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Dominant and Recessive Solutions of Self-Adjoint Matrix Systems on Time Scales

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Abstract: In this study, linear second-order self-adjoint delta-nabla matrix systems on time scales are considered with the motivation of extending the analysis of dominant and recessive solutions from the differential and discrete cases to any arbitrary dynamic equations on time scales. These results emphasize the case when the system is non-oscillatory.

Keywords: *time scales; self-adjoint; matrix equations; second-order; non-oscillation; linear.*

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1 Introduction

To motivate this study of dominant and recessive solutions, consider the self-adjoint second-order scalar differential equation

$$(px')'(t) + q(t)x(t) = 0.$$

According to the classical formulation by Kelley and Peterson [1, Section 5.6], a solution u is recessive at ω and a second, linearly-independent solution v is dominant at ω if the conditions

$$\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0, \qquad \int_{t_0}^{\omega} \frac{1}{p(t)u^2(t)}dt = \infty, \qquad \int_{t_0}^{\omega} \frac{1}{p(t)v^2(t)}dt < \infty$$

all hold; see also a related discussion for three-term difference equations in Ahlbrandt [2], Ahlbrandt and Peterson [3, Section 5.10], Ma [4], and scalar dynamic equations in Bohner

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and Peterson [5, Section 4.3], Messer [6], and [7, Section 4.5]. It is the purpose of this work to introduce a robust treatment of these types of solutions for the corresponding self-adjoint second-order matrix dynamic equation on time scales. Dynamic equations on time scales have been introduced by Hilger and Aulbach [8, 9] to unify, extend, and generalize the theory of ordinary differential equations, difference equations, quantum equations, and all other differential systems defined over nonempty closed subsets of the real line. We use this overarching theory to extend from the discrete case [3, 4] the matrix difference system

$$\Delta\left(P(t)\Delta X(t-1)\right)+Q(t)X(t)=0, \tag{1.1}$$

for $q>1$ the quantum system [10]

$$D^q\left(PD_qX\right)(t)+Q(t)X(t)=0, \tag{1.2}$$

and the continuous case developed by Reid [11–15]

$$\left(PX'\right)'(t)+Q(t)X(t)=0, \tag{1.3}$$

to the general time scale setting, which admits the self-adjoint delta-nabla matrix system

$$\left(PX^\Delta\right)^\nabla(t)+Q(t)X(t)=0. \tag{1.4}$$

Only recently has (formal) self-adjointness been investigated for arbitrary time scales, even in the scalar case, by Messer [6], Anderson, Guseinov and Hoffacker [16], and Atici and Guseinov [17]; self-adjoint matrix systems on time scales are relatively unexplored at this time [18]. More commonly authors Bohner and Peterson [5, Chapter 5] and Erbe and Peterson [19] focus on

$$\left(PX^\Delta\right)^\Delta(t)+Q(t)X^\sigma(t)=0, \tag{1.5}$$

which they term "self-adjoint" since it admits a Lagrange identity. Thus, these results connected to the self-adjoint system (1.4) extend and generalize the results related to (1.1), (1.2) and (1.3), and are different from those worked out for (1.5).

2 Technical Results on Time Scales

Any arbitrary nonempty closed subset of the reals $\mathbb{R}$ can serve as a time scale $\mathbb{T}$; see the books by Bohner and Peterson [5, 7] and the papers by Hilger and Aulbach [8, 9]. Here and in the sequel we assume a working knowledge of basic time-scale notation and the time-scale calculus. In addition, the following results will prove to be useful.

Theorem 2.1 *If f is delta differentiable at $t\in\mathbb{T}^\kappa$, then $f^\sigma(t)=f(t)+\mu(t)f^\Delta(t)$. If f is nabla differentiable at $t\in\mathbb{T}_\kappa$, then $f^\rho(t)=f(t)-\nu(t)f^\nabla(t)$.*

Theorem 2.2 *Let $f:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ be a continuous function of two variables $(t,s)\in\mathbb{T}\times\mathbb{T}$, and $a\in\mathbb{T}$. Assume that f has continuous derivatives f^Δ and f^∇ with respect to t. Then the following formulas hold:*

(i) $\left(\int_a^t f(t,s)\Delta s\right)^\Delta=f(\sigma(t),t)+\int_a^t f^\Delta(t,s)\Delta s,$

(ii) $\left(\int_a^t f(t,s)\Delta s\right)^\nabla=f(\rho(t),\rho(t))+\int_a^t f^\nabla(t,s)\Delta s,$

(iii) $\left(\int_a^t f(t,s)\nabla s\right)^\Delta = f(\sigma(t),\sigma(t)) + \int_a^t f^\Delta(t,s)\nabla s$,

(iv) $\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f^\nabla(t,s)\nabla s$.

The following sets and statement [6, Theorem 2.6] (see also [17]) will play an important role in many of our calculations.

Definition 2.1 Let the time-scale sets A and B be given by

$$A := \{t \in \mathbb{T} : t \text{ is a left-dense and right-scattered point}\}, \tag{2.1}$$

and

$$B := \{t \in \mathbb{T} : t \text{ is a right-dense and left-scattered point}\}. \tag{2.2}$$

It follows that for $t \in A$,

$$\lim_{s\to t^-} \sigma(s) = t,$$

and for $t \in \mathbb{T}\backslash A$, $\sigma(\rho(t)) = t$. Likewise for $t \in B$,

$$\lim_{s\to t^+} \rho(s) = t,$$

and for $t \in \mathbb{T}\backslash B$, $\rho(\sigma(t)) = t$.

Theorem 2.3 *Let the sets A and B be given as in* (2.1) *and* (2.2)*, respectively.*

(i) If $f : \mathbb{T} \to \mathbb{R}$ is Δ differentiable on $\mathbb{T}^\kappa$ and f^Δ is right-dense continuous on $\mathbb{T}^\kappa$, then f is ∇ differentiable on $\mathbb{T}_\kappa$, and

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & : t \in \mathbb{T}\backslash A, \\ \lim_{s\to t^-} f^\Delta(s) & : t \in A. \end{cases}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is ∇ differentiable on $\mathbb{T}_\kappa$ and f^∇ is left-dense continuous on $\mathbb{T}_\kappa$, then f is Δ differentiable on $\mathbb{T}^\kappa$, and

$$f^\Delta(t) = \begin{cases} f^\nabla(\sigma(t)) & : t \in \mathbb{T}\backslash B, \\ \lim_{s\to t^+} f^\nabla(s) & : t \in B. \end{cases}$$

The statements of the previous theorem can be formulated as $\left(f^\Delta\right)^\rho = f^\nabla$ and $\left(f^\nabla\right)^\sigma = f^\Delta$ provided that f^Δ and f^∇ are continuous, respectively.

3 Self-Adjoint Matrix Equations

All of the results in this section are from Anderson and Buchholz [18]. Let P and Q be Hermitian $n \times n$-matrix-valued functions on a time scale $\mathbb{T}$ such that $P > 0$ (positive definite) and Q are continuous for all $t \in \mathbb{T}$. (A matrix M is *Hermitian* iff $M^* = M$, where * indicates conjugate transpose.) In this section we are concerned with the second-order (formally) self-adjoint matrix dynamic equation

$$LX = 0, \quad \text{where} \quad LX(t) := \left(PX^\Delta\right)^\nabla(t) + Q(t)X(t), \quad t \in \mathbb{T}_\kappa^\kappa. \tag{3.1}$$

Definition 3.1 Let $\mathbb{D}$ denote the set of all $n \times n$ matrix-valued functions X defined on $\mathbb{T}$ such that X^Δ is continuous on $\mathbb{T}^\kappa$ and $(PX^\Delta)^\nabla$ is left-dense continuous on $\mathbb{T}^\kappa_\kappa$. Then X is a solution of (3.1) on $\mathbb{T}$ provided $X \in \mathbb{D}$ and $LX(t) = 0$ for all $t \in \mathbb{T}^\kappa_\kappa$.

Definition 3.2 (Regressivity) An $n \times n$ matrix-valued function M on a time scale $\mathbb{T}$ is *regressive* with respect to $\mathbb{T}$ provided

$$I + \mu(t)M(t) \text{ is invertible for all } t \in \mathbb{T}^\kappa, \tag{3.2}$$

and the class of all such regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}).$$

Theorem 3.1 *Let $a \in \mathbb{T}^\kappa$ be fixed and X_a, X_a^Δ be given constant $n \times n$ matrices. Then the initial boundary value problem*

$$\left(PX^\Delta\right)^\nabla(t) + Q(t)X(t) = 0, \quad X(a) = X_a, \quad X^\Delta(a) = X_a^\Delta$$

has a unique solution.

Definition 3.3 If $X, Y \in \mathbb{D}$, then the (generalized) *Wronskian matrix* of X and Y is given by

$$W(X,Y)(t) = X^*(t)P(t)Y^\Delta(t) - [P(t)X^\Delta(t)]^*Y(t)$$

for $t \in \mathbb{T}^\kappa$.

Theorem 3.2 (Lagrange identity) *If $X, Y \in \mathbb{D}$, then*

$$W(X,Y)^\nabla(t) = X^*(t)(LY)(t) - (LX(t))^*\,Y(t), \quad t \in \mathbb{T}^\kappa_\kappa.$$

Definition 3.4 Define the inner product of $n \times n$ matrices M and N on $[a,b]_\mathbb{T}$ for $a < b$ to be

$$\langle M, N\rangle = \int_a^b M^*(t)N(t)\nabla t, \quad M, N \in C_{ld}(\mathbb{T}), \quad a, b \in \mathbb{T}^\kappa. \tag{3.3}$$

Corollary 3.1 (Self-adjoint operator) *The operator L in* (3.1) *is formally self adjoint with respect to the inner product* (3.3)*; that is, the identity*

$$\langle LX, Y\rangle = \langle X, LY\rangle$$

holds provided $X, Y \in \mathbb{D}$ and X, Y satisfy $W(X,Y)(t)\big|_a^b = 0$, called the self-adjoint boundary conditions.

Corollary 3.2 (Abel's formula) *If X, Y are solutions of* (3.1) *on $\mathbb{T}$, then*

$$W(X,Y)(t) \equiv C, \quad t \in \mathbb{T}^\kappa_\kappa,$$

where C is a constant matrix.

From Abel's formula we get that if $X \in \mathbb{D}$ is a solution of (3.1) on $\mathbb{T}$, then

$$W(X,X)(t) \equiv C, \quad t \in \mathbb{T}^\kappa_\kappa,$$

where C is a constant matrix. With this in mind we make the following definition.

Definition 3.5 Let $X, Y \in \mathbb{D}$ and W be given as in (3.3).

(i) $X \in \mathbb{D}$ is a prepared (conjoined, isotropic) solution of (3.1) iff X is a solution of (3.1) and

$$W(X, X)(t) \equiv 0, \quad t \in \mathbb{T}^\kappa.$$

(ii) $X, Y \in \mathbb{D}$ are normalized prepared bases of (3.1) iff X, Y are two prepared solutions of (3.1) with

$$W(X, Y)(t) \equiv I, \quad t \in \mathbb{T}^\kappa.$$

Theorem 3.3 *Assume that $X \in \mathbb{D}$ is a solution of* (3.1) *on $\mathbb{T}$. Then the following are equivalent:*

(i) X is a prepared solution;

(ii) $X^*(t)P(t)X^\Delta(t)$ is Hermitian for all $t \in \mathbb{T}^\kappa$;

(iii) $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}^\kappa$.

Note that one can easily get prepared solutions of (3.1) by taking initial conditions at $t_0 \in \mathbb{T}$ so that $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian.

In the Sturmian theory for (3.1) the matrix function X^*PX^σ is important. We note the following result.

Lemma 3.1 *Let X be a solution of* (3.1)*. If X is prepared, then*

$$X^*(t)P(t)X^\sigma(t) \quad \textit{is Hermitian for all} \quad t \in \mathbb{T}^\kappa.$$

Conversely, if there is $t_0 \in \mathbb{T}^\kappa$ such that $\mu(t_0) > 0$ and $X^*(t_0)P(t_0)X^\sigma(t_0)$ is Hermitian, then X is a prepared solution of (3.1)*. Moreover, if X is an invertible prepared solution, then*

$$P(t)X^\sigma(t)X^{-1}(t),\ P(t)X(t)(X^\sigma)^{-1}(t),\ \textit{and}\ Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$

are Hermitian for all $t \in \mathbb{T}^\kappa$.

Lemma 3.2 *Assume that X is a prepared solution of* (3.1) *on $\mathbb{T}$. Then the following are equivalent:*

(i) $(X^*)^\sigma PX = X^*PX^\sigma > 0$ on $\mathbb{T}^\kappa$;

(ii) X is invertible and $PX^\sigma X^{-1} > 0$ on $\mathbb{T}^\kappa$;

(iii) X is invertible and $PX(X^\sigma)^{-1} > 0$ on $\mathbb{T}^\kappa$.

Theorem 3.4 (Reduction of order I) *Let $t_0 \in \mathbb{T}^\kappa$, and assume X is a prepared solution of* (3.1) *with X invertible on $\mathbb{T}$. Then a second prepared solution Y of* (3.1) *is given by*

$$Y(t) := X(t)\int_{t_0}^{t} \left(X^*PX^\sigma\right)^{-1}(s)\Delta s, \quad t \in \mathbb{T}^\kappa$$

such that X, Y are normalized prepared bases of (3.1).

Lemma 3.3 *Assume $X, Y \in \mathbb{D}$ are normalized prepared bases of* (3.1)*. Then $U := XE + YF$ is a prepared solution of* (3.1) *for constant $n \times n$ matrices E, F if and only if F^*E is Hermitian. If $F = I$, then X, U are normalized prepared bases of* (3.1) *if and only if E is a constant Hermitian matrix.*

Theorem 3.5 (Reduction of order II) *Let $t_0 \in \mathbb{T}^\kappa$, and assume X is a prepared solution of* (3.1) *with X invertible on $\mathbb{T}$. Then U is a second $n \times n$ matrix solution of* (3.1) *iff U satisfies the first-order matrix equation*

$$(X^{-1}U)^\Delta(t) = (X^*PX^\sigma)^{-1}(t)F, \quad t \in \mathbb{T}^\kappa, \quad t \geq t_0, \tag{3.4}$$

for some constant $n \times n$ matrix F iff U is of the form

$$U(t) = X(t)E + X(t)\left(\int_{t_0}^{t} \left(X^*PX^\sigma\right)^{-1}(s)\Delta s\right)F, \quad t \in \mathbb{T}, \quad t \geq t_0, \tag{3.5}$$

where E and F are constant $n \times n$ matrices. In the latter case,

$$E = X^{-1}(t_0)U(t_0), \qquad F = W(X,U)(t_0), \tag{3.6}$$

such that U is a prepared solution of (3.1) *iff $F^*E = E^*F$.*

4 Factorization of the Self-Adjoint Operator

In this section we introduce the Pólya factorization for the self-adjoint matrix-differential operator L defined in (3.1).

Theorem 4.1 (Pólya factorization) *If* (3.1) *has a prepared solution $U > 0$ (positive definite) on an interval $\mathcal{I} \subset \mathbb{T}$ such that $U^*PU^\sigma > 0$ on $\mathcal{I}$, then for any $X \in \mathbb{D}$ we have on $\mathcal{I}$ a Pólya factorization*

$$LX = M_1^*\left\{M_2(M_1X)^\Delta\right\}^\nabla, \quad M_1 := U^{-1} > 0, \quad M_2 := U^*PU^\sigma > 0.$$

Proof Assume $U > 0$ is a prepared solution of (3.1) on $\mathcal{I} \subset \mathbb{T}$ such that $U^*PU^\sigma > 0$ on $\mathcal{I}$, and let $X \in \mathbb{D}$. Then U is invertible and

$$\begin{aligned}
LX &\overset{\text{Thm 3.2}}{=} (U^*)^{-1}W(U,X)^\nabla \\
&\overset{\text{Def 3.3}}{=} (U^*)^{-1}\left\{U^*PX^\Delta - U^{\Delta *}PX\right\}^\nabla \\
&= M_1^*\left\{U^*[PX^\Delta - (U^*)^{-1}U^{\Delta *}PX]\right\}^\nabla \\
&\overset{\text{Thm 3.1}}{=} M_1^*\left\{U^*[PX^\Delta - PU^\Delta U^{-1}X]\right\}^\nabla \\
&= M_1^*\left\{M_2[(U^\sigma)^{-1}X^\Delta - (U^\sigma)^{-1}U^\Delta U^{-1}X]\right\}^\nabla \\
&= M_1^*\left\{M_2[(U^\sigma)^{-1}X^\Delta + (U^{-1})^\Delta X]\right\}^\nabla \\
&= M_1^*\left\{M_2(U^{-1}X)^\Delta\right\}^\nabla \\
&= M_1^*\left\{M_2(M_1X)^\Delta\right\}^\nabla,
\end{aligned}$$

for M_1 and M_2 as defined in the statement of the theorem. □

5 Dominant and Recessive Solutions

Throughout the rest of the paper assume $a \in \mathbb{T}$, and set $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. We focus on extending the analysis of dominant and recessive solutions developed in the case of difference system (1.1), quantum system (1.2), and differential system (1.3) to the general time-scale setting in (3.1).

Definition 5.1 A solution X of (3.1) is a basis iff $\operatorname{rank}\begin{pmatrix} X(t_0) \\ (PX^{\Delta})(t_0) \end{pmatrix} = n$ for some $t_0 \geq a$. A solution V of (3.1) is dominant at ω iff V is a prepared basis and there exists a $t_0 \in [a, \omega)_{\mathbb{T}}$ such that V is invertible on $[t_0, \omega)_{\mathbb{T}}$ and

$$\int_{t_0}^{\omega} \left(V^* P V^{\sigma}\right)^{-1}(t)\Delta t$$

converges to a Hermitian matrix with finite entries.

Lemma 5.1 *Assume the self-adjoint equation $LX = 0$ has a dominant solution V at ω. If X is any other $n \times n$ solution of* (3.1)*, then*

$$\lim_{t\to\omega} V^{-1}(t)X(t) = K$$

for some $n \times n$ constant matrix K.

Proof Since V is a dominant solution at ω of (3.1), there exists a $t_0 \in [a, \omega)_{\mathbb{T}}$ such that V is invertible on $[t_0, \omega)_{\mathbb{T}}$. By the second reduction of order theorem, Theorem 3.5,

$$X(t) = V(t)V^{-1}(t_0)X(t_0) + V(t)\left(\int_{t_0}^{t} \left(V^* P V^{\sigma}\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

Multiplying on the left by V^{-1} we have

$$V^{-1}(t)X(t) = V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{t} \left(V^* P V^{\sigma}\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

Since V is dominant at ω, the following limit exists:

$$\lim_{t\to\omega} V^{-1}(t)X(t) = K := V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{\omega} \left(V^* P V^{\sigma}\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

□

Definition 5.2 A solution U of (3.1) is recessive at ω iff U is a prepared basis and whenever X is any other $n \times n$ solution of (3.1) such that $W(X, U)$ is invertible, X is eventually invertible and

$$\lim_{t\to\omega} X^{-1}(t)U(t) = 0.$$

Lemma 5.2 *If U is a solution of* (3.1) *which is recessive at ω, then for any invertible constant matrix K, the solution UK of* (3.1) *is recessive at ω as well.*

Proof The proof follows from the definition. □

Lemma 5.3 *If U is a solution of* (3.1) *which is recessive at ω, and V is a prepared solution of* (3.1) *such that $W(V,U)$ is invertible, then V is dominant at ω.*

Proof By the definition of recessive, $W(V,U)$ invertible implies that V is invertible on $[t_0,\omega)_{\mathbb{T}}$ for some $t_0 \in [a,\omega)_{\mathbb{T}}$, and

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0. \tag{5.1}$$

Let $K := W(V,U)$; by assumption K is invertible, and by Definition 3.3

$$K = (V^*PV^\sigma)(V^\sigma)^{-1}U^\Delta - (V^{\Delta *}PV)V^{-1}U$$

for all $t \in [t_0,\omega)_{\mathbb{T}}$. Since V is prepared,

$$(V^*PV^\sigma)^{-1}K = (V^\sigma)^{-1}U^\Delta - (V^\sigma)^{-1}V^\Delta V^{-1}U = \left(V^{-1}U\right)^\Delta.$$

Delta integrating from t_0 to ω and using (5.1) yields that

$$\int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(t)\Delta t = -V^{-1}(t_0)U(t_0)K^{-1}$$

converges. Thus V is dominant at ω. □

Theorem 5.1 *Assume* (3.1) *has a solution V which is dominant at ω. Then*

$$U(t) := V(t)\int_t^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s$$

is a solution of (3.1) *which is recessive at ω and $W(V,U) = -I$.*

Proof Since V is dominant at ω, U is a well-defined function and can be written as

$$U(t) = V(t)\left[\int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s - \left(\int_{t_0}^{t} (V^*PV^\sigma)^{-1}(s)\Delta s\right) I\right];$$

by the second reduction of order theorem, Theorem 3.5, U is a solution of (3.1) of the form (3.5) with

$$E = \int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s, \qquad F = -I.$$

From (3.6), $W(V,U) = F = -I$. Since

$$E^*F = -\int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s$$

is Hermitian, U is a prepared solution of (3.1), and $W(-V,U) = I$ implies that U and $-V$ are normalized prepared bases. Let X be an $n \times n$ matrix solution of $LX = 0$ such that $W(X,U)$ is invertible. By the second reduction of order theorem,

$$\begin{aligned} X(t) &= V(t)\left[V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{t} (V^*PV^\sigma)^{-1}(s)\Delta s\right) W(V,X)\right] \\ &= V(t)C_1 + U(t)C_2, \end{aligned} \tag{5.2}$$

where

$$C_1 := V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s\right) W(V,X)$$

and

$$C_2 := -W(V,X).$$

Note that

$$W(X,U) = C_1^* W(V,U) + C_2^* W(U,U) = -C_1^*.$$

As $W(X,U)$ is invertible by assumption, C_1 is invertible. From (5.2),

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \lim_{t\to\omega} \left(C_1 + V^{-1}(t)U(t)C_2\right)$$
$$= \lim_{t\to\omega} \left(C_1 + \int_t^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s C_2\right) = C_1$$

is likewise invertible. Consequently for large t, $X(t)$ is invertible. Lastly,

$$\begin{aligned}\lim_{t\to\omega} X^{-1}(t)U(t) &= \lim_{t\to\omega} \left[V(t)C_1 + U(t)C_2\right]^{-1} U(t)\\ &= \lim_{t\to\omega} \left[C_1 + V^{-1}(t)U(t)C_2\right]^{-1} V^{-1}(t)U(t) = \left[C_1 + 0\right]^{-1} 0 = 0.\end{aligned}$$

Therefore U is a recessive solution at ω. □

Theorem 5.2 *Assume* (3.1) *has a solution U which is recessive at ω, and $U(t_0)$ is invertible for some $t_0 \in [a,\omega)_{\mathbb{T}}$. Then U is uniquely determined by $U(t_0)$, and* (3.1) *has a solution V which is dominant at ω.*

Proof Assume $U(t_0)$ is invertible; let V be the unique solution of the initial value problem

$$LV = 0, \quad V(t_0) = 0, \quad V^\Delta(t_0) = I.$$

Then V is a prepared basis and

$$W(V,U) = W(V,U)(t_0) = (V^*PU^\Delta)(t_0) - (PV^\Delta)^*(t_0)U(t_0) = -P(t_0)U(t_0)$$

is invertible. It follows from Lemma 5.3 that V is dominant at ω. Let Γ be an arbitrary but fixed $n \times n$ constant matrix. Let X solve the initial value problem

$$LX = 0, \quad X(t_0) = I, \quad X^\Delta(t_0) = \Gamma.$$

By Theorem 5.1,

$$\lim_{t\to\omega} V^{-1}(t)X(t) = K,$$

where K is an $n\times n$ constant matrix; note that K is independent of the recessive solution U. By using the initial conditions at t_0, by uniqueness of solutions it is easy to see that there exist constant $n \times n$ matrices C_1 and C_2 such that

$$U(t) = X(t)C_1 + V(t)C_2,$$

where $C_1 = U(t_0)$ is invertible. Consequently, using the recessive nature of U, we have

$$0 = \lim_{t\to\omega} V^{-1}(t)U(t) = \lim_{t\to\omega} \left(V^{-1}(t)X(t)U(t_0) + C_2\right) = KU(t_0) + C_2,$$

so that $C_2 = -KU(t_0)$. Thus the initial condition for U^Δ is

$$U^\Delta(t_0) = (\Gamma - K)U(t_0),$$

and the recessive solution U is uniquely determined by its initial value $U(t_0)$. □

Theorem 5.3 *Assume* (3.1) *has a solution U which is recessive at ω and a solution V which is dominant at ω. If U and $\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s$ are both invertible for large $t \in \mathbb{T}$, then there exists an invertible constant matrix K such that*

$$U(t) = V(t)\left(\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s\right)K$$

for large t. In addition, $W(U,V)$ is invertible and

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0.$$

Proof For sufficiently large $t \in \mathbb{T}$ define

$$Y(t) = V(t)\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s.$$

By Theorem 5.1 Y is also a recessive solution of (3.1) at ω and $W(V,Y) = -I$. Because U and $\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s$ are both invertible for large $t \in \mathbb{T}$, Y is likewise invertible for large t, and

$$\lim_{t\to\omega} V^{-1}(t)Y(t) = 0$$

by the recessive nature of Y. Choose $t_0 \in [a,\omega)_\mathbb{T}$ large enough to ensure that U and Y are invertible in $[t_0,\omega)_\mathbb{T}$. By Lemma 5.2 the solution given by

$$X(t) := Y(t)Y^{-1}(t_0)U(t_0), \quad t \in [t_0,\omega)_\mathbb{T}$$

is yet another recessive solution at ω. Since U and X are recessive solutions at ω and $U(t_0) = X(t_0)$, we conclude from the uniqueness established in Theorem 5.2 that $X \equiv U$. Thus for $t \in [t_0,\omega)_\mathbb{T}$ we have

$$U(t) = Y(t)Y^{-1}(t_0)U(t_0) = V(t)\left(\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s\right)K,$$

where $K := Y^{-1}(t_0)U(t_0)$ is an invertible constant matrix. $\square$

The next result, when $\mathbb{T} = \mathbb{Z}$, relates the convergence of infinite series, the convergence of certain continued fractions, and the existence of recessive solutions; for more see [3] and the references therein.

Theorem 5.4 (Connection theorem) *Let X and V be solutions of* (3.1) *determined by the initial conditions*

$$X(t_0) = I, \quad X^\Delta(t_0) = P^{-1}(t_0)K, \quad \textit{and} \quad V(t_0) = 0, \quad V^\Delta(t_0) = P^{-1}(t_0),$$

respectively, where $t_0 \in [a,\omega)_\mathbb{T}$ and K is a constant Hermitian matrix. Then X, V are normalized prepared bases of (3.1)*, and the following are equivalent:*

(i) V is dominant at ω;

(ii) V is invertible for large $t \in \mathbb{T}$ and $\lim_{t\to\omega} V^{-1}(t)X(t)$ exists as a Hermitian matrix $\Omega(K)$ with finite entries;

(iii) there exists a solution U of (3.1) *which is recessive at ω, with $U(t_0)$ invertible.*

If (i), (ii), and (iii) hold then

$$U^{\Delta}(t_0)U^{-1}(t_0) = X^{\Delta}(t_0) - V^{\Delta}(t_0)\Omega(K) = -P^{-1}(t_0)\Omega(0).$$

Proof Since $V(t_0) = 0$, V is a prepared solution of (3.1). Also,

$$W(X,X) = W(X,X)(t_0) = (X^*PX^{\Delta} - X^{\Delta *}PX)(t_0) = IK - K^*I = 0$$

as K is Hermitian, making X a prepared solution of (3.1) as well. Checking

$$W(X,V) = W(X,V)(t_0) = (X^*PV^{\Delta} - X^{\Delta *}PV)(t_0) = I - 0 = I,$$

we see that X, V are normalized prepared bases of (3.1). Now we show that (i) implies (ii). If V is a dominant solution of (3.1) at ω, then there exists a $t_1 \in [a,\omega)_{\mathbb{T}}$ such that $V(t)$ is invertible for $t \in [t_1,\omega)_{\mathbb{T}}$, and the delta integral

$$\int_{t_1}^{\omega} (V^*PV^{\sigma})^{-1}(s)\Delta s$$

converges to a Hermitian matrix with finite entries. By the second reduction of order theorem,

$$X(t) = V(t)E + V(t)\left(\int_{t_1}^{t} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s\right)F, \tag{5.3}$$

where

$$E = V^{-1}(t_1)X(t_1), \qquad F = W(V,X)(t_1) = -W(X,V)^* = -I.$$

Since X is prepared, $E^*F = -E^*$ is Hermitian, whence E is Hermitian. As a result, by (5.3)

$$\lim_{t\to\omega} V^{-1}(t)X(t) = E - \int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s$$

converges to a Hermitian matrix with finite entries, and (ii) holds. Next we show that (ii) implies (iii). If V is invertible on $[t_1,\omega)_{\mathbb{T}}$ and

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \Omega \tag{5.4}$$

exists as a Hermitian matrix, then from (5.3) and (5.4),

$$\Omega = \lim_{t\to\omega} V^{-1}(t)X(t) = E - \int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s;$$

in other words,

$$\int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s = E - \Omega.$$

Define

$$U(t) := X(t) - V(t)\Omega. \tag{5.5}$$

Then

$$\begin{aligned} W(U,U) &= W(X - V\Omega, X - V\Omega) \\ &= W(X,X) - W(X,V)\Omega - \Omega^*W(V,X) + \Omega^*W(V,V)\Omega \\ &= -\Omega + \Omega^* = 0, \end{aligned}$$

and $U(t_0) = X(t_0) = I$, making U a prepared basis for (3.1). If X_1 is an $n \times n$ matrix solution of $LX = 0$ such that $W(X_1, U)$ is invertible, then

$$X_1(t) = V(t)C_1 + U(t)C_2 \tag{5.6}$$

for some constant matrices C_1 and C_2 determined by the initial conditions at t_0. It follows that

$$\begin{aligned} W(X_1, U) &= W(VC_1 + UC_2, U) = C_1^* W(V, U) + C_2^* W(U, U) \\ &= C_1^* W(V, U) = C_1^* W(V, U)(t_0) = -C_1^* \end{aligned}$$

by (5.5), so that C_1 is invertible. From (5.4) and (5.5) we have that

$$\lim_{t \to \omega} V^{-1}(t)U(t) = \lim_{t \to \omega} \left[V^{-1}(t)X(t) - \Omega\right] = 0,$$

resulting in

$$\lim_{t \to \omega} V^{-1}(t)X_1(t) = \lim_{t \to \omega} \left[C_1 + V^{-1}(t)U(t)C_2\right] = C_1,$$

which is invertible. Thus $X_1(t)$ is invertible for large $t \in \mathbb{T}$, and

$$\begin{aligned} \lim_{t \to \omega} X_1^{-1}(t)U(t) &= \lim_{t \to \omega} [V(t)C_1 + U(t)C_2]^{-1}U(t) \\ &= \lim_{t \to \omega} [C_1 + V^{-1}(t)U(t)C_2]^{-1}V^{-1}(t)U(t) \\ &= C_1^{-1}(0) = 0. \end{aligned}$$

Hence U is a recessive solution of (3.1) at ω and (iii) holds. Finally we show that (iii) implies (i). If U is a recessive solution of (3.1) at ω with $U(t_0)$ invertible, then

$$W(V, U) = W(V, U)(t_0) = -U(t_0)$$

is also invertible. Hence by Lemma 5.3, V is a dominant solution of (3.1) at ω.

To complete the proof, assume (i), (ii), and (iii) hold. It can be shown via initial conditions at t_0 that

$$U(t) = X(t)U(t_0) + V(t)C$$

for some suitable constant matrix C. By (ii),

$$\lim_{t \to \omega} V^{-1}(t)X(t) = \Omega(K),$$

and thus

$$V^{-1}(t)U(t) = V^{-1}(t)X(t)U(t_0) + C.$$

As U is a recessive solution at ω by (iii),

$$0 = \lim_{t \to \omega} \left(V^{-1}(t)X(t)U(t_0) + C\right) = \Omega(K)U(t_0) + C,$$

yielding $U(t) = [X(t) - V(t)\Omega(K)]\,U(t_0)$. Delta differentiation at t_0 gives

$$U^\Delta(t_0)U^{-1}(t_0) = X^\Delta(t_0) - V^\Delta(t_0)\Omega(K).$$

Now let Y be the unique solution of the initial value problem

$$LY = 0, \quad Y(t_0) = I, \quad Y^\Delta(t_0) = 0.$$

Using the initial conditions at t_0 we see that $X(t) = Y(t) + V(t)K$. Consequently,

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \lim_{t\to\omega} V^{-1}(t)Y(t) + K$$

implies, by (ii) and the fact that $X = Y$ when $K = 0$, that $\Omega(K) = \Omega(0) + K$. Therefore

$$X^{\Delta}(t_0) - V^{\Delta}(t_0)\Omega(K) = -V^{\Delta}(t_0)\Omega(0) = -P^{-1}(t_0)\Omega(0).$$

Thus the proof is complete. □

Theorem 5.5 (Variation of parameters) *Let H be an $n\times n$ matrix function that is left-dense continuous on $[t_0,\omega)_{\mathbb{T}}$. If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t\in[t_0,\omega)_{\mathbb{T}}$, then the nonhomogeneous equation $LY = H$ has a solution $Y\in\mathbb{D}$ given by*

$$\begin{aligned} Y(t) &= X(t)X^{-1}(t_0)Y(t_0) + X(t)\int_{t_0}^{t} \left(X^*PX^{\sigma}\right)^{-1}(\tau)\Delta\tau W(X,Y)(t_0) \\ &\quad + X(t)\int_{t_0}^{t}\left(\left(X^*PX^{\sigma}\right)^{-1}(\tau)\int_{t_0}^{\tau} X^*(s)H(s)\nabla s\right)\Delta\tau. \end{aligned}$$

Proof Let $Y\in\mathbb{D}$ and assume X is a prepared solution of (3.1) invertible on $[t_0,\omega)_{\mathbb{T}}$. As in Theorem 4.1, we factor LY to get

$$H = LY = X^{*-1}\left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)^{\nabla}.$$

Multiplying by X^* and nabla integrating from t_0 to t we arrive at

$$\left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)(t) - W(X,Y)(t_0) = \int_{t_0}^{t} X^*(s)H(s)\nabla s,$$

where $W(X,Y)(t_0) = \left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)(t_0)$ since X is prepared. This leads to

$$(X^{-1}Y)^{\Delta}(t) = (X^*PX^{\sigma})^{-1}(t)\left(W(X,Y)(t_0) + \int_{t_0}^{t} X^*(s)H(s)\nabla s\right),$$

which is then delta integrated from t_0 to t to obtain the form for Y given in the statement of the theorem. Clearly the right-hand side of the form of Y above reduces to $Y(t_0)$ at t_0, and since X is an invertible prepared solution, by Theorem 3.1 the delta derivative reduces to $Y^{\Delta}(t_0)$ at t_0. □

Corollary 5.1 *Let H be an $n\times n$ matrix function that is left-dense continuous on $[t_0,\omega)_{\mathbb{T}}$. If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t\in[t_0,\omega)_{\mathbb{T}}$, then the nonhomogeneous initial value problem*

$$LY = (PY^{\Delta})^{\nabla} + QY = H, \quad Y(t_0) = Y_0, \quad Y^{\Delta}(t_0) = Y_0^{\Delta} \tag{5.7}$$

has a unique solution.

Proof By Theorem 5.5, the nonhomogeneous initial value problem (5.7) has a solution. Suppose Y_1 and Y_2 both solve (5.7). Then $X = Y_1 - Y_2$ solves the homogeneous initial value problem

$$LX = 0, \quad X(t_0) = 0, \quad X^{\Delta}(t_0) = 0;$$

by Theorem 3.1, this has only the trivial solution $X = 0$. □

We will also be interested in analyzing the self-adjoint vector dynamic equation

$$Lx = 0, \quad \text{where} \quad Lx(t) := \left(Px^{\Delta}\right)^{\nabla}(t) + Q(t)x(t), \quad t \in [a, \omega)_{\mathbb{T}}, \tag{5.8}$$

where x is an $n \times 1$ vector-valued function defined on $\mathbb{T}$ such that x^{Δ} is continuous and $(Px^{\Delta})^{\nabla}$ is left-dense continuous on $[a, \omega)_{\mathbb{T}}$. We will see interesting relationships between the so-called unique two-point property (defined below) of the nonhomogeneous vector equation $Lx = h$, disconjugacy of $Lx = 0$, and the construction of recessive solutions to the matrix equation $LX = 0$. The following theorem can be proven by modifying the proof of Theorem 5.5 and its corollary.

Theorem 5.6 *Let h be an $n \times 1$ vector function that is left-dense continuous on $[t_0, \omega)_{\mathbb{T}}$. If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_{\mathbb{T}}$, then the nonhomogeneous vector initial value problem*

$$Ly = (Py^{\Delta})^{\nabla} + Qy = h, \quad y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_0^{\Delta} \tag{5.9}$$

has a unique solution.

Definition 5.3 Assume h is an $n \times 1$ left-dense continuous vector function on $[t_0, \omega)_{\mathbb{T}}$. Then the vector dynamic equation $Lx = h$ has the unique two-point property on $[t_0, \omega)_{\mathbb{T}}$ provided given any $t_0 \le t_1 < t_2$ in $\mathbb{T}$, if u and v are solutions of $Lx = h$ with $u(t_1) = v(t_1)$ and $u(t_2) = v(t_2)$, then $u \equiv v$ on $[t_0, \omega)_{\mathbb{T}}$.

Theorem 5.7 *If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_{\mathbb{T}}$, and if the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_{\mathbb{T}}$, then the boundary value problem*

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta,$$

where $t_0 \le t_1 < t_2$ in $\mathbb{T}$ and $\alpha, \beta \in \mathbb{C}^n$, has a unique solution on $[t_0, \omega)_{\mathbb{T}}$.

Proof If t_1 is a right-scattered point and $t_2 = \sigma(t_1)$, then the boundary value problem is an initial value problem and the result holds by Theorem 5.6. Assume $t_2 > \sigma(t_1)$. Let $X(t, t_1)$ and $Y(t, t_1)$ be the unique $n \times n$ matrix solutions of (3.1) determined by the initial conditions

$$X(t_1, t_1) = 0, \quad X^{\Delta}(t_1, t_1) = I, \quad \text{and} \quad Y(t_1, t_1) = I, \quad Y^{\Delta}(t_1, t_1) = 0;$$

by variation of constants, Theorem 5.5,

$$X(t, t_1) = X(t) \int_{t_1}^{t} (X^* P X^{\sigma})^{-1}(\tau) \Delta\tau X^*(t_1) P(t_1)$$

and

$$Y(t, t_1) = X(t) X^{-1}(t_1) - X(t) \int_{t_1}^{t} (X^* P X^{\sigma})^{-1}(\tau) \Delta\tau X^{\Delta *}(t_1) P(t_1).$$

Then a general solution of (5.8) is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta, \tag{5.10}$$

for $\gamma, \delta \in \mathbb{C}^n$, as $x(t_1) = \delta$ and $x^\Delta(t_1) = \gamma$. By the unique two-point property the homogeneous boundary value problem

$$Lx = 0, \quad x(t_1) = 0, \quad x(t_2) = 0$$

has only the trivial solution. For x given by (5.10), the boundary condition at t_1 implies that $\delta = 0$, and the boundary condition at t_2 yields

$$X(t_2, t_1)\gamma = 0;$$

by uniqueness and the fact that x is trivial, $\gamma = 0$ is the unique solution, meaning $X(t_2, t_1)$ is invertible. Next let v be the solution of the initial value problem

$$Lv = h, \quad v(t_1) = 0, \quad v^\Delta(t_1) = 0.$$

Then the general solution of $Lx = h$ is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta + v(t).$$

We now show that the boundary value problem

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta$$

has a unique solution. The boundary condition at t_1 implies that $\delta = \alpha$. The condition at t_2 leads to the equation

$$\beta = X(t_2, t_1)\gamma + Y(t_2, t_1)\alpha + v(t_2);$$

since $X(t_2, t_1)$ is invertible, this can be solved uniquely for γ. □

Corollary 5.2 *If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_\mathbb{T}$, and if the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_\mathbb{T}$, then the matrix boundary value problem*

$$LX = 0, \quad X(t_1) = M, \quad X(t_2) = N$$

has a unique solution, where M and N are given constant $n \times n$ matrices.

Proof Modify the proof of Theorem 5.7 to get existence and uniqueness. □

Theorem 5.8 *Assume the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_\mathbb{T}$, and the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_\mathbb{T}$. Further assume U is a solution of* (3.1) *which is recessive at ω with $U(t_0)$ invertible. For each fixed $s \in (t_0, \omega)_\mathbb{T}$, let $Y(t, s)$ be the solution of the boundary value problem*

$$LY(t, s) = 0, \quad Y(t_0, s) = I, \quad Y(s, s) = 0.$$

Then the recessive solution $U(t)U^{-1}(t_0)$ is uniquely determined by

$$U(t)U^{-1}(t_0) = \lim_{s \to \omega} Y(t, s). \tag{5.11}$$

Proof Assume U is a solution of (3.1) which is recessive at ω with $U(t_0)$ invertible. Let V be the unique solution of the initial value problem

$$LV = 0, \quad V(t_0) = 0, \quad V^{\Delta}(t_0) = P^{-1}(t_0).$$

By the connection theorem, Theorem 5.4, V is invertible for large t. By checking boundary conditions at t_0 and s for s large, we get that

$$Y(t,s) = -V(t)V^{-1}(s)U(s)U^{-1}(t_0) + U(t)U^{-1}(t_0).$$

Then

$$W(V,U) = W(V,U)(t_0) = (V^*PU^{\Delta} - V^{\Delta *}PU)(t_0) = -U(t_0)$$

is invertible, and by the recessive nature of U,

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0.$$

As a result,

$$\lim_{s\to\omega} Y(t,s) = 0 + U(t)U^{-1}(t_0),$$

and the proof is complete. $\square$

Definition 5.4 A prepared vector solution x of (5.8) has a generalized zero at a iff $x(a) = 0$, and x has a generalized zero at $t_0 > a$ iff $x(t_0) = 0$, or if t_0 is a left-scattered point and $x^{*\rho}(t_0)P^{\rho}(t_0)x(t_0) < 0$. Equation (5.8) is disconjugate on $[a,\omega)_{\mathbb{T}}$ iff no nontrivial prepared vector solution of (5.8) has two generalized zeros in $[a,\omega)_{\mathbb{T}}$.

Definition 5.5 A prepared basis X of (3.1) has a generalized zero at a iff $X(a)$ is noninvertible, and X has a generalized zero at $t_0 \in (a,\omega)_{\mathbb{T}}$ iff $X(t_0)$ is noninvertible, or $X^{*\rho}(t_0)P^{\rho}(t_0)X(t_0)$ is invertible but $X^{*\rho}(t_0)P^{\rho}(t_0)X(t_0) \le 0$.

Lemma 5.4 *If a prepared basis X of* (3.1) *has a generalized zero at $t_0 \in [a,\omega)_{\mathbb{T}}$, then there exists a vector $\gamma \in \mathbb{C}^n$ such that $x = X\gamma$ is a nontrivial prepared solution of* (5.8) *with a generalized zero at t_0.*

Proof The proof follows from Definitions 5.4 and 5.5. $\square$

Lemma 5.5 *If f and g are continuous on $[t_0,\omega)_{\mathbb{T}}$, then*

$$\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s = \int_{t_0}^{t} f(s)g^{\sigma}(s)\Delta s, \qquad t \in [t_0,\omega)_{\mathbb{T}}.$$

Proof Set

$$F(t) := \int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s - \int_{t_0}^{t} f(s)g^{\sigma}(s)\Delta s;$$

clearly $F(t_0) = 0$, and

$$F^{\Delta}(t) = \left[\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s\right]^{\Delta} - f(t)g^{\sigma}(t).$$

Using Theorem 2.2 (iii) and the set B in (2.2),

$$\left[\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s\right]^{\Delta} = \begin{cases} (f^{\rho}g)(\sigma(t)) & : t \in \mathbb{T}\backslash B, \\ \lim_{s\to t^+}(f^{\rho}g)(s) & : t \in B. \end{cases}$$

For $t \in \mathbb{T}\backslash B$, $\rho(\sigma(t)) = t$, so that $(f^{\rho}g)(\sigma(t)) = (fg^{\sigma})(t)$. For $t \in B$, $t = \sigma(t)$ and $\lim_{s\to t^+}\rho(s) = t$, yielding

$$\lim_{s\to t^+}(f^{\rho}g)(s) = (fg)(t) = (fg^{\sigma})(t).$$

Thus in either case $F^{\Delta}(t) = 0$. By the uniqueness property, $F \equiv 0$, and the result follows. $\square$

Theorem 5.9 *If the vector equation* (5.8) *is disconjugate on* $[\rho(t_0),\omega)_{\mathbb{T}}$*, then the matrix equation* (3.1) *has a solution* V *which is dominant at* ω *and a solution* U *which is recessive at* ω*, with* V *and* U *both invertible such that* $PV^{\Delta}V^{-1} > PU^{\Delta}U^{-1}$ *on* $(\sigma(t_0),\omega)_{\mathbb{T}}$.

Proof Let X be the solution of the initial value problem

$$LX = 0, \quad X^{\rho}(t_0) = 0, \quad X^{\Delta\rho}(t_0) = I.$$

If X is not invertible on $(t_0,\omega)_{\mathbb{T}}$, then there exists a $t_1 > t_0$ such that $X(t_1)$ is singular. But then there exists a nontrivial vector $\delta \in \mathbb{C}^n$ such that $X(t_1)\delta = 0$. If $x(t) := X(t)\delta$, then x is a nontrivial prepared solution of (5.8) with

$$x^{\rho}(t_0) = 0, \quad x(t_1) = 0,$$

a contradiction of disconjugacy. Hence X is invertible in $(t_0,\omega)_{\mathbb{T}}$. We next claim that

$$(X^{*\rho}P^{\rho}X)(t) > 0, \quad t \in (\sigma(t_0),\omega)_{\mathbb{T}}; \tag{5.12}$$

if not, there exists $t_2 \in (\sigma(t_0),\omega)_{\mathbb{T}}$ such that

$$(X^{*\rho}P^{\rho}X)(t_2) \not> 0.$$

It follows that there exists a nontrivial vector γ such that $x(t) := X(t)\gamma$ is a nontrivial prepared vector solution of $Lx = 0$ with a generalized zero at t_2. Using the initial condition for X, however, we have $x^{\rho}(t_0) = 0$, another generalized zero, a contradiction of the assumption that the vector equation (5.8) is disconjugate on $[\rho(t_0),\omega)_{\mathbb{T}}$. Thus (5.12) holds, in particular for any $t_2 \in (\sigma(t_0),\omega)_{\mathbb{T}}$. Define for $t \in [t_2,\omega)_{\mathbb{T}}$

$$V(t) := X(t)\left[I + \int_{t_2}^{t}(X^*PX^{\sigma})^{-1}(s)\Delta s\right] = X(t)\left[I + \int_{t_2}^{t}(X^{*\rho}P^{\rho}X)^{-1}(s)\nabla s\right],$$

where the second equality follows from Lemma 5.5. By Theorem 3.5, V is a prepared solution of $LV = 0$ with $W(X,V) = I$. Note that V is also invertible on $[t_2,\omega)_{\mathbb{T}}$, so that by the reduction of order theorem again,

$$X(t) = V(t)\left[I - \int_{t_2}^{t}(V^*PV^{\sigma})^{-1}(s)\Delta s\right], \quad t \in [t_2,\omega)_{\mathbb{T}}.$$

Consequently,

$$I=[V^{-1}(t)X(t)][X^{-1}(t)V(t)]=\left[I-\int_{t_2}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s\right]\left[I+\int_{t_2}^{t}(X^*PX^\sigma)^{-1}(s)\Delta s\right].$$

Since the second factor is strictly increasing and bounded below by I, the first factor is positive definite and strictly decreasing, ensuring the existence of a limit, in other words, we have

$$0 \le I-\int_{t_2}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s < I-\int_{t_2}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s \le I.$$

It follows that

$$0 \le \int_{t_2}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s < \int_{t_2}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s \le I, \quad t\in[t_2,\omega)_{\mathbb{T}}, \tag{5.13}$$

and V is a dominant solution of (3.1) at ω. Set

$$U(t) := V(t)\int_{t}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s.$$

By Theorem 5.1, U is a recessive solution of (3.1) at ω, and $W(U,V)=I$. Since

$$U(t)=V(t)\left[\int_{t_2}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s-\int_{t_2}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s\right],$$

V is invertible on $[t_2,\omega)_{\mathbb{T}}$, and the difference in brackets is positive definite on $[t_2,\omega)_{\mathbb{T}}$, we get that U is invertible on $[t_2,\omega)_{\mathbb{T}}$ as well. Then on $[t_2,\omega)_{\mathbb{T}}$, we have

$$\begin{aligned}
PV^\Delta V^{-1}-PU^\Delta U^{-1} &= U^{*-1}U^*PV^\Delta V^{-1}-X^{*-1}X^{\Delta *}PVV^{-1}\\
&= U^{*-1}\left[U^*PV^\Delta-U^{\Delta *}PV\right]V^{-1}\\
&= U^{*-1}\left[W(U,V)\right]V^{-1}UU^{-1}\\
&= U^{*-1}\left[V^{-1}U\right]U^{-1}\\
&= U^{*-1}\left[\int_{t}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s\right]U^{-1}>0
\end{aligned}$$

by (5.13). Since t_2 in $(\sigma(t_0),\omega)_{\mathbb{T}}$ arbitrary, the conclusions of the theorem follow. $\square$

Corollary 5.3 *Assume the vector equation* (5.8) *is disconjugate on* $[\rho(t_0),\omega)_{\mathbb{T}}$, *and* K *is a constant Hermitian matrix. Let* U,V *be the matrix solutions of* $LX=0$ *satisfying the initial conditions*

$$U(t_2)=I, \quad U^\Delta(t_2)=P^{-1}(t_2)K, \quad \text{and} \quad V(t_2)=0, \quad V^\Delta(t_2)=P^{-1}(t_2)$$

for any $t_2\in(\sigma(t_0),\omega)_{\mathbb{T}}$. *Then* V *is invertible in* $(\sigma(t_2),\omega)_{\mathbb{T}}$, V *is a dominant solution of* (3.1) *at* ω, *and*

$$\lim_{t\to\omega}V^{-1}(t)U(t)$$

exists as a Hermitian matrix.

Proof By Theorem 5.9, the matrix equation (3.1) has a solution U which is recessive at ω with $U(t)$ invertible for $t \in [t_2, \omega)_{\mathbb{T}}$. Thus (iii) of the connection theorem, Theorem 5.4 holds; by (i), then, V is a dominant solution of (3.1) at ω, and by (ii),

$$\lim_{t \to \omega} V^{-1}(t)U(t)$$

exists as a Hermitian matrix. Since $V(t_2) = 0$ and the vector equation (5.8) is disconjugate on $[\rho(t_0), \omega)_{\mathbb{T}}$,

$$\left(V^{*\rho} P^{\rho} V\right)(t) > 0, \quad t \in (\sigma(t_2), \omega)_{\mathbb{T}}.$$

In particular, V is invertible in $(\sigma(t_2), \omega)_{\mathbb{T}}$. □

Theorem 5.10 *If the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*, then* $Lx(t) = h(t)$ *has the unique two-point property in* $[t_0, \omega)_{\mathbb{T}}$*. In particular, every boundary value problem of the form*

$$Lx(t) = h(t), \qquad x(\tau_1) = \alpha, \qquad x(\tau_2) = \beta,$$

where $\tau_1, \tau_2 \in [t_2, \omega)_{\mathbb{T}}$ *for* $t_2 \in (\sigma(t_0), \omega)_{\mathbb{T}}$ *with* $\tau_1 < \tau_2$*, and where* α, β *are given* n*-vectors, has a unique solution.*

Proof By Theorem 5.9, disconjugacy of (5.8) implies the existence of a prepared, invertible matrix solution of (3.1). Thus by Theorem 5.7, it suffices to show that (5.8) has the unique two-point property in $[t_2, \omega)_{\mathbb{T}}$. To this end, assume u, v are solutions of $Lx = 0$, and there exist points $s_1, s_2 \in \mathbb{T}$ such that $t_2 \le s_1 < s_2$ and

$$u(s_1) = v(s_1), \qquad u(s_2) = v(s_2).$$

If s_1 is a right-scattered point and $s_2 = \sigma(s_1)$, then u and v satisfy the same initial conditions and $u \equiv v$ by uniqueness; hence we assume $s_2 > \sigma(s_1)$. Setting $x = u - v$, we see that x solves the initial value problem

$$Lx = 0, \qquad x(\tau_1) = 0, \qquad x(\tau_2) = 0.$$

Since $Lx = 0$ is disconjugate and x is a prepared solution with two generalized zeros, it must be that $x \equiv 0$ in $[t_2, \omega)_{\mathbb{T}}$. Consequently, $u = v$ and the two-point property holds. □

Corollary 5.4 (Construction of the recessive solution) *Assume the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*. For each* $s \in (t_0, \omega)_{\mathbb{T}}$*, let* $U(t, s)$ *be the solution of the boundary value problem*

$$LU(\cdot, s) = 0, \quad U(t_0, s) = I, \quad U(s, s) = 0.$$

Then the solution U *with* $U(t_0) = I$ *which is recessive at* ω *is given by*

$$U(t) = \lim_{s \to \omega} U(t, s),$$

satisfying

$$(U^{*\rho} P^{\rho} U)(t) > 0, \quad t \in [t_0, \omega)_{\mathbb{T}}. \tag{5.14}$$

Proof By Theorem 5.9 and Theorem 5.10, $LX = 0$ has a recessive solution and $Lx = h$ has the unique two-point property. The conclusion then follows from Theorem 5.8, except for (5.14). From the boundary condition $U(s,s) = 0$ and the fact that $Lx = 0$ is disconjugate, it follows that $U^*(\rho(t), s)P^\rho(t)U(t,s) > 0$ holds in $[t_0, s)_\mathbb{T}$. Again from Theorem 5.8,

$$\lim_{s\to\omega} U(t,s) = U(t)U^{-1}(t_0) = U(t),$$

so that U invertible on $[t_0, \omega)_\mathbb{T}$ and (5.14) holds. □

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