $\widetilde{\mathcal{P}}(\rho,\delta,t)$ expressions given by Eqs. (12) and (15).

6 Null-Control Vector Design

The choice of the null-control vector y affects neither realizability of the attitude deviation norm measure dynamics given by Eq. (7) nor steady state spacecraft response. However, the choice of the null-control vector y affects both of spacecraft internal dynamics and spacecraft transient response. Hence, it provides a freedom that can be utilized to stabilize internal states of the spacecraft. Internal dynamics stability and stability robustness against controls coefficient singularity are the most important factors to be considered in designing the null-control vector y.

The structure of the control law τ given by Eqs. (10) has a special feature, namely the affinity of its auxiliary part in y, which provides a pointwise-linear parametrization to the nonlinear control law. Hence, let y be chosen as

$$y = K\omega,$$

where $K\in\mathbb{R}^{3\times3}$ is to be determined. With this choice of y, a class of control laws that globally realize the attitude deviation norm measure dynamics given by Eq. (7) is given by

$$\begin{aligned}\tau &= \mathcal{A}^{+}(\rho,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K\omega\\ &= [\mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K]\,\omega + \mathcal{H}_2(\rho,t),\end{aligned} \tag{18}$$

where

$$\begin{aligned}\mathcal{H}_1(\rho,\omega,t) = -\mathcal{A}^{+}(\rho,t)z^{T}(\rho,t)\Big[\dot{G}(\rho,\omega) + G(\rho)J^{-1}\omega^{\times}J + c_1G(\rho)\Big]\\ - \mathcal{A}^{+}(\rho,t)\Big[G(\rho)\omega - \dot{\rho}_d(t)\Big]^{T}G(\rho)\end{aligned} \tag{19}$$

and

$$\mathcal{H}_2(\rho,t) = -\frac{c_2}{2}\mathcal{A}^+(\rho,t)z^T(\rho,t)z(\rho,t) + \mathcal{A}^+(\rho,t)z^T(\rho,t)\Big[\ddot{\rho}_d(t) + c_1\dot{\rho}_d(t)\Big] - \mathcal{A}^+(\rho,t) \parallel \dot{\rho}_d(t) \parallel_2^2 . \quad (20)$$

Hence, a class of closed loop dynamical subsystems realizing the dynamics given by Eq. (7) is obtained by substituting the control law given by Eqs. (18) in Eqs. (2), and it takes the form

$$\dot{\omega} = \left[J^{-1}\omega^\times J + \mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K\right]\omega + \mathcal{H}_2(\rho,t). \quad (21)$$

The term $\mathcal{H}_2(\rho,t)$ in the above closed loop dynamical subsystem can be viewed as a forcing term that drives the internal dynamics of the spacecraft to realize the desired attitude deviation norm measure dynamics.

7 Spacecraft Internal Stability

The cascaded nature of the spacecraft mathematical model given by Eqs. (1) and (2) implies that coupling between the spacecraft kinematics and dynamics is unidirectional, i.e., the open loop spacecraft dynamical subsystem is independent of the attitude parameters. This allows to independently analyze dynamical subsystem stability by using the following squared Euclidean norm of the spacecraft angular velocity vector as a control Liapunov function

$$V = \parallel \omega \parallel^2 .$$

Differentiating V along the trajectories of the unforced part of the closed loop dynamical subsystem Eqs. (21) obtained by setting $\mathcal{H}_2(\rho,t) = \mathbf{0}_{3\times1}$ and noticing skew-symmetry of $\omega^\times$ yields

$$\begin{aligned}\dot{V} &= 2\omega^T\left[J^{-1}\omega^\times J + \mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K\right]\omega \\ &= \omega^T\big[\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t)\big]\omega,\end{aligned}$$

where the matrix gain K is chosen to be symmetric. Global exponential stability of the unforced part of the closed loop dynamical subsystem given by Eqs. (21) at $\omega = \mathbf{0}_{3\times1}$ is guaranteed if $\dot{V}$ remains negative-definite as the spacecraft dynamics evolves in time, which implies the existence of a positive-definite constant matrix $Q \in \mathbb{R}^{3\times3}$ such that the Liapunov equation

$$\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t) + Q = 0 \quad (22)$$

is satisfied for all $t \geq 0$. Lemma 5.1 implies that Eq. (22) can be written as

$$\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t) + Q = 0. \quad (23)$$

To solve the above matrix equation for the matrix gain K, the individual terms in the equation are vectorized by stacking their columns above each others such that [5]

$$\text{vec}\Big[\widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t)K\Big] + \text{vec}\Big[K\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t)\Big] = -\text{vec}\left[\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + Q\right].$$

Employing the relation between the matrix vectorizing operation and the Kronecker product of matrices yields [5]

$$\left\{I_{3\times 3}\otimes \widetilde{\mathcal{P}}(\rho,\delta,t)\right\}\mathrm{vec}\left[\mathcal{P}(\rho,t)K\right]+\left\{\widetilde{\mathcal{P}}(\rho,\delta,t)\otimes I_{3\times 3}\right\}\mathrm{vec}\left[K\mathcal{P}(\rho,t)\right]= \\ -\,\mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right].$$

Therefore, the unique matrix gain solution of Liapunov equation (22) for $\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)$ is obtained as

$$\begin{aligned}\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t) &= -\mathrm{vec}^{-1}\Big\{\left[I_{3\times 3}\otimes \widetilde{\mathcal{P}}(\rho,\delta,t)+\widetilde{\mathcal{P}}(\rho,\delta,t)\otimes I_{3\times 3}\right]^{-1} \\ &\qquad \mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right]\Big\} \\ &= -\mathrm{vec}^{-1}\Big\{\left[\widetilde{\mathcal{P}}(\rho,\delta,t)\oplus\widetilde{\mathcal{P}}(\rho,\delta,t)\right]^{-1} \\ &\qquad \mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right]\Big\} \end{aligned} \tag{24}$$

and the control law

$$\tau=\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\right]\omega$$

renders the equilibrium point $\omega=\mathbf{0}_{3\times 1}$ for the unforced part of the closed loop spacecraft dynamical subsystem equations (21) given by

$$\dot{\omega}=\left[J^{-1}\omega^{\times}J+\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\right]\omega \tag{25}$$

globally exponentially stable, where $\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)$ is given by Eqs. (24).

8 Controls Coefficient Singularity Analysis

If the controls coefficient $\mathcal{A}(\rho,t)$ is singular at specific values of ρ and t, i.e., has zero elements, then its MPGI $\mathcal{A}^{+}(\rho,t)$ given by Eqs. (11) is infinite. The following proposition relates global realizability of the linear attitude deviation norm measure dynamics to controls coefficient singularity.

Proposition 8.1 *Given a desired spacecraft attitude vector $\rho_d(t)$ satisfying the smoothness assumption, a control law τ globally realizes the linear attitude deviation norm measure dynamics given by Eq. (7) by the spacecraft equations of motion (1) and (2) only if*

$$\lim_{t\to\infty}\mathcal{A}(\rho,t)=\mathbf{0}_{1\times 3}.$$

Proof Because of the equivalency of linear attitude deviation norm measure dynamics given by Eq. (7) and its quasi-linear form given by Eq. (8), global realizability of of the first implies the existence of a control law that drives ϕ according to the dynamics given by Eq. (7) at all ρ and t such that $z(\rho,t)\neq\mathbf{0}_{3\times 1}$. The norm property of ϕ implies that $z(\rho,t)=\mathbf{0}_{3\times 1}$ if and only if $\phi=0$. Therefore, global realizability of the stable dynamics given by Eq. (7) implies that

$$\lim_{t\to\infty}\phi=0 \quad\text{and}\quad \lim_{t\to\infty}z(\rho,t)=\mathbf{0}_{3\times 1}.$$

Since the matrix $G(\rho)$ is nonsingular for all finite values of ρ, Eq. (9) implies that

$$\lim_{t\to\infty} z(\rho,t) = \mathbf{0}_{3\times 1} \quad \text{if and only if} \quad \lim_{t\to\infty} \mathcal{A}(\rho,t) = \mathbf{0}_{1\times 3}.$$

With the expression of $\mathcal{A}(\rho,t)$ given by Eq. (9), the MPGI controls coefficient given by Eq. (11) can be written as

$$\mathcal{A}^{+}(\rho,t) = \frac{G^T(\rho)z(\rho,t)}{\| G^T(\rho)z(\rho,t) \|^2}.$$

Therefore,

$$\| \mathcal{A}^{+}(\rho,t) \| = \frac{\| G^T(\rho)z(\rho,t) \|}{\| G^T(\rho)z(\rho,t) \|^2} = \frac{1}{\| G^T(\rho)z(\rho,t) \|}. \tag{26}$$

Since $G(\rho)$ is finite for all finite values of ρ, Eq. (26) implies that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \| \mathcal{A}^{+}(\rho,t) \| = \infty.$$

In other words, unbounded CCGI $\mathcal{A}^{+}(\rho,t)$ in a control law given by Eqs. (10) is indispensable to globally realize the associated attitude deviation norm measure dynamics. For the purpose of controlling the growth of $\mathcal{A}^{+}(\rho,t)$, a limited-growth modified CCGI is introduced next.

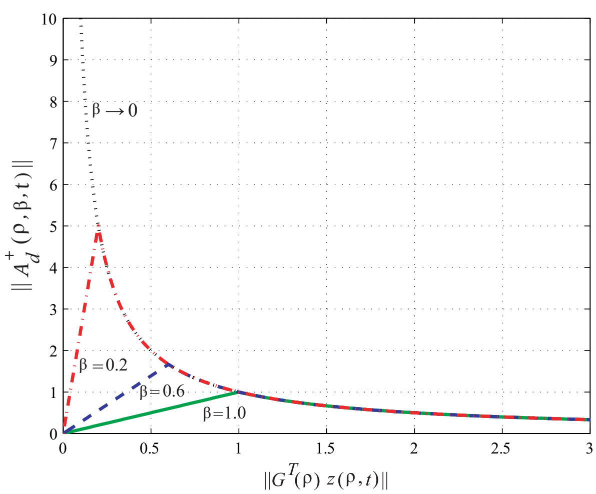

Figure 8.1: Damped CCGI.

Definition 8.1 The damped CCGI $\mathcal{A}_d^{+}(\rho,\beta,t)$ is defined as

$$\mathcal{A}_d^{+}(\rho,\beta,t) := \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\|\mathcal{A}(\rho,t)\|^2} & : \quad \| \mathcal{A}(\rho,t) \| \geqslant \beta, \\ \frac{\mathcal{A}^T(\rho,t)}{\beta^2} & : \quad \| \mathcal{A}(\rho,t) \| < \beta, \end{cases}$$

where the scalar β is a positive *generalized inverse damping factor*.

The above definition implies that

$$\|\mathcal{A}_d^+(\rho,\beta,t)\| \leqslant \frac{1}{\beta}$$

and that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times1}} \| \mathcal{A}_d^+(\rho,\beta,t) \| = \lim_{z(\rho,t)\to\mathbf{0}_{3\times1}} \frac{1}{\beta^2}\|G^T(\rho)z(\rho,t)\| = 0$$

and that $\mathcal{A}_d^+(\rho,\beta,t)$ pointwise converges to $\mathcal{A}^+(\rho,t)$ as β vanishes (see Figure 8.1). Accordingly, we define $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$ and $\mathcal{H}_{2d}(\rho,\beta,t)$ by replacing the CCGI $\mathcal{A}^+(\rho,t)$ in the $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ expressions given by Eqs. (19) and (20) with the damped CCGI $\mathcal{A}_d^+(\rho,\beta,t)$. Consequently, $K_d(\rho,\omega,\beta,\delta,t)$ is defined by replacing $\mathcal{H}_1(\rho,\omega,t)$ in the expression of $K(\rho,\omega,\delta,t)$ given by Eqs. (24) with $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$.

9 Generalized Inversion-Based Attitude Tracking Control Law

Theorem 9.1 *The control law*

$$\tau_d = \mathcal{A}_d^+(\rho,\beta,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K_d(\rho,\omega,\beta,\delta,t)\omega \tag{27}$$

renders the trajectory tracking errors of the closed loop system given by Eqs. (1) and (2) globally uniformly ultimately bounded. Furthermore, any closed loop spacecraft attitude control trajectory with initial condition $\rho(0) \in \mathbb{R}^3$ enters the domain defined by

$$\|z(\rho,t)\| < \frac{\beta}{\sigma(G(\rho))} \tag{28}$$

in finite time and remains in it for all future time, where $\sigma(G(\rho))$ is the three times-repeated singular value of $G(\rho)$.

Proof Let ϕ_d be a norm measure function of the attitude deviation obtained by applying the control law given by Eqs. (27) to the spacecraft equations of motion (1) and (2), and let $\dot{\phi}_d$, $\ddot{\phi}_d$ be its first two time derivatives. Hence,

$$\begin{aligned} \phi_d &:= \phi_d(\rho,t) = \phi(\rho,t), \\ \dot{\phi}_d &:= \dot{\phi}_d(\rho,\omega,t) = \dot{\phi}(\rho,\omega,t), \\ \ddot{\phi}_d &:= \ddot{\phi}_d(\rho,\omega,\tau_d,t) = \ddot{\phi}(\rho,\omega,\tau,t) + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau, \end{aligned} \tag{29}$$

where τ is given by

$$\tau = \mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\omega.$$

Adding $c_1\dot{\phi}_d + c_2\phi_d$ to both sides of Eq. (29) yields

$$\ddot{\phi}_d + c_1\dot{\phi}_d + c_2\phi_d = \ddot{\phi} + c_1\dot{\phi} + c_2\phi + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau = \mathcal{A}(\rho,t)[\tau_d - \tau].$$

Therefore, let the state vector $\Phi_d \in \mathbb{R}^{2\times1}$ be defined as

$$\Phi_d := \begin{bmatrix}\phi_d & \dot{\phi}_d\end{bmatrix}^T.$$

The attitude deviation norm measure closed loop dynamics becomes

$$\dot{\Phi}_d = \Lambda_1\Phi_d + \Delta_1(\rho,\beta,t), \tag{30}$$

where the asymptotically stable system matrix $\Lambda_1 \in \mathbb{R}^{2\times 2}$ is

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -c_2 & -c_1 \end{bmatrix}$$

and the input matrix valued function $\Delta_1 : \mathbb{R}^{5\times 1} \to \mathbb{R}^{2\times 1}$ is

$$\Delta_1(\rho,\omega,\beta,t) = \begin{cases} \mathbf{0}_{2\times 1} & : \quad \| \mathcal{A}(\rho,t) \| \geqslant \beta, \\ \begin{bmatrix} 0 \\ \frac{1}{\beta^2}\mathcal{B}(\rho,\omega,t) - \mathcal{B}(\rho,\omega,t) \end{bmatrix} & : \quad \| \mathcal{A}(\rho,t) \| < \beta. \end{cases}$$

On the other hand, the control law given by Eqs. (27) can be written as

$$\tau_d = [\mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K_d]\,\omega + \mathcal{H}_{2d}(\rho,\beta,t).$$

Using τ_d with the dynamical subsystem given by Eqs. (2) results in the closed loop dynamical subsystem

$$\dot{\omega} = \Lambda_2(\rho,\omega,\beta,\delta,t)\omega + \Delta_2(\rho,\beta,t), \tag{31}$$

where

$$\Lambda_2(\rho,\omega,\beta,\delta,t) = \left[J^{-1}\omega^{\times}J + \mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K_d\right]$$

and

$$\Delta_2(\rho,\beta,t) = \mathcal{H}_{2d}(\rho,\beta,t).$$

Let the augmented state space vector ξ be defined as

$$\xi := \begin{bmatrix}\Phi_d^T & \omega^T\end{bmatrix}^T,$$

then Eqs. (30) and (31) form the augmented state space model

$$\dot{\xi} = \Lambda(\rho,\omega,\beta,\delta,t)\xi + \Delta(\rho,\omega,\beta,t), \tag{32}$$

where

$$\Lambda(\rho,\omega,\beta,\delta,t) = \begin{bmatrix} \Lambda_1 & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{3\times 2} & \Lambda_2(\rho,\omega,\beta,\delta,t) \end{bmatrix}, \qquad \Delta(\rho,\omega,\beta,t) = \begin{bmatrix} \Delta_1(\rho,\omega,\beta,t) \\ \Delta_2(\rho,\beta,t) \end{bmatrix}.$$

Now consider the unforced system

$$\dot{\xi}_p = \Lambda(\rho,\omega,\beta,\delta,t)\xi_p$$

and consider the positive definite $V_a(\xi) = \|\Phi_d\|^2 + \|\omega/\omega_0\|^2$, where ω_0 is a nondimensionalizing scalar. It can easily be verified that $\dot{V}_a$ is negative definite along the trajectories of ξ_p satisfying $\|\mathcal{A}(\rho,t)\| \geqslant \beta$, and that $\Delta(\rho,\omega,\beta,t)$ is a norm bounded nonvanishing perturbation vector. Therefore, the trajectories of the augmented dynamical system given by Eqs. (32) are globally uniformly ultimately bounded ([6], pp. 347). Furthermore, since $\Delta_1 = \mathbf{0}_{2\times 1}$ in the domain defined by $\|\mathcal{A}(\rho,t)\| \geqslant \beta$, it follows from Liapunov theory that the closed loop attitude trajectories move in the direction of decreasing $V_a(\xi)$ and must cross in finite time the boundary of the domain to its open complement domain defines by $\| \mathcal{A}(\rho,t) \| < \beta$, which becomes an invariant set. Moreover, $G(\rho)$ satisfies

$$\sigma_{min}(G(\rho)) = \sigma_{max}(G(\rho)) = \sigma(G(\rho)).$$

Therefore,

$$\|\mathcal{A}(\rho,t)\| = \|z^T(\rho,t)G(\rho)\| = \sigma(G(\rho))\|z(\rho,t)\|,$$

and the bound estimate of the attitude deviation vector norm given by Eq. (28) follows.

10 Numerical Simulations

The spacecraft model selected has inertia parameters $I_1 = 200$ Kg-m^2, $I_2 = 150$ Kg-m^2, $I_3 = 175$ Kg-m^2. Values of $c_1 = 0.9$ and $c_2 = 0.3$ are chosen, and the desired MRPs

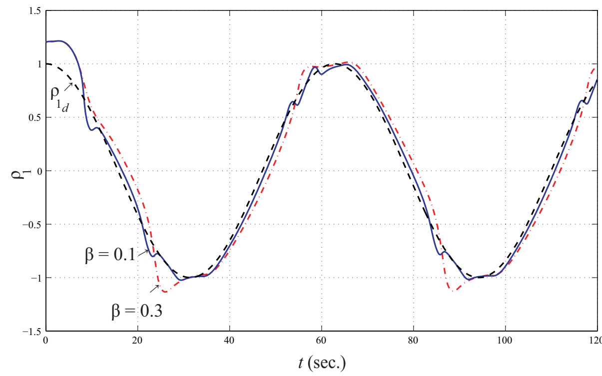

Figure 10.1: MRP ρ_1 vs. t: $\beta = 0.1, 0.3$.

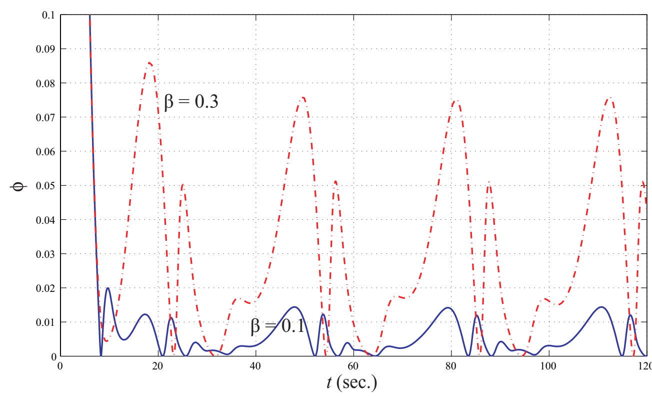

Figure 10.2: Attitude deviation norm measure ϕ vs. t: $\beta = 0.1, 0.3$.

trajectories are chosen to be $\rho_{d_i}(t) = \cos 0.1t$, $i = 1, 2, 3$, and $Q = I_{3\times3}$. All figures correspond to $\delta = 0.01$ and two values of $\beta = 0.1, 0.3$. Figure 10.1 shows the response of $\rho_1(t)$. Similar figures are obtained for $\rho_2(t)$ and $\rho_2(t)$, but are not shown. Figures 10.2 and 10.3 reveal the tradeoff between generalized inversion stability robustness against singularity and closed loop system tracking performance. The effect of changing δ on the closed loop response is minor.

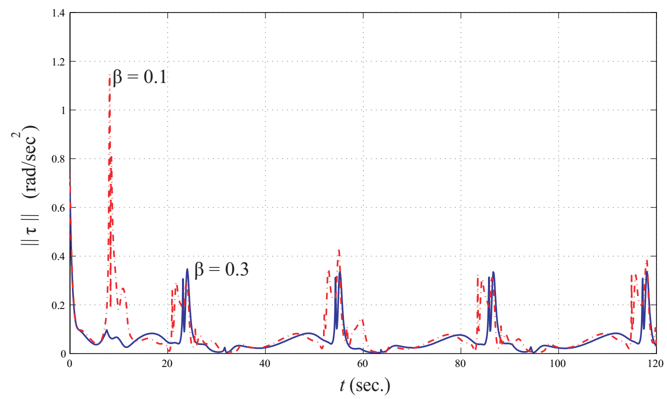

Figure 10.3: Scaled controls moments norm $\| \tau(\rho,\omega,t) \|_2$ vs. t: $\beta = 0.1, 0.3$.

References

[1] Bajodah, Abdulrahman H., Hodges, Dewey H., and Chen, Ye-Hwa. Inverse dynamics of servoconstraints based on the generalized inverse. *Nonlinear Dynamics* **39**(1-2) (2005) 179–196.

[2] Mitropolskii, Yu.A., Borne, P., and Martynyuk, A.A. Personage in Science: Alexander Mikhaylovich Liapunov. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 113–120.

[3] Shuster, M. D. A survey of attitude representation. *Journal of Astronautical Sciences* **41**(4)(1993) 439–517.

[4] Udwadia, Firdaus E. and Kalaba, Robert E. *Analytical Dynamics: A New Approach*. Cambridge University Press, 2003.

[5] Bernstein, Dennis S. *Matrix Mathematics*. Princeton University Press, 2005.

[6] Khalil, Hassan K. *Nonlinear Systems*. Prentice–Hall Inc., 3rd edition, 2002.

Nonlinear Dynamics and Systems Theory, **9**(2) (2009) 161–169

Some Linear and Nonlinear Integral Inequalities on Time Scales in Two Independent Variables

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Abstract: We establish some linear and nonlinear integral inequalities of Gronwall–Bellman–Bihari type for functions with two independent variables on general time scales. The results are illustrated with examples, obtained by fixing the time scales to concrete ones. An estimation result for the solution of a partial delta dynamic equation is given as an application.

Keywords: *integral inequalities; Gronwall–Bellman–Bihari inequalities; time scales; two independent variables.*

Mathematics Subject Classification (2000): 26D15, 45K05.

1 Introduction

Inequalities have always been of great importance for the development of several branches of mathematics. For instance, in approximation theory and numerical analysis, linear and nonlinear inequalities, in one and more than one variable, play an important role in the estimation of approximation errors [12].

Time scales, which are defined as nonempty closed subsets of the real numbers, are the basic but fundamental ingredient that permits to define a rich calculus that encompasses both differential and difference tools [8, 9]. At the same time one gains more (cf., e.g., Corollary 3.1). For an introduction to the calculus on time scales we refer the reader to [6] and [4, 5], respectively for functions of one and more than one independent variables.

Integral inequalities of Gronwall–Bellman–Bihari type for functions of a single variable on a time scale can be found in [2, 3, 7, 11, 14]. To the best of the authors knowledge, no such results exist in the literature of time scales when functions of two independent variables are considered. It is our aim to obtain here a first insight on this type of inequalities.

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2 Linear Inequalities

Throughout the text we assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales with at least two points and consider the time scales intervals $\tilde{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\tilde{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$, and $a_2 \in \mathbb{T}_2$. We also use the notations $\mathbb{R}_0^+ = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $e_p(t, s)$ denotes the usual exponential function on time scales with $p \in \mathcal{R}$, i.e., p is a regressive function [6].

Theorem 2.1 *Let $u(t_1, t_2), a(t_1, t_2), f(t_1, t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$ with $a(t_1, t_2)$ nondecreasing in each of its variables. If*

$$u(t_1, t_2) \leq a(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2 \tag{1}$$

for $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t_1, t_2) \leq a(t_1, t_2) e_{\int_{a_2}^{t_2} f(t_1, s_2) \Delta_2 s_2}(t_1, a_1), \quad (t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \tag{2}$$

Proof Since $a(t_1, t_2)$ is nondecreasing on $(t_1, t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, inequality (1) implies, for an arbitrary $\varepsilon > 0$, that

$$r(t_1, t_2) \leq 1 + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(s_1, s_2) r(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2,$$

where $r(t_1, t_2) = \frac{u(t_1, t_2)}{a(t_1, t_2) + \varepsilon}$. Define $v(t_1, t_2)$ by the right hand side of the last inequality. Then,

$$\frac{\partial}{\Delta_2 t_2} \left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right) = f(t_1, t_2) r(t_1, t_2) \leq f(t_1, t_2) v(t_1, t_2), \ (t_1, t_2) \in \tilde{\mathbb{T}}_1^k \times \tilde{\mathbb{T}}_2^k. \tag{3}$$

From (3), and taking into account that $v(t_1, t_2)$ is positive and nondecreasing, we obtain

$$\frac{v(t_1, t_2) \frac{\partial}{\Delta_2 t_2} \left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right)}{v(t_1, t_2) v(t_1, \sigma_2(t_2))} \leq f(t_1, t_2),$$

from which it follows that

$$\frac{v(t_1, t_2) \frac{\partial}{\Delta_2 t_2} \left(\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \right)}{v(t_1, t_2) v(t_1, \sigma_2(t_2))} \leq f(t_1, t_2) + \frac{\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} \frac{\partial v(t_1, t_2)}{\Delta_2 t_2}}{v(t_1, t_2) v(t_1, \sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2} \left(\frac{\frac{\partial v(t_1, t_2)}{\Delta_1 t_1}}{v(t_1, t_2)} \right) \leq f(t_1, t_2).$$

Delta integrating with respect to the second variable from a_2 to t_2 (we observe that t_2 can be the maximal element of $\tilde{\mathbb{T}}_2$, if it exists), and noting that $\frac{\partial v(t_1, t_2)}{\Delta_1 t_1} |_{(t_1, a_2)} = 0$, we have

$$\frac{\frac{\partial v(t_1, t_2)}{\Delta_1 t_1}}{v(t_1, t_2)} \leq \int_{a_2}^{t_2} f(t_1, s_2) \Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \le \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 v(t_1,t_2).$$

Fixing $t_2 \in \tilde{\mathbb{T}}_2$ arbitrarily, we have that $p(t_1) := \int_{a_2}^{t_2} f(t_1,s_2)\Delta_2 s_2 \in \mathcal{R}^+$. Because $v(a_1,t_2)=1$, by [2, Theorem 5.4] $v(t_1,t_2) \le e_p(t_1,a_1)$. Inequality (2) follows from

$$u(t_1,t_2) \le [a(t_1,t_2)+\varepsilon]v(t_1,t_2)$$

and the arbitrariness of ε. □

Corollary 2.1 (cf. Lemma 2.1 of [10]) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and assume that the functions $u(x,y), a(x,y), f(x,y) \in C([x_0,\infty)\times[y_0,\infty), \mathbb{R}_0^+)$ with $a(x,y)$ nondecreasing in its variables. If*

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(t,s)u(t,s)dtds$$

for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$, then

$$u(x,y) \le a(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} f(t,s)dtds\right)$$

for $(x,y) \in [x_0,\infty)\times[y_0,\infty)$.

Corollary 2.2 (cf. Theorem 2.1 of [13]) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and assume that the functions $u(m,n), a(m,n), f(m,n)$ are nonnegative and that $a(m,n)$ is nondecreasing for $m \in [m_0,\infty)\cap\mathbb{Z}$ and $n \in [n_0,\infty)\cap\mathbb{Z}$, $m_0, n_0 \in \mathbb{Z}$. If*

$$u(m,n) \le a(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)u(s,t)$$

for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$, then

$$u(m,n) \le a(m,n)\prod_{s=m_0}^{m-1}\left[1+\sum_{t=n_0}^{n-1} f(s,t)\right]$$

for all $(m,n) \in [m_0,\infty)\cap\mathbb{Z}\times[n_0,\infty)\cap\mathbb{Z}$.

Remark 2.1 We note that, following the same steps of the proof of Theorem 2.1, one can obtained other bound on the function u, namely

$$u(t_1,t_2) \le a(t_1,t_2)e_{\int_{a_1}^{t_1} f(s_1,t_2)\Delta_1 s_1}(t_2,a_2). \tag{4}$$

When $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then the bounds in (2) and (4) coincide (see Corollary 2.1). If, for example, we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the bounds obtained can be different. Moreover, at different points one bound can be sharper than the other and vice-versa (see Example 2.1).

Example 2.1 *Let $f(t_1,t_2)$ be a function defined by $f(0,0) = 1/4$, $f(1,0) = 1/5$, $f(2,0) = 1$, $f(0,1) = 1/2$, $f(1,1) = 0$, and $f(2,1) = 5$. Set $a_1 = a_2 = 0$. Then, from (2) we get*

$$u(2,1) \leq a(2,1)\frac{3}{2}, \quad u(3,2) \leq a(3,2)\frac{147}{10},$$

while from (4) we get

$$u(2,1) \leq a(2,1)\frac{29}{20}, \quad u(3,2) \leq a(3,2)\frac{637}{40}.$$

Other interesting corollaries can be obtained from Theorem 2.1.

Corollary 2.3 *Let $\mathbb{T}_1 = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, for some $q > 1$, and $\mathbb{T}_2 = \mathbb{R}$. Assume that the functions $u(t,x)$, $a(t,x)$ and $f(t,x)$ satisfy the hypothesis of Theorem 2.1 for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$ with $a_1 = 1$ and $a_2 = 0$. If*

$$u(t,x) \leq a(t,x) + \sum_{s=1}^{t/q}(q-1)s\int_0^x f(s,\tau)u(s,\tau)d\tau$$

for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t,x) \leq a(t,x)\prod_{s=1}^{t/q}\left[1 + (q-1)s\int_0^x f(s,\tau)d\tau\right]$$

for all $(t,x) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$.

We now generalize Theorem 2.1. If in Theorem 2.2 we let $f \equiv 1$ and g not depending on the first two variables, then we obtain Theorem 2.1.

Theorem 2.2 *Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with a and f nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2) \in C(S, \mathbb{R}_0^+)$ be nondecreasing in t_1 and t_2, where $S = \{(t_1,t_2,s_1,s_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 : a_1 \leq s_1 \leq t_1, a_2 \leq s_2 \leq t_2\}$. If*

$$u(t_1,t_2) \leq a(t_1,t_2) + f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2) \leq a(t_1,t_2)e_{\int_{a_2}^{t_2} f(t_1,t_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1)\,, \quad (t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2. \qquad (5)$$

Proof We start by fixing arbitrary numbers $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$, and considering the following function defined on $[a_1,t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2,t_2^*] \cap \tilde{\mathbb{T}}_2$ for an arbitrary $\varepsilon > 0$:

$$v(t_1,t_2) = a(t_1^*,t_2^*) + \varepsilon + f(t_1^*,t_2^*)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1^*,t_2^*,s_1,s_2)u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2\,.$$

From our hypothesis we see that

$$u(t_1,t_2) \leq v(t_1,t_2), \text{ for all } (t_1,t_2) \in [a_1,t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2,t_2^*] \cap \tilde{\mathbb{T}}_2.$$

Moreover, delta differentiating with respect to the first variable and then with respect to the second, we obtain

$$\begin{aligned}\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) &= f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)u(t_1,t_2)\\ &\le f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)v(t_1,t_2),\end{aligned}$$

for all $(t_1,t_2)\in[a_1,t_1^*]^k\cap\tilde{\mathbb{T}}_1\times[a_2,t_2^*]^k\cap\tilde{\mathbb{T}}_2$. From this last inequality, we can write

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}\le f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)\,.$$

Hence,

$$\frac{v(t_1,t_2)\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}\le f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2)+\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\frac{\partial v(t_1,t_2)}{\Delta_2 t_2}}{v(t_1,t_2)v(t_1,\sigma_2(t_2))}.$$

The previous inequality can be rewritten as

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)}\right)\le f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,t_2).$$

Delta integrating with respect to the second variable from a_2 to t_2 and noting that $\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\mid_{(t_1,a_2)}=0$, we have

$$\frac{\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}}{v(t_1,t_2)}\le\int_{a_2}^{t_2}f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2,$$

that is,

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\le\int_{a_2}^{t_2}f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2 v(t_1,t_2).$$

Fix $t_2=t_2^*$ and put $p(t_1):=\int_{a_2}^{t_2^*}f(t_1^*,t_2^*)g(t_1^*,t_2^*,t_1,s_2)\Delta_2 s_2\in\mathcal{R}^+$. By [2, Theorem 5.4]

$$v(t_1,t_2^*)\le(a(t_1^*,t_2^*)+\varepsilon)e_p(t_1,a_1).$$

Letting $t_1=t_1^*$ in the above inequality, and remembering that t_1^*, t_2^* and ε are arbitrary, it follows (5). □

3 Nonlinear Inequalities

Theorem 3.1 *Let $u(t_1,t_2)$ and $f(t_1,t_2)\in C(\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2,\mathbb{R}_0^+)$. Moreover, let $a(t_1,t_2)\in C(\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2,\mathbb{R}^+)$ be a nondecreasing function in each of the variables. If p and q are two positive real numbers such that $p\ge q$ and if*

$$u^p(t_1,t_2)\le a(t_1,t_2)+\int_{a_1}^{t_1}\int_{a_2}^{t_2}f(s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 \tag{6}$$

for $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2)\le a^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2}f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}},\ (t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2. \tag{7}$$

Proof Since $a(t_1,t_2)$ is positive and nondecreasing on $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, inequality (6) implies that

$$u^p(t_1,t_2)\le a(t_1,t_2)\left(1+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2\right).$$

Define $v(t_1,t_2)$ on $\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$ by

$$v(t_1,t_2)=1+\int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2.$$

Then

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)=f(t_1,t_2)\frac{u^q(t_1,t_2)}{a(t_1,t_2)}\le f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v^{\frac{q}{p}}(t_1,t_2)\,,$$

and noting that $v^{\frac{q}{p}}(t_1,t_2)\le v(t_1,t_2)$ we conclude that

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right)\le f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,t_2)v(t_1,t_2).$$

We can now follow the same procedure as in the proof of Theorem 2.1 to obtain

$$v(t_1,t_2)\le e_p(t_1,a_1),$$

where $p(t_1)=\int_{a_2}^{t_2} f(t_1,s_2)a^{\frac{q}{p}-1}(t_1,s_2)\Delta_2 s_2$. Noting that

$$u(t_1,t_2)\le a^{\frac{1}{p}}(t_1,t_2)v^{\frac{1}{p}}(t_1,t_2),$$

we obtain the desired inequality (7). $\square$

Theorem 3.2 *Let $u(t_1,t_2),a(t_1,t_2),f(t_1,t_2)\in C(\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2,\mathbb{R}_0^+)$, with a and f nondecreasing in each of the variables and $g(t_1,t_2,s_1,s_2)\in C(S,\mathbb{R}_0^+)$ be nondecreasing in t_1 and t_2, where $S=\{(t_1,t_2,s_1,s_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2\times\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2 : a_1\le s_1\le t_1, a_2\le s_2\le t_2\}$. If p and q are two positive real numbers such that $p\ge q$ and if*

$$u^p(t_1,t_2)\le a(t_1,t_2)+f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)u^q(s_1,s_2)\Delta_1 s_1\Delta_2 s_2 \tag{8}$$

for all $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, then

$$u(t_1,t_2)\le a^{\frac{1}{p}}(t_1,t_2)\left[e_{\int_{a_2}^{t_2} f(t_1,t_2)a^{\frac{q}{p}-1}(t_1,s_2)g(t_1,t_2,t_1,s_2)\Delta_2 s_2}(t_1,a_1)\right]^{\frac{1}{p}}$$

for all $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$.

Proof Since $a(t_1,t_2)$ is positive and nondecreasing on $(t_1,t_2)\in\tilde{\mathbb{T}}_1\times\tilde{\mathbb{T}}_2$, inequality (8) implies that

$$u^p(t_1,t_2)\le a(t_1,t_2)\left(1+f(t_1,t_2)\int_{a_1}^{t_1}\int_{a_2}^{t_2} g(t_1,t_2,s_1,s_2)\frac{u^q(s_1,s_2)}{a(s_1,s_2)}\Delta_1 s_1\Delta_2 s_2\right).$$

Fix $t_1^* \in \tilde{\mathbb{T}}_1$ and $t_2^* \in \tilde{\mathbb{T}}_2$ arbitrarily and define a function $v(t_1, t_2)$ on $[a_1, t_1^*] \cap \tilde{\mathbb{T}}_1 \times [a_2, t_2^*] \cap \tilde{\mathbb{T}}_2$ by

$$v(t_1,t_2) = 1 + f(t_1^*, t_2^*) \int_{a_1}^{t_1} \int_{a_2}^{t_2} g(t_1^*, t_2^*, s_1, s_2) \frac{u^q(s_1,s_2)}{a(s_1,s_2)} \Delta_1 s_1 \Delta_2 s_2.$$

Then

$$\begin{aligned}\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) &= f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) \frac{u^q(t_1,t_2)}{a(t_1,t_2)} \\ &\leq f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1,t_2) v^{\frac{q}{p}}(t_1,t_2).\end{aligned}$$

Since $v^{\frac{q}{p}}(t_1,t_2) \leq v(t_1,t_2)$, we have that

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial v(t_1,t_2)}{\Delta_1 t_1}\right) \leq f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, t_2) a^{\frac{q}{p}-1}(t_1,t_2) v(t_1,t_2).$$

We can follow the same steps as done before to reach the inequality

$$\frac{\partial v(t_1,t_2)}{\Delta_1 t_1} \leq \int_{a_2}^{t_2} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1,s_2) \Delta_2 s_2 v(t_1,t_2).$$

Fix $t_2 = t_2^*$ and put $p(t_1) := \int_{a_2}^{t_2^*} f(t_1^*, t_2^*) g(t_1^*, t_2^*, t_1, s_2) a^{\frac{q}{p}-1}(t_1,s_2) \Delta_2 s_2 \in \mathcal{R}^+$. Again, an application of [2, Theorem 5.4] gives

$$v(t_1, t_2^*) \leq e_p(t_1, a_1),$$

and putting $t_1 = t_1^*$ we obtain the desired inequality. $\square$

We end this section by considering a particular time scale. Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers and let

$$t_0^\alpha \in \mathbb{R}, \quad t_k^\alpha = t_0^\alpha + \sum_{n=1}^{k} \alpha_n, \ k \in \mathbb{N},$$

where we assume that $\lim_{k\to\infty} t_k^\alpha = \infty$. Then, we define the following time scale: $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$. For $p \in \mathcal{R}$ we have (cf. [1, Example 4.6]):

$$e_p(t_k^\alpha, t_0^\alpha) = \prod_{n=1}^{k} (1 + \alpha_n p(t_{n-1})), \text{ for all } k \in \mathbb{N}_0. \tag{9}$$

Given two sequences $\{\alpha_k, \beta_k\}_{k\in\mathbb{N}}$ and two numbers $t_0^\alpha, t_0^\beta \in \mathbb{R}$ as above, we define the two time scales $\mathbb{T}^\alpha = \{t_k^\alpha : k \in \mathbb{N}_0\}$ and $\mathbb{T}^\beta = \{t_k^\beta : k \in \mathbb{N}_0\}$. We state now our last corollary.

Corollary 3.1 *Let $u(t,s)$, $a(t,s)$, and $f(t,s)$, defined on $\mathbb{T}^\alpha \times \mathbb{T}^\beta$, be nonnegative with a and f nondecreasing. Further, let $g(t,s,\tau,\xi)$, where $(t,s,\tau,\xi) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta \times \mathbb{T}^\alpha \times \mathbb{T}^\beta$ with $\tau \leq t$ and $\xi \leq s$, be nonnegative and nondecreasing in the first two variables t and s. If p and q are two positive real numbers such that $p \geq q$ and if*

$$u^p(t,s) \leq a(t,s) + f(t,s) \sum_{\tau\in[t_0^\alpha,t)} \sum_{\xi\in[t_0^\beta,s)} \mu^\alpha(\tau)\mu^\beta(\xi) g(t,s,\tau,\xi) u^q(\tau,\xi) \tag{10}$$

for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$*, where* μ^α *and* μ^β *are the graininess functions of* $\mathbb{T}^\alpha$ *and* $\mathbb{T}^\beta$*, respectively, then*

$$u(t,s) \le a^{\frac{1}{p}}(t,s)\left[e_{\int_{t_0^\beta}^{s} f(t,s)a^{\frac{q}{p}-1}(t,\xi)g(t,s,t,\xi)\Delta^\beta\xi}(t,t_0^\alpha)\right]^{\frac{1}{p}}$$

for all $(t,s) \in \mathbb{T}^\alpha \times \mathbb{T}^\beta$*, where* e *is given by (9).*

Remark 3.1 In (10) we are slightly abusing on notation by considering $[t_0^\alpha, t) = [t_0^\alpha, t) \cap \mathbb{T}^\alpha$ and $[t_0^\beta, t) = [t_0^\beta, t) \cap \mathbb{T}^\beta$.

4 An Application

Let us consider the partial delta dynamic equation

$$\frac{\partial}{\Delta_2 t_2}\left(\frac{\partial u^2(t_1,t_2)}{\Delta_1 t_1}\right) = F(t_1,t_2,u(t_1,t_2)) \tag{11}$$

under given initial boundary conditions

$$u^2(t_1,0) = g(t_1),\ u^2(0,t_2) = h(t_2),\ g(0)=0,\ h(0)=0, \tag{12}$$

where we are assuming $a_1 = a_2 = 0$, $F \in C(\tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \mathbb{R}_0^+, \mathbb{R}_0^+)$, $g \in C(\tilde{\mathbb{T}}_1, \mathbb{R}_0^+)$, $h \in C(\tilde{\mathbb{T}}_2, \mathbb{R}_0^+)$, with g and h nondecreasing functions and positive on their domains except at zero.

Theorem 4.1 *Assume that on its domain,* F *satisfies*

$$F(t_1,t_2,u) \le t_2 u.$$

If $u(t_1,t_2)$ *is a solution of the IBVP (11)-(12) for* $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$*, then*

$$u(t_1,t_2) \le \sqrt{(g(t_1)+h(t_2))}\left[e_{\int_0^{t_2} s_2(g(t_1)+h(s_2))^{-\frac{1}{2}}\Delta_2 s_2}(t_1,0)\right]^{\frac{1}{2}} \tag{13}$$

for $(t_1,t_2) \in \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$*, except at the point* $(0,0)$*.*

Proof Let $u(t_1,t_2)$ be a solution of the IBVP (11)-(12). Then, it satisfies the following delta integral equation:

$$u^2(t_1,t_2) = g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} F(s_1,s_2,u(s_1,s_2))\Delta_1 s_1 \Delta_2 s_2.$$

The hypothesis on F imply that

$$u^2(t_1,t_2) \le g(t_1) + h(t_2) + \int_0^{t_1}\int_0^{t_2} s_2 u(s_1,s_2)\Delta_1 s_1 \Delta_2 s_2.$$

An application of Theorem 3.1 with $a(t_1,t_2) = g(t_1)+h(t_2)$ and $f(t_1,t_2) = t_2$ gives (13). □

References

[1] Agarwal, R., Bohner, M., O'Regan, D. and Peterson, A. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.* **141** (1-2) (2002) 1–26.

[2] Agarwal, R., Bohner, M. and Peterson, A. Inequalities on time scales: a survey. *Math. Inequal. Appl.* **4** (4) (2001) 535–557.

[3] Akin-Bohner, E., Bohner, M. and Akin, F. Pachpatte inequalities on time scales. *JIPAM. J. Inequal. Pure Appl. Math.* **6** (1) (2005) Article 6, 23 pp. (electronic)

[4] Bohner, M. and Guseinov, G.Sh. Partial differentiation on time scales. *Dynam. Systems Appl.* **13** (3-4) (2004) 351–379.

[5] Bohner, M. and Guseinov, G.Sh. Multiple integration on time scales. *Dynam. Systems Appl.* **14** (3-4) (2005) 579–606.

[6] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales. An Introduction with Applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.

[7] Ferreira, R.A.C. and Torres, D.F.M. Generalizations of Gronwall–Bihari inequalities on time scales. *J. Difference Equ. Appl.* (in press)

[8] Hilger, S. Analysis on measure chains — a unified approach to continuous and discrete calculus. *Results Math.* **18**(1-2) (1990) 18–56.

[9] Hilger, S. Differential and difference calculus — unified. *Proceedings of the Second World Congress of Nonlinear Analysts*, Part 5 (Athens, 1996). *Nonlinear Anal.* **30** (5) (1997) 2683–2694.

[10] Khellaf, H. On integral inequalities for functions of several independent variables. *Electron. J. Differential Equations* **2003** (123) (2003) 12 pp.

[11] Özgün, S.A., Zafer, A. and Kaymakçalan, B. Gronwall and Bihari type inequalities on time scales. *Advances in difference equations* (Veszprém, 1995), 481–490, Gordon and Breach, Amsterdam, 1997.

[12] Pachpatte, B.G. *Integral and finite difference inequalities and applications*. North–Holland Mathematics Studies, 205. Elsevier Science B.V., Amsterdam, 2006.

[13] Salem, Sh. and Raslan, K.R. Some new discrete inequalities and their applications. *JIPAM. J. Inequal. Pure Appl. Math.* **5** (1) (2004) Article 2, 9 pp. (electronic)

[14] Wong, F.-H., Yeh, C.-C. and Hong, C.-H. Gronwall inequalities on time scales. *Math. Inequal. Appl.* **9** (1) (2006) 75–86.

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An LMI Criterion for the Global Stability Analysis of Nonlinear Polynomial Systems

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Abstract: This paper presents an original practical criterion of global stability analysis of nonlinear polynomial systems. This criterion derived from the application of the Lyapunov direct method with a quadratic function generalizes the famous Lyapunov stability condition for linear systems. Useful mathematical transformations have allowed the formulation of the obtained conditions as an LMI (Linear Matrix Inequalities) problem according to the polynomial system parameters.

Keywords: *nonlinear polynomial systems; Lyapunov methods; global stability analysis; LMI approach.*

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1 Introduction

The problem of stability analysis of nonlinear systems has received considerable attention in the field of research in automatic control and different approaches have been proposed in the literature related with this subject [1]– [16]. The polynomial technique of studying stability of nonlinear systems is one of the most important developed approaches. It is based on the modeling of the considered nonlinear analytical systems by a polynomial system [17]– [27]. Notice that the class of polynomial systems is large enough to include the description of numerous physical processes such as electrical machines and robot manipulators [28]. Moreover, the description of polynomial systems can be simplified using the Kronecker product and power of vectors and matrices [17, 29, 30].

In previous works, sufficient algebraic conditions of global asymptotic stability of polynomial systems have been derived using the direct Lyapunov method with a quadratic

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function [17, 18, 25, 29, 31, 32] or non quadratic function as polynomial or monomial Lyapunov functions [33, 34]. The advantage of the proposed criteria is that they are expressed according to the studied polynomial system parameters, generalizing the famous Lyapunov condition known for the linear systems. However, the implementation of the general form of the derived stability conditions of polynomial systems requires the resolution of nonlinear matrix inequalities [35, 36]. To overcome this difficulty, we propose in this paper a new development which leads to the formulation of a practical LMI stability condition for polynomial systems.

This paper is organized as follows: Section 2 is concerned with the description of the studied systems and some useful notations. Then, in the third section we present the derived global stability condition for polynomial systems. The fourth section shows how the obtained condition can be implemented as an LMI problem. Section 5 is devoted to a numerical example which illustrates the availability of the proposed approach.

2 System Description and Notations

2.1 System description

The considered nonlinear polynomial systems are described by the following state equation:

$$\dot{X} = f(X), \tag{2.1}$$

where $f(X)$ is a polynomial vector function of X.

$$f(X) = \sum_{i=1}^{r} A_i X^{[i]} = \sum_{i=1}^{r} \tilde{A}_i \tilde{X}^{[i]} \tag{2.2}$$

with $X = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $X^{[i]}$ is the Kronecker power of the vector X defined as:

$$\begin{cases} X^{[0]} = 1, \\ X^{[i]} = X^{[i-1]} \otimes X = X \otimes X^{[i-1]} \quad for \quad i \geq 1, \end{cases} \tag{2.3}$$

$\otimes$ designates the symbol of the Kronecker product [30], $\tilde{X}^{[i]}_{i=1,\ldots,r} \in \mathbb{R}^{n_i}$, $n_i = \begin{pmatrix} n+i-1 \\ i \end{pmatrix}$ is the nun-redundant Kronecker power of the state vector X defined as:

$$\begin{aligned} &\tilde{X}^{[1]} = X^{[1]} = X, \\ &\forall\, i \geq 2,\ \tilde{X}^{[i]} = [x_1^i,\ x_1^{i-1}x_2,\ \ldots,\ x_1^{i-1}x_n,\ \ldots,\ x_1^{i-2}x_n^2,\ \ldots,\ x_1^{i-3}\,x_2^3,\ \ldots,\ x_n^i]^T, \end{aligned} \tag{2.4}$$

where the repeated components of the redundant (ith-power) $X^{[i]}$ are omitted, $A_{i,i=1,\ldots,r} \in \mathbb{R}^{n \times n^i}$ (resp. $\tilde{A}_i \in \mathbb{R}^{n \times n_i}$) are constant matrices. The polynomial order r is considered odd: $r = 2s - 1$, with $s \in \mathbb{N}^*$. Let's recall that this class of systems describes a large set of processes as electrical machines and robot manipulators and that any analytical system can be approached by a polynomial model.

2.2 Notations

In this section, we introduce some useful notations and needed rules and functions. Let the matrices and vectors be of the following dimensions: $A(p \times q)$, $B(r \times s)$, $C(q \times f)$, $E(n \times p)$, $X(n \times 1)$, $Y(m \times 1)$.

(i) We consider the following notations: I_n is an $(n \times n)$ identity matrix; $0_{n \times m}$ is an $(n \times m)$ zero matrix; 0 is a zero matrix of convenient dimension; A^T is a transpose of matrix A; $A > 0 (A \geq 0)$ is a symmetric positive definite (semi-definite) matrix; e_k^q is a q dimensional unit vector which has 1 in the kth element and zero elsewhere.

(ii) The relation between the redundant and the nun-redundant Kronecker power of the state vector X can be stated as follows:

$$\left\{ \begin{array}{lccc} \forall i \in \mathbb{N} \ \exists & T_i & \in & \mathbb{R}^{n^i \times n_i} \\ & X^{[i]} & = & T_i \tilde{X}^{[i]} \end{array} \right\}, \tag{2.5}$$

where (n_i) stands for the binomial coefficient. A procedure of the determination of the matrix T_i is given in [37].

(iii) The permutation matrix denoted by $(U_{n \times m})$ is defined as:

$$U_{n \times m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(e_i^n \cdot e_j^{mT} \right) \otimes \left(e_j^m \cdot e_i^{nT} \right). \tag{2.6}$$

This matrix is square $(nm \times nm)$ and has precisely a single 1 in each row and in each column. Among the main properties of this matrix presented in [30], we recall the following useful ones:

$$(B \otimes A) = U_{r \times p} (A \otimes B) U_{q \times s}, \tag{2.7}$$

$$(X \otimes Y) = U_{n \times m} (Y \otimes X), \tag{2.8}$$

$$\forall i \leq k \quad X^{[k]} = U_{n^i \times n^{k-i}} X^{[k]}. \tag{2.9}$$

(iv) An important vector valued function of matrix denoted by $vec(.)$ was defined in [30] as follows:

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_q \end{bmatrix} \in \mathbb{R}^{p \times q}, \quad A_i \in \mathbb{R}^p, \quad i \in \{1, ..., q\},$$

$$vec(A) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix} \in \mathbb{R}^{pq}.$$

We recall the following useful rules [30] of *vec-function*:

$$vec(EAC) = (C^T \otimes E) vec(A), \tag{2.10}$$

$$vec(A^T) = U_{p \times q} vec(A). \tag{2.11}$$

(v) A special function $mat_{(n,m)}(.)$ can be defined as follows:
If V is a vector of dimension $p = nm$ then $M = mat_{(n,m)}(V)$ is the $(n \times m)$ matrix verifying: $V = vec(M)$.

(vi) For a polynomial vectorial function:

$$f(X) = \sum_{i=1}^{r} A_i X^{[i]}, \tag{2.12}$$

where $X \in \mathbb{R}^n$, $A_{i,\, i=1,\ldots,r}$ are $(n \times n^i)$ constant matrices and $r = 2s-1$, $s \in \mathbb{N}^*$, $\mathcal{M}(f)$ designates the set of matrices defined by:

$$\mathcal{M}(f) = \{\mathcal{M}_\lambda(f) \in \mathbb{R}^{\upsilon \times \upsilon} \quad ; \quad \lambda = [\lambda_{ij}] \in \mathbb{R}^{s \times s}\} \tag{2.13}$$

such that:

$$\mathcal{M}_\lambda(f) = \begin{bmatrix} \lambda_{11}M_{11} & \lambda_{12}M_{12} & \ldots & \lambda_{1k}M_{1k} & \ldots & \lambda_{1s}M_{1s} \\ \lambda_{21}M_{21} & \lambda_{22}M_{22} & \ldots & \lambda_{2k}M_{2k} & \ldots & \lambda_{2s}M_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{k1}M_{k1} & \lambda_{k2}M_{k2} & \vdots & \lambda_{kk}M_{kk} & \vdots & \lambda_{ks}M_{ks} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{s1}M_{s1} & \lambda_{s2}M_{s2} & \ldots & \lambda_{sk}M_{sk} & \ldots & \lambda_{ss}M_{ss} \end{bmatrix}, \tag{2.14}$$

$\upsilon = n + n^2 + \ldots + n^s$, and

- for $k = 1, \ldots, r = 2s-1$,
- for $j = g_k, \ldots, h_k$ where $g_k = \sup(1, k+1-s)$ and $h_k = \inf(s, k)$

we have:

$$M_{k+1-j,j} = \begin{bmatrix} mat_{(n^{k-j}, n^j)}\left(A_k^{1T}\right) \\ mat_{(n^{k-j}, n^j)}\left(A_k^{2T}\right) \\ \vdots \\ mat_{(n^{k-j}, n^j)}\left(A_k^{nT}\right) \end{bmatrix}, \tag{2.15}$$

A_k^i is the i^{th} row of the matrix A_k:

$$A_k = \begin{bmatrix} A_k^1 \\ A_k^2 \\ \vdots \\ A_k^n \end{bmatrix}. \tag{2.16}$$

Notice that, for all integer numbers i and j such that $1 \leq i, j \leq s$, there exist $k \in \mathbb{N}^*$ such that $1 \leq k \leq 2s-1$, $i = k+1-j$ and $g_k \leq j \leq h_k$.
λ_{ij} are arbitrary reals verifying:

$$\sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} = 1. \tag{2.17}$$

(vii) We introduce the matrix $\mathcal{R}$ defined by:

$$\mathcal{R} = \tau_1^{+[2]} \cdot \mathcal{U} \cdot \mathcal{H} \cdot \tau_2, \tag{2.18}$$

where

$$\tau_1 = \begin{bmatrix} T_1 & & & & \\ & T_2 & & 0 & \\ & & T_3 & & \\ & 0 & & \ddots & \\ & & & & T_s \end{bmatrix}, \tag{2.19}$$

$$\tau_2 = \begin{bmatrix} T_2 & & & 0 \\ & T_3 & & \\ & & \ddots & \\ 0 & & & T_{2s} \end{bmatrix}, \tag{2.20}$$

$$\mathcal{U} = \begin{bmatrix} U_{n\times\eta_0} & & & 0 \\ & U_{n^2\times\eta_0} & & \\ & & \ddots & \\ 0 & & & U_{n^s\times\eta_0} \end{bmatrix}, \tag{2.21}$$

$$\mathcal{H} = \begin{bmatrix} I_{\eta_1} & & & & 0 \\ 0_{\eta_2\times\eta_1} & I_{\eta_2} & & & \\ 0_{\eta_3\times(\eta_1+\eta_2)} & & I_{\eta_3} & & \\ \vdots & & & \ddots & \\ 0_{\eta_s\times(\eta_1+\eta_2+\ldots+\eta_{s-1})} & & & & I_{\eta_s} \end{bmatrix}, \tag{2.22}$$

for $j = 1, \ldots.., s, :\ \eta_j = n^j \cdot \left(\sum_{i=1}^{s} n^i \right)$,

τ_1^+ is the Moore-Penrose pseudo-inverse of τ_1.

We note Γ is the matrix defined by:

$$\Gamma = \left(I_{\eta^2} + U_{\eta\times\eta} \right) \left(\mathcal{R}^{+\mathrm{T}} \mathcal{R}^T - I_{\eta^2} \right) \tag{2.23}$$

with $\eta = \sum_{j=1}^{s} n_j = \sum_{j=1}^{s} \begin{pmatrix} n+j-1 \\ j \end{pmatrix}$ and $\mathcal{R}^+$ is the Moore-Penrose pseudo-inverse of $\mathcal{R}$.

$\beta = rank(\Gamma)$ and $C_{i,\, i=1,\ldots,\beta}$ are β linearly independent columns of Γ.

(iix) For a $(n \times n)$ matrix P, we define the $(\upsilon \times \upsilon)$ matrix $\mathcal{D}_s(P)$ as:

$$\mathcal{D}_s(P) = \begin{bmatrix} P & & & 0 \\ & P \otimes I_n & & \\ & & \ddots & \\ & 0 & & P \otimes I_{n^{s-1}} \end{bmatrix}. \tag{2.24}$$

Notice that if P is a definite symmetric positive matrix then so is $\mathcal{D}_s(P)$.

3 Stability Criterion of Polynomial Systems

We consider the analytical nonlinear autonomous systems described by the following polynomial state-space equation:

$$\dot{X} = f(X) = \sum_{k=1}^{r} A_k X^{[k]}, \quad r = 2s-1. \tag{3.1}$$

The Lyapunov's direct method leads to a sufficient condition of the global asymptotic stability of the equilibrium ($X = 0$) of the polynomial system (3.1). This condition is stated in the following theorem.

Theorem 1 *Consider the nonlinear polynomial system defined by the equation (3.1) where the integer r is odd: $r = 2s-1$. If there exist:*

- *an $(n \times n)$-symmetric positive definite matrix P,*
- *an $(s \times s)$-matrix $\lambda = [\lambda_{ij}]$ verifying $\sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} = 1$,*
- *arbitrary parameters $\mu_{i,i=1,\dots,\beta} \in \mathbb{R}$*

such that the $(\eta \times \eta)$ symmetric matrix $\mathcal{Q}$ defined by:

$$\mathcal{Q} \quad = \quad -\tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1 + \sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) \tag{3.2}$$

is positive definite, then the equilibrium $X = 0$ of the considered system (3.1) is globally asymptotically stable.

Proof Consider the quadratic Lyapunov function:

$$V(X) = X^T P X. \tag{3.3}$$

Differentiating $V(X)$ along the trajectory of the system (3.1), one obtains:

$$\dot{V}(X) = \sum_{k=1}^{r} (X^T P A_k X^{[k]} + X^{[k]^T} A_k^T P X) = 2\sum_{k=1}^{r} X^T P A_k X^{[k]}. \tag{3.4}$$

Using the rule of the vec-function (2.10), the relation (3.4) can then be written as:

$$\dot{V}(X) = 2\sum_{k=1}^{r} V_k^T X^{[k+1]}, \tag{3.5}$$

where

$$V_k = vec(PA_k). \tag{3.6}$$

To ensure the global asymptotic stability of the equilibrium $(X = 0)$ of the system (3.1), it is sufficient to have $\dot{V}(X)$ negative definite for $\forall X \in \mathbb{R}^n$.

Let the following notations be used for $k = 1, \dots, 2s-1$ and $j = g_k, \dots, h_k$

$$N_{k+1-j,\,j} = mat_{(n^{k+1-j},\,n^j)}(V_k). \tag{3.7}$$

Then, using the relation (3.7), we can write:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} X^{[k+1-j]^T} N_{k+1-j} X^{[j]} \tag{3.8}$$

such that $\sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} = 1$. It can be shown [17] that one has:

$$N_{k+1-j,j} = mat_{(n^{k+1-j},n^j)}(Vec(PA_k)) = U_{n^{k-j}\times n}(P \otimes I_{n^{k-j}}).M_{k+1-j,j}, \tag{3.9}$$

where $M_{k+1-j,j}$ is defined in (2.15).

Using the result (3.9) and the relation (2.9), we can write:

$$\begin{aligned} X^{[k+1-j]^T} N_{k+1-j,j} X^{[j]} &= X^{[k+1-j]^T} U_{n^{k-j}\times n}(P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]} \\ &= X^{[k+1-j]^T} (P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]}. \end{aligned} \tag{3.10}$$

Consequently, we obtain:

$$V_k^T X^{[k+1]} = \sum_{j=g_k}^{h_k} \lambda_{k+1-j,j} X^{[k+1-j]^T} (P \otimes I_{n^{k-j}}) M_{k+1-j,j} X^{[j]} = \mathcal{X}^T \mathcal{D}_S(P) \mathcal{M}_k(\lambda) \mathcal{X}$$

with

$$\mathcal{X} = \begin{bmatrix} X^T & X^{[2]^T} & \cdots & X^{[s]^T} \end{bmatrix}^T \tag{3.11}$$

and

$$\mathcal{M}_k(\lambda) = \begin{bmatrix} 0 & & \lambda_{1k} M_{1k} \\ & \lambda_{k-1,2} M_{k-1,2} \;\; \cdot^{\cdot^{\cdot}} & \\ \lambda_{k1} M_{k1} & & 0 \end{bmatrix}. \tag{3.12}$$

Then $\dot{V}(X)$ can be written as:

$$\dot{V}(X) = 2 \sum_{k=1}^{2s-1} V_k^T X^{[k+1]} = \mathcal{X}^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^T \mathcal{D}_S(P)) \mathcal{X}, \tag{3.13}$$

where $\mathcal{M}_\lambda(f) = \sum_{k=1}^{r} \mathcal{M}_k(\lambda)$ is defined in (2.14).

Using the nun-redundant form, the vector $\mathcal{X}$ can be written as:

$$\mathcal{X} = \tau_1 \tilde{\mathcal{X}}, \tag{3.14}$$

where $\widetilde{\mathcal{X}} \in \mathbb{R}^\eta$, $\eta = \sum_{j=1}^{s} n_j$ and τ_1 is defined in (2.19).

Then $\dot{V}(X)$ can be written in the following form:

$$\dot{V}(X) = \widetilde{\mathcal{X}}^{\mathrm{T}} \tau_1^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}} \mathcal{D}_S(P)) \tau_1 \widetilde{\mathcal{X}}. \tag{3.15}$$

A sufficient condition of the global asymptotic stability of the equilibrium ($X = 0$) is that the quadratic form $\dot{V}(X)$ should be negative definite. This condition can be ensured if there exists a symmetric positive definite $\mathcal{Q} \in \mathbb{R}^{\eta\times\eta}$ such that:

$$\widetilde{\mathcal{X}}^{\mathrm{T}} \tau_1^T (\mathcal{D}_S(P) \mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}} \mathcal{D}_S(P)) \tau_1 \widetilde{\mathcal{X}} = -\widetilde{\mathcal{X}}^{\mathrm{T}} \mathcal{Q} \widetilde{\mathcal{X}}. \tag{3.16}$$

Using the *vec*-function, the equality (3.16) can be expressed as:

$$vec^{\mathrm{T}}\left(\tau_1^{\mathrm{T}}(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1+\mathcal{Q})\right)\widetilde{\mathcal{X}}^{[2]}=0 \tag{3.17}$$

But, it can be easily checked that $\widetilde{\mathcal{X}}^{[2]}$ can be written as

$$\widetilde{\mathcal{X}}^{[2]}=\mathcal{R}\widetilde{\mathcal{X}}_2, \tag{3.18}$$

where

$$\widetilde{\mathcal{X}}_2=\begin{bmatrix}\widetilde{\mathcal{X}}^{[2]}\\ \vdots\\ \widetilde{\mathcal{X}}^{[s+1]}\\ \widetilde{\mathcal{X}}^{[s+2]}\\ \vdots\\ \widetilde{\mathcal{X}}^{[2s]}\end{bmatrix} \tag{3.19}$$

and $\mathcal{R}$ is the matrix defined in (2.18). Therefore the equality (3.17) yields the following equation:

$$\mathcal{R}^{\mathrm{T}}vec(S)=0 \tag{3.20}$$

with $S=\tau_1^T\left(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P)\right)\tau_1+\mathcal{Q}$. The η^2-vector $vec(S)$ which is a solution of (3.20) can be expressed as:

$$vec(S)=\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^{\mathrm{T}}-I_{\eta^2}\right)\mathcal{Y}, \tag{3.21}$$

where $\mathcal{Y}$ is an arbitrary vector of $\mathbb{R}^{\eta^2}$. On the other hand, the matrix S is symmetric since $\mathcal{Q}$ is symmetric, then we have

$$S=\tfrac{1}{2}(S+S^T) \tag{3.22}$$

and using the property (2.11) yields

$$vec(S)=\tfrac{1}{2}(I_{\eta^2}+U_{\eta\times\eta})vec(S)=\sum_{i=1}^{\beta}\mu_iC_i, \tag{3.23}$$

where $\beta=rank\left[\left(I_{\eta^2}+U_{\eta\times\eta}\right)\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^T-I_{\eta^2}\right)\right]$, $C_{i,i=1,\dots,\beta}$ are β linearly independent columns of

$$\left(I_{\eta^2}+U_{\eta\times\eta}\right)\left(\mathcal{R}^{+\mathrm{T}}\mathcal{R}^T-I_{\eta^2}\right), \tag{3.24}$$

$\mu_{i,i=1,\dots,\beta}$ are arbitrary values. Consequently, the matrix $\mathcal{Q}$ verifying (3.20) is of the following form:

$$\mathcal{Q}=-\tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f)+\mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1+\sum_{i=1}^{\beta}\mu_i mat_{(\eta,\eta)}(C_i) \tag{3.25}$$

which ends the proof.

Remark. For $r=1$, the system (3.1) becomes linear ($\dot{X}=AX$) and by (3.25) we obtain the famous Lyapunov stability condition for linear system: *The asymptotic stability of the origin equilibrium of the system $\dot{X}=AX$ is ensured iff there exist symmetric positive definite matrices P and Q such that $A^TP+PA=-Q$.* Thus, the criterion stated in Theorem 1 generalizes this linear stability Lyapunov condition for polynomial systems.

4 LMI Formulation of the Global Stability Criterion of Polynomial Systems

In this section we show how the stated stability conditions of Theorem 1 can be formulated as LMI conditions. Les us notice that the proved stability condition can be presented as the following matrix inequality feasibility problem. Find:

- a $(n \times n)$ matrix P;
- $\lambda = [\lambda_{ij}] \in \mathbb{R}^{s \times s}$ verifying the relation (2.17);
- real parameters $\mu_{i,i=1,\ldots,\beta}$;

such that:

$$\begin{cases} P > 0, \\ \tau_1^T(\mathcal{D}_S(P)\mathcal{M}_\lambda(f) + \mathcal{M}_\lambda(f)^{\mathrm{T}}\mathcal{D}_S(P))\tau_1 - \sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) < 0. \end{cases} \tag{4.1}$$

However, these inequalities are nonlinear with respect of the unknown parameters P,λ_{ij} and μ_i, since the second inequality of (4.1) is bilinear on (P, λ_{ij}). To overcome this problem we make use of the separation lemma [38] and we exploit the generalized Schur's complement [35], in order to transform the BMI problem into an LMI one.

Let us remark that the coefficients λ_{ij} of the matrix λ verify the relations (2.17) which implies that

$$\lambda_{11} = 1, \quad \lambda_{ss} = 1, \tag{4.2}$$

and the matrix $\mathcal{M}_\lambda(f)$ can be written as

$$\mathcal{M}_\lambda(f) = \mathcal{N}(f) + \mathcal{N}_\lambda(f), \tag{4.3}$$

where:

$$\mathcal{N}(f) = \begin{bmatrix} M_{11} & & & \mathbf{0} \\ & M_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & M_{ss} \end{bmatrix} \tag{4.4}$$

and

$$\mathcal{N}_\lambda(f) = \begin{bmatrix} 0 & \lambda_{12}M_{12} & \cdots & \cdots & \lambda_{1s}M_{1s} \\ \lambda_{21}M_{21} & \alpha_2 M_{22} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{s-1}M_{s-1,s-1} & \lambda_{s-1s}M_{s-1,s} \\ \lambda_{s1}M_{s1} & \cdots & \cdots & \lambda_{ss-1}M_{s,s-1} & 0 \end{bmatrix} \tag{4.5}$$

for $k = 2, ..., s-1$,

$$\alpha_k = - \sum_{\substack{1 \le i,j \le s \\ i+j = 2k \\ i \ne j}} \lambda_{ij}. \tag{4.6}$$

According to the relation (4.3), the second inequality of (4.1) becomes:

$$\begin{aligned} &- \sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ &+ [\mathcal{D}_S(P)\tau_1]^T [\mathcal{N}_\lambda(f)\tau_1] + [\mathcal{N}_\lambda(f)\tau_1]^T [\mathcal{D}_S(P)\tau_1] < 0. \end{aligned} \tag{4.7}$$

Making use of the following separation lemma.

Lemma 1 *[38]: For any matrices A and B with appropriate dimensions and for any positive scalar $\epsilon > 0$, one has: $A^T B + B^T A \leq \epsilon A^T A + \epsilon^{-1} B^T B$.*

Then, the inequality (4.7) is satisfied if there exists a real $\epsilon > 0$ such that

$$\begin{array}{l} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ +\epsilon \left[\mathcal{D}_S(P)\tau_1\right]^T \left[\mathcal{D}_S(P)\tau_1\right] + \frac{1}{\epsilon} \left[\mathcal{N}_\lambda(f)\tau_1\right]^T \left[\mathcal{N}_\lambda(f)\tau_1\right] < 0. \end{array} \tag{4.8}$$

This inequality (4.8) can be put as

$$\begin{array}{l} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 \\ -\left[\mathcal{D}_S(P)\tau_1\right]^T (-\epsilon I) \left[\mathcal{D}_S(P)\tau_1\right] - \left[\mathcal{N}_\lambda(f)\tau_1\right]^T \left(-\frac{1}{\epsilon} I\right) \left[\mathcal{N}_\lambda(f)\tau_1\right] < 0 \end{array} \tag{4.9}$$

Using Schur complement, inequality (4.9) holds if and only if

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & \left[\mathcal{D}_S(P)\tau_1\right]^T & \left[\mathcal{N}_\lambda(f)\tau_1\right]^T \\ \mathcal{D}_S(P)\tau_1 & -\frac{1}{\epsilon} I & 0 \\ \mathcal{N}_\lambda(f)\tau_1 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{4.10}$$

Multiplying $diag\left(I, I, \epsilon^{-1} I\right)$ for both sides of (4.10), we have

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & \left[\mathcal{D}_S(P)\tau_1\right]^T & \left[\widetilde{\mathcal{N}}_\lambda(f)\tau_1\right]^T \\ \mathcal{D}_S(P)\tau_1 & -\epsilon^{-1} I & 0 \\ \widetilde{\mathcal{N}}_\lambda(f)\tau_1 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0, \tag{4.11}$$

where $\widetilde{\lambda}_{ij} = \epsilon^{-1}\lambda_{ij}$. This new inequality (4.11) is linear on the decision variables, and then we can state the following theorem.

Theorem 2 *The equilibrium ($X = 0$) of the system (3.1) is globally asymptotically stable if there exist:*

- *a $(s \times s)$-matrix $\widetilde{\lambda} = [\widetilde{\lambda}_{ij}]$ verifying $\sum_{j=g_k}^{h_k} \widetilde{\lambda}_{k+1-j,j} = 1$;*
- *a $(n \times n)$-symmetric positive definite matrix P ;*
- *arbitrary parameters $\mu_{i,i=1,\ldots,\beta} \in \mathbb{R}$;*
- *a real $\epsilon > 0$;*

such that:

$$P > 0 \tag{4.12}$$

and

$$\begin{bmatrix} -\sum_{i=1}^{\beta} \mu_i mat_{(\eta,\eta)}(C_i) + \tau_1^T(\mathcal{D}_S(P)\mathcal{N}(f) + \mathcal{N}(f)^T\mathcal{D}_S(P))\tau_1 & \left[\mathcal{D}_S(P)\tau_1\right]^T & \left[\widetilde{\mathcal{N}}_\lambda(f)\tau_1\right]^T \\ \mathcal{D}_S(P)\tau_1 & -\epsilon^{-1} I & 0 \\ \widetilde{\mathcal{N}}_\lambda(f)\tau_1 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0. \tag{4.13}$$

The stability analysis of polynomial systems using Theorem 2, can be carried out using Matlab software.

5 Illustrative Example

To illustrate the availability of the proposed method we consider the stability study of the origin equilibrium of the following second order polynomial systems:

$$\left\{\begin{array}{lll}\dot{x}_1 & = & -x_1 - x_2 + x_1^2 + x_1x_2 - x_1^3 + x_1^2x_2 - x_1x_2^2 + 2x_2^3, \\ \dot{x}_2 & = & -x_1 - 1.5x_2 - 1.1x_1^2 + 0.3x_1x_2 - 1.8x_1^3 - 5.6x_1^2x_2 - 5.3x_1x_2^2 - 0.7x_2^3.\end{array}\right. \tag{5.1}$$

This system can be written in the following form:

$$\dot{X} = A_1X + A_2X^{[2]} + A_3X^{[3]} \tag{5.2}$$

with

$$A_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1.1 & 0.3 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ -1.8 & 0.9 & -5.2 & -1.8 & -1.3 & 4.3 & -8.3 & -0.7 \end{bmatrix}.$$

Solving the optimization problem formulated by Theorem 2, we obtain:

$$\left\{\begin{array}{l}\mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 3.8529\end{array}\right., \quad \left\{\begin{array}{l}\lambda_{11} = 1 \\ \lambda_{12} = 0.1419 \\ \lambda_{21} = 0.8581 \\ \lambda_{22} = 1\end{array}\right., \quad \epsilon = 0.1864, \quad P = \begin{bmatrix} 1.9551 & -0.1723 \\ -0.1723 & 1.1529 \end{bmatrix},$$

which ensure the global asymptotic stability of the equilibrium $X = 0$.

6 Conclusion

In this paper, we have presented an original practical criterion for global stability analysis of nonlinear polynomial systems. This criterion is stated as sufficient conditions derived from a quadratic Lyapunov function. Furthermore, useful mathematical transformations have allowed the formulation of the obtained conditions as an LMI problem, which has facilitated the numerical implementation of the proposed criterion using Matlab LMI toolboxes.

Let's notice that the obtained results presented in this paper are developed with a quadratic Lyapunov function, but they can be easily extended for the case of polynomial Lyapunov functions. Also, we point out that a similar method can be elaborated for the stabilization and robust control of polynomial systems.

References

[1] Boyd, S. and Yang, Q. Structured and simultaneous Lyapunov functions for system stability problems. *Int. J. Control* **49**(6) (1989) 2215–2240.

[2] Banks, S. and Yi, Z. On the stability analysis of nonlinear systems. *Journal of Mathematical Control and Information* **8**(3) (1991) 275–285.

[3] Kiong, N.S. and Fu, M. Global quadratic stabilization of a class of nonlinear systems. *Int. J. Robust Nonlinear Control.* **8** (1998) 483–497.

[4] Rios-Patron, E. and Braatz, R. Global stability analysis for discrete-time nonlinear systems. *American Control Conference* **1** (1998) 338–342.

[5] Gleria, I., Figueiredo, A. and Filho, T.R. A numerical method for the stability analysis of quasi-polynomial vector fields. *Nonlinear Analysis: Theory, Methods and Applications* **52**(1) (2003) 329–342.

[6] Borne, P., Richard, J.P. and Radhy, N.E. Stability, stabilization, regulation using vector norm. *Nonlinear Systems, (Chapter 2)* **2** (1996) 45–90.

[7] Perruquetti, W., Richard, J.P., Grujic, Lj.T. and Borne, P. On practical stability with the settling time via vector norm. *International Journal of Control* **62**(1) (1995) 173–189.

[8] Borne, P. and Gentina, J.C. On the stability of large nonlinear systems structured and simultaneous lyapunov for system stability problems. *Joint. Aut. Cont. Conf, Austin, Texas*, 1974.

[9] Borne, P. and Benrejeb, M. On the stability of a class of interconnected systems. application to the forced working conditions. *Actes 4th IFAC Symposium MTS Frederiction, Canada*, 1977.

[10] Papachristodoulou, A. and Prajna, S. On the construction of Lyapunov function using sum of squares decomposition. *In Proceeding of IEEE Conference on Descion and Control, CDC'02*, 2002.

[11] Parrilo, P.A. Exploiting structure in sum of squares programs. *In Proceeding of the IEEE Conference on Descion and Control,CDC'03*, December 2003.

[12] Henrion, D. and Garulli, A. *Positive polynomials in Control*. Springler LNCIS, January 2005.

[13] Rotella, F. and Tanguy, G. Non linear systems: identification and optimal control. *Int. J. Control* **48**(2) (1988) 525–544.

[14] Martynyuk, A.A. and Slyn'ko, V.I. Stability Results for Large-Scale Difference Systems via Matrix-Valued Liapunov Functions. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 217–224.

[15] Liu, B. and Lui, F. Robusty Global Exponential Stability of Time-varying Linear Impulsive Systems with Uncertainty. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 187–196.

[16] Radu Balan. An Extension of Barbashin–Krasovskii–LaSalle Theorem to a class of Nonautonomous Systems. *Nonlinear Dynamics and Systems Theory* **8**(3) (2008) 255–268.

[17] Benhadj Braiek, N., Rotella, F. and Benrejeb, M. Algebraic criteria for global stability analysis of nonlinear systems. *Journal of Systems Analysis Modelling and Simulation, Gordon and Breach Science Publishers* **17** (1995) 221–227.

[18] Benhadj Braiek, N. On the global stability of nonlinear polynomial systems. *IEEE Conference On Decision and Control, CDC'96*, December 1996.

[19] Benhadj Braiek, N. and Rotella, F. Design of observers for nonlinear time variant systems. *IEEE Syst. Man and Cybernetics Conference* **4** (1993) 219–225.

[20] Benhadj Braiek, N. and Rotella, F. State observer design for analytical nonlinear systems. *IEEE Syst. Man and Cybernetics Conference* **3** (1994) 2045–2050.

[21] Belkhiria, H. and Benhadj Braiek, N. On the robust stability analysis of uncertain polynomial systems : an LMI Approach. *17th IMACS World Congress, Scientific Computation, Applied Mathematics and Simulation, Paris*, Juillet, 2005.

[22] Mtar, R., Belkhiria, H. and Benhadj Braiek, N. Robust stability of polynomial systems under nonlinear perturbations: an LMI Approach. *Fourth IEEE International Multi-Conference on Systems, Signals and Devices, SSD'2007*, Hammamet–Tunisie, 19–22 Mars, 2007.

[23] Mtar, R., Belhaouane, M., Belkhiria, H. and Benhadj Braiek, N. H_∞-Performance Analysis for Nonlinear Polynomial Systems: an LMI Approach. *Journées Tunisiennes d'Electrotechnique et d'Automatique, JTEA'2008*, Hammamet–Tunisie, 02–04 Mai, 2008.

[24] Benhadj Braiek, N, Bacha, A. and Jerbi, H. A Technique of a Stability Domain Determination for Nonlinear Discrete Polynomial System. *IFAC World Congress*,Seoul 6-11 Juillet 2008.

[25] Benhadj Braiek, N. Feedback stabilization and stability domain estimation of nonlinear systems. *Journal of the Franklin Institute* **332** (2) (1995) 183–193.

[26] Parrilo, P.A. Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. *PhD thesis, California Institue of Technologie, Pasadena*, CA, 2000.

[27] Zachary, W.J.W. Lyapunov based analysis and controllers synthesis for polynomial systems using sum of squares optimization. *PhD thesis, University of California, Berkeley*, 2001.

[28] Benhadj Braiek, N and Rotella, F. Robot model simplification by means of an identification method. In: *Robotics and Flexible Manufacturing Systems* (Eds. Gentina, J.C. and Tzafesta, S.G.). Elsevier Science Publishes B. V, North Holand (1992) 217–227.

[29] Benhadj Braiek, N and Rotella, F. Stabilization of nonlinear systems using a Kronecker product approach. *European Control Conference ECC'95*, September 1995, 2304–2309.

[30] Brewer, J. Kronecker product and matrix calculus in system theory. *IEEE Trans. Circ. Sys* **CAS-25** (1978) 722–781.

[31] Khalil, H.K. *Non-linear Systems*. 3rd ed. New York, Prentice–Hall, 2000.

[32] Bohner, M. and Martynyuk, A.A. Elements of Stability of A.M. Liapunov for Dynamic Equations on Time Scales. *Nonlinear Dynamics and Systems Theory* **7**(3) (2007) 225–252.

[33] Zelentsovsky, A.L. Non quadratic Lyapunov functions for robust stability analysis of linear uncertain systems. *IEEE Trans. Automat. Control* **39**(1) (1994).

[34] Bouzaouache, H. and Benhadj Braiek, N. On the stability analysis of nonlinear systems using polynomial Lyapunov functions. *Mathematics and Computers in Simulation* **76**(5–6) (2008) 316–329.

[35] Boyd, S., Ghaoui, L. and Balakrishnan, F. Linear Matrix Inequalities in System and Control Theory. *SIAM*, 1994.

[36] Karimi, H.R., Lohmann, B. and Buskens, C. An LMI Approach to H∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs. *Nonlinear Dynamics and Systems Theory* **7**(4) (2007) 351–368.

[37] Benhadj Braiek, N and Rotella, F. Logic: a nonlinear systems identification software. *Modelling and Simulation of Systems, Scientific Publishing Co.* (1990) 211–218.

[38] Zhou, K. and Khargonedkar, P. P. Robust stabilization of linear systems with normbounded time-varying uncertainty. *Sys. Contr. Letters* **10** (1988) 17–20.

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Positive Solutions of a Second Order m-point BVP on Time Scales

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Abstract: In this study, we are concerned with proving the existence of multiple positive solutions of a general second order nonlinear m-point boundary value problem (m-PBVP)

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda h(t)f(t,u) = 0, \ \ t \in [0,1],$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

on time scales. The proofs are based on the fixed point theorems in a Banach space. We present an example to illustrate how our results work.

Keywords: *m-point boundary value problems, positive solutions, fixed point theorems, time scales.*

Mathematics Subject Classification (2000): 39A10, 34B18, 34B40, 45G10.

1 Introduction

The theory of dynamic equations on time scales unifies the well-known analogies in the concept of difference equations and differential equations. Some basic definitions and theorems on time scales can be found in the books [3, 4]. In the past few years starting with Il'in and Mossiev [8] and Gupta [6], the existence of positive solutions for nonlinear high-order and second order boundary value problems have been studied by many authors by using the coincidence degree theory and fixed point theorems in cones (see [1, 2, 7, 9, 11, 12, 15] and references therein).

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The m-point boundary value problems for dynamic equations on time scales arise in a variety of different areas of applied mathematics, physics and engineering. Recently Yaslan [14], Sun and Lee [13] obtained some existence results for three point and multi-point boundary value problems on time scales.

In 2003, Ma and Wang [12] studied the nonlinear boundary value problem

$$u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0,1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1)$$

and obtained some existence results if f satisfies either superlinear and sublinear conditions by applying fixed point theorems in cones. We generalized the results of Ma and Wang in three aspects: (a) we generalized the three point BVP to m-point BVP with a dynamic equation; (b) we study the eigenvalue problem; (c) we obtain the existence of at least three positive solutions.

In this paper we deal with the determining the value of λ for which the following m-point BVP has a positive solution:

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda h(t)f(t,u) = 0, \ t \in [0,1], \tag{1}$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{2}$$

where $0 < \eta_i < 1, \ \forall i = 1,2,\dots,m-2, \ h, f, a$ and b satisfy:

(H1) $f \in \mathcal{C}([\rho(0),\sigma(1)] \times [0,\infty),[0,\infty))$;

(H2) $h \in \mathcal{C}([0,1],[0,\infty))$ and there exists $t_0 \in [0,1]$ such that $h(t_0) > 0$;

(H3) $a \in \mathcal{C}([0,1],[0,\infty)), \ b \in \mathcal{C}([0,1],(-\infty,0])$.

This paper is organized as follows. In Section 2, starting with some preliminary lemmas we state the Krasnosel'skii and Legget-Williams fixed point theorems. In Section 3, we give the main results which state the sufficient conditions for the m-point BVP (1)-(2) to have at least one or at least three solutions.

2 Preliminaries and Fixed Point Theorems

In this section we state the preliminary information that we need the prove the main results.

Lemma 2.1 *Assume that* (H3) *holds. Let* ϕ_1 *and* ϕ_2 *be the solutions of*

$$\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t) = 0, \tag{3}$$

$$\phi_1(\rho(0)) = 0, \ \phi_1(\sigma(1)) = 1, \tag{4}$$

$$\phi_2^{\Delta\nabla}(t) + a(t)\phi_2^{\Delta}(t) + b(t)\phi_2(t) = 0, \tag{5}$$

$$\phi_2(\rho(0)) = 1, \ \phi_2(\sigma(1)) = 0 \tag{6}$$

respectively. Then
(i) ϕ_1 *is strictly increasing on* $[\rho(0),1]$, **(ii)** ϕ_2 *is strictly decreasing on* $[\rho(0),1]$.

Lemma 2.2 *Assume that* (H3) *holds. Then* (3)-(4) *and* (5)-(6) *have unique solutions respectively.*

The proofs of the Lemma 2.1 and Lemma 2.2 can be obtained easily by generalizing the proofs of Lemma 2.1 and Lemma 2.2 in [12] to time scales.

For the rest of the paper we need the following assumption

(H4) $0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) < 1.$

In the following lemma we express the Green's function and the form of the solution of the linear m-point BVP corresponding to (1)-(2).

Lemma 2.3 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}[\rho(0), \sigma(1)]$. *Then the problem*

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + y(t) = 0, \ t \in [0,1], \tag{7}$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{8}$$

is equivalent to the integral equation

$$u(t) = \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)y(s)\nabla s + A\phi_1(t), \tag{9}$$

where

$$A = \frac{1}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s \right), \tag{10}$$

$$p(t) = e_a(\rho(t), \rho(0)), \tag{11}$$

$$G(t,s) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \begin{cases} \phi_1(t)\phi_2(s), & s \geq t, \\ \phi_1(s)\phi_2(t), & t \geq s. \end{cases} \tag{12}$$

Proof First we show that the unique solution of (7)-(8) can be represented by (9). From Lemma 2.1, we know that the homogenous part of (7) has two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$ since

$$\begin{vmatrix} \phi_1(\rho(0)) & \phi_1^{\Delta}(\rho(0)) \\ \phi_2(\rho(0)) & \phi_2^{\Delta}(\rho(0)) \end{vmatrix} = -\phi_1^{\Delta}(\rho(0)) \neq 0.$$

Now by the method of variations of constants, we can obtain the unique solution of (7)-(8) which can be represented by (9) where A and G are as in (10) and (12) respectively. Next we check the function defined in (9) is the solution of the BVP (7)-(8). For this purpose we first show that (9) satisfies (7). From the definition of the Green's function (12), we get

$$u(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \left(\int_{\rho(0)}^{t} \phi_1(s)\phi_2(t)p(s)y(s)\nabla s + \int_{t}^{\sigma(1)} \phi_1(t)\phi_2(s)p(s)y(s)\nabla s \right) + A\phi_1(t).$$

Hence the derivatives u^{Δ} and $u^{\Delta\nabla}$ are as follows:

$$u^{\Delta}(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\left(\phi_2^{\Delta}(t)\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s + \phi_1^{\Delta}(t)\int_{t}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s\right) + A\phi_1^{\Delta}(t)$$

and

$$\begin{aligned} u^{\Delta\nabla}(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(&\phi_2^{\Delta\nabla}(t)\int_{\rho(0)}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s + \phi_2^{\Delta}(t)\phi_1(t)p(t)y(t) \\ &+\phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s - \phi_1^{\Delta}(t)\phi_2(t)p(t)y(t)\Big) + A\phi_1^{\Delta\nabla}(t). \end{aligned}$$

Replacing the derivatives in (7), we deduce

$$\begin{aligned} u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) &= A\big(\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t)\big) \\ &+\Big(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s\Big)\big(\phi_2^{\Delta\nabla}(t) + a(t)\phi_2^{\Delta}(t) + b(t)\phi_2(t)\big) \\ &+\Big(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{t}^{\sigma(1)}\phi_s(s)p(s)y(s)\nabla s\Big)\big(\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t)\big) \\ &+\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta\nabla}(t)\int_{t}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s + \phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{t}\phi_2(s)p(s)y(s)\nabla s\Big) \\ &+\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\Big)p(t)y(t) \\ &= \frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta\nabla}(t)(\rho(t)-t)\phi_1(t)p(t)y(t) - \phi_1^{\Delta\nabla}(t)(\rho(t)-t)\phi_2(t)p(t)y(t) \\ &\qquad +\phi_2^{\Delta}(t)\phi_1(t)y(t) - \phi_1^{\Delta}(t)\phi_2(t)y(t)\Big) \\ &= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\big) \\ &\qquad -\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)(\rho(t)-t)\big(\phi_1^{\Delta\nabla}(t)\phi_2(t) - \phi_2^{\Delta\nabla}(t)\phi_1(t)\big) \\ &= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big\{\big(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\big) \\ &\qquad +(\rho(t)-t)\big(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\big)^{\nabla}\big\} \\ &= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big(\phi_2^{\Delta}(\rho(t))\phi_1(\rho(t)) - \phi_1^{\Delta}(\rho(t))\phi_2(\rho(t))\big) \\ &= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)e_{\ominus a}(\rho(t),\rho(0))\big(-\phi_1^{\Delta}(\rho(0))\big) \\ &= -y(t). \end{aligned}$$

Therefore the function defined in (9) satisfies (7). Further we obtain that (8) is satisfied by (9). The first boundary condition of (8) follows from (9), (10) and (12). Now we verify the second boundary condition. Since

$$G(\sigma(1), s) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\phi_1(s)\phi_2(\sigma(1)) = 0,$$

we obtain

$$u(\sigma(1)) = \int_{\rho(0)}^{\sigma(1)} G(\sigma(1), s)p(s)y(s)\nabla s + A\phi_1(\sigma(1)) = A. \tag{13}$$

On the other hand, by using equation (10) we find

$$\begin{aligned}
\sum_{i=2}^{m-2} \alpha_i u(\eta_i) &= \sum_{i=2}^{m-2} \alpha_i \left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s + A\phi_1(\eta_i)) \right) \\
&= \sum_{i=2}^{m-2} \alpha_i \left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s + \frac{\sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) \int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \right) \\
&= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s \right) = A. \qquad (14)
\end{aligned}$$

Combining the equations (13) and (14) finishes the proof. □

In this study we consider the Banach space $\mathcal{B}$ of continuous functions defined on $[\rho(0), \sigma(1)]$ with the supremum norm. Now we set

$$q(t) = \min \left\{ \frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||} \right\}. \tag{15}$$

Lemma 2.4 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}([\rho(0), \sigma(1)], [0, \infty))$. *Then the unique solution of* (7)-(8) *satisfies* $u(t) \geq ||u||q(t)$.

Proof Let t_0 be the point in $(\rho(0), \sigma(1))$ such that $||u|| = u(t_0)$. Next we verify

$$G(t, s) \geq G(t_0, s)q(t). \tag{16}$$

For this purpose, we consider the following four cases:

(i) $t, t_0 \leq s$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(t)}{\phi_1(t_0)} \geq \frac{\phi_1(t)}{||\phi_1||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(ii) $t, t_0 \geq s$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_2(t)}{\phi_2(t_0)} \geq \frac{\phi_2(t)}{||\phi_2||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(iii) $t_0 \leq s \leq t$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(s)\phi_2(t)}{\phi_1(t_0)\phi_2(s)} \geq \frac{\phi_2(t)}{\phi_2(s)} \geq \frac{\phi_2(t)}{||\phi_2||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(iv) $t \leq s \leq t_0$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(t)\phi_2(s)}{\phi_1(s)\phi_2(t_0)} \geq \frac{\phi_1(t)}{\phi_1(s)} \geq \frac{\phi_1(t)}{||\phi_1||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

In the third and the fourth cases we make use of Lemma 2.1. It follows from the fact $1 \geq \phi_1(t) \geq q(t),\ \forall t \in [\rho(0), \sigma(1)]$ and the inequality (16) that

$$\begin{aligned} u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)y(s)\nabla s + A\phi_1(t)\Big\} \geq \lambda\Big\{q(t)\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\phi_1(t)\Big\} \\ &\geq \lambda q(t)\Big(\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\Big) \geq \lambda q(t)\Big(\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\phi_1(t_0)\Big) \\ &= q(t)u(t_0) = q(t)||u||. \ \Box \end{aligned}$$

Assume that $\xi := \inf\{t \in \mathbb{T} : t > \rho(0)\},\ w := \sup\{t \in \mathbb{T} : t < \sigma(1)\}$ both exist and are included in $[\rho(0), \sigma(1)]$, and also satisfy $\rho(0) < \xi < w < \sigma(1)$. Also assume that $\sigma(w) < \sigma(1)$ and $\rho(\xi) > \rho(0)$ hold.

Lemma 2.5 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}([\rho(0), \sigma(1)], [0, \infty))$. *Then there exists* $\gamma > 0$ *such that unique solution of* (7)–(8) *satisfies* $u(t) > \gamma||u||$.

Proof Choose

$$\gamma = \min\{q(t) : t \in [\xi, w]\}. \tag{17}$$

It is clear that $\gamma > 0$ and $u(t) \geq q(t)||u|| > \gamma||u||,\ \forall t \in [\xi, w]$. $\Box$

To make use of the fixed point theorems we consider the cone

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) > 0, t \in [\rho(0), \sigma(1)], \min_{t \in [\xi, w]} u(t) \geq \gamma||u||\} \tag{18}$$

on the Banach space $\mathcal{B}$, and set $\mathcal{P}_r = \{x \in \mathcal{P} : ||x|| < r\}$.

Theorem 2.1 [5] *(Krasnosel'skii Fixed Point Theorem) Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open, bounded subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1, \|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1, \|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.2 [11] *(Legget–Williams Fixed Point Theorem) Let $\mathcal{P}$ be a cone in a real Banach space E. Set*

$$\mathcal{P}(\psi, a, b) := \{x \in \mathcal{P} : a \leq \psi(x), \|x\| \leq b\}.$$

Suppose $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$ be a completely continuous operator and ψ be a nonnegative, continuous, concave functional on $\mathcal{P}$ with $\psi(u) \leq \|u\|$ for all $u \in \overline{\mathcal{P}_r}$. If there exist $0 < p < q < l \leq r$ such that the following conditions hold:

(i) $\{u \in \mathcal{P}(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ *and* $\psi(Au) > q$ *for all* $u \in \mathcal{P}(\psi, q, l)$,

(ii) $\|Au\| < p$ *for all* $\|u\| \leq p$,

(iii) $\psi(Au) > q$ *for* $u \in \mathcal{P}(\psi, q, r)$ *with* $\|Au\| > l$.

Then A has at least three positive solutions u_1, u_2 and u_3 in $\overline{\mathcal{P}_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \ \text{ with } \psi(u_3) < q.$$

3 Main Results

We are concerned with determining values of λ, for which there exist positive solutions of m-point boundary value problem (1)-(2). We use Krasnosel'skii fixed point theorem and Legget-Williams fixed point theorem to prove the main results. From Lemma 2.3, it is clear that the solutions of (1)-(2) are the fixed points of the operator

$$\Phi_\lambda u(t) = \lambda\Big\{ \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s + A\phi_1(t)\Big\}. \tag{19}$$

To state the main results we need to define the following extended real numbers:

$$f_0 = \lim_{u\to 0^+}\inf \min_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{20}$$

$$f^0 = \lim_{u\to 0^+}\sup \max_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{21}$$

$$f_\infty = \lim_{u\to \infty}\inf \min_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{22}$$

$$f^\infty = \lim_{u\to \infty}\sup \max_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}. \tag{23}$$

Let K and L be defined by

$$K = \min_{t\in[\xi,w]} \int_{\xi}^{w} G(t,s)p(s)h(s)\nabla s, \tag{24}$$

$$L = \max_{t\in[\rho(0),\sigma(1)]} \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)\nabla s \le \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s. \tag{25}$$

In the following three main results, we state the criteria on λ to make sure the existence of positive solutions of (1)-(2).

Theorem 3.1 *Assume that* (H1)-(H4) *are satisfied. Then for each λ satisfying either one of the following conditions*

$$\textbf{\textit{(a)}}\ \frac{1}{\gamma \mathrm{K} \mathrm{f}_\infty} < \lambda < \frac{1}{\mathrm{L}\mathrm{f}^0}\Bigg(\frac{1-\sum\limits_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}\phi_1(\eta_{\mathrm{i}})}{1+\sum\limits_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}}\Bigg); \quad \textbf{\textit{(b)}}\ \frac{1}{\gamma \mathrm{K} \mathrm{f}_0} < \lambda < \frac{1}{\mathrm{L}\mathrm{f}^\infty}\Bigg(\frac{1-\sum\limits_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}\phi_1(\eta_{\mathrm{i}})}{1+\sum\limits_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}}\Bigg),$$

there exists at least one positive solution of (1)-(2).

Proof We claim that $\Phi_\lambda : \mathcal{P} \to \mathcal{P}$ Let $u \in \mathcal{P}$. First from the nonnegativity of G and from the assumptions (H2) and (H3), it is clear that $\Phi_\lambda u(t) \ge 0$ for $t \in [\rho(0), \sigma(1)]$.

Next by using (16) and (15), we get

$$\begin{aligned}\min_{t\in[\xi,w]}\Phi_\lambda u(t) &= \min_{t\in[\xi,w]}\lambda\Big\{\int_{\rho(0)}^{\sigma(1)}G(t,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\geq \lambda\Big\{\int_{\rho(0)}^{\sigma(1)}q(t)G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\geq q(t)\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\Big\}\\ &\geq \gamma\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\Big\}\\ &\geq \gamma\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t_0)\Big\}=\gamma||\Phi_\lambda u||.\end{aligned}$$

Thus $\Phi_\lambda u\in\mathcal{P}$. Also complete continuity of $\Phi_\lambda u(t)$ can be obtained easily by the analysis methods. Now we seek for the fixed points of $\Phi_\lambda u(t)$ which belongs to $\mathcal{P}$.

Assume **(a)** holds. Since $\lambda<\frac{1}{Lf^0}\Big(\frac{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}{1+\sum_{i=1}^{m-2}\alpha_i}\Big)$ there exists $\epsilon>0$ such that

$$\lambda L(f^0+\epsilon)\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\leq 1.$$

The use of the definition of f^0 guarantees that there exists $r_1>0$, sufficiently small such that

$$\frac{f(t,u)}{u}<f^0+\epsilon,\quad \forall u\in[0,r_1].$$

It follows that $f(t,u)<(f^0+\epsilon)u$ for $0\leq u\leq r_1$ and $t\in[\rho(0),\sigma(1)]$. If $u\in\partial\mathcal{P}_{r_1}$ then using the fact $G(t,s)\leq G(s,s)$ we obtain

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)}G(t,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\leq \lambda\Big(1+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\int_{\rho(0)}^{\sigma(1)}G(s,s)p(s)h(s)f(s,u(s))\nabla s\\ &\leq \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\int_{\rho(0)}^{\sigma(1)}G(s,s)p(s)h(s)f(s,u(s))\nabla s\end{aligned}$$

$$\le \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)(f^0+\epsilon)||u||\int_{\rho(0)}^{\sigma(1)}G(s,s)h(s)\nabla s$$
$$= \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)(f^0+\epsilon)||u||L \le ||u||.$$

Hence if we define the open bounded set

$$\Omega_1 = \{u \in \mathcal{P} : ||u|| < r_1\}, \tag{26}$$

then

$$||\Phi_\lambda u|| \le ||u||, \ \forall u \in \partial\mathcal{P}_{r_1} = \mathcal{P}\cap\partial\Omega_1. \tag{27}$$

Now we use the other part of the inequality in part **(a)**, $\frac{1}{\gamma K f_\infty} < \lambda$. We distinguish this part of the proof into two parts and first consider the case $f_\infty < \infty$. In this case, we pick ϵ_1 such that $\gamma K(f_\infty - \epsilon_1) \ge 1$. The use of the definition of f_∞ guarantees that there exists $r > r_1$, sufficiently large so that

$$\frac{f(t,u)}{u} > f_\infty - \epsilon_1, \quad \forall u \ge r.$$

Therefore, $f(t,u) > (f_\infty - \epsilon_1)u$ for $(t,u) \in [\rho(0),\sigma(1)] \times [0,r_1]$. We pick r_2 such that $r_2 \ge \frac{r}{\gamma} > r_1$ and define

$$\Omega_2 = \{u \in \mathcal{P} : ||u|| < r_2\}. \tag{28}$$

If $u \in \partial\mathcal{P}_{r_2}$, then Lemma 2.5 leads us to have

$$\begin{aligned}
\Phi_\lambda u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s + A\phi_1(t)\Big\} \\
&\ge \lambda\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s \\
&\ge \lambda(f_\infty-\epsilon_1)\gamma||u||\int_{\xi}^{w} G(t,s)p(s)h(s)\nabla s \\
&\ge \lambda(f_\infty-\epsilon_1)\gamma||u||K \\
&\ge ||u||.
\end{aligned} \tag{29}$$

Consequently, we consider the case $f_\infty = \infty$ for which the second part of the inequality in part **(a)** becomes $\lambda > 0$. If we choose M sufficiently large so that

$$\lambda M\gamma\int_{\xi}^{w} G(t,s)p(s)h(s)\nabla s \ge 1 \quad (or \ \ \lambda M\gamma K \ge 1)$$

for any $t \in [\rho(0), \sigma(1)]$, then there exists $r > r_1$ so that $f(t,u) > Mu$ for $u \geq r_1$. Let r_2 be defined as above and let $u \in \partial\mathcal{P}_{r_2}$. Then for all $t \in [\rho(0), \sigma(s)]$, we have

$$\begin{aligned}\Phi_\lambda u(t) &\geq \lambda M \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)u(s)\nabla s \\ &\geq \lambda M\gamma ||u|| \int_{\xi}^{w} G(t,s)p(s)h(s)\nabla = \lambda M\gamma K||u|| \leq ||u|| \end{aligned} \tag{30}$$

From the inequalities (29) and (30)

$$||\Phi_\lambda u|| \geq ||u||, \ \forall u \in \partial\mathcal{P}_{r_2} = \mathcal{P} \cap \partial\Omega_2. \tag{31}$$

Inequalities (27) and (31) show that the conditions of Krasnosel'skii fixed point theorem (Theorem 2.1) are fulfilled. Thus from Theorem 2.1, we conclude that $\Phi_\lambda u$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$. □

The following result states the existence of at least one positive solution of problem (1)-(2) in a different manner and also bounds the positive solution.

Theorem 3.2 *Let $f(t,u)$ satisfy* (H1). *Assume that there exist two positive constants $r_2 > r_1 > 0$ such that the following conditions are satisfied:*

(H5) $f(t,u) \leq \dfrac{Mr_2}{\lambda}$ *for* $(t,u) \in [\rho(0), \sigma(1)] \times [0, r_2]$,

(H6) $f(t,u) \geq \dfrac{Nr_1}{\lambda}$ *for* $(t,u) \in [\rho(0), \sigma(1)] \times [0, r_1]$,

where

$$M = \frac{1 - \sum\limits_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)}{1 + \sum\limits_{i=1}^{m-2} \alpha_i} \Big(\int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s \Big)^{-1}, \tag{32}$$

$$N = \Big(\gamma \int_{\xi}^{w} G(t_0,s)p(s)h(s)\nabla s \Big)^{-1} \tag{33}$$

and $t_0 \in (\rho(0), \sigma(1))$ such that $||u|| = u(t_0)$. Then the problem (1)-(2) *has at least one positive solution u satisfying $r_1 \leq ||u|| \leq r_2$.*

Proof Let Ω_2 be defined as in (28). If $u \in \partial\Omega_2$ then $||u|| = r_2$.

$$\begin{aligned}\Phi_\lambda u(t) &\leq \lambda \Big(\frac{1 + \sum\limits_{i=1}^{m-2} \alpha_i}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \Big) \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)f(s,u(s))\nabla s \\ &\leq \lambda \Big(\frac{1 + \sum\limits_{i=1}^{m-2} \alpha_i}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \Big) \frac{Mr_2}{\lambda} \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s = r_2.\end{aligned}$$

Therefore,

$$||\Phi_\lambda u|| \leq ||u||, \quad \forall u \in \partial\Omega_2. \tag{34}$$

Let Ω_1 be defined as in (26). Using (16) and (17) we obtain

$$\Phi_\lambda u \geq \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s \geq \lambda q(t) \int_{\xi}^{w} G(t_0,s)p(s)h(s)f(s,u(s))\nabla s$$
$$\geq \lambda\gamma \int_{\xi}^{w} G(t_0,s)p(s)h(s)f(s,u(s))\nabla s \geq \lambda\gamma \frac{Nr_1}{\lambda} \int_{\xi}^{w} G(t_0,s)p(s)h(s)\nabla s = r_1.$$

Therefore,

$$||\Phi_\lambda u|| \geq ||u||, \quad \forall u \in \partial\Omega_1. \tag{35}$$

Inequalities (35) and (34) imply that the conditions of Theorem 2.1 hold. Hence $\Phi_\lambda u$ has at least one fixed point i.e., (1)-(2) has at least one positive solution in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying $r_1 \leq ||u|| \leq r_2$. □

Theorem 3.3 *Let $f(t,u)$ satisfy* (H1) *and there exist constants $0 < r_1 < r_2 < r_3$ such that the following assumptions hold:*

(H7) $f(t,u) < \lambda^{-1}Mr_1$ *for all* $(t,u) \in [\rho(0),\sigma(1)] \times [0,r_1]$,

(H8) $f(t,u) \geq \lambda^{-1}Nr_2$ *for all* $(t,u) \in [\xi,w] \times [r_2,r_3]$,

(H9) $f(t,u) \leq \lambda^{-1}Mr_3$ *for all* $(t,u) \in [\rho(0),\sigma(1)] \times [\rho(0),r_3]$.

Then (1)-(2) *has at least three positive solutions u_1, u_2 and u_3 satisfying*

$$||u_1|| < r_1, \quad r_2 < \min_{t\in[\xi,w]} |u_2(t)| \leq r_3, \quad r_1 < ||u_3|| \leq r_3 \quad \text{and} \quad \min_{t\in[\xi,w]} |u_3(t)| < r_2.$$

Proof We verify that the conditions of Legget-Williams fixed point theorem (Theorem 2.2) are satisfied. For this purpose we first define the nonnegative, continuous, concave functional $\psi : \mathcal{P} \to [0, \infty)$ to be $\psi(u) := \min_{t\in[\xi,w]} |u(t)|$, the cone $\mathcal{P}$ is as in (18), M as in (32) and N as in (33). Then $\psi(u) \leq \|u\|$ for all $u \in \mathcal{P}$.
If $u \in \overline{\mathcal{P}}_{r_3}$, then $\|u\| \leq r_3$. So by using assumption (H9) and the similar calculations as in Theorem (3.2), we get

$$\Phi_\lambda u(t) \leq \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\mu_i)}\Big) \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)f(s,u(s))\nabla s$$
$$\leq \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\mu_i)}\Big)\lambda^{-1}M^{-1}r_3 \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s = r_3.$$

Hence $\Phi_\lambda : \overline{\mathcal{P}}_{r_3} \to \overline{\mathcal{P}}_{r_3}$.

In the same way, if $u \in \overline{\mathcal{P}}_{r_1}$, i.e. $||u|| \leq r_1$ assumption (H7) yields $||\Phi_\lambda u|| < r_1$. Therefore **(ii)** of Theorem 2.2 is satisfied.

To check the condition **(i)** of Theorem 2.2 we choose $u(t) = r_3, \quad \forall t \in [\rho(0),\sigma(1)]$. It is clear that $u(t) = r_3 \in \mathcal{P}(\phi, r_2, r_3)$. Consequently, since $\phi(u) = \phi(r_3) = r_3 > r_2$ then

$\{u \in \mathcal{P}(\phi, r_2, r_3) : \phi(u) > r_2\} \neq \emptyset$. Moreover by taking assumption (H8) and Lemma 2.5 into account, we obtain

$$\begin{aligned}\phi(\Phi_\lambda u) &= \min_{t\in[\xi,w]} |\Phi_\lambda u(t)| \geq \lambda\gamma \int_\xi^w G(t_0,s)p(s)h(s)f(s,u(s))\nabla s\\ &\geq \lambda\gamma\lambda^{-1}Nr_2 \int_\xi^w G(t_0,s)p(s)h(s)\nabla s = r_2.\end{aligned}$$

Therefore **(i)** of Theorem 2.2 holds.

Similarly **(iii)** of Theorem 2.2 is satisfied. Hence $\Phi_\lambda u$ has at least three fixed points $u_1,\ u_2$ and u_3 satisfying

$$||u_1|| < r_1, \quad r_2 < \min_{t\in[\xi,w]} |u_2(t)| \leq r_3, \quad r_1 < ||u_3|| \leq r_3 \;\text{ and }\; \min_{t\in[\xi,w]} |u_3(t)| < r_2. \;\Box$$

To illustrate how our results can be used in practice we present an example.

Example 3.1 Let $\mathbb{T} = \Big\{0, \dfrac{1}{4}, \dfrac{2}{4}, \dfrac{3}{4}, 1, \dfrac{5}{4}, ...\Big\}$. We consider the following four point boundary value problem:

$$\begin{gathered}u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) + 10^{-3}(35+u)e_1(t,0) = 0, \quad t\in[0,1],\\ u(0) = 0, \quad u(\frac{5}{4}) = \frac{1}{2}u(\frac{1}{4}) + \frac{1}{4}u(\frac{1}{2}).\end{gathered}$$

This problem can be regarded as a BVP of the form (1)-(2), where $a(t) = 12/5, b(t) = -16/5, \lambda = 10^{-3}, h(t) = 1$ and $f(t,u) = (35+u)e_1(t,0)$. Clearly (H1)-(H3) are satisfied. Let $\phi_1(t)$ and $\phi_2(t)$ be the solutions of the following linear BVP's respectively.

$$\begin{aligned}u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) = 0 \quad t\in[0,1], \quad u(0) = 0, \quad u(\frac{5}{4}) = 1,\\ u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) = 0 \quad t\in[0,1], \quad u(0) = 1, \quad u(\frac{5}{4}) = 0.\end{aligned}$$

It is evident (from the the Corollaries 4.24 and 4.25 and Theorem 4.28 of [4]) that

$$\phi_1(t) = \frac{(\frac{5}{4})^{4t} - (\frac{1}{2})^{4t}}{(\frac{5}{4})^5 - (\frac{1}{2})^5} \quad \text{and} \quad \phi_2(t) = \frac{(\frac{5}{4})^5(\frac{1}{2})^{4t} - (\frac{1}{2})^5(\frac{5}{4})^{4t}}{(\frac{5}{4})^5 - (\frac{1}{2})^5}.$$

Also $\phi_1(t)$ satisfies (H4). The Green's function is of the following form:

$$G(t,s) = \frac{1024}{9279}\begin{cases} \{(\frac{5}{4})^{4t} - (\frac{1}{2})^{4t}\}\{(\frac{5}{4})^5(\frac{1}{2})^{4s} - (\frac{1}{2})^5(\frac{5}{4})^{4s}\}, & s \geq t,\\ \{(\frac{5}{4})^{4s} - (\frac{1}{2})^{4s}\}\{(\frac{5}{4})^5(\frac{1}{2})^{4t} - (\frac{1}{2})^5(\frac{5}{4})^{4t}\}, & t \geq s.\end{cases}$$

$p(t) = (\dfrac{2}{5})^{4t-1}$ follows from $e_\alpha(t,t_0) = \Big(1+\alpha h\Big)^{\frac{t-t_0}{h}}$ on $\mathbb{T} = h\mathbb{N}$. Furthermore we obtain $\gamma \approx 0,106$,

$$\int_0^{\frac{5}{4}} G(s,s)\Big(\frac{2}{5}\Big)^{4s-1}\nabla s \approx 0,44 \quad \text{and} \quad \int_{\frac{1}{2}}^{\frac{3}{4}} G(s,s)\Big(\frac{2}{5}\Big)^{4s-1}\nabla s \approx 0,0025.$$

and thus $M \approx 19,84$ and $N \approx 3650$.
If we choose $r_2 > r_1 > 0$ such that $r_1 = 5\cdot 10^{-6}$ and $r_2 = 0,1$, then it is straightforward from Theorem 3.2 that the four point BVP has at least one positive solution satisfying $5\cdot 10^{-6} \leq \|u\| \leq 0,1$.

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References

[1] Anderson, D.R. Solutions to second order three-point problems on time scales. *J. Difference Equations*, **47** (2004) 1–12.

[2] Avery, R.I. and Henderson, J. Two positive fixed points of nonlinear operators on ordered Banach spaces. *Comm. Appl. Nonlinear Anal.* **8** (2001) 27–36.

[3] Bohner, M. and Peterson, A. *Dynamic Equations on time scales, An Introduction with Applications*. Birkhäuser, Boston, 2001.

[4] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.

[5] Guo, D. and Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*. Academic Press, San Diego, 1988.

[6] Gupta, C.P. Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation. *J. Math. Anal. Appl.* **168** (1992) 540–551.

[7] Gupta, C.P. A generalized multi-point boundary value problem for a second order ordinary differential equations. *Appl. Math. Comput.* **89** (1998) 133–146.

[8] Il'in, V.A. and Moiseev, E.I. Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. *Differential Equations* **23** (1987) 803–810.

[9] Jiang, W. and Guo, Y. Multiple positive solutions for second-order m-point boundary value problems. *J. Math. Anal. Appl.* **327** (2007) 415–424.

[10] Krasnosel'skii, M.A. *Positive Solutions of Operator Equations*. Moskow, Fizmatgiz, 1962.

[11] Legget, R.W. and Williams, L.R. Multiple positive fixed points of nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **28** (1979) 673–688.

[12] Ma, R. and Wang, H. Positive solutions of nonlinear three-point boundary value problems. *J. Math. Anal. Appl.* **279** (2003) 216–227.

[13] Sun, H.R. and Li, W.T. Positive solutions for nonlinear three-point boundary value problems on time scales. *J. Math. Anal. Appl.* **299** (2004) 508–524.

[14] Yaslan, I. Existence of positive solutions for nonlinear three-point boundary value problems on time scales. *J. Comput. Appl. Math.* **206** (2007) 888–897.

[15] Zhong, X., Liang, J., Shi, Y., Wang, D. and Ge, L. Existence of nonoscillatory solution of high-order nonlinear difference equation. *Nonlinear Dynamics and System Theory* **6**(2) (2006) 205–210.

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Global Robust Dissipativity of Neural Networks with Variable and Unbounded Delays

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Abstract: In this paper, the global robust dissipativity of a class of neural networks with variable and unbounded delays is investigated. Several criteria are obtained by constructing radically unbounded and positive definite Lyapunov functionals and using analytic techniques. Some numerical examples are given to compare our results with previous robust dissipativity results derived in the literature. It is shown that our results extend and improve earlier ones.

Keywords: *dissipativity; neural networks; attractive set; integro-differential models.*

Mathematics Subject Classification (2000): 92B20, 93D09, 34K35.

1 Introduction

In recent years, the stability of dynamical neural networks has received much attention and has been used in signal processing, pattern recognition, associative memory and optimization problems [1–10]. However, it is possible that there are no equilibrium points of dynamical systems in some situations. As pointed in [11–15], the global dissipativity is a more general concept and is of great importance to study in dynamical neural networks. It has found applications in the areas such as stability theory, chaos and synchronization theory and robust control [12]. The authors of [12] analyzed the global dissipation of neural networks with both variable and unbounded delays. In [11], some conditions for globally robust dissipativity of neural networks with time-varying delays are derived.

In this paper, motivated by the above discussions, we obtain several new sufficient conditions for the global robust dissipativity of integro-differential models of neural networks with variable and unbounded delays. The results compared with those presented in [11] can be checked easily. Some numerical examples illustrate the proposed conditions may provide useful and less conservative results for the problem.

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2 System Description

In this paper, we consider the model of neural network with variable and unbounded delays as follows [11]:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds + u_i. \quad (1)$$

for $i = 1, 2, ..., n$, where n denotes the number of the neurons in the neural network, $x_i(t)$ is the state of the ith neuron at time t, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))]^T$ is the activation function of the jth neuron at time t, $D = \mathrm{diag}(d_1, d_2, ..., d_n)$ is a positive definite diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ are the feedback matrix and the delayed feedback matrix, respectively, $u = (u_1, u_2, ..., u_n)^T$ is a constant external input vector. The assumption on the transmission delay $\tau(t)$ is proposed as $0 < \tau_{ij}(t) \le \sigma$, $\tau(t)$ is a differential function such that $\frac{d\tau_{ij}(t)}{dt} \le \tau^* \le 1$, for $i, j = 1, 2, ...n$.

The delay kernel function $k(\cdot) = (K_{ij}(\cdot))_{n\times n}, i, j = 1, 2, ..., n$ is assumed to satisfy the following conditions simultaneously:

(1) $K_{ij} : [0, \infty) \to [0, \infty)$;

(2) K_{ij} are bounded and continuous on $[0, \infty)$;

(3) $\int_0^\infty K_{ij}(s)ds = 1$;

(4) there exists a positive number ε such that $\int_0^\infty K_{ij}(s)e^{\varepsilon s}ds < \infty$,

(5) $\int_0^\infty e^{\beta s} K_{ij}(s)ds = p_{ij}(\beta)$, for $i, j = 1, 2, ..., n$, where $p_{ij}(\beta)$ is continuous function in $[0, \delta), \delta > 0$, and $p_{ij}(0) = 1$.

The initial conditions associated with the system (1) are given by $x_i(s) = \phi_i(s), -\sigma \le s \le 0, i = 1, 2, ..., n$, where $\phi_i(\cdot)$ is bounded and continuous on $[-\sigma, 0]$.

Throughout this paper, we will employ the following classes of activation functions :

(1) The set of bounded activation functions is defined as

$$\Gamma = \{f(x)||f_i(x_i)| \le k_i, i = 1, 2, ..., n\}.$$

(2) The set of Lipschitz-continuous activation functions is defined as

$$\Psi = \{f(x)|0 \le \frac{f_i(x_i) - f_i(y_i)}{x_i - y_i} \le l_i, l_i > 0, \forall x_i, y_i \in R, x_i \ne y_i, i = 1, 2, ..., n\}.$$

(3) The general set of monotone non-decreasing activation functions is defined as

$$\Phi = \{f(x)|D^+ f_i(x_i) \ge 0, i = 1, 2, ..., n\}.$$

(4) There exist constants $\vartheta_i > 0$ such that $|f_i| \le \vartheta_i |x|, i = 1, 2, ..., n, \forall x \in R$. This class of functions will be denoted by $f(x) \in \Upsilon$.

The quantities d_i, a_{ij}, b_{ij} and c_{ij} may be considered as intervals as follows [15]:

$$\begin{aligned} D_I : &= \{D = \mathrm{diag}(d_i) : \underline{D} \le D \le \overline{D}, i.e., \underline{d}_i \le d_i \le \overline{d}_i, i = 1, ..., n, \forall D \in D_I\}, \\ A_I : &= \{A = (a_{ij})_{n\times n} : \underline{A} \le A \le \overline{A}, i.e., \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i, j = 1, ..., n, \forall A \in A_I\}, \\ B_I : &= \{B = (b_{ij})_{n\times n} : \underline{B} \le B \le \overline{B}, i.e., \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, i, j = 1, ..., n, \forall B \in B_I\}, \\ C_I : &= \{C = (c_{ij})_{n\times n} : \underline{C} \le C \le \overline{C}, i.e., \underline{c}_{ij} \le c_{ij} \le \overline{c}_{ij}, i, j = 1, ..., n, \forall C \in C_I\}. \quad (2) \end{aligned}$$

Similar to [11], we give the following definitions.

Definition 2.1 The neural network definied by (1) is said to be a dissipative system, if there exists a compact set $S \subset R^n$, such that $\forall x_0 \in R^n, \exists T > 0$, when $t \geq t_0 + T, x(t, t_0, x_0) \subseteq S$, where $x(t, t_0, x_0)$ denotes the solution of Eq. (1) from initial state x_0 and initial time t_0. In this case, S is called a globally attractive set. A set S is called positive invariant if $\forall x_0 \in S$ implies $x(t, t_0, x_0) \subseteq S$ for $t \geq t_0$.

Definition 2.2 If $R \to R$ is a continuous function, then the upper right derivative $\frac{D^+ f(t)}{dt}$ of $f(t)$ is defined as

$$D^+ f(t) = \lim_{\theta \to 0^+} \frac{f(t+\theta) - f(t)}{\theta}. \tag{3}$$

Lemma 2.1 *[16] Let D, S and P be real matrices of appropriate dimensions with $P > 0$. Then for any vectors x, y with appropriate dimensions,*

$$2x^T DSy \leq x^T DPD^T x + y^T S^T P^{-1} Sy.$$

3 Main Results

Theorem 3.1 *Let $f(x) \in \Gamma$, then neural network system (1) is a robust dissipative system and the set S_1 is a positive invariant and globally attractive set, where*

$$S_1 = \{x || x_i| \leq d_i^{-1} \sum_{j=1}^{n} [(a_{ij}^* + b_{ij}^* + c_{ij}^*) k_j + |u_i|)], i = 1, 2, ..., n\}, \tag{4}$$

$a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij}), b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij})$ *and* $c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$.

Proof Let us use a radically unbounded and positive definite Lyapunov functional

$$V(x) = \sum_{i=1}^{n} \frac{1}{r} |x_i|^r.$$

Computing $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned} \frac{dV}{dt} &= \sum_{i=1}^{n} |x_i|^{r-1} sgn(x_i) \frac{dx_i}{dt} \\ &\leq \sum_{i=1}^{n} [-\underline{d}_i |x_i|^r + \sum_{j=1}^{n} (a_{ij}^* + b_{ij}^* + c_{ij}^*) k_j |x_i|^{r-1} + |u_i||x_i|^{r-1}] \\ &= -\sum_{i=1}^{n} |x_i|^{r-1} [\underline{d}_i |x_i| - \sum_{j=1}^{n} [(a_{ij}^* + b_{ij}^* + c_{ij}^*) k_j + |u_i|] < 0, \end{aligned} \tag{5}$$

where $x \in R^n \backslash S_1$, i.e., $x \overline{\in} S_1$. Eq. (5) implies that the neural network system (1) is a robust dissipative system and S_1 is a positive invariant and globally attractive set. □

Theorem 3.2 *Let $f(x) \in \Psi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If*

$$\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*} \overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1) I \leq 0,$$

where $C^* = (c_{ji}^*)_{n \times n}$, then the neural network system (1) is a robust dissipative system and the set $S_2 = \{x || f_i(x_i(t))| \leq \frac{l_i |u_i|}{\underline{d}_i}, i = 1, 2, ..., n$ is a positive invariant and globally attractive set.

Proof We use the following positive definite and unbounded Lyapunov functional:

$$V(x(t)) = 2\sum_{i=1}^{n}\int_{0}^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} f_i^2(x_i(s))ds + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ji}(s)(\int_{t-s}^{t} f_i^2(x_i(\xi)d\xi)ds.$$

Computing $\frac{dV}{dt}$ along the positive half trajectory of (1), we can conclude that

$$\begin{aligned}
\frac{dV}{dt} &= -2\sum_{i=1}^{n} d_i f_i(x_i(t))x_i(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(x_i(t)) f_j(x_j(t)) + \sum_{i=1}^{n} f_i^2(x_i(t)) \\
&+ 2\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} f_i(x_i(t)) f_j(x_j(t-\tau_j(t))) - \sum_{i=1}^{n}(1-\frac{d\tau_i(t)}{dt}) f_i^2(x_i(t-\tau_i(t))) \\
&+ 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_i(x_i(t)) \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds + 2\sum_{i=1}^{n} f_i(x_i(t))u_i \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds \\
&\le -2\sum_{i=1}^{n} \frac{\underline{d}_i}{l_i} f_i^2(x_i(t)) + f^T(x(t))(A+A^T)f(x(t)) + 2f^T(x(t))Bf(x(t-\tau(t))) \\
&+ f^T(x(t))f(x(t)) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^* \int_0^\infty K_{ij} f_i(x_i(t)) f_j(x_j(t-s))ds \\
&+ 2\sum_{i=1}^{n} |u_i||f_i(x_i(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* f_j^2(x_j(t)) \\
&- (1-\tau^*) f^T(x(t-\tau(t))) f(x(t-\tau(t))) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ij}(s) f_j^2(x_j(t-s))ds.
\end{aligned} \tag{6}$$

Using Lemma 1 and inequality technique, we have

$$\begin{aligned}
\frac{dV}{dt} &\le -2\sum_{i=1}^{n} \frac{\underline{d}_i}{l_i} |f_i(x_i(t))|[|f_i(x_i(t))| - \frac{l_i|u_i|}{\underline{d}_i}] + f^T(x(t))[\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*}\overline{B}\overline{B}^T \\
&+ (1 + \|C^*\|_\infty + \|C^*\|_1)I]f(x(t)) \le -2\sum_{i=1}^{n} \frac{\underline{d}_i}{l_i} |f_i(x_i(t))|[|f_i(x_i(t))| - \frac{l_i|u_i|}{\underline{d}_i}] < 0
\end{aligned} \tag{7}$$

when $x \in R^n \backslash S_2$. Eq. (7) implies that the set S_2 is a positive invariant and globally attractive set. □

Theorem 3.3 *Let $f(x) \in \Psi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, ..., p_n)$ such that the matrix*

$$Q = P(\overline{A} - \underline{D}L^{-1}) + (\overline{A}^T - \underline{D}L^{-1})P + \frac{1}{1-\tau^*}P\overline{B}\overline{B}^T P + (1 + \|PC^*\|_\infty + \|PC^*\|_1)I$$

is negative definite, then the neural network system (1) is a robust dissipative system and the set

$$S_3 = \left\{ x | \sum_{i=1}^{n} (f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)})^2 \leq \sum_{i=1}^{n} (\frac{p_i u_i}{\lambda_M(Q)})^2, i = 1, 2, ..., n \right\}$$

is a positive invariant and globally attractive set, where $L = \mathrm{diag}(L_1, L_2, ..., L_n), P = \mathrm{diag}(p_1, p_2, ..., p_n)$ *and* $\lambda_M(Q)$ *is the maximum eigenvalue of the matrix* Q.

Proof We employ the following positive definite and radially unbounded Lyapunov functional:

$$V(x(t)) = 2\sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} f_i^2(x_i(\xi))d\xi$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s)(\int_{t-s}^{t} f_i^2(x_i(\xi))d\xi)ds.$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we obtain that

$$\begin{aligned}
\frac{dV}{dt} &= 2\sum_{i=1}^{n} p_i f_i(x_i(t))[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + u_i \\
&+ \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds] - \sum_{i=1}^{n} (1 - \frac{d\tau_i(t)}{dt}) f_i^2(x_i(t-\tau_i(t))) \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds + \sum_{i=1}^{n} f_i^2(x_i(t)) \\
&\leq -2\sum_{i=1}^{n} \frac{p_i d_i}{l_i} f_i^2(x_i(t)) + f^T(x(t))(P\overline{A} + \overline{A}^T P) f(x(t)) + f^T(x(t)) f(x(t)) \\
&+ 2\sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ij}^* \int_0^{\infty} K_{ij}(s) f_i(x_i(t)) f_j(x_j(t-s))ds + 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) \\
&- (1-\tau^*) f^T(x(t-\tau(t))) f(x(t-\tau(t))) + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* f_i^2(x_i(t)) \\
&- \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s) f_i^2(x_i(t-s))ds + 2f^T(x(t)) PB f(x(t-\tau(t))).
\end{aligned} \quad (8)$$

From Lemma 1 and inequality technique, we can write the following inequalities:

$$\begin{aligned}
\frac{dV}{dt} &\leq -2\sum_{i=1}^{n} \frac{p_i d_i}{l_i} f_i^2(x(t)) + f^T(x(t))(P\overline{A} + \overline{A}^T P) f(x(t)) \\
&+ \frac{1}{1-\tau^*} f^T(x(t)) P\overline{B}\overline{B}^T P f(x(t)) + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ij}^* f_i^2(x_i(t))
\end{aligned}$$

$$\begin{aligned}
&+ \quad f^T(x(t))f(x(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i c_{ij}^* f_j^2(x_j(t)) + 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) \\
&= \quad 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) + f^T(x(t))Qf(x(t)) \\
&\leq \quad 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) + \lambda_M(Q)\sum_{i=1}^{n} f_i^2(x_i(t)) \\
&= \quad \lambda_M(Q)\sum_{i=1}^{n}[(f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)})^2 - (\frac{p_i u_i}{\lambda_M(Q)})^2] < 0, \qquad (9)
\end{aligned}$$

when $x \in R^n \backslash S_3$. Eq. (9) implies that the set S_3 is a positive invariant and globally attractive set. □

Theorem 3.4 *Let $f(x) \in \Phi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If the following condition holds:*

$$\overline{A} + \overline{A}^T + \overline{B} + \frac{1}{1-\tau^*}\overline{B}^T + (\|C^*\|_\infty + \|C^*\|_1)I \leq 0,$$

where $C^* = (c_{ji}^*)_{n\times n}$, then the neural network system (1) is a robust dissipative system and the set $S_4 = \{x||x_i(t) \leq \frac{|u_i|}{\underline{d}_i}, i = 1, 2, ..., n\}$ is a positive invariant and globally attractive set.

Proof Let us use the following positive definite and radially unbounded Lyapunov functional:

$$V(x(t)) = 2\sum_{i=1}^{n}\int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ji}(t)}^{t} b_{ji}^* f_i^2(x_i(s))ds$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^*\int_0^{\infty} K_{ji}(s)(\int_{t-s}^{t} f_i^2(x_i(\xi))d\xi)ds.$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned}
\frac{dV}{dt} \quad &= \quad 2\sum_{i=1}^{n} f_i(x_i(t))[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + u_i \\
&+ \quad \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds] - \sum_{i=1}^{n}\sum_{j=1}^{n}(1 - \frac{d\tau_{ji}(t)}{dt}) f_i^2(x_i(t-\tau_{ji}(t))) \\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}^* f_i^2(x_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^*\int_0^{\infty} K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds \\
&\leq \quad -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + f^T(x(t))(\overline{A} + \overline{A}^T)f(x(t)) + 2\sum_{i=1}^{n} u_i f_i(x_i(t))
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{*}\int_{0}^{\infty} K_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))ds + 2f^T(x(t))Bf(x(t-\tau(t)))\\
&- \quad (1-\tau^*)f^T(x(t-\tau(t)))Bf(x(t-\tau(t))) + f^T(x(t))Bf(x(t))\\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}f_i^2(x_i(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}\int_{0}^{\infty} K_{ji}(s)f_i^2(x_i(t-s))ds. \qquad (10)
\end{aligned}
$$

From Lemma 1, it follows that

$$
\begin{aligned}
2f^T(x(t))Bf(x(t-\tau(t))) \quad &\le \quad \frac{1}{1-\tau^*}f^T(x(t))B^Tf(x(t))\\
&+ \quad (1-\tau^*)f^T(x(t-\tau(t)))B^TB^{-T}Bf(x(t-\tau(t)))\\
&= \quad \frac{1}{1-\tau^*}f^T(x(t))B^Tf(x(t))\\
&+ \quad (1-\tau^*)f^T(x(t-\tau(t)))Bf(x(t-\tau(t))) \qquad (11)
\end{aligned}
$$

By using the inequality $2ab \le a^2+b^2$ for any $a, b \in R$, we have

$$
2\int_{0}^{\infty} K_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))ds \le \int_{0}^{\infty} K_{ij}(s)f_i^2(x_i(t))ds + \int_{0}^{\infty} K_{ij}(s)f_j^2(x_j(t-s))ds. \quad (12)
$$

From (10) to (12), we get

$$
\begin{aligned}
\frac{dV}{dt} \quad &\le \quad -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i|\\
&+ \quad f^T(x(t))(\overline{A}+\overline{A}^T+\overline{B}+\frac{1}{1-\tau^*}\overline{B}^T+(\|C^*\|_\infty+\|C^*\|_1)I)f(x(t))\\
&\le \quad -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| < 0, \qquad (13)
\end{aligned}
$$

when $x \in R^n \backslash S_4$. Eq. (13) implies that the set S_4 is a positive invariant and globally attractive set. □

Theorem 3.5 *Let $f(x) \in \Upsilon, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If the following condition holds:*

$$
\sum_{j=1}^{n}(\overline{a}_{ij} + \frac{1}{1-\tau^*}b_{ij}^* + c_{ij}^*) < 0,
$$

where $a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij}), b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij}), c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$, then the neural network system (1) is a robust dissipative system and the set $S_5 = \{x||x_i(t) \le \frac{|u_i|}{\underline{d}_i}, i = 1,2,...,n\}$ is a positive invariant and globally attractive set.

Proof Let us use the following positive definite and radially unbounded Lyapunov functional:

$$
V(x(t)) = x_i(t) + \frac{1}{1-\tau^*}\sum_{j=1}^{n} b_{ij}^* \int_{t-\tau_j(t)}^{t} |f_j(x_j(s))|ds + \sum_{j=1}^{n} c_{ij}^*\vartheta_j \int_{0}^{\infty} K_{ij}(s)\int_{t-s}^{t} |x_j(\xi)|d\xi\, ds.
$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned}\frac{dV}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau(t))) + u_i \\ &+ \frac{1}{1-\tau^*} \sum_{j=1}^{n} b_{ij}^* f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} \int_0^\infty K_{ij}(s) f_j(x_j(t-s)) ds \\ &- \frac{1}{1-\tau^*}(1 - \frac{d\tau(t)}{dt}) \sum_{j=1}^{n} b_{ij}^* f_j(x_j(t-\tau(t))) \\ &+ \sum_{j=1}^{n} c_{ij}^* \vartheta \int_0^\infty K_{ij}(s)|x_j(t)| ds - \sum_{j=1}^{n} c_{ij}^* \vartheta_j \int_0^\infty K_{ij}(s)|x_j(t-s)| ds \\ &\le -\underline{d}_i |x_i(t)| + \sum_{j=1}^{n} \overline{a}_{ij} \vartheta_j |x_j| + \sum_{j=1}^{n} b_{ij} \vartheta_j |x_j(t-\tau(t))| + |u_i| + \frac{1}{1-\tau^*} \sum_{j=1}^{n} b_{ij}^* \vartheta_j x_j(t) \\ &- \sum_{j=1}^{n} b_{ij} \vartheta_j x_j(t-\tau(t))) + \sum_{j=1}^{n} c_{ij}^* \vartheta_j |x_j| \\ &= -\underline{d}_i |x_i(t)| + |u_i| + \sum_{j=1}^{n} \vartheta_j (\overline{a}_{ij} + \frac{1}{1-\tau^*} b_{ij}^* + c_{ij}^*)|x_j| < 0, \end{aligned} \tag{14}$$

when $x \in R^n \backslash S_5$. Eq. (14) implies that the set S_5 is a positive invariant and globally attractive set. □

Remark 3.1 Our activation functions are more general than those in [11]. Hence, our results improve and generalizes the earlier results.

Remark 3.2 Our methods used in this paper, such as Lyapunov functional and matrix inequalities used in Theorem 4, are different to those in [11].

Remark 3.3 The neural network system in [14] can be seen as a special case for model (1). Therefore, the global robust dissipativity of that system can be studied similarly.

4 Comparison and Examples

To compare with [11], we restated Theorem 1 of [11].

Theorem 4.1 *Let $f(x) \in \Upsilon, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$, the neural network defined by (1) is a robust dissipative system and the set $S_6 = \{x||x_i(t) \le \frac{|u_i|}{\underline{d}_i}, i = 1, 2, ..., n\}$ is a positive invariant and globally attractive set, if there exist positive constants $p_i > 0, i = 1, 2, ..., n$ such that*

$$p_i(-\overline{a}_{ii} - \frac{1}{1-\tau^*} b_{ii}^* - c_{ii}^*) - \sum_{j=1, j\neq i}^{n} p_j(a_{ji}^* + \frac{1}{1-\tau^*} b_{ji}^* + c_{ji}^*) \ge 0, \tag{15}$$

where $i = 1, 2, ..., n$, $a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij})$, $b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij})$, $c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$.

Example 4.1 Consider the system (1) with delays: $\tau_{ij}(t) = 1$ for $i, j = 1, 2$,

$$\underline{D} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1.2 \end{bmatrix}, \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}, \underline{A} = \begin{bmatrix} -2 & 0.7 \\ -0.9 & -3 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} -1.5 & 0.4 \\ 0.3 & -1.5 \end{bmatrix}, \underline{B} = \begin{bmatrix} 0.25 & -0.5 \\ -0.2 & -0.7 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0.5 & 0.25 \\ 0 & -0.5 \end{bmatrix},$$

$$\underline{C} = \overline{C} = 0, u_1 = 1.5, u_2 = -2, \sigma = 1, \tau^* = 0, p_i = 1.$$

The initial values of system (1) is assumed as $\phi(s) = 0.5, t \in [-1, 0)$. Since

$$\begin{cases} \overline{a}_{11} + \overline{a}_{12} + \frac{1}{1-\tau^*}b^*_{11} + \frac{1}{1-\tau^*}b^*_{12} + c^*_{11} + c^*_{12} = -0.1 < 0, \\ \overline{a}_{21} + \overline{a}_{22} + \frac{1}{1-\tau^*}b^*_{21} + \frac{1}{1-\tau^*}b^*_{22} + c^*_{21} + c^*_{22} = -0.3 < 0, \end{cases}$$

the condition of Theorem 5 in this paper is satisfied; the neural network system (1) is a globally robust dissipative system, and the set $S_5 = \{(x_1(t), x_2(t)) || x_1(t)| \le \frac{15}{4}, |x_2(t)| \le \frac{5}{3}\}$ is positive invariant and globally attractive. Since

$$\begin{cases} -\overline{a}_{11} - \frac{1}{1-\tau^*}b^*_{11} - c^*_{11} - (a^*_{21} + \frac{1}{1-\tau^*}b^*_{21} + c^*_{21}) = -0.1 < 0, \\ -\overline{a}_{22} - \frac{1}{1-\tau^*}b^*_{22} - c^*_{22} - (a^*_{12} + \frac{1}{1-\tau^*}b^*_{12} + c^*_{12}) = -0.4 < 0, \end{cases}$$

the condition of Theorem 6 is not satisfied, one can not determine the dissipativity of the neural network (1). Therefore, our obtained criteria for the global robust dissipativity of neural networks with variable and unbounded delays are new.

Example 4.2 Consider the system (1) with delays: $\tau_{ij}(t) = 1$ for $i, j = 1, 2$,

$$\underline{D} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1.2 \end{bmatrix}, \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}, \underline{A} = \begin{bmatrix} -3.3 & -0.25 \\ \frac{1}{3} & -3 \end{bmatrix}, \overline{A} = \begin{bmatrix} -3 & 0.25 \\ 0.5 & -4 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \overline{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \underline{C} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix}, \overline{C} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix},$$

and $u_1 = 1.5, u_2 = -2, \sigma = 1, \tau^* = 0$. The initial values of system (1) are assumed as $\phi(s) = 0.5, t \in [-1, 0)$. Since that

$$\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*}\overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1)I = \begin{bmatrix} -2 & \frac{7}{4} \\ \frac{7}{4} & -4 \end{bmatrix} \le 0,$$

then the conditions of Theorem 2 are satisfied, and the neural networks system (1) is a globally robust dissipative system, and the set

$$S_2 = \{f_1(x_1(t)), f_2(x_2(t)) || f_1(x_1(t))| \le \frac{15}{4}l_1, |f_2(x_2(t))| \le \frac{5}{3}l_2\}$$

is positive invariant and globally attractive.

5 Conclusion

This paper studies the global robust dissipativity of a class of neural networks with variable and unbounded delays. Several sufficient conditions are presented to characterize the global dissipation together with their sets of attraction. Our results would make good effects in studying the uniqueness of equilibria, global asymptotic stability, instability and the exsitence of periodic solutions. In addition, several examples are given to demonstrate the improvements and correctness of our results.

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References

[1] Roska, T. and Chua, L.O. Cellular neural networks with nonlinear and delay-type template. *Int. J. Circuit Theory Appl.* **20** (1992) 469–481.

[2] Arik, S. Stability analysis of delayed neural networks. *IEEE Trans. Circuits and Systems I.* **47** (7) (2000) 1089–1092.

[3] Cao,J.D. A set of stability criteria for delayed cellular neural networks. *IEEE Trans. Circuits and Systems I.* **48** (4) (2001) 494–498.

[4] Arik, S. An improved global stability result for delayed cellular neural networks. *IEEE Trans Circ Sys I.* **49** (2002) 1211–1214.

[5] Singh, V. Robust stability of cellular neural networks with delay: linear matrix inequality approach. *IEEE Proc* **15** (2004) 125–129.

[6] Singh, V. A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks. *IEEE Trans Neural Network.* **15** (1) (2004) 223–225.

[7] Arik, S. Global robust stability of delayed neural networks. *IEEE Trans Circ Syst I* **50** (1) (2003) 156–160.

[8] Cao, J.D. and Wang, J. Global asymptotic and robust stability of recurrent neural networks with time delays. *IEEE Trans Circ Syst I* **52** (2) (2005) 417–426.

[9] Zhang, H.B. and Liao, X.F. LMI-based robust stability analysis of neural networks with time-varying delay. *Neurocomputing* **67** (2005) 306–312.

[10] Singh, V. A novel global robust stability criterion for neural networks with delay. *Physics Letters A* **337** (2005) 369–373.

[11] Lou, X.Y. and Cui, B.T. Global robust dissipativity for integro-differential systems modeling neural networks with delays. *Chaos, Solitons and Fractals* **36** (2006) 469–478.

[12] Song, Q.K. and Zhao, Z.J. Global dissipativity of neural networks with both variable and unbounded delays. *Chaos, Solitons and Fractals* **25** (2005) 393–401.

[13] Liao, X.X. and Wang, J. Global dissipativity of continuous-time recurrent neural networks with time delay. it Physical Review E **68** (2003) 1–7.

[14] Arik, S. On the global dissipativity of dynamical neural networks with time delays. *Physics Letters A* **326** (2004) 126–132.

[15] Cao, J.D., Yuan, K., Ho, D.W.C. and Lam, J. Global point dissipativity of neural networks with mixed time-varying delays. *Chaos* **16** (2006) 1–9.

[16] Xu, S., Lam, J., Ho, D.W.C. and Zou, Y. Global robust exponential stability analysis for interval recurrent neural networks. *Physics Letters A* **325** (2004) 124–133.

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Frequent Oscillatory Solutions of a Nonlinear Partial Difference Equation

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Abstract: This paper is concerned with a class of nonlinear delay partial difference equations with variable coefficients, which may change sign. By making use of frequency measures, some new oscillatory criteria are established.

Keywords: *partial difference equations; frequency oscillatory; frequency measures; nonlinear.*

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1 Introduction

Let Z be the set of integers, $Z[k,l] = \{i \in Z | i = k, k+1, ..., l\}$ and $Z[k,\infty) = \{i \in Z | i = k, k+1, ...\}$.

In [1], authors considered oscillations of the partial difference equation with several nonlinear terms of the form

$$u_{m+1,n} + u_{m,n+1} - u_{m,n} + \sum_{i=1}^{h} p_i(m,n) \left|u_{m-k_i,n-l_i}\right|^{\alpha_i} \operatorname{sgn} u_{m-k_i,n-l_i} = 0.$$

In this paper, we investigate the equation of the following form

$$u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} - u_{m,n} + \sum_{i=1}^{h} p_i(m,n) \left|u_{m-k_i,n-l_i}\right|^{\alpha_i} \operatorname{sgn} u_{m-k_i,n-l_i} = 0, \quad (1)$$

where $m,n \in Z[0,\infty)$, $P_i(m,n) \geq 0$ $(i = 1, 2, \cdots, h)$ and

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(H_1) $\alpha_h > \alpha_{h-1} > \cdots > \alpha_k > 1 > \alpha_{k-1} > \cdots > \alpha_1 > 0$;

(H_2) k_i, l_i $(i = 1, 2, \cdots, h)$ are nonnegative integers.

Such an equation arises in several mathematical models (see e.g.[3]) including interconnected neuron units placed on an arbitrary large board, heat transfer in lattice of molecules, population migration among cities, and discrete simulation of the heat equation *et al.*

The usual concepts of oscillation or stability of steady state solutions do not catch all their fine details, and it is necessary to use the concept of frequency measures introduced in [2] to provide better descriptions. In this paper, by employing frequency measures, some new oscillatory criteria of (1) are established.

In addition to (H_1) and (H_2), we also assume

(H_3) $p_i = \{p_i(m,n)\}_{m,n\in Z[0,\infty)}$ $(i = 1, 2, \cdots, h)$ are real double sequences;

(H_4) Suppose there exists $a_i > 0$ $(i = 1, 2, \cdots, h)$ such that $\sum_{i=1}^h a_i = 1$ and $\sum_{i=1}^h a_i\alpha_i = 1$;

(H_5) If $p_i = \{p_i(m,n)\}$ has negative components, then a_i is chosen such that a_i is a quotient of odd positive integers.

Let

$$\overline{k} = \max_{1\le i\le h}\{k_i\} > 0,\ \overline{l} = \max_{1\le i\le h}\{l_i\} > 0,\ \underline{k} = \min_{1\le i\le h}\{k_i\},\ \underline{l} = \min_{1\le i\le h}\{l_i\}$$

and

$$\gamma = \min\left\{\frac{1}{a_1}, \cdots, \frac{1}{a_h}\right\}.$$

Since $0 < a_i < 1$, we see that $\gamma > 1$.

Our plan is as follows. In the next section, we recall some of the terminologies and basic results related to the frequency measures. Then we derive several criteria for all solutions of (1) to be frequently oscillatory or unsaturated. In the final section, we give some examples to illustrate our results.

For the sake of convenience, $Z[-\overline{k},\infty) \times Z[-\overline{l},\infty)$ will be denoted by Ω in the sequel. Given a double sequence $\{u_{m,n}\}$, the partial differences $u_{m+1,n} - u_m$ and $u_{m,n+1} - u_{m,n}$ will be denoted by $\Delta_1 u_{m,n}$ and $\Delta_2 u_{m,n}$ respectively.

2 Preliminaries

The union, intersection and difference of two sets A and B will be denoted by $A + B$, $A \cdot B$ and $A\backslash B$ respectively. The number of elements of a set S will be denoted by $|S|$. Let Φ be a subset of Ω. Then

$$X^m\Phi = \{(i+m, j) \in \Omega |\, (i,j) \in \Phi\},\quad Y^m\Phi = \{(i, j+m) \in \Omega |\, (i,j) \in \Phi\}$$

are the translations of Φ. Let α, β, λ and δ be integers satisfying $\alpha \le \beta$ and $\lambda \le \delta$. The union $\sum_{i=\alpha}^{\beta}\sum_{j=\lambda}^{\delta} X^iY^j\Phi$ will be denoted by $X_\alpha^\beta Y_\lambda^\delta\Phi$. Clearly,

$$(i,j) \in \Omega\backslash X_\alpha^\beta Y_\lambda^\delta\Phi \Leftrightarrow (i-s, j-t) \in \Omega\backslash\Phi$$

for $\alpha \le s \le \beta$ and $\lambda \le t \le \delta$.

For any $m, n \in Z[0,\infty)$, we set $\Phi^{(m,n)} = \left\{(i,j) \in \Phi |\, -\overline{k} \le i \le m, -\overline{l} \le j \le n\right\}$. If

$$\limsup_{m,n\to\infty} \frac{\left|\Phi^{(m,n)}\right|}{mn}$$

exists, then the superior limit, denoted by $\mu^*(\Phi)$, will be called the upper frequency measure of Φ. Similarly, if

$$\liminf_{m,n\to\infty} \frac{\left|\Phi^{(m,n)}\right|}{mn}$$

exists, then the inferior limit, denoted by $\mu_*(\Phi)$, will be called the lower frequency measure of Φ. If $\mu_*(\Phi) = \mu^*(\Phi)$, then the common limit is denoted by $\mu(\Phi)$ and is called the frequency measure of Φ.

Clearly, $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and $0 \le \mu_*(\Phi) \le \mu^*(\Phi) \le 1$ for any subset Φ of Ω, furthermore if Φ is finite, then $\mu(\Phi) = 0$.

The following results are concerned with the frequency measures and their proofs are similar to those in [3].

Lemma 2.1 *Let Φ and Γ be subsets of Ω. Then $\mu^*(\Phi + \Gamma) \le \mu^*(\Phi) + \mu^*(\Gamma)$. Furthermore, if Φ and Γ are disjoint, then*

$$\mu_*(\Phi) + \mu_*(\Gamma) \le \mu_*(\Phi + \Gamma) \le \mu_*(\Phi) + \mu^*(\Gamma) \le \mu^*(\Phi + \Gamma) \le \mu^*(\Phi) + \mu^*(\Gamma),$$

so that

$$\mu_*(\Phi) + \mu^*(\Omega\backslash\Phi) = 1.$$

Lemma 2.2 *Let Φ be a subset of Ω and α, β, λ and δ be integers such that $\alpha \le \beta$ and $\lambda \le \delta$. Then*

$$\mu^*\left(X_\alpha^\beta Y_\lambda^\delta \Phi\right) \le (\beta - \alpha + 1)(\delta - \lambda + 1)\mu^*(\Phi)$$

and

$$\mu_*\left(X_\alpha^\beta Y_\lambda^\delta \Phi\right) \le (\beta - \alpha + 1)(\delta - \lambda + 1)\mu_*(\Phi).$$

Lemma 2.3 *Let $\Phi_1, ..., \Phi_n$ be subsets of Ω. Then*

$$\mu^*\left(\sum_{i=1}^n \Phi_i\right) \le \sum_{i=1}^n \mu^*(\Phi_i) - (n-1)\mu^*\left(\prod_{i=1}^n \Phi_i\right)$$

and

$$\mu_*\left(\sum_{i=1}^n \Phi_i\right) \le \mu_*(\Phi_1) + \mu^*\left(\sum_{i=2}^n \Phi_i\right) - (n-1)\mu^*\left(\prod_{i=1}^n \Phi_i\right).$$

Lemma 2.4 *Let Φ and Γ be subsets of Ω. If $\mu_*(\Phi) + \mu^*(\Gamma) > 1$, then the intersection $\Phi \cdot \Gamma$ is infinite.*

For any real double sequence $\{v_{i,j}\}$ defined on a subset of Ω, the level set $\{(i,j) \in \Omega |\, v_{i,j} > c\}$ is denoted by $(v > c)$. The notations $(v \ge c)$, $(v < c)$, $(v \le c)$ are similarly defined. Let $u = \{u_{i,j}\}_{(i,j)\in\Omega}$ be a real double sequence. If $\mu^*(u \le 0) = 0$, then u is said to be frequently positive, and if $\mu^*(u \ge 0) = 0$, then u is said to be frequently negative.

u is said to be frequently oscillatory if it is neither frequently positive nor frequently negative.. If $\mu^*(u > 0) = \omega \in (0, 1)$, then u is said to have unsaturated upper positive part, and if $\mu_*(u > 0) = \omega \in (0, 1)$, then u is said to have unsaturated lower positive part. u is said to have unsaturated positive part if $\mu^*(u > 0) = \mu_*(u > 0) = \omega \in (0, 1)$.

The concepts of frequently oscillatory and unsaturated double sequences were introduced in [2-6]. It was also observed that if a double sequence $u = \{u_{i,j}\}_{(i,j)\in\Omega}$ is frequently oscillatory or has unsaturated positive part, then it is oscillatory, that is, u is not positive for all large m and n, nor negative for all large m and n. Thus if we can show that every solution of (1) is frequently oscillatory or has unsaturated positive part, then every solution of (1) is oscillatory.

3 Frequently Oscillatory Solutions

An inequality, which can be found in [7], will be used in deriving the following results:

$$\sum_{i=1}^{h} \sigma_i x_i \geq \prod_{i=1}^{h} x_i^{\sigma_i}, \tag{2}$$

where $\sigma_i > 0, \sum_{i=1}^{h} \sigma_i = 1, x_i \geq 0, i = 1, 2, \cdots, h$.

Lemma 3.1 *Suppose there exist* $m_0 \geq 2\overline{k}$ *and* $n_0 \geq 2\overline{l}$ *such that*

$$p_i(m,n) \geq 0 \;\; \text{for } (m,n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1], \;\; i = 1, 2, \cdots, h.$$

Let $\{u_{m,n}\}$ *be a solution of (1). If* $u_{m,n} \geq 0$ *for* $(m,n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$, *then*

$$\Delta_1 u_{m,n} \leq 0, \Delta_2 u_{m,n} \leq 0 \;\; \text{for } (m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0],$$

and if $u_{m,n} \leq 0$ *for* $(m,n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$, *then*

$$\Delta_1 u_{m,n} \geq 0, \Delta_2 u_{m,n} \geq 0 \;\; \text{for } (m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0].$$

Proof If $u_{m,n} \geq 0$ for $(m,n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$, it follows from (1) that

$$\begin{aligned} u_{m,n} &= u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} + \sum_{i=1}^{h} p_i(m,n) u_{m-k_i,n-l_i}^{\alpha_i} \\ &\geq u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} \\ &\geq u_{m+1,n} + u_{m,n+1}. \end{aligned}$$

Hence $\Delta_1 u_{m,n} \leq 0$, $\Delta_1 u_{m,n} \leq 0$ for $(m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0]$.

Similarly, we also have $\Delta_1 u_{m,n} \geq 0, \Delta_2 u_{m,n} \geq 0$ for $(m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0]$. Let

$$\prod_{i=1}^{h} p_i^{a_i} = \left\{ \prod_{i=1}^{h} p_i^{a_i}(m,n) \right\}_{m,n \in Z[0,\infty)}.$$

Under the assumption (H_5), $\prod_{i=1}^{h} p_i^{a_i}$ is well defined. We remark that if $p_i(m,n) \geq 0$, the assumption (H_5) is not needed.

Theorem 3.1 *Suppose there exist constants* ω_i $(i = 1, 2, \cdots, h)$ *and* ω *such that*

$$\mu^*(p_i < 0) = \omega_i \;\; (i = 1, 2, \cdots, h), \;\; \mu_* \left(\prod_{i=1}^{h} (p_i < 0) \right) = \omega,$$

$$\mu_* \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1 \right) > 4(\overline{k} + 1)(\overline{l} + 1) \left(\sum_{i=1}^{h} \omega_i - (h-1)\,\omega \right).$$

Then every nontrivial solution of (1) is frequently oscillatory.

Proof Suppose to the contrary that $u = \{u_{m,n}\}$ is a frequently positive solution of (1). Then $\mu^*(u \le 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}
1 &= \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} \\
&\quad + \mu_*\left\{X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} \\
&\le \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} \\
&\quad + 4(\overline{k}+1)(\overline{l}+1)\left\{\mu_*\left(\sum_{i=1}^{h}(p_i < 0)\right) + \mu^*(u \le 0)\right\} \\
&\le \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} + 4(\overline{k}+1)(\overline{l}+1)\left(\sum_{i=1}^{h}\omega_i - (h-1)\,\omega\right) \\
&< \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} + \mu_*\left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right).
\end{aligned}$$

Therefore by Lemma 4, the intersection

$$\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \le 0)\right]\right\} \cdot \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right)$$

is infinite. This implies that there exist $m_0 \ge 2\overline{k}$ and $n_0 \ge 2\overline{l}$ such that

$$\gamma \prod_{i=1}^{h} p_i^{a_i}(m_0, n_0) > 1 \tag{3}$$

and

$$p_i(m,n) \ge 0 \ (i = 1, 2, \cdots, h), u_{m,n} > 0. \tag{4}$$

for $(m,n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$. In view of (4) and Lemma 3.1, we may then see that $\Delta_1 u_{m,n} \le 0$ and $\Delta_2 u_{m,n} \le 0$ for $(m,n) \in Z[m_0 - \overline{k}, m_0] \times Z[n_0 - \overline{l}, n_0]$, and hence $u_{m_0 - k_i, n_0 - l_i} \ge u_{m_0 - \underline{k}, n_0 - \underline{l}} \ge u_{m_0, l_0}$ $(i = 1, 2, \cdots, h)$, so that by (2) and (4),

$$\begin{aligned}
0 &\ge u_{m_0+1,n_0+1} + u_{m_0+1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} + \sum_{i=1}^{h} p_i(m_0,n_0) u_{m_0-\underline{k},n_0-\underline{l}}^{\alpha_i} \\
&\ge u_{m_0+1,n_0+1} + u_{m_0+1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} + \gamma \prod_{i=1}^{h} p_i^{a_i}(m_0,n_0) u_{m_0,n_0} \\
&\ge \left(\gamma \prod_{i=1}^{h} p_i^{a_i}(m_0,n_0) - 1\right) u_{m_0,n_0} > 0,
\end{aligned}$$

which is a contradiction.

In a similar manner, if $u = \{u_{m,n}\}$ is a frequently negative solution of (1) such that $\mu^*(u \geq 0) = 0$, then we may show that

$$\left\{\Omega \backslash X_{-1}^{2k_1} Y_{-1}^{2l_1}\left[\sum_{i=1}^{h}(p_i < 0) + (u \geq 0)\right]\right\} \cdot \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right)$$

is infinite. Again we may arrive at a contradiction as above. The proof is complete.

Theorem 3.2 *Suppose there exist constants* ω_i $(i = 1, 2, \cdots, h)$ *and* ω *such that*

$$\mu^*(p_i < 0) = \omega_i \ (i = 1, 2, \cdots, h), \ \mu^*\left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) = \omega,$$

$$\mu_*\left(\prod_{i=1}^{h}(p_i < 0) \cdot \left(\gamma \prod_{j=1}^{h} p_j^{a_j} \leq 1\right)\right) > \frac{\sum_{i=1}^{h}\omega_i + \omega}{h} - \frac{1}{4h(\overline{k}+1)(\overline{l}+1)}.$$

Then every nontrivial solution of (1) is frequently oscillatory.

Proof Suppose to the contrary that $u = \{u_{m,n}\}$ be an eventually positive solution of (1). Then $\mu^*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we get

$$\begin{aligned}
&\mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\} \\
= \ & 1 - \mu_*\left\{X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\} \\
\geq \ & 1 - 4(\overline{k}+1)(\overline{l}+1)\left\{\mu_*\left[\sum_{i=1}^{h}(p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right)\right] + \mu^*(u \leq 0)\right\} \\
\geq \ & 1 - 4(\overline{k}+1)(\overline{l}+1)\left[\sum_{i=1}^{h}\mu^*(p_i < 0) + \mu^*\left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right)\right. \\
& \left. - h\mu_*\left(\prod_{i=1}^{h}(p_i < 0) \cdot \left(\gamma \prod_{j=1}^{h} p_j^{a_j} \leq 1\right)\right)\right] > 0.
\end{aligned}$$

Thus, by Lemma 2.4, the intersection

$$\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \leq 0)\right]\right\}$$

is infinite. This implies that there exist $m_0 \geq 2\overline{k}$ and $n_0 \geq 2\overline{l}$ such that (3) and

$$p_i(m,n) \geq 0 \ (i = 1, 2, \cdots, h), \ u_{m,n} > 0$$

hold for $(m, n) \in Z[m_0 - 2\overline{k}, m_0 + 1] \times Z[n_0 - 2\overline{l}, n_0 + 1]$. By similar discussions as in the proof of Theorem 3.1, we may arrive at a contradiction against (3).

In case $u = \{u_{m,n}\}$ is eventually negative, then $\mu^*(u \geq 0) = 0$. In an analogous manner, we may see that

$$\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + \left(\gamma \prod_{i=1}^{h} p_i^{a_i} \leq 1\right) + (u \geq 0)\right]\right\}$$

is infinite. This can lead to a contradiction again. The proof is complete.

4 Unsaturated Solutions

The methods used in the above proofs can be modified to obtain the following results for unsaturated solutions.

Theorem 4.1 *Suppose there exist constants ω_i $(i = 1, 2, \cdots, h), \omega$ and $\omega_0 \in (0, 1)$ such that*

$$\mu^*(p_i < 0) = \omega_i \ (i = 1, 2, \cdots, h), \ \mu_*\left(\prod_{i=1}^{h}(p_i < 0)\right) = \omega,$$

$$\mu_*\left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right) > 4(\overline{k}+1)(\overline{l}+1)\left(\sum_{i=1}^{h}\omega_i + \omega_0 - (h-1)\,\omega\right).$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Proof Let $u = \{u_{m,n}\}$ be a nontrivial solution of (1). We assert that $\mu^*(u > 0) \in (\omega_0, 1)$. Suppose not, then $\mu^*(u > 0) \leq \omega_0$ or $\mu^*(u > 0) = 1$. In the former case, applying arguments similar to the proof of Theorem 3.1, we may then arrive at the fact that

$$\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u > 0)\right]\right\} \cdot \left(\gamma \prod_{i=1}^{h} p_i^{a_i} > 1\right)$$

is infinite and a subsequent contradiction. In the latter case, we have $\mu_*(u \leq 0) = 0$. By Lemmas 2.1–2.3, we have

$$\begin{aligned}
1 &= \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \leq 0)\right]\right\} \\
&\quad + \mu_*\left\{X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \leq 0)\right]\right\} \\
&\leq \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \leq 0)\right]\right\} \\
&\quad + 4\left(\overline{k}+1\right)(\overline{l}+1)\mu_*\left[\sum_{i=1}^{h}(p_i < 0) + (u \leq 0)\right] \\
&\leq \mu^*\left\{\Omega \backslash X_{-1}^{2\overline{k}} Y_{-1}^{2\overline{l}} \left[\sum_{i=1}^{h}(p_i < 0) + (u \leq 0)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 4\left(\overline{k}+1\right)\left(\overline{l}+1\right)\left\{\mu^*\left[\sum_{i=1}^{h}(p_i<0)\right]+\mu_*\left(u\le 0\right)\right\}\\
&\le \quad \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}\\
&+ \quad 4\left(\overline{k}+1\right)\left(\overline{l}+1\right)\left(\sum_{i=1}^{h}\omega_i+\omega_0-(h-1)\,\omega\right)\\
&< \quad \mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}+\mu_*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right).
\end{aligned}
$$

Therefore by Lemma 2.4, we know that the set

$$
\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+(u\le 0)\right]\right\}\cdot\left(\gamma\prod_{i=1}^{h}p_i^{a_i}>1\right)
$$

is infinite. Then by discussions similar to those in the proof of Theorem 3.1 again, we may arrive at a contradiction. This completes the proof. Combining Theorem 3.2 and 4.1, we have the following theorem the proof of which is omitted.

Theorem 4.2 *Suppose there exist constants* ω_i $(i=1,2,\cdots,h),\omega$ *and* $\omega_0\in(0,1)$ *such that*

$$
\mu^*(p_i<0)=\omega_i\ (i=1,2,\cdots,h),\ \mu^*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)=\omega,
$$

$$
\mu_*\left(\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{j=1}^{h}p_j^{a_j}\le 1\right)\right)>\frac{\sum_{i=1}^{h}\omega_i+\omega+\omega_0}{h}-\frac{1}{4h(\overline{k}+1)(\overline{l}+1)}.
$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Theorem 4.3 *Suppose there exist constants* ω_i $(i=1,2,\cdots,h),\omega',\omega''$ *and* $\omega_0\in(0,1)$ *such that*

$$
\mu^*(p_i<0)=\omega_i\ (i=1,2,\cdots,h),\ \mu^*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)=\omega',
$$

$$
\mu_*\left(\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{j=1}^{h}p_j^{a_j}\le 1\right)\right)=\omega'',\ 4(\overline{k}+1)(\overline{l}+1)\left(\sum_{i=1}^{h}\omega_i+\omega'+\omega_0-h\omega''\right)<1.
$$

Then every nontrivial solution of (1) has unsaturated upper positive part.

Proof We claim that $\mu^*(u>0)\in(\omega_0,1)$. First, we prove that $\mu^*(u>0)>\omega_0$.

Otherwise, if $\mu^*(u>0)\le\omega_0$, by Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}
&\mu_*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}+\mu^*\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}\\
=\;& 2-\mu_*\left\{X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}-\mu_*\left\{X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}\\
\ge\;& 2-4(\overline{k}+1)(\overline{k}+1)\left\{\sum_{i=1}^{h}\mu^*(p_i<0)+\mu_*\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)+\mu_*(u>0)\right.\\
&\left.-h\mu_*\left[\prod_{i=1}^{h}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}>1.
\end{aligned}$$

Hence, by Lemma 2.4, we see that

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}\cdot\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u>0)\right]\right\}$$

is infinite. Then there exist $m_0\ge 2\overline{k}$ and $n_0\ge 2\overline{l}$ such that (3) and

$$p_i(m,n)\ge 0\ (1,2,\cdots,h),\ \ u_{m,n}\le 0$$

hold for $(m,n)\in Z[m_0-2\overline{k},m_0+1]\times Z[n_0-2\overline{l},n_0+1]$. Applying similar discussions as in the proof of Theorem 3.1, we can get a contradiction. Next, we prove that $\mu^*(u>0)<1$. Otherwise, $\mu_*(u\le 0)=0$. Analogously, we see that

$$\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[\sum_{i=1}^{h}(p_i<0)+\left(\gamma\prod_{i=1}^{h}p_i^{a_i}\le 1\right)\right]\right\}\cdot\left\{\Omega\backslash X_{-1}^{2\overline{k}}Y_{-1}^{2\overline{l}}\left[(u\le 0)\right]\right\}$$

is infinite. Then, we can also come to a contradiction. The proof is complete. We remark that very nontrivial solution of (1) has unsaturated lower positive part under the same conditions as in Theorem 4.1, Theorem 4.2 or Theorem 4.3.

5 Examples

We give two examples to illustrate our previous results.

Example 5.1 Consider the partial difference equation

$$\begin{aligned}
&u_{m+1,n+1}+u_{m+1,n}+u_{m,n+1}-u_{m,n}+p_1(m,n)|u_{m-4,n-3}|^{\frac14}\mathrm{sgn}u_{m-4,n-3}\\
&+p_2(m,n)|u_{m-3,n-2}|^{\frac12}\mathrm{sgn}u_{m-3,n-2}+p_3(m,n)|u_{m-1,n-1}|^{\frac32}\mathrm{sgn}u_{m-1,n-1}=0, \qquad (5)
\end{aligned}$$

where $p_1(m,n)=2^{\frac14(n-1)}+2^{\frac14(5n-3)}+2^{\frac14(3n+7)}$, $p_2(m,n)=p_3(m,n)=1$. Obviously, $\alpha_1=1/4,\alpha_2=1/2,\alpha_3=3/2$. Let $a_1=1/5,a_2=1/4,a_3=11/20$. It is easy to see that $\sum_{i=1}^{3}a_i\alpha_i=1,\ \gamma=20/11$. It is clear that

$$\mu_*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}>1\right)=1,\quad \mu_*\left(\prod_{i=1}^{3}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\le 1\right)\right)=0,$$

$$\mu^*(p_1<0)=\mu^*(p_2<0)=\mu^*(p_3<0)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\right)=\mu^*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\leq 1\right)=0.$$

Therefore, by Theorem 3.1 or 3.2, every nontrivial solution of (5) is frequently oscillatory. Furthermore, let $\omega_0\in(0,1/80)$, we see that all conditions in Theorem 4.1, 4.2 or 4.3 are satisfied. Thus, every nontrivial solution of (5) has unsaturated upper positive part. Indeed, $u=\{(-1)^m 2^n\}$ is such a solution with $\mu^*(u>0)=1/2$.

Example 5.2 Consider the partial difference equation

$$u_{m+1,n+1}+u_{m+1,n}+u_{m,n+1}-u_{m,n}+p_1(m,n)|u_{m-3,n-3}|^{\frac{1}{3}}\text{sgn}u_{m-3,n-3}$$
$$+p_2(m,n)|u_{m-3,n-2}|^{\frac{1}{2}}\text{sgn}u_{m-3,n-2}+p_3(m,n)|u_{m-1,n-1}|^{2}\text{sgn}u_{m-1,n-1}=0, \quad (6)$$

where

$$p_1(m,n)=p_3(m,n)=1,\quad p_2(m,n)=\begin{cases}-1, & m=10s \text{ and } n=13t,\ s,t\in Z[0,\infty),\\ 1, & \text{otherwise.}\end{cases}$$

Choose $a_1=3/10$, $a_2=1/3$, $a_3=11/30$. It is easy to see that $\sum_{i=1}^{3}a_i=1$, $\sum_{i=1}^{3}a_i\alpha_i=1$ and $\gamma=30/11$. Clearly,

$$\mu^*\left(p_1<0\right)=\mu^*\left(p_3<0\right)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\right)=\mu_*\left(\prod_{i=1}^{3}(p_i<0)\cdot\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\leq 1\right)\right)=0,$$

$$\mu^*\left(p_2<0\right)=\ \mu^*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}\leq 1\right)=\frac{1}{130},\quad \mu_*\left(\gamma\prod_{i=1}^{3}p_i^{a_i}>1\right)=\frac{129}{130}.$$

Then by Theorem 3.1 or 3.2, every nontrivial solution of (6) is frequently oscillatory. Furthermore, when given $\omega_0=1/4161$, applying Theorem 4.1, 4.2 and 4.3, we may see that every nontrivial solution of (6) has unsaturated upper positive part.

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References

[1] Zhang, B.G. and Xing, Q.J. Oscillation of certain partial difference equations. *J. Math. Anal. Appl.* **329** (1) (2007) 567–580.

[2] C. J. Tian, S. L. Xie and S. S. Cheng. Measures for oscillatory sequences. *Comput. Math. Appl.* **36** (1998) 149–161.

[3] Cheng, S.S. *Partial Difference Equations.* London, New York, Taylor and Francis, 2003.

[4] Zhu, Z.Q. and Cheng, S.S. Frequent oscillation in a neutral difference equation. *Southeast Asian Bull. Math.* **29** (3) (2005) 627–634.

[5] Zhu, Z.Q. and Cheng, S.S. Frequently oscillatory solutions for multi-level partial difference equations. *International Math. Forum* **1** (31) (2006) 1497–1509.

[6] Zhu, Z.Q. and Cheng, S.S. Unsaturated solutions for partial difference equations with forcing terms. *Central European J. Math.,* to appear.

[7] Beckenbach, Edwin F. and Bellman, R. *Inequalities.* Berlin, Springer–Verlag, 1961.